An LMI-based Stability Analysis for Adaptive Controllers

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***Abstract*— We develop a Linear Matrix Inequality (LMI) tool for analyzing the stability and performance of adaptive controllers that employ $\sigma-$modification. The formulation involves recasting the error dynamics composed of the tracking error and the weight estimator error into a linear parameter varying form. We show how stability, convergence rate, domain of attraction, and the transient and steady state behavior of the adaptive control system can be analyzed using the developed LMI tool. It is guaranteed that less conservative estimates for the convergence rate and the size of the ultimate bound for the tracking error are obtained compared to the standard analysis in the literature.**

I. Introduction

Despite numerous practical applications and success stories of adaptive control, adaptive control is still not well accepted in safety-critical applications with its main criticism being the lack of an appropriate analysis tool to quantify stability characteristics, such as convergence rate, domain of attraction, transient behavior of the tracking error, and the size of the ultimate bound on the tracking error in the presence of bounded external disturbances. Unlike linear system theory which is based on exponential stability, adaptive controllers can only guarantee asymptotic convergence of the tracking error, and its robustness to uncertainties that are not perfectly parametrized is in question [1]. Unlike exponential stability, asymptotic stability does not guarantee the closed loop system remains asymptotically stable when disturbed. This situation makes it essential to employ additional modifications to adaptive laws in practice because industrial applications can have variety of uncertainties not supported by the parametrization assumed in adaptive control. The lack of exponential stability has also been an major obstacle in applying LMI analysis for adaptive control because conventional LMI methods require local exponential stability for being applied to nonlinear systems [10].

In this paper, we employ $\sigma-$modification term [2] as an essential ingredient and develop a Linear Matrix Inequalities (LMI) tool for analysis of adaptive systems. A key procedure in this study is to cast the error dynamics, composed of the tracking error and the weight estimation error, into a linear parameter varying (LPV) form in which an *exponentially stable* system is perturbed by a constant external disturbance. In particular, we show that affine parameterization is possible. By this formulation, we reveal that a series of standard LMI analysis tools can also be employed for the stability analysis of adaptive controllers, which include a guaranteed convergence rate, domain of attraction, and the size of an ultimate bound on the tracking error. Moreover, by showing that the standard analysis in the literature is a special solution to formulated LMIs, it is establshied that the formulated LMIs are guaranteed to provide less conservative analysis for adaptive controllers.

The paper is organized as follows. We go through a control architecture in Section II in which nominal linear controllers are augmented by adaptive controllers as in [3]–[5]. In Section III, we present the standard results on the quadratic stability. In Section IV, we show how

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adaptive control system can be cast into a LPV form and present several LMI analysis tools. We conclude the paper in Section V. Throughout the manuscript, $\|\cdot\|$ means Euclidean norm for a vector and the induced 2-norm for a matrix.

II. Adaptive Control with $\sigma-$ modification

Consider a single-input single-output system described by:

$$\begin{aligned}\dot{\boldsymbol{x}} =& A\boldsymbol{x} + \boldsymbol{b}(u + \boldsymbol{W}^\top \boldsymbol{\phi}(\boldsymbol{x}))\\ y =& \boldsymbol{c}^\top \boldsymbol{x},\end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $\boldsymbol{W} \in \mathbb{R}^N$ is a uncertain parameter vector, $\boldsymbol{\phi}(\boldsymbol{x}) \in \mathbb{R}^N$ is a known set of smooth basis functions, and the system matrices $A, \boldsymbol{b}, \boldsymbol{c}^\top$ are known. A nominal linear controller:

$$u_{nom} = -\boldsymbol{K}_x^\top \boldsymbol{x} + K_r r(t), \tag{2}$$

is assumed to be designed such that the resulting closed-loop system with $\boldsymbol{W} = 0$ defines the following reference model for the desired system behavior,

$$\begin{aligned}\dot{\boldsymbol{x}}_m =& A_m \boldsymbol{x}_m + \boldsymbol{b}_m r(t)\\ y_m =& \boldsymbol{c}^\top \boldsymbol{x}_m,\end{aligned} \tag{3}$$

where $A_m = A - \boldsymbol{b}\boldsymbol{K}_x^\top$ is Hurwitz, $\boldsymbol{b}_m = \boldsymbol{b}K_r$, $r(t)$ is a bounded reference command, and the subscript m is used to represent the reference model.

