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Analysis and Design of Controllers for a Double Inverted Pendulum

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Abstract

A physical control problem is studied with the μ methodology. The issues of modelling, uncertainty modelling, performance specification, controller design and laboratory implementation are discussed. The laboratory experiment is a double inverted pendulum placed on a cart. The limitations in the system with respect to performance are the limitation in the control signal and the limitation of the movement of the cart. It is shown how these performance limitation will effect the design of μ controller for the system.

1 Introduction

The pendulum system is one of the classical examples used in connection with feedback control. The single inverted pendulum is a standard example in many text books dealing with classical as well as modern control. The reason is that the system is quite simple, non-linear and unstable. In connection with the classical control, the single inverted pendulum system has among other thing been used to show that the system cannot be stabilized by using just a P controller. In spite of that the system is unstable, the design of stabilizing controllers for the system can be done reasonable easy. However, this is not the case when considering the quite more complicated double inverted pendulum system. It is quite more complicated to design/tune stabilizing controllers for the system. Therefore, more advanced controller architectures and advanced design methods can be applied with advantage. This involved different types of model based controllers designed by using e.g. $\mathcal{H}_2$ based methods, $\mathcal{H}_\infty$ based methods and μ based methods.

The main problem in the design of stabilizing controllers for the double inverted pendulum system is the trade-off between robust stability and performance. This trade-off is quite limited, there is not much space for reduction of the robustness to increase the performance of the system. The reason is the nonlinearities in the system together with the limitations/saturations in the system. The limitations in the system are e.g. maximal power to the motor (maximal acceleration of the cart), maximal length of the track for mention the two most important limitations. In spite of this limited trade-off between robustness and performance of the system, it is possible to design controllers that can handle this trade-off in a systematic way. Design methods as e.g. $\mathcal{H}_2$ based methods, $\mathcal{H}_\infty$ based methods and μ based methods can be applied for handling this trade-off in a systematic way.

This paper describe a complete design procedure for the design of advanced stabilizing controllers for the double inverted pendulum system together with an implementation of the controllers on a laboratory system. This lead to the following items that will be considered in the following: System modelling, system analysis, design problem formulation, uncertainty modelling, controller design, analysis of the closed loop system, implementation of controllers on a microcontroller, validation of the closed loop system on the laboratory system.

Due to the space limitation, the first three items will only be considered shortly. A more detailed description of these parts can be found in [3].

2 Model of a Double Inverted Pendulum

In the following, a short description of the double inverted pendulum system is given. Both the nominal as well as a real laboratory model are considered. A more detailed description can be found in [3]. The construction of the pendulum system is described in [1].

2.1 Description of the System
The double inverted pendulum consist of a cart placed on a track, and two aluminium arms connected to each other. These are constrained to rotate within a single

plane. The axis of the rotation is perpendicular to the direction of the motion of the cart. The cart is attached to the bottom of the pendulum, and moving along a linear low friction track. The cart is moved by an exerting force by a servo motor system. A principal structure of the pendulum system is shown in Figure 1, where the forces acting on the system has been included.
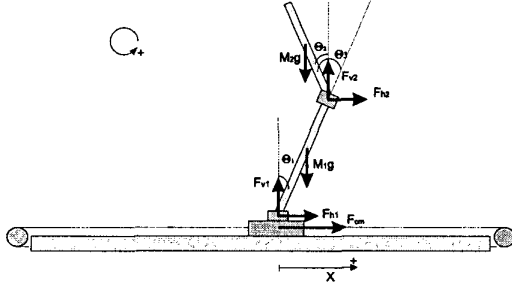

Figure 1: Principal diagram of the double inverted pendulum system.

Some data for the complete system are as follows:
mass of cart, $m = 0.81$kg, length of track, $l_t = 1.34$m, mass of lower arm, $m_1 = 0.548$kg, length of lower arm, $l_1 = 0.535$m, length of lower arm from bottom to center of mass, $l_{1cm} = 0.355$m, inertia of lower arm around the lower joint, $I_1 = 2.678e^{-2}kgm^2$, mass of upper arm, $m_2 = 0.21$kg, length of upper arm, $l_2 = 0.512$m, length of upper arm from bottom to center of mass, $l_{2cm} = 0.12$m, inertia of upper arm around the lower joint, $I_2 = 5.217e^{-3}kgm^2$.

