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Closed-Form Solution for a Class of Discrete-Time Algebraic Riccati Equations

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***Abstract*— In the present paper we obtain a closed-form solution for the class of discrete-time algebraic Riccati equations (ARE) with vanishing state weighting, whenever the unstable eigenvalues are distinct. The AREs in such a class solve a minimum energy control problem for a single-input single-output (SISO) system. The obtained closed-form solution gives insight on issues such as loss of controllability and it might also prove competitive in terms of numerical precision over current solving algorithms.**

I. Introduction

The area of Control over Networks has been a growing topic of increased interest in recent years; see for example [1], [2] and references therein. A line of research reported in

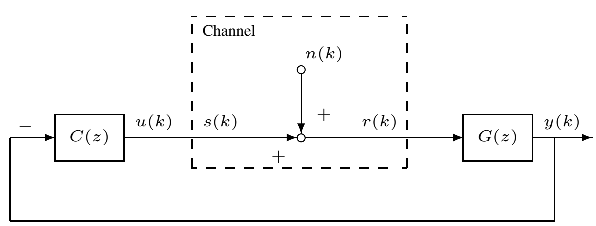

Fig. 1. General problem setting.

[3], [4] (and related work in [5], [6]), introduced a framework to study the fundamental limitation in stabilisability of a single-input single-output (SISO) feedback loop over channels with a signal to noise ratio (SNR) constraint. A distinctive characteristic of the SNR approach is that it is a linear formulation.

In Figure 1 we observe the particular case of an additive white Gaussian noise (AWGN) channel located between the controller and the plant. The channel is defined by $\mathcal{P}$, the channel input power constraint

$$\mathcal{P} > \mathcal{E}\left\{u^2\right\},$$

where $\mathcal{E}$ is the expectation, and by the channel additive white Gaussian noise $n(k)$, with zero-mean and variance σ^2. In [3] it has been proved that the AWGN channel infimal SNR for stabilisability satisfies

$$\frac{\mathcal{P}}{\sigma^2} > \prod_{i=1}^{m} \rho_i^2 - 1 \tag{1}$$

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where ρ_i are the (possibly repeated) unstable poles of the plant model $G(z)$. The same result also holds for output feedback, [3, Theorem III.2], when $G(z)$ is minimum phase and of relative degree one and for state-feedback, [3, Theorem III.1]. In the state-feedback case it is shown that the infimal SNR for stabilisability result is linked to the solution $\mathbf{P}$ of an algebraic Riccati equation (ARE) with vanishing state weighting for a state-space representation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ of $G(z)$

$$\frac{\mathcal{P}}{\sigma^2} > \mathbf{B}^T\mathbf{P}\mathbf{B}.$$

Both results are equivalent, suggesting that it might be possible to express $\mathbf{P}$ in closed-form.

In [4] the infimal SNR for stabilisability result is extended to include a discrete-time unstable (non)-minimum phase plant $G(z)$ with distinct unstable poles, over an additive coloured Gaussian noise (ACGN) channel with memory. For the AWGN channel (which can be seen as a particular case of an ACGN channel) [4, Theorem 2] reduces to

$$\frac{\mathcal{P}}{\sigma^2} > \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{r_i r_j}{\rho_i\rho_j - 1},$$

where

$$r_i = (1-\rho_i^2)\prod_{\substack{j=1\\ j\neq i}}^{m} \frac{1-\rho_i\rho_j}{\rho_i - \rho_j},\ \forall i = 1,\cdots,m.$$

Thus, motivated by the insight that the infimal SNR for stabilisability problem can be stated as a minimum energy problem, we analyse in the present paper the class of discrete-time AREs with vanishing state weighting.

The main contribution of the present paper, based on the previous infimal SNR for stabilisability results above, is a closed-form solution for such a class of discrete-time AREs with non repeated unstable eigenvalues. To the best knowledge of the author the closed-form solution obtained here is novel. As a result of the closed-form nature of the solution, we obtain further insights on the structure of the minimum energy problem.

The paper is organised as follow: in Section II we derive the closed-form solution for the class of ARE with vanishing state weighting, when the unstable poles of the plant are all distinct. In Section III we present the final remarks for the present work and future directions. For completeness we

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present in the Appendix two lemmas required for the proof of the main result.

Terminology: let $\mathbb{C}$ denote the complex plane. Let $\mathbb{D}^-$, $\bar{\mathbb{D}}^-$, $\mathbb{D}^+$ and $\bar{\mathbb{D}}^+$ denote respectively the open unit-disk, closed unit-disk, open unit disk complement and closed unit disk complement in the complex plane $\mathbb{C}$, with $\partial\mathbb{D}$ the unit-disk itself. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{R}_o^+$ the set of non-negative real numbers and $\mathbb{R}^-$ the set of real negative numbers. Let $\mathbb{Z}^+$ denote the set of positive integers. We use bold notation to represent a generic matrix $\mathbf{A}$. Similarly $\mathbf{0}$ stands for a matrix, of suitable dimensions, with all its entries set to zero. For the product of a pair of matrices $\mathbf{A}$ and $\mathbf{B}$, the transpose of the product is given by $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

II. Closed-Form Solution to the Minimum Energy Algebraic Riccati Equation

Consider the closed loop system over a SNR constrained AWGN channel shown in Figure 1 and the following set of assumptions

1) The plant model $G(z)$ has a minimal realisation $(\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ 0)$ such that
$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_2\end{bmatrix},\quad \mathbf{B} = \begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix},\quad \mathbf{C} = \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}, \tag{2}$$
where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times 1}$, $\mathbf{C} \in \mathbb{R}^{1\times n}$, $\mathbf{A}_1 \in \mathbb{R}^{m\times m}$, $\mathbf{B}_1 \in \mathbb{R}^{m\times 1}$, $\mathbf{C}_1 \in \mathbb{R}^{1\times m}$.
2) The eigenvalues of $\mathbf{A}_1$ are all in $\mathbb{D}^+$ and they are all distinct.
3) $\mathbf{A}_1$ is diagonal and $\mathbf{B}_1 = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}^T$.
4) The eigenvalues of $\mathbf{A}_2$ are all in $\bar{\mathbb{D}}^-$.

