

# Decentralized Control of Discrete-Time Linear Time Invariant Systems with Input Saturation

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**Abstract**— We study decentralized stabilization of discrete-time linear time invariant (LTI) systems subject to actuator saturation, using LTI controllers. The requirement of stabilization under both saturation constraints and decentralization impose obvious necessary conditions on the open-loop plant, namely that its eigenvalues are in the closed unit disk and further that the eigenvalues on the unit circle are not decentralized fixed modes. The key contribution of this work is to provide a broad sufficient condition for decentralized stabilization under saturation. Specifically, we show through an iterative argument that stabilization is possible whenever 1) the open-loop eigenvalues are in the closed unit disk, 2) the eigenvalues on the unit circle are not decentralized fixed modes, and 3) these eigenvalues on the unit circle have algebraic multiplicity 1.

## I. INTRODUCTION

The result presented here contributes to our ongoing study of the stabilization of decentralized systems subject to actuator saturation. The eventual goal of this study is the *design* of controllers for saturating decentralized systems that achieve not only stabilization but also high performance. As a first step toward this design goal, we are currently looking for tight conditions on a decentralized plant with input saturation, for the *existence* of stabilizing controllers. Even this check for the existence of stabilizing controllers turns out to be extremely intricate: we have yet to obtain necessary and sufficient conditions for stabilization, but have obtained a broad sufficient condition in our earlier work [4]. This article further contributes to the study of the existence of stabilizing controllers, by describing an analogous sufficient condition for discrete-time decentralized plants.

To motivate and introduce the main result in the article, let us briefly review foundational studies on both decentralized control and saturating control systems. We recall that a necessary and sufficient condition for stabilization of a decentralized system using LTI state-space controllers is given in Wang and Davison's classical work [5]. They obtain

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that stabilization is possible if and only if all *decentralized fixed modes* of a plant are in the open left half plane, and give specifications of and methods for finding these decentralized fixed modes. Numerous further characterizations of decentralized stabilization (and fixed modes) have been given, see for instance the work of Corfmat and Morse [2]. In complement, for centralized control systems subject to actuator saturation, not only conditions for stabilization but also practical designs have been obtained, using the low-gain and low-high-gain methodology. For a background on the results for centralized systems subject to input saturation we refer to two special issues [1], [3]. Of importance here, we recall that a necessary and sufficient condition for *semi-global* stabilization of LTI plants with actuator saturation is that their open-loop poles are in the closed left half plane. Combining this observation with Wang and Davison's result, one might postulate that that stabilization of a saturating linear decentralized control system is possible if and only of 1) the open-loop plant poles are in the closed left half plane (respectively, closed unit disk, for discrete-time systems), and 2) the poles on the imaginary axis (respectively, unit circle) are not decentralized fixed modes. The necessity of the two requirements is immediate, but we have not yet been able to determine whether the requirements are also sufficient. As a first step for continuous-time plants, we showed in [4] that decentralized stabilization under saturation is possible when 1) the plant's open-loop poles are in the CLHP with imaginary axis poles non-repeated, and 2) the imaginary axis poles are not decentralized fixed modes. Here, we develop an analogous result for discrete-time plants, in particular showing that decentralized stabilization under saturation is possible if 1) the plant's open-loop poles are in the closed unit disk with unit-circle poles non-repeated, and 2) the unit-circle poles are not decentralized fixed modes.

## II. PROBLEM FORMULATION

Consider the LTI discrete-time systems subject to actuator saturation,

$$\Sigma : \begin{cases} x(k+1) &= Ax(k) + \sum_{i=1}^{\nu} B_i \text{sat}(u_i(k)) \\ y_i(k) &= C_i x(k), \quad i = 1, \dots, \nu, \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$  is state,  $u_i \in \mathbb{R}^{m_i}$ ,  $i = 1, \dots, \nu$  are control inputs,  $y_i \in \mathbb{R}^{p_i}$ ,  $i = 1, \dots, \nu$  are measured outputs, and 'sat' denotes the standard saturation element.

Here we are looking for  $\nu$  controllers of the form,

$$\Sigma_i : \begin{cases} z_i(k+1) &= K_i z_i(k) + L_i y_i(k), z_i \in \mathbb{R}^{s_i} \\ u_i(k+1) &= M_i z_i(k) + N_i y_i(k). \end{cases} \quad (2)$$

**Problem 1:** Let the system (1) be given. The **semi-global stabilization problem via decentralized control** is said to be solvable if for all compact sets  $\mathcal{W}$  and  $\mathcal{S}_1, \dots, \mathcal{S}_\nu$  there exists  $\nu$  controllers of the form (2) such that the closed loop system is asymptotically stable with the set

$$\mathcal{W} \times \mathcal{S}_1 \times \dots \times \mathcal{S}_\nu$$

contained in the domain of attraction.

The main objective of this paper is to develop necessary and sufficient conditions such that the semi-global stabilization problem via decentralized control is solvable. This objective has not yet been achieved. However, we obtain necessary conditions as well as sufficient conditions which are quite close.