A nonlinear model for the complete system can be derived by using Newtons 2. low and 3. low on every part of the system. A detailed description of the nonlinear model can be found in [3]. Based on this nonlinear model, a linear model can be derived by a linearization of the nonlinear model around the working point. The linear model Σ_G for the complete system can be described by the following state space description

$$\Sigma_G : \begin{cases} \dot{x} = Ax + B_w w + B_u u \\ y = C_y x + D_{yw} w + D_{yu} u \end{cases}$$
(2.1)

where x is the state, w is the exogenous inputs, u is the control input and y is the measurement output. The linear model is of order 7 with the following states:

$$x = [\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2 \ x_c \ \dot{x}_c \ i]^T$$
(2.2)

where θ_1 is the angle between vertical and the lower arm, $\dot{\theta}_1$ is the angular velocity related to θ_1, θ_2 is the angle between vertical and the upper arm, $\dot{\theta}_2$ is the angular velocity related to θ_2, x_c is the cart position, $\dot{x}_c$ is the velocity of the cart and i is the motor current.

The exogenous input vector are given by

$$w = [r_c \ M_{d_1} \ M_{d_2} \ n_1 \ n_3 \ M_{dm} \ n_x]^T$$
(2.3)

where r_c is the cart position reference, M_{d_1} and M_{d_2} are the torque disturbance on the joint on the lower arm and on the upper arm, respectively, n_1 and n_3 are noise signal in measuring θ_1 and $\theta_3 = \theta_1 - \theta_2$, respectively, M_{dm} is the torque disturbance on the motor, and n_x is the noise signal in the measuring of the cart position x_c.

The measurement vector y is given by

$$y = [e_c \ \theta_1 \ \theta_3]^T$$
(2.4)

where e_c is the cart position error $r_c - x_c$.

The pendulum system is an unstable system with the following open loop poles:

$$\text{poles} = \{0, \ -251, \ -7441, \ -6.4, \ 6.4, \ -4.1, \ 4.1\}$$

2.2 Formulation of Nominal Design Problem
Based on the system setup in Section 2.1, we can now formulate the following main conditions to the pendulum system.

- The cart position have to be close to a given reference signal r_c, with limit cycles as small as possible. This means that the goal is to minimize $e_x = r_c - x$.

- The lower arm have to be as close as possible to vertical, while still allowing the cart to move. This means that the angle θ_1 are to be minimized.

- The upper arm are to be as close to vertical as possible. Meaning that the angle θ_2 are minimized, which in terms of system outputs means that $\theta_1 - \theta_3$ are minimized.

This type of control problem is often referred to as a tracking problem, as in [4]. When designing a controller the performance and the bandwidth available have a major influent, and therefore it is important to include the controller signal U as an exogenous output. Selecting the signal U as an output gives the possibility to limit the bandwidth and magnitude of the designed controllers.

Since the motor system is an isolated closed loop system it is not easy to influence its dynamics with the higher level controller. The only way to give the controller access to the motor system dynamics, is to select one of its signals as an exogenous output. A good reason to select the current i as the output is that it is proportional to the generated motor torque.

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The exogenous output vector is selected as

$$z = \begin{bmatrix} e_x & \theta_1 & \theta_2 & U & i \end{bmatrix}^T \qquad (2.5)$$

When the exogenous outputs are included in the state space setup the nominal system Σ_P are defined by

$$\Sigma_P : \begin{cases} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ v &= C_2 x + D_{21} w + D_{22} u \end{cases} \qquad (2.6)$$

where the exogenous input w is define in Section 2.1.

2.3 Model Uncertainties

The model of the real system include a number of uncertainties. These uncertainties can be split up into two groups:

- parameter uncertainty

- neglected linear, nonlinear and unmodelled dynamics uncertainty

It is easy to identify a number of uncertain parameters. The parameter uncertainties can be caused by parameters are difficult or impossible to get a precise measure of or that the parameters tent to vary as function of time, temperature etc. For this pendulum system, the main uncertain parameters are the Coulomb friction constants.

With respect to neglected dynamics, the pendulum system does have unmodelled dynamics like bearings, track inclination, all kinds of high frequency dynamics etc. Further, the system include also nonlinear elements/dynamic. The nonlinear dynamic appear from the special the *sine* and *cosine* functions in the nonlinear model. However, the system will only work with small angles, which will reduce the nonlinear effect from the *sine* and *cosine* functions.