Notice that assumption 1) also implies that the pair $\mathbf{A}_1$ and $\mathbf{B}_1$ is controllable. Also notice that the choice of $\mathbf{A}_1$ and $\mathbf{B}_1$ in 3) is not restrictive. Indeed from 1) we have that
$$G(z) = \underbrace{\mathbf{C}_1\left(z\mathbf{I}-\mathbf{A}_1\right)^{-1}\mathbf{B}_1}_{G_1(z)} + \underbrace{\mathbf{C}_2\left(z\mathbf{I}-\mathbf{A}_2\right)^{-1}\mathbf{B}_2}_{G_2(z)} \tag{3}$$
where $G_1(z)$ contains all the unstable distinct poles of $G(z)$ and similarly $G_2(z)$ all the stable poles of $G(z)$. With the choice of $\mathbf{A}_1$ and $\mathbf{B}_1$ in 3), the coefficients of $\mathbf{C}_1$ can be identified from a partial fraction expansion of $G_1(z)$.

Example 1: consider the plant model $G(z) = \frac{(z-0.3)}{(z-0.1)(z-2)(z-7)}$. We recognise $G_1(z)$ and $G_2(z)$ as
$$G_1(z) = \frac{(0.0153z+0.8639)}{(z-2)(z-7)},\quad G_2(z) = \frac{-0.0153}{(z-0.1)} \tag{4}$$
From imposing $\mathbf{A}_1$ and $\mathbf{B}_1$ as in assumption 3) we have
$$\mathbf{A}_1 = \begin{bmatrix}2 & 0\\ 0 & 7\end{bmatrix},\quad \mathbf{B}_1 = \begin{bmatrix}1\\ 1\end{bmatrix}, \tag{5}$$
and finally from the partial fraction expansion of $G_1(z)$
$$G_1(z) = \frac{-0.1789}{(z-2)} + \frac{0.1942}{(z-7)}, \tag{6}$$
we obtain $\mathbf{C}_1$ as $\begin{bmatrix}-0.1789 & 0.1942\end{bmatrix}$. □

A discrete-time ARE is given by
$$\mathbf{P} = \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}\left(R+\mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A} + \mathbf{Q}. \tag{7}$$
In the present paper we consider a particular class of such discrete-time AREs, namely the class with vanishing state weighting, that is $\mathbf{Q} = \mathbf{0}$. This is a class of AREs that we refer to as the discrete-time minimum energy ARE
$$\mathbf{P} = \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}\left(R+\mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}. \tag{8}$$
Under the assumptions for $\mathbf{A}$, $\mathbf{B}$ and a dual of the continuous-time argument in [7, Lemma 2], the unique symmetric positive definite solution of (8) satisfies
$$\mathbf{P} = \begin{bmatrix}\mathbf{P}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}. \tag{9}$$
Thus, the discrete-time minimum energy ARE becomes
$$\mathbf{P}_1 = \mathbf{A}_1^T\mathbf{P}_1\mathbf{A}_1 - \mathbf{A}_1^T\mathbf{P}_1\mathbf{B}_1\left(R+\mathbf{B}_1^T\mathbf{P}_1\mathbf{B}_1\right)^{-1}\mathbf{B}_1^T\mathbf{P}_1\mathbf{A}_1. \tag{10}$$
We introduce now a closed-form characterisation of $\mathbf{P}_1$, the non-trivial solution to the minimum energy ARE in (10), when $R = 1$.

Proposition 1: (**Closed-form Solution for** $R = 1$) the closed-form solution to the minimum energy ARE in (10) with $R = 1$ is given by
$$\mathbf{P}_1^{cf} = \begin{bmatrix}\frac{r_1^2}{\rho_1^2-1} & \frac{r_1r_2}{\rho_1\rho_2-1} & \cdots & \frac{r_1r_m}{\rho_1\rho_m-1}\\ \frac{r_2r_1}{\rho_2\rho_1-1} & \frac{r_2^2}{\rho_2^2-1} & \cdots & \frac{r_2r_m}{\rho_2\rho_m-1}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{r_mr_1}{\rho_m\rho_1-1} & \frac{r_mr_2}{\rho_m\rho_2-1} & \cdots & \frac{r_m^2}{\rho_m^2-1}\end{bmatrix}, \tag{11}$$
with r_i defined as
$$r_i = (1-\rho_i^2)\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{1-\rho_i\rho_j}{\rho_i-\rho_j},\ \forall i = 1,\cdots,m. \tag{12}$$