### III. REVIEW OF DISCRETE-TIME LTI DECENTRALIZED SYSTEMS AND STABILIZATION

Before we tackle the problem introduced in Section II, let us first review the necessary and sufficient conditions for the decentralized stabilization of the linearized model of the given system  $\bar{\Sigma}$ ,

$$\bar{\Sigma} : \begin{cases} x(k+1) &= Ax(k) + \sum_{i=1}^{\nu} B_i u_i(k) \\ y_i(k) &= C_i x(k), \quad i = 1, \dots, \nu, \end{cases} \quad (3)$$

The decentralized stabilization problem for  $\bar{\Sigma}$  is to find LTI dynamic controllers  $\Sigma_i$ ,  $i = 1, \dots, \nu$ , of the form (2) such that the poles of the closed loop system are in the desired locations in the open unit disc.

Given system  $\bar{\Sigma}$  and controllers  $\Sigma_i$ , defined by (3) and (2) respectively, let us first define the following matrices in order to provide an easier bookkeeping:

$$\begin{aligned} B &= [B_1 \ \dots \ B_\nu], & C &= [C'_1 \ \dots \ C'_\nu]' \\ K &= \text{diag}[K_1, \dots, K_\nu], & L &= \text{diag}[L_1, \dots, L_\nu] \\ M &= \text{diag}[M_1, \dots, M_\nu], & N &= \text{diag}[N_1, \dots, N_\nu] \end{aligned}$$

**Definition 1:** Consider system  $\bar{\Sigma}$ ,  $\lambda \in \mathbb{C}$  is called a decentralized fixed mode if for all block diagonal matrices  $H$  we have

$$\det(\lambda I - A - BHC) = 0$$

We look at eigenvalues that can be moved by static decentralized controllers. However, it is known that if we cannot move an eigenvalue by static decentralized controllers then we cannot move the eigenvalue by dynamic decentralized controllers either.

**Lemma 1:** Necessary and sufficient condition for the existence of a decentralized feedback control law for the system  $\bar{\Sigma}$  such that the closed loop system is asymptotically stable is that all the fixed modes of the system be asymptotically stable (in the unit disc).

**Proof:** We first establish necessity.

Assume local controllers  $\Sigma_i$  together stabilize  $\bar{\Sigma}$  then for any  $|\lambda| \geq 1$  there exists a  $\delta$  such that  $(\lambda + \delta)I - K$  is invertible and the closed loop system replacing  $K$  with  $K - \delta I$  is still asymptotically stable. This choice is possible because if  $\lambda I - K$  is invertible obviously we can choose  $\delta = 0$ . If

$\lambda I - K$  is not invertible, by small enough choice of  $\delta$  we can make sure that  $(\lambda + \delta)I - K$  is invertible and the closed loop system replacing  $K$  with  $K - \delta I$  is still asymptotically stable. But the closed loop system when  $K - \delta I$  is in the loop is asymptotically stable. In particular, it can not have a pole in  $\lambda$ . So

$$\det(\lambda I - A - B[M(\lambda I - (K - \delta I))^{-1}L + N]C) \neq 0$$

Hence the block diagonal matrix

$$S = M(\lambda I - (K - \delta I))^{-1}L + N$$

has the property that

$$\det(\lambda I - A - BSC) \neq 0$$

thus  $\lambda$  is not a fixed mode. Since this argument is true for any  $\lambda$  on or outside the unit disc, this implies that all the fixed modes must be inside the unit disc. This proves the necessity of the Lemma 1.

Next, we establish sufficiency. The papers [2], [5] showed that if the decentralized fixed modes of a strongly connected system are stable, we can find a stabilizing controller for the system. However, these papers are based on continuous-time results. For completeness we present the proof for discrete-time which is a straightforward modification of [5]. We first claim that decentralized fixed modes are invariant under preliminary output injection. But this is obvious from our necessity proof since a trivial modification shows that no dynamic controller can move a fixed mode. To prove that we can actually stabilize the system, we use a recursive argument. Assume the system has an unstable eigenvalue in  $\mu$ . Since  $\mu$  is not a fixed mode there exists  $N_i$  such that

$$A + \sum_{i=1}^{\nu} B_i N_i C_i$$

no longer has an eigenvalue in  $\mu$ . Let  $k$  be the smallest integer such that an unstable eigenvalue of  $A$  is no longer an eigenvalue of

$$A + \sum_{i=1}^k B_i N_i C_i$$

while  $N_i$  can be chosen small enough not to introduce additional unstable eigenvalues. Then for the system

$$\left( A + \sum_{i=1}^{k-1} B_i N_i C_i, B_k, C_k \right)$$

an unstable eigenvalue is both observable and controllable. But this implies that there exists a dynamic controller which moves this eigenvalue in the open unit disc without introducing new unstable eigenvalues. Through a recursion, we can move all eigenvalues one-by-one in the open unit disc and in this way find a decentralized controller which stabilizes the system. This proves the sufficiency of the lemma 1.

#### IV. MAIN RESULTS

In this section, we present the main results of this paper.