From simulations, it was found that the system is especially sensitive to offsets on the angular measurements and in general any kind of disturbance on the measurements. Since it is not possible to make an uncertainty model of offsets, because of its nonlinear nature, it has been modelled as a general uncertainty. This implies that it will be reasonable to model the uncertainty as multiplicative output uncertainty. Thereby trying to model all the uncertainty that influence the measured angles with a simple model; this is called lumping uncertainty. The multiplicative output uncertainty is described by

$$G_p = (I + W_o \Delta_o) G \qquad (2.7)$$

where the perturbation matrix Δ_o is given as either structured or unstructured and satisfies $\|\Delta_o\|_\infty \leq 1$ as in [4]

and where W_o is a weight that indicate a percentage of error as function of frequency. A unstructured perturbation matrix is a matrix with complex numbers in all elements. Where as a structured perturbation matrix is a block diagonal matrix with either complex or real elements. Due to the fact that all uncertainties are lumped into a single block, it will not give any meaning to consider specific parameter uncertainties, neglected dynamics etc.

3 Controller Design

Controllers has been designed by using the μ synthesis. The setup for the designs are based on the nominal setup described in Section 2.2 and the uncertain description for the system given in Section 2.3. Combining the the nominal design setup with the uncertain model description gives the complete design setup shown in Figure 2, where W_P is the weight matrix for the performance specification and W_o is the weight matrix for the multiplicative output uncertainty.
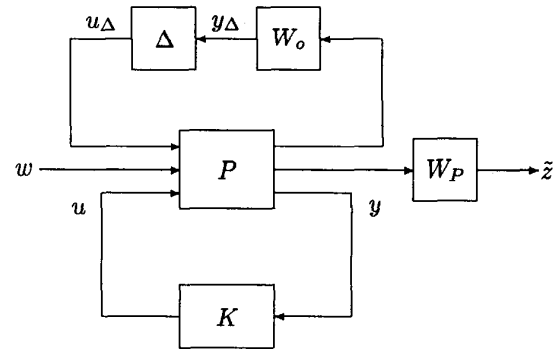

Figure 2: The complete system setup for design of robust feedback controllers

Note that only weights at the outputs are applied in connection with the performance specifications (i.e. weight at the output z) are included in Figure 2. However, weights at the external input w can be included without any problem if needed.

The two weight matrices are given as diagonal matrices, i.e.

$$\begin{aligned} W_P &= diag(W_e, W_{\theta_1}, W_{\theta_2}, W_U, W_i) \\ W_o &= diag(W_{o_1}, W_{o_2}) \end{aligned}$$

The weight matrices for the μ controller design has been

selected as follows:

$$W_e = \frac{75}{30s+1} \quad W_{\theta_1} = \frac{68}{s+10}$$

$$W_{\theta_2} = \frac{40}{s+10} \quad W_U = 0.1\frac{s+100}{0.01s+100}$$

$$W_i = 1.0 \quad W_{o_1} = 0.05$$

$$W_{o_2} = 0.05$$

The controller design based on the weight selection given above result a μ controller of order 19 (after 4 iterations).

Note that constant weights has been applied for the multiplicative uncertainty in the system.

4 Controller Analysis

The μ controller has been designed by using the DK-iteration, [5]. The weights for the μ design is so the performance of the closed loop system is optimized. The μ value for the four iterations in the DK-iteration is shown in Figure 3. It is shown clearly in Figure 3, that the μ value is reduced from the first iteration to the last iteration with a final value of μ below 1.
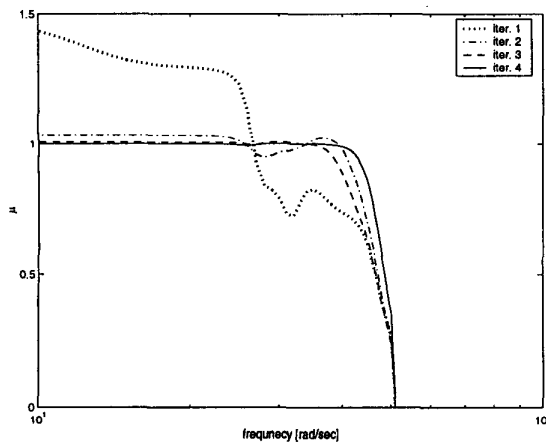

Figure 3: The μ - values for the four steps in the DK-iteration.