Proof: consider the minimum energy ARE in (10) with $R = 1$ and rewrite it as
$$\mathbf{A}_1^T\mathbf{P}_1\mathbf{A}_1 - \mathbf{P}_1 = \mathbf{A}_1^T\mathbf{P}_1\mathbf{B}_1\left(1+\mathbf{B}_1^T\mathbf{P}_1\mathbf{B}_1\right)^{-1}\mathbf{B}_1^T\mathbf{P}_1\mathbf{A}_1. \tag{13}$$
Due to assumption 3 we have that
$$\mathbf{A}_1 = \begin{bmatrix}\rho_1 & 0 & \cdots & 0\\ 0 & \rho_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \rho_m\end{bmatrix},\quad \mathbf{B}_1 = \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}, \tag{14}$$

where $\rho_i,\ i=1,\cdots,m$ are the eigenvalues of $\mathbf{A}_1$. Replace in (13) $\mathbf{P}_1^{cf}$ as in (11), $\mathbf{A}_1$ and $\mathbf{B}_1$ as in (14). Notice that the LHS in (13) is then given by

$$\begin{bmatrix} r_1^2 & r_1r_2 & \cdots & r_1r_m \\ r_2r_1 & r_2^2 & \cdots & r_2r_m \\ \vdots & \vdots & \ddots & \vdots \\ r_mr_1 & r_mr_2 & \cdots & r_m^2 \end{bmatrix}, \tag{15}$$

whilst the RHS is given by

$$\begin{bmatrix} r_1^2\frac{\left(\rho_1\sum_{l=1}^m\frac{r_l}{\rho_1\rho_l-1}\right)^2}{1+\sum_{i=1}^m\sum_{j=1}^m\frac{r_ir_j}{\rho_i\rho_j-1}} & \cdots & r_1r_m\frac{\left(\rho_1\sum_{l=1}^m\frac{r_l}{\rho_1\rho_l-1}\right)\left(\rho_m\sum_{l=1}^m\frac{r_l}{\rho_m\rho_l-1}\right)}{1+\sum_{i=1}^m\sum_{j=1}^m\frac{r_ir_j}{\rho_i\rho_j-1}} \\ \vdots & \ddots & \vdots \\ r_mr_1\frac{\left(\rho_m\sum_{l=1}^m\frac{r_l}{\rho_m\rho_l-1}\right)\left(\rho_1\sum_{l=1}^m\frac{r_l}{\rho_1\rho_l-1}\right)}{1+\sum_{i=1}^m\sum_{j=1}^m\frac{r_ir_j}{\rho_i\rho_j-1}} & \cdots & r_m^2\frac{\left(\rho_m\sum_{l=1}^m\frac{r_l}{\rho_m\rho_l-1}\right)^2}{1+\sum_{i=1}^m\sum_{j=1}^m\frac{r_ir_j}{\rho_i\rho_j-1}} \end{bmatrix}. \tag{16}$$

Recall now from [3, Proof of Theorem III.1] (see also [8, §5.2.1]) that

$$\sum_{i=1}^m\sum_{j=1}^m\frac{r_ir_j}{\rho_i\rho_j-1}=\mathbf{B}_1^T\mathbf{P}_1\mathbf{B}_1=\prod_{i=1}^m\rho_i^2-1. \tag{17}$$

Also by means of Lemma 2 in the Appendix we have

$$\frac{\left(\rho_i\sum_{l=1}^m\frac{r_l}{\rho_i\rho_l-1}\right)\left(\rho_j\sum_{l=1}^m\frac{r_l}{\rho_j\rho_l-1}\right)}{1+\sum_{i=1}^m\sum_{j=1}^m\frac{r_ir_j}{\rho_i\rho_j-1}}= \frac{\left((-1)^m\prod_{i=1}^m\rho_i\right)\left((-1)^m\prod_{i=1}^m\rho_i\right)}{1+\prod_{i=1}^m\rho_i^2-1}=1, \tag{18}$$

Thus substituting in (16) we can see that (16) is equal to (15). Therefore we conclude that $\mathbf{P}_1^{cf}$ is indeed the solution to (10) when $R=1$, which completes the proof. ■

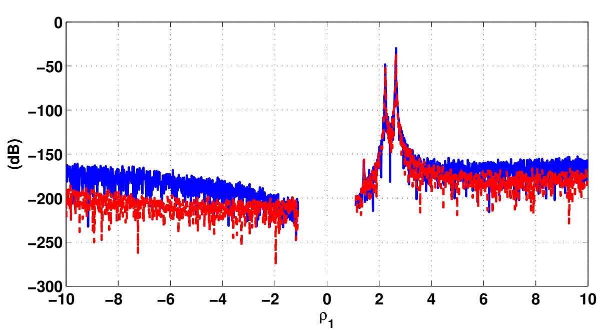

Fig. 2. Numerical error e_{ARE} upon replacing $\mathbf{P}_1$ obtained from Matlab, solid line, or in closed-form, dashed line.

We now extend the result of Proposition 1 to the general case of $R=\lambda$.

Corollary 1: (**Closed-form Solution for** $R=\lambda$) the closed-form solution to the minimum energy ARE in (10) with weight $R=\lambda$ is given by

$$\tilde{\mathbf{P}}_1^{cf}=\lambda\begin{bmatrix} \frac{r_1^2}{\rho_1^2-1} & \frac{r_1r_2}{\rho_1\rho_2-1} & \cdots & \frac{r_1r_m}{\rho_1\rho_m-1} \\ \frac{r_2r_1}{\rho_2\rho_1-1} & \frac{r_2^2}{\rho_2^2-1} & \cdots & \frac{r_2r_m}{\rho_2\rho_m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r_mr_1}{\rho_m\rho_1-1} & \frac{r_mr_2}{\rho_m\rho_2-1} & \cdots & \frac{r_m^2}{\rho_m^2-1} \end{bmatrix}, \tag{19}$$

with $r_i,\ \forall i=1,\cdots,m$, defined as in (12).