Let

$$u = u_{nom} - u_{ad}, \tag{4}$$

where u_{ad} is an adaptive signal to approximately cancel the uncertainty $\boldsymbol{W}^\top \boldsymbol{\phi}(\boldsymbol{x})$ that is given by:

$$u_{ad} = \widehat{\boldsymbol{W}}(t)^\top \boldsymbol{\phi}(\boldsymbol{x}), \tag{5}$$

whose estimate $\widehat{\boldsymbol{W}}(t)$ for the unknown parameter vector $\boldsymbol{W}$ in (1) is updated using:

$$\dot{\widehat{\boldsymbol{W}}} = -\gamma \boldsymbol{\phi}(\boldsymbol{x}) \boldsymbol{e}^\top P \boldsymbol{b}, \tag{6}$$

where $\gamma > 0 \in \mathbb{R}$ is the adaptation gain, and $P > 0$ is obtained by solving the following Lyapunov function with a chosen $Q > 0$:

$$A_m^\top P + P A_m + Q = 0. \tag{7}$$

The standard stability result of adaptive control has been associated with the tracking error:

$$\boldsymbol{e} = \boldsymbol{x}_m - \boldsymbol{x}, \tag{8}$$

whose dynamics are described by:

$$\dot{\boldsymbol{e}} = A_m \boldsymbol{e} + \boldsymbol{b}\widetilde{\boldsymbol{W}}(t)^\top \boldsymbol{\phi}(\boldsymbol{x}), \tag{9}$$

where $\widetilde{\boldsymbol{W}}(t) = \widehat{\boldsymbol{W}}(t) - \boldsymbol{W}$ is the weight estimation error.

Let

$$\boldsymbol{\zeta} = [\boldsymbol{e}^\top, \widetilde{\boldsymbol{W}}^\top]^\top. \tag{10}$$

Then the error dynamics composed of the tracking error and the weight estimation error are described by:

$$\dot{\boldsymbol{\zeta}} = \begin{bmatrix} A_m & \boldsymbol{b}\boldsymbol{\phi}(\boldsymbol{x})^\top \\ -\gamma \phi(\boldsymbol{x})\boldsymbol{b}^\top P & 0 \end{bmatrix} \boldsymbol{\zeta}. \tag{11}$$

A stability analysis for the system in (11) is typically carried out by considering the following Lyapunov candidate function:

$$V_0(\boldsymbol{\zeta}) = \boldsymbol{e}^\top P \boldsymbol{e} + \frac{1}{\gamma}\widetilde{\boldsymbol{W}}(t)^\top \widetilde{\boldsymbol{W}}(t) = \boldsymbol{\zeta}^\top X_0 \boldsymbol{\zeta}, \tag{12}$$

where

$$X_0 = \begin{bmatrix} P & 0 \\ 0 & \gamma^{-1} I_N \end{bmatrix}. \tag{13}$$

The time derivative of V along with (9) and (6) becomes:

$$\dot{V}_0(\boldsymbol{\zeta}) = -\boldsymbol{e}^\top Q \boldsymbol{e} \le 0. \tag{14}$$