The four transfer functions between the uncertainty model input signal u_Δ and the uncertainty model output signal y_Δ when the μ controller is applied are shown in Figure 4.

A simulation of the closed loop system is shown in Figure 5. From Figure 5, it can be seen that the cart position reference tracking is slow. It does not settle around $x = 0$ within the 5 sec. of the simulation. The system
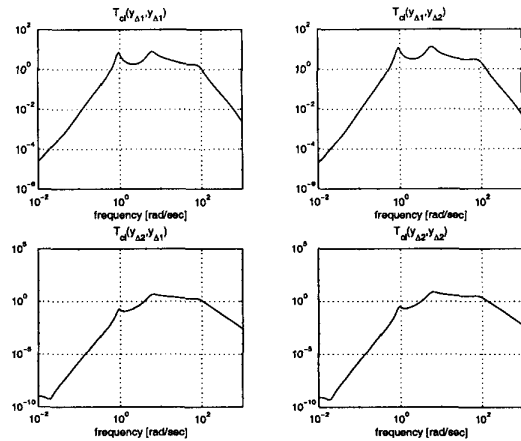

Figure 4: Transfer functions for the angles between the uncertainty model input signal u_Δ and the uncertainty model output signal y_Δ when the μ controller is applied.

is also very sensitive to the angles, showing large overshots but very fast settling time. The limit cycles are very small.
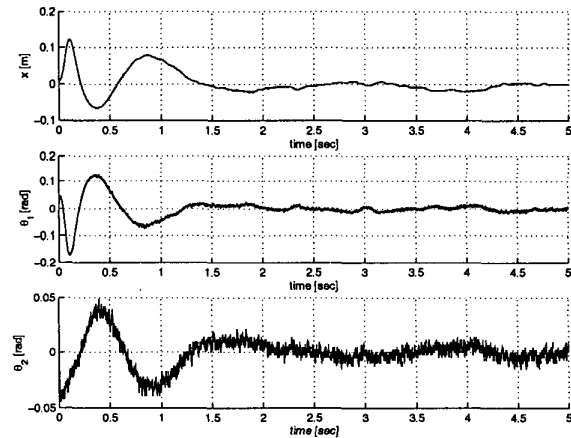

Figure 5: Simulation of the nonlinear system with the μ controller. The initial conditions are: $\theta_1 = 0.05rad$ and $\theta_2 = -0.04rad$, similar to what would happen for the lab. model.

A simulation of the initial angles from which the μ controller stabilize the system has been derived. The result is shown in Figure 6.

It turns out that the final μ controller is of order 19. Therefore, a model reduction is needed to reduce the order of the controller. The reduction of the controller is derived by using the optimal Hankel norm approxi-
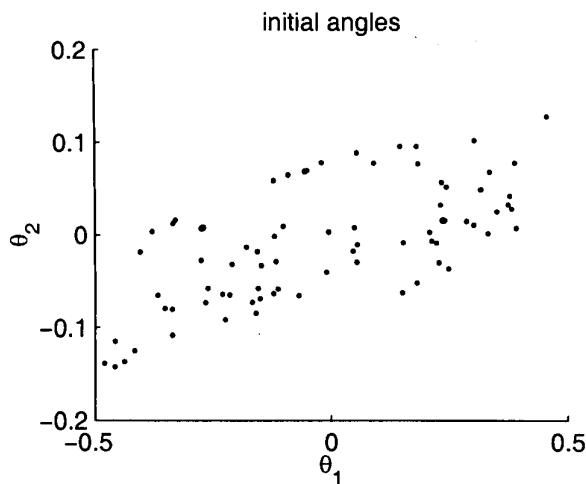
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Figure 6: Simulation of the initial angles from which the μ controller can stabilize the system. The contour formed by the dots, shows the area from where the closed loop system can be started in order to be stabilized.

mation, see e.g. [4]. The result of the reduction of the controller order is shown in Figure 7. A reduction of the controller order to order 6 or 7 does not have any serious effect on the performance of the system. A 7th order controller is implemented on the laboratory model.