Proof: From Proposition 1 we have that $\mathbf{P}_1^{cf}$ satisfies the minimum energy ARE in (10) with $R=1$

$$\mathbf{P}_1^{cf}=\mathbf{A}_1^T\mathbf{P}_1^{cf}\mathbf{A}_1 - \mathbf{A}_1^T\mathbf{P}_1^{cf}\mathbf{B}_1\left(1+\mathbf{B}_1^T\mathbf{P}_1^{cf}\mathbf{B}_1\right)^{-1}\mathbf{B}_1^T\mathbf{P}_1^{cf}\mathbf{A}_1,$$

which, since λ is a scalar, is equivalent to

$$\mathbf{P}_1^{cf}=\mathbf{A}_1^T\mathbf{P}_1^{cf}\mathbf{A}_1 - \mathbf{A}_1^T\mathbf{P}_1^{cf}\mathbf{B}_1\left(\lambda+\mathbf{B}_1^T\lambda\mathbf{P}_1^{cf}\mathbf{B}_1\right)^{-1}\mathbf{B}_1^T\lambda\mathbf{P}_1^{cf}\mathbf{A}_1.$$

Now multiply both sides of the above expression by λ and observe that $\tilde{\mathbf{P}}_1^{cf}=\lambda\mathbf{P}_1^{cf}$

$$\tilde{\mathbf{P}}_1^{cf}=\mathbf{A}_1^T\tilde{\mathbf{P}}_1^{cf}\mathbf{A}_1 - \mathbf{A}_1^T\tilde{\mathbf{P}}_1^{cf}\mathbf{B}_1\left(\lambda+\mathbf{B}_1^T\tilde{\mathbf{P}}_1^{cf}\mathbf{B}_1\right)^{-1}\mathbf{B}_1^T\tilde{\mathbf{P}}_1^{cf}\mathbf{A}_1,$$

and thus $\tilde{\mathbf{P}}_1^{cf}$ satisfies (10) with $R=\lambda$, which concludes the proof. ■

We present next an example of a direct application of Proposition 1.

Example 2: consider in this example $\mathbf{A}_1$ and $\mathbf{B}_1$ as

$$\mathbf{A}_1=\begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix},\quad \mathbf{B}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \tag{20}$$

where $\rho_1\in[-10,-1.1]\cup[1.1,10]$, $\rho_2=\sqrt{2}$, $\rho_3=\sqrt{5}$ and $\rho_4=\sqrt{7}$. The closed-form solution $\mathbf{P}_1$ for the minimum energy ARE (10) for this example is given by

$$\mathbf{P}_1^{cf}=\begin{bmatrix} \frac{r_1^2}{\rho_1^2-1} & \frac{r_1r_2}{\rho_1\rho_2-1} & \frac{r_1r_3}{\rho_1\rho_3-1} & \frac{r_1r_4}{\rho_1\rho_4-1} \\ \frac{r_2r_1}{\rho_2\rho_1-1} & \frac{r_2^2}{\rho_2^2-1} & \frac{r_2r_3}{\rho_2\rho_3-1} & \frac{r_2r_4}{\rho_2\rho_4-1} \\ \frac{r_3r_1}{\rho_3\rho_1-1} & \frac{r_3r_2}{\rho_3\rho_2-1} & \frac{r_3^2}{\rho_3^2-1} & \frac{r_3r_4}{\rho_3\rho_4-1} \\ \frac{r_4r_1}{\rho_4\rho_1-1} & \frac{r_4r_2}{\rho_4\rho_2-1} & \frac{r_4r_3}{\rho_4\rho_3-1} & \frac{r_4^2}{\rho_4^2-1} \end{bmatrix}, \tag{21}$$

with

$$\begin{aligned}
r_1&=(1-\rho_1^2)\left(\tfrac{1-\rho_1\rho_2}{\rho_1-\rho_2}\right)\left(\tfrac{1-\rho_1\rho_3}{\rho_1-\rho_3}\right)\left(\tfrac{1-\rho_1\rho_4}{\rho_1-\rho_4}\right),\\
r_2&=(1-\rho_2^2)\left(\tfrac{1-\rho_2\rho_1}{\rho_2-\rho_1}\right)\left(\tfrac{1-\rho_2\rho_3}{\rho_2-\rho_3}\right)\left(\tfrac{1-\rho_2\rho_4}{\rho_2-\rho_4}\right),\\
r_3&=(1-\rho_3^2)\left(\tfrac{1-\rho_3\rho_1}{\rho_3-\rho_1}\right)\left(\tfrac{1-\rho_3\rho_2}{\rho_3-\rho_2}\right)\left(\tfrac{1-\rho_3\rho_4}{\rho_3-\rho_4}\right),\\
r_4&=(1-\rho_4^2)\left(\tfrac{1-\rho_4\rho_1}{\rho_4-\rho_1}\right)\left(\tfrac{1-\rho_4\rho_2}{\rho_4-\rho_2}\right)\left(\tfrac{1-\rho_4\rho_3}{\rho_4-\rho_3}\right).
\end{aligned} \tag{22}$$