**Theorem 1** Consider the system  $\Sigma$ . There exists non-negative integers  $s_1, \dots, s_\nu$  such that for any given collection of compact sets  $W \subset \mathbb{R}^n$  and  $S_i \subset \mathbb{R}^{s_i}$ ,  $i = 1, \dots, \nu$ , there exists  $\nu$  controllers of the form (2) such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes  $W \times S_1 \times \dots \times S_\nu$  only if

- All fixed modes are in the open unit disc.
- All eigenvalues of  $A$  are in the closed unit disc.

**Proof:** There exists an open neighborhood containing the origin for the closed loop system of  $\Sigma$  with the controllers  $\Sigma_i$  is identical to the closed loop system of  $\bar{\Sigma}$  with the controllers  $\bar{\Sigma}_i$ . Hence asymptotic stability of one closed loop system is equivalent to asymptotic stability of the other closed loop system. But then it is obvious from Lemma 1 that the first item of Theorem 1 is necessary for the existence of controllers of the form (2) for  $\bar{\Sigma}$  such that the origin of the resulting closed loop system is asymptotically stable.

To prove the necessity of the second item of Theorem 1, assume that  $\lambda$  is an eigenvalue of  $A$  outside the unit disc with associated left eigenvector  $p$ . We obtain:

$$px(k+1) = \lambda px(k) + v(k)$$

where

$$v(k) := \sum_{i=1}^{\nu} pB_i \text{sat}(u_i(k)).$$

Because of the saturation elements, there exists an  $\tilde{M} > 0$  such that  $|v(k)| \leq \tilde{M}$  for all  $k \geq 0$ . But then we have

$$px(k) = \lambda^k px(0) + \sum_{i=0}^{k-1} \lambda^{k-1-i} v(i) = \lambda^k (px(0) + S_k), \quad (4)$$

where  $S_k = \sum_{i=0}^{k-1} \frac{v(i)}{\lambda^{i+1}}$ . We find that

$$|S_k| \leq \tilde{M} \sum_{i=1}^k \frac{1}{|\lambda|^i} = \tilde{M} \cdot \frac{1 - \frac{1}{|\lambda|^k}}{|\lambda| - 1} < \frac{\tilde{M}}{|\lambda| - 1}$$

and then from (4) we find

$$|px(k)| > |\lambda|^k \left( |px(0)| - \frac{\tilde{M}}{|\lambda| - 1} \right) \quad \forall k \geq 1.$$

Hence  $|px(k)|$  does not converge to zero independent of our choice for a controller if we choose the initial condition  $x(0)$  such that  $|px(0)| > \frac{\tilde{M}}{|\lambda| - 1}$  because of the fact that  $|\lambda| > 1$ . However, the system was semi-globally stabilizable and hence there exists a controller which contains this initial condition in its domain of attraction and hence  $|px(k)| \rightarrow 0$  which yields a contradiction. This proves the second item of Theorem 1.

We now proceed to the next theorem which gives a sufficient condition for semi-global stabilizability of (1) when the set of controllers given by (2) are utilized.

**Theorem 2** Consider the system  $\Sigma$ . There exists non-negative integers  $s_1, \dots, s_\nu$  such that for any given collection of compact sets  $W \subset \mathbb{R}^n$  and  $S_i \subset \mathbb{R}^{s_i}$ ,  $i = 1, \dots, \nu$ ,

there exists  $\nu$  controllers of the form (2) such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes  $W \times S_1 \times \dots \times S_\nu$  if

- All fixed modes are in the open unit disc,
- All eigenvalues of  $A$  are in the closed unit disc with those eigenvalues on the unit circle having algebraic multiplicity equal to one.

To prove this theorem we will exploit the following lemma which follows directly from classical results of eigenvalues and eigenvectors and the results of perturbations of the matrix on those eigenvalues and eigenvectors.

**Lemma 2:** Let  $A_\delta \in \mathbb{R}^{n \times n}$  be a sequence of matrices parametrized by  $\delta$  and a matrix  $A \in \mathbb{R}^{n \times n}$  such that  $A_\delta \rightarrow A$  as  $\delta \rightarrow 0$ . Let  $A$  be a matrix with all eigenvalues in the closed unit disc and with  $p$  eigenvalues on the unit disc with all of them having multiplicity 1. Also assume that  $A_\delta$  has all its eigenvalues in the closed unit disc. Let matrix  $P > 0$  be such that  $A'PA - P \leq 0$  is satisfied. Then for small  $\delta > 0$  there exists a family of matrices  $P_\delta > 0$  such that

$$A'_\delta P_\delta A_\delta - P_\delta \leq 0$$

and  $P_\delta \rightarrow P$  as  $\delta \rightarrow 0$

**Proof:** We first observe that there exists a matrix  $S$  such that

$$S^{-1}AS = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

where all eigenvalues of  $A_{11}$  are on the unit circle while the eigenvalues of  $A_{22}$  are in the open unit disc. Since  $A_\delta \rightarrow A$  and the eigenvalues of  $A_{11}$  and  $A_{22}$  are distinct, there exists a parametrized matrix  $S_\delta$  such that for sufficiently small  $\delta$

$$S_\delta^{-1}A_\delta S_\delta = \begin{pmatrix} A_{11,\delta} & 0 \\ 0 & A_{22,\delta} \end{pmatrix}$$

where  $S_\delta \rightarrow S$ ,  $A_{11,\delta} \rightarrow A_{11}$  and  $A_{22,\delta} \rightarrow A_{22}$  as  $\delta \rightarrow 0$ .