This guarantees that $\boldsymbol{\zeta} \in \mathcal{L}_\infty$. Further analysis shows that 1) $\boldsymbol{e} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and the tracking error converges to the origin by Barbalat's lemma ($\boldsymbol{e}(t) \to 0$ as $t \to \infty$) 2) the weight errors remain bounded $\widetilde{\boldsymbol{W}}(t) \in \mathcal{L}_\infty$, and $\lim_{t\to\infty} \dot{\widetilde{\boldsymbol{W}}}(t) = \lim_{t\to\infty} \dot{\widehat{\boldsymbol{W}}}(t) = 0$. This result implies that the only guaranteed stability results for adaptive control are asymptotic convergence of the tracking error and boundedness of the weight estimate, which is much weaker than exponential stability. One consequence is that adaptive control systems are not guaranteed to remain bounded when disturbed. A prominent result in this regard is the possibility of parameter drift in the presence of a bounded external disturbance [6]. Whereas stable linear systems remain bounded in the presence of bounded disturbances, adaptive control systems may exhibit unbounded parameter drift even for bounded disturbances. This has led to various modifications [6] for guaranteeing boundedness of closed-loop signals.

In this paper, we employ $\sigma-$modification [2] and make use of LMIs to study the stability properties of the resulting adaptive control system. Towards this end, the update law (6) is modified to:

$$\dot{\widehat{\boldsymbol{W}}}(t) = -\gamma \boldsymbol{\phi}(\boldsymbol{x}) \boldsymbol{e}^\top P \boldsymbol{b} - \sigma \widehat{\boldsymbol{W}}(t), \tag{15}$$

which is equivalent to:

$$\dot{\widetilde{\boldsymbol{W}}}(t) = -\gamma \boldsymbol{\phi}(\boldsymbol{x}) \boldsymbol{e}^\top P \boldsymbol{b} - \sigma \widetilde{\boldsymbol{W}}(t) - \sigma \boldsymbol{W}. \tag{16}$$

Due to modification, the combined error $\boldsymbol{\zeta}(t)$ evolves according to:

$$\dot{\boldsymbol{\zeta}} = \underbrace{\begin{bmatrix} A_m & \boldsymbol{b}\boldsymbol{\phi}(\boldsymbol{x})^\top \\ -\gamma\phi(\boldsymbol{x})\boldsymbol{b}^\top P & -\sigma I_N \end{bmatrix}}_{\bar{A}(\boldsymbol{x})} \boldsymbol{\zeta} + \underbrace{\begin{bmatrix} 0 \\ -I_N \end{bmatrix}}_{\bar{B}} \sigma\boldsymbol{W}$$
$$\boldsymbol{e} = \underbrace{\begin{bmatrix} I_n & \mathbf{0}_{n\times N} \end{bmatrix}}_{\bar{C}} \boldsymbol{\zeta}. \tag{17}$$

Note that the tracking error $\boldsymbol{e}$ is considered as a performance variable in the dynamics in (17). Stability analysis using the Lyapunov candidate in (12) leads to:

$$\begin{aligned} \dot{V}_0(\boldsymbol{\zeta}) &= -\boldsymbol{e}^\top Q\boldsymbol{e} - 2\frac{\sigma}{\gamma}\widetilde{\boldsymbol{W}}^\top\widehat{\boldsymbol{W}} \\ &= -\boldsymbol{e}^\top Q\boldsymbol{e} - \frac{\sigma}{\gamma}[\left\|\widetilde{\boldsymbol{W}}\right\|^2 + \left\|\widehat{\boldsymbol{W}}\right\|^2 - \|\boldsymbol{W}\|^2] \\ &\le -\lambda_{\min}(Q)\|\boldsymbol{e}\|^2 - \frac{\sigma}{\gamma}\left\|\widetilde{\boldsymbol{W}}\right\|^2 + \frac{\sigma}{\gamma}\|\boldsymbol{W}\|^2 \\ &\le -c_1\|\boldsymbol{\zeta}\|^2 + \frac{\sigma}{\gamma}\|\boldsymbol{W}\|^2, \end{aligned} \tag{18}$$

where $c_1 = \min\{\lambda_{\min}(Q), \frac{\sigma}{\gamma}\}$. Whenever $\|\boldsymbol{\zeta}\| \ge \sqrt{\frac{\sigma}{\gamma c_1}}\|\boldsymbol{W}\|$, $\dot{V}_0 \le 0$. Hence, $\boldsymbol{\zeta}(t)$ is uniformly ultimately bounded (UUB). Note that by employing $\sigma-$modification, the resulting stability proof has weakened to UUBness of the closed-loop signals. However, with σ-modification it is possible to show that the adaptive system remains bounded in the presence of bounded external disturbances because the analysis in (18) ensures that $\bar{A}(\boldsymbol{x})$ in (17) is exponentially stable.