We compare the expression in (21) with the solution resulting from the Matlab command dare, based on [9], by executing the line

$$\texttt{dare}(\texttt{A}_1,\texttt{B}_1,\texttt{zeros}(4),1),$$

in Matlab (7.5.0.342 (R2007b)) with $\mathbf{A}_1$ and $\mathbf{B}_1$ as in (20). To quantify the difference between the closed-form solution and the Matlab solution, we propose the following error function

$$e_{ARE}=\left\{\begin{bmatrix}1&1&1&1\end{bmatrix}[\mathbf{F}_1-\mathbf{F}_2]\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}^2 \tag{23}$$

with

$$\begin{aligned}
\mathbf{F}_1 &=\mathbf{P}_1^{cf}\\
\mathbf{F}_2 &=\mathbf{A}_1^T\mathbf{P}_1^{cf}\mathbf{A}_1-\\
&\qquad \mathbf{A}_1^T\mathbf{P}_1^{cf}\mathbf{B}_1(1+\mathbf{B}_1^T\mathbf{P}_1^{cf}\mathbf{B}_1)^{-1}\mathbf{B}_1^T\mathbf{P}_1^{cf}\mathbf{A}_1.
\end{aligned} \tag{24}$$

Notice that e_{ARE} is the square difference between the LHS and RHS of (10), that is our proposed error function is quantifying how precisely each $\mathbf{P}_1$ solution, either in closed-form or from Matlab, satisfies the minimum energy ARE (10). The result for e_{ARE} can be observed in Figure 2, where the solid line is e_{ARE} obtained with the solution using Matlab, whilst the dashed line is e_{ARE} obtained with $\mathbf{P}_1^{cf}$. For both approaches, either closed-form or Matlab, we observe how for all value of ρ_1 the error is indeed very small, in the order of -200 (dB) (where (dB)s are obtained as $10\log_{10} e_{ARE}$ accounting for the square in the definition of the error function). Also as ρ_1 approaches ρ_2, ρ_3 or ρ_4 the error tends to grow. Indeed the higher error value located in Figure 2 at ρ_2, ρ_3 and ρ_4 is signalling the loss of controllability that occurs when ρ_1 matches any of these values. In the neighbourhood of such values the quasi-loss of controllability produces P_1 solutions with very high entries in each of its elements making numerical errors all the more significant. The higher value in each of the entries of $\mathbf{P}_1$ can also be observed from the expressions for r_1, r_2, r_3 and r_4 in (22), whenever ρ_1 approaches the values of the other unstable eigenvalues, due to the factor $\rho_1-\rho_i$ with $i=2,3,4$ in each of their denominators. The error at precisely the value of ρ_2, ρ_3 and ρ_4 should grow to infinity (as the closed-form suggests), but this is avoided in Figure 2 where the spikes are only result of interpolation, since ρ_1 is considered in steps of 0.01 and it does not precisely overlap any of the other poles. Finally, as precise as both solutions are, for most values of ρ_1 the error obtained with the closed-form solution is several (dB) below the one obtained with Matlab. This suggests that the proposed closed-form solution may offer not only insight into the structure of the minimum energy ARE solution but also increased numerical precision over current algorithms. □

Corollary 2: (**Transformed Closed-form Solution**) the closed-form solution to the minimum energy ARE in (10) with weight $R=1$, subject to a nonsingular transformation of the state $\mathbf{T}$ is given by

$$\bar{\mathbf{P}}_1^{cf}=\mathbf{T}^T\begin{bmatrix}\frac{r_1^2}{\rho_1^2-1}&\frac{r_1r_2}{\rho_1\rho_2-1}&\cdots&\frac{r_1r_m}{\rho_1\rho_m-1}\\ \frac{r_2r_1}{\rho_2\rho_1-1}&\frac{r_2^2}{\rho_2^2-1}&\cdots&\frac{r_2r_m}{\rho_2\rho_m-1}\\ \vdots&\vdots&\ddots&\vdots\\ \frac{r_mr_1}{\rho_m\rho_1-1}&\frac{r_mr_2}{\rho_m\rho_2-1}&\cdots&\frac{r_m^2}{\rho_m^2-1}\end{bmatrix}\mathbf{T}, \tag{25}$$

with r_i, $\forall i=1,\cdots,m$, defined as in (12).

Proof: subject to a nonsingular transformation $\mathbf{T}$ the new matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{B}}_1$ are given by $\mathbf{T}^{-1}\mathbf{A}_1\mathbf{T}$ and $\mathbf{T}^{-1}\mathbf{B}_1$ respectively. Consider now Proposition 1 and the minimum energy ARE in (10) with $R=1$. Multiply both sides by $\mathbf{T}$ from the right and by $\mathbf{T}^T$ from the left

$$\begin{aligned}
\mathbf{T}^T\mathbf{P}_1^{cf}\mathbf{T}&=\mathbf{T}^T\mathbf{A}_1^T\mathbf{P}_1^{cf}\mathbf{A}_1\mathbf{T}-\\
&\mathbf{T}^T\mathbf{A}_1^T\mathbf{P}_1^{cf}\mathbf{B}_1\left(1+\mathbf{B}_1^T\mathbf{P}_1^{cf}\mathbf{B}_1\right)^{-1}\mathbf{B}_1^T\mathbf{P}_1^{cf}\mathbf{A}_1\mathbf{T}.
\end{aligned}$$