Given a matrix  $P > 0$  such that  $A'PA - P \leq 0$ . Let us define

$$\bar{P} = S'PS = \begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}'_{12} & \bar{P}_{22} \end{pmatrix}$$

with this definition we have

$$\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \leq 0 \quad (5)$$

Next given an eigenvector  $x_1$  of  $A_{11}$ , i.e.  $A_{11}x_1 = \lambda x_1$  with  $|\lambda| = 1$ , we have

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}^* \left[ \begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0$$

Using (5), the above implies that

$$\left[ \begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0$$

Since all the eigenvalues on the unit disc of  $A_{11} \in \mathbb{R}^{p \times p}$  are distinct we find that the eigenvectors of  $A_{11}$  span  $\mathbb{R}^p$  and hence

$$\left[ \begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} I \\ 0 \end{pmatrix} = 0$$

This results in

$$\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} = \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \leq 0$$

This implies that  $A'_{11} \bar{P}_{12} A_{22} - \bar{P}_{12} = 0$  and since eigenvalues of  $A_{11}$  are on the unit disc and eigenvalues of  $A_{22}$  are inside the unit disc, we find that  $\bar{P}_{12} = 0$  because

$$A'_{11} \bar{P}_{12} A_{22} = \bar{P}_{12} \Rightarrow (A'_{11})^k \bar{P}_{12} A_{22}^k = \bar{P}_{12}$$

where  $k$  is an arbitrary positive integer. Note that  $(A'_{11})^k$  remains bounded while  $A_{22}^k \rightarrow 0$  as  $k \rightarrow \infty$ . This means that for  $k \rightarrow \infty$ ,  $\bar{P}_{12} \rightarrow 0$  and because  $\bar{P}_{12}$  is independent of  $k$ , we find that  $\bar{P}_{12} = 0$ . Next, since  $A_{22}$  has all its eigenvalues in the open unit disc, there exists a parametrized matrix  $P_{\delta,22}$  such that for  $\delta$  small enough

$$A'_{\delta,22} P_{\delta,22} A_{\delta,22} - P_{\delta,22} = V \leq 0$$

while  $P_{\delta,22} \rightarrow P_{22}$  as  $\delta \rightarrow 0$ .

Let  $A_{11} = W \Lambda_A W^{-1}$  with  $\Lambda_A$  a diagonal matrix. Because the eigenvectors of  $A_{11}$  are distinct and  $A_{11,\delta} \rightarrow A_{11}$ , the eigenvectors of  $A_{11,\delta}$  depend continuously on  $\delta$  for  $\delta$  small enough and hence there exists a parametrized matrix  $W_\delta$  such that  $W_\delta \rightarrow W$  while  $A_{11,\delta} = W_\delta \Lambda_{A_\delta} W_\delta^{-1}$  with  $\Lambda_{A_\delta}$  diagonal. The matrix  $\bar{P}_{11}$  satisfies

$$A'_{11} \bar{P}_{11} A_{11} - \bar{P}_{11} = 0$$

This implies that  $\Lambda_P = W^* \bar{P}_{11} W$  satisfies

$$\Lambda_A^* \Lambda_P \Lambda_A - \Lambda_P = 0$$

The above equation then shows that  $\Lambda_P$  is a diagonal matrix. We know that

$$\Lambda_{A_\delta} \rightarrow \Lambda_A.$$

We know that  $\Lambda_{A_\delta}$  is a diagonal matrix the diagonal elements of which have magnitude less or equal to one while  $\Lambda_P$  is a positive definite diagonal matrix.

Using this, it can be verified that we have

$$\Lambda_{A_\delta}^* \Lambda_P \Lambda_{A_\delta} - \Lambda_P \leq 0$$

We choose  $\bar{P}_{11,\delta}$  as

$$\bar{P}_{11,\delta} = (W_\delta^*)^{-1} \Lambda_P (W_\delta)^{-1}$$

We can see that this choice of  $\bar{P}_{11,\delta}$  satisfies

$$A'_{11,\delta} \bar{P}_{11,\delta} A_{11,\delta} - \bar{P}_{11,\delta} \leq 0$$

It is easy to see that  $\bar{P}_{11,\delta} \rightarrow \bar{P}_{11}$  as  $\delta \rightarrow 0$ . Then

$$P_\delta = (S_\delta^{-1})' \begin{pmatrix} \bar{P}_{11,\delta} & 0 \\ 0 & \bar{P}_{22,\delta} \end{pmatrix} S_\delta^{-1}$$

satisfies the condition of the lemma. This completes the proof of Lemma 2.