5 Laboratory Experiments

The μ controller has been implemented as discrete-time controllers on a microcontroller. The microcontroller used here is a Motorola 68040 computer running with the real time operating system OS9. The microcontroller have an interface card with multiple 12 bit AD/DA converters. The converters have a range of ± 5V. Maximal sampling rate $f_s = 500Hz$ or a sampling period of $T_s = 2msec$.

The controller is implemented in Simulink. The AD inputs and the DA outputs are implemented in the Simulink library and can easily be used any other inputs and outputs. Using the Real-Time Workshop (RTW) Toolbox, the Simulink model is the compiled in to OS9 executable code.

The continuous-time controller is transformed into an equivalent discrete-time controller by using the bilinear Tustin transformation given by:
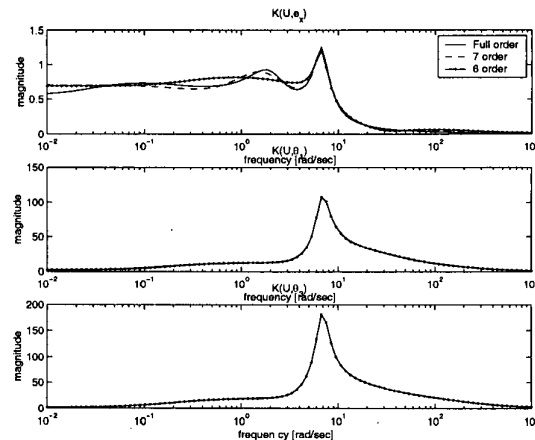
$$s = \frac{2}{T_s} \frac{z-1}{z+1}$$

Figure 7: Controller transfer functions for the full order controller and the two reduced controllers.

The fastest sampling period has been selected, i.e. $T_s = 2msec$.

The result of the laboratory experiments are shown in Figure 8 for the four different controllers. Note that in all four cases, the system is started up with very small angles, i.e. $\theta_1 \approx 0$ and $\theta_2 \approx 0$.
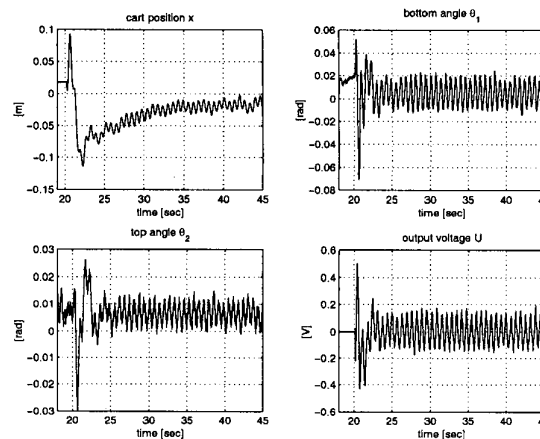

Figure 8: Sampled data from laboratory model, running with the μ controller.

The μ controlled system has a quite good performance, see Figure 8, and the robustness of the closed-loop system is also reasonable, see e.g. Figure 6. The tracking of the cart position is quite slow for the μ controlled system.

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6 Conclusion

A complete design and implementation of controllers for an unstable system has been described in this paper. A linear model of a double inverted pendulum system together with a description of the model uncertainties has been derived. A complete nonlinear model has been implemented in Simulink for use in connection with simulation of the system. A controller has been designed using the μ synthesis method. In [2, 3], three different controllers has also been designed by using the $\mathcal{H}_\infty$ and compared with the μ controller.

For simplifying the controller design, a graphical user interface (PCDT) has been derived in MATLAB. This PCDT interface handle the selection of the weights for the controller design, the controller design, simulation and animation of the system and finally, dump the controller data to the workspace for laboratory experiments. Finally, the designed controllers has been implemented on a micro controller for laboratory experiments.

The four designed controllers has been validated both by simulation and calculation of various transfer functions as well as by laboratory experiments. In both the simulation as well as in the laboratory experiments, the trade-off between performance of the closed-loop system and robustness is shown clearly. However, the trade-off between performance and robustness for this system is very limited. The unstable double inverted pendulum system need to be stabilized with a certain stability margin, else it is impossible to start the laboratory system up. This leaves only a minor freedom for the trade-off between performance and robustness.

An extended version of this paper can be found in [2] with a complete description of the four designed controllers.

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