We can observe that since $\mathbf{T}$ is nonsingular then $\mathbf{T}^{-T}\bar{\mathbf{P}}_1^{cf}\mathbf{T}^{-1}=\mathbf{P}_1^{cf}$. Replace in the above expression and rearrange terms according to the properties of transposition to obtain

$$\begin{aligned}
\bar{\mathbf{P}}_1^{cf}&=\bar{\mathbf{A}}_1^T\bar{\mathbf{P}}_1^{cf}\bar{\mathbf{A}}_1-\\
&\bar{\mathbf{A}}_1^T\bar{\mathbf{P}}_1^{cf}\bar{\mathbf{B}}_1\left(1+\bar{\mathbf{B}}_1^T\bar{\mathbf{P}}_1^{cf}\bar{\mathbf{B}}_1\right)^{-1}\bar{\mathbf{B}}_1^T\bar{\mathbf{P}}_1^{cf}\bar{\mathbf{A}}_1,
\end{aligned}$$

which concludes the proof. ■

Example 3: For $m=2$, with a generic nonsingular transformation $\mathbf{T}=\left[\begin{smallmatrix}t_{11}&t_{12}\\t_{21}&t_{22}\end{smallmatrix}\right]$, we have that $\bar{\mathbf{P}}_1^{cf}$ is given by

$$\bar{\mathbf{P}}_1^{cf}=\begin{bmatrix}\frac{r_1^2}{\rho_1^2-1}t_{11}^2+\frac{2r_1r_2}{\rho_1\rho_2-1}t_{11}t_{21}+\frac{r_2^2}{\rho_2^2-1}t_{21}^2 & \frac{r_1^2}{\rho_1^2-1}t_{11}t_{12}+\frac{r_1r_2}{\rho_1\rho_2-1}t_{21}t_{12}+\frac{r_1r_2}{\rho_1\rho_2-1}t_{11}t_{22}+\frac{r_2^2}{\rho_2^2-1}t_{21}t_{22}\\ \frac{r_1^2}{\rho_1^2-1}t_{11}t_{12}+\frac{r_1r_2}{\rho_1\rho_2-1}t_{21}t_{12}+\frac{r_1r_2}{\rho_1\rho_2-1}t_{11}t_{22}+\frac{r_2^2}{\rho_2^2-1}t_{21}t_{22} & \frac{r_1^2}{\rho_1^2-1}t_{12}^2+\frac{2r_1r_2}{\rho_1\rho_2-1}t_{12}t_{22}+\frac{r_2^2}{\rho_2^2-1}t_{22}^2\end{bmatrix},$$

with r_1 and r_2 given by

$$r_1=(1-\rho_1^2)\left(\tfrac{1-\rho_1\rho_2}{\rho_1-\rho_2}\right),\quad r_2=(1-\rho_2^2)\left(\tfrac{1-\rho_2\rho_1}{\rho_2-\rho_1}\right).$$

From the definition of r_1 and r_2 above, we can verify in this example the known fact that the property of controllability (and its loss, whenever $\rho_1 = \rho_2$) is a shared condition for both $\mathbf{P}_1^{cf}$ and $\bar{\mathbf{P}}_1^{cf}$ and does not depend on the transformation $\mathbf{T}$. □

Throughout the present work we have stressed that we do not consider the case of repeated unstable eigenvalues. To give the reader a hint of the challenges involved in extending the result of Proposition 1 to such a case we present next an example which considers the simple case of a single repeated unstable eigenvalue ρ.

Example 4: consider for the present example the simple case of one unstable eigenvalue ρ with multiplicity two. The most natural extension of Assumption 3) is to consider a Jordan block representation for $\mathbf{A}_1$ and $\mathbf{B}_1$, that is

$$\mathbf{A}_1 = \begin{bmatrix} \rho & 1 \\ 0 & \rho \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It can be seen that the extension of the infimal SNR result in [4] to the present case of one repeated unstable pole is given by

$$\frac{\mathcal{P}}{\sigma^2} > \frac{r_1^2}{\rho^2 - 1} - \frac{2 r_1 r_2 \rho}{(\rho^2-1)^2} + \frac{r_2^2(\rho^2+1)}{(\rho^2-1)^3},$$

with

$$r_1 = 2\rho(\rho^2 - 1), \quad r_2 = (\rho^2-1)^2. \tag{26}$$

Notice that the above result assumes that the plant model behind matrices $\mathbf{A}_1$ and $\mathbf{B_1}$ has relative degree one [1]. On the other hand the result from [3, Theorem III.1] is not limited by the multiplicity of the unstable eigenvalue and thus for the present example is given by

$$\frac{\mathcal{P}}{\sigma^2} > \rho^4 - 1. \tag{27}$$

As in the proof of Proposition 1, to introduce the minimum energy ARE solution in the discussion we match the term $\mathbf{B}_1^T \mathbf{P}_1 \mathbf{B}_1$ to both infimal SNR results. The first observation is that by the choice of $\mathbf{B}_1$, then $\mathbf{P}_1^{cf}(2,2)$ has to be equal to the infimal SNR, that is $\mathbf{P}_1^{cf}(2,2) = \rho^4 - 1$. Further comparison allow us to identify $\mathbf{P}_1^{cf}(1,1) = r_2^2/(\rho^2-1)$ (or equivalently $\mathbf{P}_1(1,1) = (\rho^2-1)^3$ replacing r_2 as in (26)). Finally from (10) and $\mathbf{P}_1^{cf}(1,1)$ we can identify $\mathbf{P}_1^{cf}(1,2)$ (and therefore $\mathbf{P}_1^{cf}(2,1)$) as $r_2\rho$, that is $\rho(\rho^2-1)^2$. The closed-form minimum energy ARE solution is thus

$$\mathbf{P}_1^{cf} = \begin{bmatrix} (\rho^2-1)^3 & \rho(\rho^2-1)^2 \\ \rho(\rho^2-1)^2 & \rho^4 - 1 \end{bmatrix}. \tag{28}$$