We now show a recursive algorithm that at each step moves at least one eigenvalue on the unit circle in a decentralized fashion while preserving the stability of other modes in the open unit disc in a way that the magnitude of each decentralized feedback control is assured never to exceed  $1/n$ . The algorithm will consist of at most  $n$  steps, and

therefore the overall decentralized inputs will not saturate for an appropriate choice of the initial state.

#### Algorithm:

- **Step 0:** We initialize algorithm at this step. Let  $A_0 := A$ ,  $B_{0,i} := B_i$ ,  $C_{0,i} := C_i$ ,  $n_{i,0} := 0$ ,  $N_{i,\varepsilon}^0 := 0$ ,  $i = 1, \dots, \nu$  and  $x_0 := x$ . Also let us define  $P_0^\varepsilon := \varepsilon P$ , where  $P > 0$  and satisfies  $A'PA - P \leq 0$ .
- **Step  $m$ :** For the system  $\Sigma$ , we want to design  $\nu$  parametrized decentralized feedback control laws,

$$\Sigma_i^{m,\varepsilon} : \begin{cases} p_i^m(k+1) = K_{i,\varepsilon}^m p_i^m(k) + L_{i,\varepsilon}^m y_i(k) \\ u_i(k) = M_{i,\varepsilon} p_i^m(k) + N_{i,\varepsilon}^m y_i(k) + v_i^m(k). \end{cases}$$

where  $p_i^m \in \mathbb{R}^{n_{i,m}}$  and if  $n_{i,m} = 0$ :

$$\Sigma_i^{m,\varepsilon} : \begin{cases} u_i(k) = N_{i,\varepsilon}^m y_i(k) + v_i^m(k) \end{cases}$$

The closed loop system consisting of the decentralized controller and the system  $\Sigma$  can be written as

$$\Sigma_{cl}^{m,\varepsilon} : \begin{cases} x_m(k+1) = A_m^\varepsilon x_m(k) + \sum_{i=1}^\nu B_{m,i} v_i^m(k) \\ y_i(k) = C_{m,i} x_m(k), \quad i = 1, \dots, \nu \end{cases}$$

where  $x_m \in \mathbb{R}^{n_m}$  with  $n_m = n + \sum_{i=1}^\nu n_{i,m}$  is given by

$$x_m = \begin{pmatrix} x \\ p_1^m \\ \vdots \\ p_\nu^m \end{pmatrix}$$

we can rewrite  $u_i$  as

$$u_i = F_{i,\varepsilon}^m x_m + v_i^m$$

for some appropriate matrix  $F_{i,\varepsilon}^m$ .

Our objective here is to design the decentralized stabilizers in such a way that they satisfy the following properties:

- 1) Matrix  $A_m^\varepsilon$  has all its eigenvalues in the closed unit disc, and eigenvalues on unit circle are distinct.
- 2)  $A_m^\varepsilon$  has less eigenvalues on the unit circle than  $A_{m-1}^\varepsilon$ .
- 3) There exists a family of matrices  $P_m^\varepsilon$  such that  $P_m^\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and

$$(A_m^\varepsilon)' P_m^\varepsilon A_m^\varepsilon - P_m^\varepsilon \leq 0$$

Furthermore, there exists an  $\varepsilon^*$  such that for  $\varepsilon \in (0, \varepsilon^*]$  and  $\nu_i^m = 0$  we have  $\|u_i(k)\| \leq \frac{m}{n}$  for all states with  $x_m'(k) P_m^\varepsilon x_m(k) \leq n - m + 1$ .

- **Terminal Step:** There exists a value for  $m$ , say  $l \leq n$ , such that  $A_l^\varepsilon$  has all its eigenvalues in the open unit disc, and also property 3 above is satisfied, which means that for  $\varepsilon$  small enough,  $\|u_i\| \leq 1$  for all states with  $x_l' P_l^\varepsilon x_l \leq 1$ . The decentralized control laws  $\Sigma_i^{l,\varepsilon}$ ,  $i = 1, \dots, l$  together construct our decentralized feedback law for system  $\Sigma$ .

Finally, we show that for an appropriate choice of  $\varepsilon$ , this recursive algorithm provides a set of decentralized feedbacks which satisfy the requirements of Theorem 2. We will first prove properties 1, 2 and 3 listed above by induction. It

is easy to see that the initialization step satisfies these properties. We assume that the design in the step  $m$  can be done, and then we must show that the design in the step  $m + 1$  can be done.