To further verify that the above result for $\mathbf{P}_1$ is correct we introduce an error function e_P

$$e_P = \begin{bmatrix} 1 & 1 \end{bmatrix} \left(\mathbf{P}_1^m - \mathbf{P}_1^{cf}\right)\left(\mathbf{P}_1^m - \mathbf{P}_1^{cf}\right)^T \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{29}$$

[1]An example of such a model is $G(z) = K(z-a)/(z-\rho)^2$, which gives $\mathbf{A}_1, \mathbf{B}_1$ as in the example and furthermore $\mathbf{C}_1 = \begin{bmatrix} K(\rho - a) & K \end{bmatrix}$.

where $\mathbf{P}_1^m$ is the solution obtained with Matlab (7.5.0.342 (R2007b)), and $\mathbf{P}_1^{cf}$ is the solution in (28). In Figure 3 we can see the resulting e_P function for ρ in $[-10, -1.01] \cup [1.01, 10]$. We observe that the error between the two solutions is indeed very small, indicating that $\mathbf{P}_1^m$ and $\mathbf{P}_1^{cf}$ are very close in value. The lower limit of -320 (dB)in Figure 3 is effectively in the order of the machine precision ($eps = 2.2204 \cdot 10^{-16}$) used by Matlab (7.5.0.342 (R2007b)) for the difference $\mathbf{P}_1^m - \mathbf{P}_1^{cf}$. □

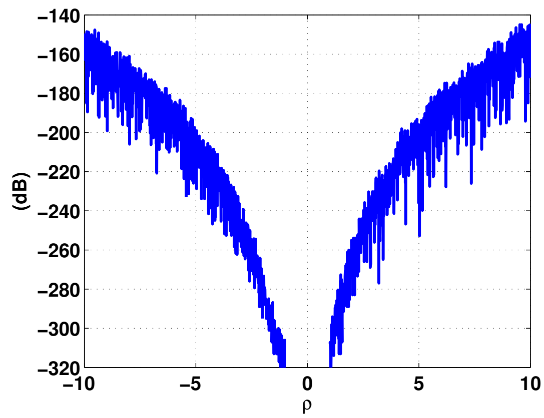

Fig. 3. Numerical error e_P between $\mathbf{P}_1$ obtained from Matlab and $\mathbf{P}_1$ obtained in closed-form.

The above example illustrates how hard it is, when considering repeated unstable eigenvalues, to gain sufficient insight into the minimum energy problem to be able to find $\mathbf{P}_1^{cf}$ in closed-form. At the very least, as a first step to pursue such extension, we will need to extend [4, Theorem 2] to repeated unstable eigenvalues.

III. Conclusion

In the present paper we have presented the discrete-time solution for a class of minimum energy AREs. This particular class is characterised by AREs which consider only distinct eigenvalues for the matrix $\mathbf{A}_1$, of dimensions $m \times m$, and only one input (that is matrix $\mathbf{B}_1$ of dimensions $m \times 1$). As an example we have compared the closed-form result to the one obtained using the command `dare` in Matlab. The closed-form result for the proposed example proved more accurate than the Matlab solution. This suggests that, beside the theoretical insight obtained from the closed-form solution $\mathbf{P}_1^{cf}$, one might also benefit from increased numerical precision by implementing the proposed closed-form solution. Future research will consider extending the class of minimum energy AREs treated here by lifting the condition of distinct eigenvalues for $\mathbf{A}_1$ and by considering a more general matrix $\mathbf{B}_1$ of dimensions $m \times n_u$ with $n_u \geq 1, n_u \in \mathbb{Z}^+$.

IV. Appendix

We present here two lemmas required in the proof of Proposition 1.

Lemma 1: the following equality holds $\forall l = 1, \cdots, k_o$,

$$\sum_{j=1}^{k_o} \frac{q_j}{\rho_l\rho_j-1}\left(\frac{1-\rho_{k_o+1}^2}{\rho_j-\rho_{k_o+1}}\right)+\frac{t_{k_o+1}}{\rho_l\rho_{k_o+1}-1}=0, \tag{30}$$

where

$$q_j=(1-\rho_j^2)\prod_{\substack{s=1\\ s\neq j}}^{k_o} \frac{1-\rho_j\rho_s}{\rho_j-\rho_s}, \tag{31}$$

and

$$t_{k_o+1}=(1-\rho_{k_o+1}^2)\prod_{s=1}^{k_o} \frac{1-\rho_{k_o+1}\rho_s}{\rho_{k_o+1}-\rho_s}. \tag{32}$$

Proof: observe that (30) can be rewritten as

$$\sum_{j=1}^{k_o} \frac{q_j}{\rho_l\rho_j-1}\left(\frac{1-\rho_{k_o+1}^2}{\rho_j-\rho_{k_o+1}}\right) + \frac{(1-\rho_{k_o+1}^2)\prod_{s=1}^{k_o}\frac{1-\rho_{k_o+1}\rho_s}{\rho_{k_o+1}-\rho_s}}{\rho_l\rho_{k_o+1}-1}. \tag{33}$$