Now assume that we are in step  $m + 1$ . The closed loop system  $\Sigma_{cl}^{m,\varepsilon}$  has properties (1), (2) and (3). Let  $\lambda$  be an eigenvalue on the unit disc of  $A_m^\varepsilon$ . We know that  $\lambda$  is not a fixed mode of the closed loop system. Thus there exist  $\bar{K}_i$  such that

$$A_m^\varepsilon + \sum_{i=1}^{\nu} B_{m,i} \bar{K}_i C_{m,i}$$

has no eigenvalue at  $\lambda$ . Therefore the determinant of the matrix  $\lambda I - A_m^\varepsilon - \delta \sum_{i=1}^{\nu} B_{m,i} \bar{K}_i C_{m,i}$ , seen as a polynomial in  $\delta$ , is non-zero for  $\delta = 1$ , which implies that it is non-zero for almost all  $\delta > 0$ . This means that for almost all  $\delta > 0$

$$A_m^\varepsilon + \delta \sum_{i=1}^{\nu} B_{m,i} \bar{K}_i C_{m,i}$$

has no eigenvalue at  $\lambda$ . Let  $j$  be the largest integer such that

$$A_m^{\varepsilon,\delta} = A_m^\varepsilon + \delta \sum_{i=1}^j B_{m,i} \bar{K}_i C_{m,i}$$

has  $\lambda$  as an eigenvalue and the same number of eigenvalues on the unit disc as  $A_m^\varepsilon$  for small enough  $\delta$ . This implies that  $A_m^{\varepsilon,\delta}$  still has all its eigenvalues in the closed unit disc.

Using Lemma 2, we know that there exists a  $\bar{P}_m^{\varepsilon,\delta}$  such that

$$(A_m^{\varepsilon,\delta})' \bar{P}_m^{\varepsilon,\delta} A_m^{\varepsilon,\delta} - \bar{P}_m^{\varepsilon,\delta} \leq 0$$

while  $\bar{P}_m^{\varepsilon,\delta} \rightarrow P_m^\varepsilon$  as  $\delta \rightarrow 0$ . Hence for small enough  $\delta$

$$x'_m(k) P_m^{\varepsilon,\delta} x_m(k) \leq n - m + \frac{1}{2} \Rightarrow x'_m(k) P_m^\varepsilon x_m(k) \leq n - m + 1$$

and also for small enough  $\delta$  we have

$$\|\delta \bar{K}_i x_m\| \leq \frac{1}{2n} \quad \forall x_m \text{ such that } x'_m P_m^{\varepsilon,\delta} x_m \leq n - m + \frac{1}{2}$$

We choose  $\delta = \delta_\varepsilon$  small enough such that the above two properties hold. Define  $K_i^\varepsilon = \delta_\varepsilon \bar{K}_i$ ,  $\bar{P}_m^\varepsilon = \bar{P}_m^{\varepsilon,\delta_\varepsilon}$  and

$$\bar{A}_m^\varepsilon := A_m^\varepsilon + \sum_{i=1}^j B_{m,i} K_i^\varepsilon C_{m,i}$$

By the definition of  $j$ , we know that

$$A_m^\varepsilon + \sum_{i=1}^{j+1} B_{m,i} K_i^\varepsilon C_{m,i}$$

either does not have  $\lambda$  as an eigenvalue or has less eigenvalues on the unit circle. This means that

$$(\bar{A}_m^\varepsilon, B_{m,j+1}, C_{m,j+1})$$

has a stabilizable and detectable eigenvalue on the unit circle. Let  $V$  be such that

$$VV' = I \text{ and } \ker V = \ker \langle C_{m,j+1} | \bar{A}_m^\varepsilon \rangle.$$

Since we might not be able to find a stable observer for the state  $x_m$  we actually construct an observer for the observable

part of the state  $Vx_m$ . Because our triplet has a stabilizable and detectable eigenvalue on the unit disc, the observable part of the state  $Vx_m$  must contain at least one eigenvalue on the unit circle that can be stabilized. This motivates the following decentralized feedback law:

$$\begin{aligned} v_i^m(k) &= K_i^\varepsilon x_m(k) + v_i^{m+1}(k), \quad i = 1, \dots, j, \\ p(k+1) &= A_s^\varepsilon p(k) + V B_{m,j+1} v_{j+1}^m(k) \\ &\quad + K(C_{m,j+1} V' p(k) - y_{j+1}(k)) \\ v_{j+1}^m(k) &= F_\rho p(k) + v_{j+1}^{m+1}(k) \\ v_i^m(k) &= v_i^{m+1}(k), \quad i = j+2, \dots, \nu. \end{aligned}$$

Here  $p \in \mathbb{R}^s$  and  $A_s^\varepsilon$  is such that  $A_s^\varepsilon V = V \bar{A}_m^\varepsilon$  and  $K$  is chosen such that  $A_s^\varepsilon + K C_{m,j+1} V'$  has all its eigenvalues in the open unit disc and does not have any eigenvalues in common with  $\bar{A}_m^\varepsilon$ . Furthermore  $F_\rho$  is chosen in a way that  $\bar{A}_m^\varepsilon + B_{m,j+1} F_\rho V$  has at least one less eigenvalue on the unit disc than  $A_m^\varepsilon$  and still all of its eigenvalues are in the closed unit disc and also  $F_\rho \rightarrow 0$  as  $\rho \rightarrow 0$ . Defining