More so the term $\frac{\prod_{s=1}^{k_o}\frac{1-\rho_{k_o+1}\rho_s}{\rho_{k_o+1}-\rho_s}}{\rho_l\rho_{k_o+1}-1}$ can be decomposed in a partial fraction expansion such as

$$\frac{\prod_{s=1}^{k_o}\frac{1-\rho_{k_o+1}\rho_s}{\rho_{k_o+1}-\rho_s}}{\rho_l\rho_{k_o+1}-1}=\sum_{j=1}^{k_o}\frac{q_j}{\rho_l\rho_j-1}\left(\frac{1}{\rho_{k_o+1}-\rho_j}\right), \tag{34}$$

which when replaced into (33) gives

$$\sum_{j=1}^{k_o} \frac{q_j}{\rho_l\rho_j-1}\left(\frac{1-\rho_{k_o+1}^2}{\rho_j-\rho_{k_o+1}}\right)+ (1-\rho_{k_o+1}^2)\sum_{j=1}^{k_o}\frac{q_j}{\rho_l\rho_j-1}\left(\frac{1}{\rho_{k_o+1}-\rho_j}\right)=$$

$$\sum_{j=1}^{k_o} \frac{q_j}{\rho_l\rho_j-1}\left(\frac{1-\rho_{k_o+1}^2}{\rho_j-\rho_{k_o+1}}\right) - \sum_{j=1}^{k_o} \frac{q_j}{\rho_l\rho_j-1}\left(\frac{1-\rho_{k_o+1}^2}{\rho_j-\rho_{k_o+1}}\right)=0, \tag{35}$$

which ends the proof. ■

Lemma 2: consider that for the case of a memoryless AWGN channel the residue factor r_i is equal to

$$r_i=(1-\rho_i^2)\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{1-\rho_i\rho_j}{\rho_i-\rho_j},\ \forall i=1,\cdots,m, \tag{36}$$

then the following equality holds

$$\rho_i\sum_{l=1}^{m}\frac{r_l}{\rho_i\rho_l-1}=(-1)^m\prod_{j=1}^{m}\rho_j,\ \forall l=1,\cdots,m. \tag{37}$$

Proof: the following proof is based on an induction argument

1) For m=1, the only selection for i is 1, thus

$$\rho_1\frac{1-\rho_1^2}{\rho_1^2-1}=-\rho_1. \tag{38}$$

2) Assume the case $m = k_o$

$$\rho_i\sum_{l=1}^{k_o}\frac{q_l}{\rho_i\rho_l-1}=(-1)^{k_o}\prod_{j=1}^{k_o}\rho_j, \tag{39}$$

to be true.

3) To study the case $m = k_o{+}1$ for $i = 1, \cdots, k_o$ consider

$$\rho_i\sum_{l=1}^{k_o}\frac{q_l}{\rho_i\rho_l-1}=(-1)^{k_o}\prod_{j=1}^{k_o}\rho_j, \tag{40}$$

and by means of Lemma 1

$$-\rho_{k_o+1}\rho_i\sum_{l=1}^{k_o}\frac{q_l}{\rho_i\rho_l-1}+ \rho_i\left(\sum_{l=1}^{k_o}\frac{q_l}{\rho_i\rho_j-1}\left(\frac{1-\rho_{k_o+1}^2}{\rho_j-\rho_{k_o+1}}\right)+\frac{t_{k_o+1}}{\rho_i\rho_{k_o+1}-1}\right) =(-1)^{k_o+1}\prod_{j=1}^{k_o+1}\rho_j. \tag{41}$$

Now rearrange terms to observe

$$\begin{aligned}\rho_i\sum_{l=1}^{k_o}\frac{t_l}{\rho_i\rho_l-1}+\rho_i\frac{t_{k_o+1}}{\rho_i\rho_{k_o+1}-1}&=(-1)^{k_o+1}\prod_{j=1}^{k_o+1}\rho_j\\ \rho_i\sum_{l=1}^{k_o+1}\frac{t_l}{\rho_i\rho_l-1}&=(-1)^{k_o+1}\prod_{j=1}^{k_o+1}\rho_j.\end{aligned} \tag{42}$$

4) Finally for the case $m = k_o{+}1$ and $i = k_o{+}1$, consider the following constructive argument

$$\begin{aligned}&\rho_{k_o+1}\sum_{l=1}^{k_o+1}\frac{t_l}{\rho_{k_o+1}\rho_l-1}\\ &=\rho_{k_o+1}\sum_{l=1}^{k_o}\frac{t_l}{\rho_{k_o+1}\rho_l-1}+\rho_{k_o+1}\frac{t_{k_o+1}}{\rho_{k_o+1}^2-1}\\ &=\rho_{k_o+1}\sum_{l=1}^{k_o}\frac{q_l}{\rho_l-\rho_{k_o+1}}+ \rho_{k_o+1}\left(\sum_{l=1}^{k_o}\frac{q_l}{\rho_{k_o+1}-\rho_j}+(-1)^{k_o+1}\prod_{j=1}^{k_o}\rho_j\right)\\ &=(-1)^{k_o+1}\prod_{j=1}^{k_o+1}\rho_j,\end{aligned} \tag{43}$$

which ends the proof. ■

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