$$\bar{x}_{m+1} = \begin{pmatrix} x_m \\ p - Vx_m \end{pmatrix},$$

we have

$$\begin{aligned} \bar{x}_{m+1}(k+1) &= \\ &\begin{pmatrix} \bar{A}_m^\varepsilon + B_{m,j+1} F_\rho V & B_{m,j+1} F_\rho \\ 0 & A_s^\varepsilon + K C_{m,j+1} V' \end{pmatrix} \bar{x}_{m+1}(k) \\ &\quad + \sum_{i=1}^{\nu} \bar{B}_{m+1,i} v_i^{m+1}(k) \end{aligned} \quad (6)$$

$$y_i(k) = \bar{C}_{m+1,i} \bar{x}_{m+1}(k) \quad i = 1, \dots, \nu,$$

where

$$\bar{B}_{m+1,i} = \begin{pmatrix} B_{m,i} \\ -V B_{m,i} \end{pmatrix}, \quad \bar{C}_{m+1,i} = (C_{m,i} \quad 0)$$

for  $i \neq j+1$  and

$$\bar{B}_{m+1,j+1} = \begin{pmatrix} B_{m,j+1} \\ 0 \end{pmatrix}, \quad \bar{C}_{m+1,j+1} = \begin{pmatrix} C_{m,j+1} & 0 \\ V & I \end{pmatrix}$$

It is easy to check that the above feedback laws satisfy the properties (1) and (2). What remains is to show that they satisfy property (3). Also we need to show that the control laws can be written in the form mentioned in step  $m$  for step  $m + 1$ .

For any  $\varepsilon$  there exists a  $R_m^\varepsilon > 0$  with

$$(A_s^\varepsilon + K C_{m,j+1} V')' R_m^\varepsilon (A_s^\varepsilon + K C_{m,j+1} V') - R_m^\varepsilon < 0$$

such that  $R_m^\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Because  $F_\rho \rightarrow 0$  as  $\rho \rightarrow 0$ , for each  $\varepsilon$ , for small enough  $\rho$  we have

$$\|F_\rho e\| < \frac{1}{2n} \quad \forall e \mid e' R_m^\varepsilon e \leq n - m + \frac{1}{2}.$$

Note that  $\bar{A}_m^\varepsilon + B_{m,j+1} F_\rho V$  has at least one less eigenvalue on the unit disc than  $\bar{A}_m^\varepsilon$  and has all its eigenvalues in the close unit disc. Applying Lemma (2), for small  $\rho$  we have

$$(\bar{A}_m^\varepsilon + B_{m,j+1} F_\rho V)' \bar{P}_\rho^\varepsilon (\bar{A}_m^\varepsilon + B_{m,j+1} F_\rho V) - \bar{P}_\rho^\varepsilon \leq 0$$

with  $\bar{P}_\rho^\varepsilon \rightarrow \bar{P}_m^\varepsilon$  as  $\rho \rightarrow 0$

Now note that  $\bar{A}_m^\varepsilon$  and  $A_s^\varepsilon + KC_{m,j+1}V'$  have disjoint eigenvalues we find that for small  $\rho$ , the matrices  $\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V$  and  $A_s^\varepsilon + KC_{m,j+1}V'$  have disjoint eigenvalues since  $F_\rho \rightarrow 0$  as  $\rho \rightarrow 0$ . But then there exists a  $W_{\varepsilon,\rho}$  such that

$$B_{m,j+1}F_\rho + (\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V)W_{\varepsilon,\rho} - W_{\varepsilon,\rho}(A_s^\varepsilon + KC_{m,j+1}V') = 0$$

while  $W_{\varepsilon,\rho} \rightarrow 0$  as  $\rho \rightarrow 0$ . Now if we define  $\bar{P}_{m+1}^{\varepsilon,\rho}$  to be

$$\bar{P}_{m+1}^{\varepsilon,\rho} = \begin{pmatrix} I & 0 \\ -W_{\varepsilon,\rho}' & I \end{pmatrix} \begin{pmatrix} \bar{P}_\rho^\varepsilon & 0 \\ 0 & R_m^\varepsilon \end{pmatrix} \begin{pmatrix} I & -W_{\varepsilon,\rho} \\ 0 & I \end{pmatrix}$$

We define

$$\bar{A}_{m+1}^{\varepsilon,\rho} = \begin{pmatrix} \bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V & B_{m,j+1}F_\rho \\ 0 & A_s^\varepsilon + KC_{m,j+1}V' \end{pmatrix}$$

We will have the following properties

$$(\bar{A}_{m+1}^{\varepsilon,\rho})' \bar{P}_{m+1}^{\varepsilon,\rho} \bar{A}_{m+1}^{\varepsilon,\rho} - \bar{P}_{m+1}^{\varepsilon,\rho} \leq 0$$

and

$$\lim_{\rho \rightarrow 0} \bar{P}_{m+1}^{\varepsilon,\rho} = \begin{pmatrix} \bar{P}_m^\varepsilon & 0 \\ 0 & R_m^\varepsilon \end{pmatrix}$$

Now consider  $\bar{x}_{m+1}$  such that

$$\bar{x}_{m+1}' \bar{P}_{m+1}^{\varepsilon,\rho} \bar{x}_{m+1} \leq n - m$$

Then with small enough choice of  $\rho$  we can have

$$x_m' \bar{P}_m^\varepsilon x_m \leq n - m + \frac{1}{2} \quad \text{and}$$

$$(p - Vx_m)' R_m^\varepsilon (p - Vx_m) \leq n - m + \frac{1}{2}$$

Next for each  $\varepsilon$  we choose  $\rho = \rho_\varepsilon$  such that the above holds and we have

$$\|F_\rho Vx_m\| < \frac{1}{2n} \quad \forall x_m \mid x_m' \bar{P}_m^\varepsilon x_m \leq n - m + \frac{1}{2}$$

Next we must check the bounds on the inputs in step  $m+1$ . For  $i = 1, \dots, j$ , we have

$$\|u_i\| = \|F_{i,\varepsilon}^m x_m + K_i^\varepsilon x_m\| \leq \frac{m}{n} + \frac{1}{2n} \leq \frac{m+1}{n}$$

For  $i = j+1$ , we have:

$$\begin{aligned} \|u_i\| &= \|F_{i,\varepsilon}^m x_m + F_{\rho_\varepsilon} p\| \\ &= \|F_{i,\varepsilon}^m x_m + F_{\rho_\varepsilon} Vx_m + F_{\rho_\varepsilon} (p - Vx_m)\| \\ &\leq \frac{m}{n} + \frac{1}{2n} + \frac{1}{2n} = \frac{m+1}{n} \end{aligned}$$

Finally, for  $i = j+2, \dots, \nu$ , we have:

$$\|u_i\| = \|F_{i,\varepsilon}^m x_m\| \leq \frac{m}{n} \leq \frac{m+1}{n}$$

Now for  $i \neq j+1$  we set  $n_{i,m+1} = n_{i,m}$  and for  $i = j+1$  we set  $n_{i,m+1} = n_{i,m} + s$ .

If  $n_{i,m} > 0$  we choose

$$p_i^{m+1} = \begin{pmatrix} p_i^m \\ p \end{pmatrix}$$

and if  $n_{i,m} = 0$  we choose  $p_i^{m+1} = p$ . Now we are able to the system in terms of  $x_{m+1}$ . We introduce a basis transformation  $T_{m+1}$  such that  $\bar{x}_{m+1} = T_{m+1}x_{m+1}$ . Next, we define

$$P_{m+1}^\varepsilon = T_{m+1}' \bar{P}_{m+1}^{\varepsilon,\rho_\varepsilon} T_{m+1}.$$

Now for  $i = 1, \dots, \nu$  depending on the value of  $n_{i,m+1}$  we can rewrite the control laws in the desired form and subsequently the properties (1), (2) and (3) are obtained.

We know that there exists a value of  $m$ , say  $\ell \leq n$ , such that  $A_\ell^\varepsilon$  has all its eigenvalues in the open unit disc. We set  $\nu_i^\ell = 0$  for  $i = 1, 2, 3, \dots, \ell$ . Then the decentralized control laws  $\Sigma_i^{\ell,\varepsilon}$ ,  $i = 1, 2, 3, \dots, \ell$  together represent a decentralized semi global feedback law for the system  $\Sigma$ . In other words, we claim that for any given compact sets  $W \subset \mathbb{R}^n$  and  $S_i \subset \mathbb{R}^{n_i,\ell}$  for  $i = 1, 2, 3, \dots, \ell$ , there exists an  $\varepsilon^*$  such that the origin of the closed loop system is exponentially stable for any  $0 < \varepsilon < \varepsilon^*$  and the compact set  $W \times S_1 \times \dots \times S_\nu$  is within the domain of attraction. Furthermore for all initial conditions within  $W \times S_1 \times \dots \times S_\nu$ , the closed loop system behaves like a linear system, that is the saturation is not activated.

We know that for  $\varepsilon$  small enough, the set

$$\Omega_1^\varepsilon := \{x_\ell \in \mathbb{R}^{n_\ell} \mid x_\ell' P_\ell^\varepsilon x_\ell \leq 1\}$$

is inside the domain of attraction of the equilibrium point of the closed loop system comprising the given system  $\Sigma$  and the decentralized control laws  $\Sigma_i^{\ell,\varepsilon}$ ,  $i = 1, 2, 3, \dots, \ell$  because for all initial conditions within  $\Omega_1^\varepsilon$ , it is obvious that  $\|u_i\| \leq 1$ ,  $i = 1, 2, 3, \dots, \ell$  which means that the closed loop system behaves like a linear system, that is the saturation is not activated. Furthermore since all of the eigenvalues of  $A_\ell^\varepsilon$  are in the open unit disc, this linear system is asymptotically stable. In addition because of the fact that  $P_\ell^\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we find that  $W \times S_1 \times \dots \times S_\nu$  is inside  $\Omega_1^\varepsilon$  for  $\varepsilon$  sufficiently small. This concludes that the decentralized control laws  $\Sigma_i^{\ell,\varepsilon}$ ,  $i = 1, 2, 3, \dots, \ell$  are semi-globally stabilizing.

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