III. Quadratic Stability of Linear Affine-Parameter Models

The viewpoint taken in our effort is to consider the system in (17) as a linear parameter varying (LPV) system and employ LMI tools of analysis. A major complexity of a stability analysis for LPV systems is associated with how parameters are characterized [7]. Before we describe the dynamics in (17) in LPV form, we go over standard results on affine quadratic stability in this section.

Proposition 1: ([8]) Let $f : \mathcal{S} \to \mathbb{R}$ be a convex function where $\mathcal{S} := co(\mathcal{S}_0)$. Then $f(x) \le \gamma$ for all $x \in \mathcal{S}$ iff $f(x) \le \gamma$ for all $x \in \mathcal{S}_0$.

Consider a system described by

$$\dot{\boldsymbol{\zeta}} = \bar{A}(\boldsymbol{\rho}(t))\boldsymbol{\zeta}, \tag{19}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^d$ is the state vector, and the state matrix $\bar{A}(\boldsymbol{\rho})$ is a function of a real valued parameter vector $\boldsymbol{\rho} := (\rho_1, \ldots, \rho_k) \in \mathbb{R}^k$.

Definition 1: ([8]) The system (19) is *quadratically stable for perturbation* $\mathcal{P}$ if there exists a matrix $X = X^\top$ such that

$$\bar{A}(\boldsymbol{\rho}(t))^\top X + X\bar{A}(\boldsymbol{\rho}(t)) < 0 \tag{20}$$

for all perturbations $\boldsymbol{\rho} \in \mathcal{P}$.
Quadratic stability for perturbations $\mathcal{P}$ is equivalent to the existence of a quadratic Lyapunov function $V(\boldsymbol{\zeta}) = \boldsymbol{\zeta}^\top X\boldsymbol{\zeta}$, $X > 0$ such that $\dot{V} = \boldsymbol{\zeta}^\top[\bar{A}(\boldsymbol{\rho})^\top X + X\bar{A}(\boldsymbol{\rho})]\boldsymbol{\zeta} < 0$ for all $\boldsymbol{\rho} \in \mathcal{P}$. Note that in general quadratic stability of the system for an uncertainty class $\mathcal{P}$ places an infinite number of constraints on the symmetric matrix X. For the quadratic stability problem to be numerically tractable, additional assumptions on the way the uncertainty enters the system need to be introduced. Suppose that $\bar{A}(\boldsymbol{\rho})$ is an affine function of the parameter vector $\boldsymbol{\rho}$. That is, suppose that there exits real matrices $A_0, \ldots, A_k$, all of dimension $d \times d$, such that

$$\bar{A}(\boldsymbol{\rho}) = A_0 + \rho_1(t)A_1 + \ldots + \rho_k(t)A_k \tag{21}$$

for all $\boldsymbol{\rho} \in \mathcal{P}$. Suppose that the uncertain parameters $\rho_j(t), j = 1, \ldots, k, t \in \mathbb{R}$ assume their values in an interval $[\underline{\rho}_j, \overline{\rho}_j]$, i.e.,

$$\rho_j(t) \in [\underline{\rho}_j, \overline{\rho}_j].$$

Define the set of *corners* of the uncertainty region as

$$\mathcal{P}_0 := \{\boldsymbol{\rho} = (\rho_1, \ldots, \rho_k) \;:\; \rho_j \in \{\underline{\rho}_j, \overline{\rho}_j\}, j = 1, \ldots, k\}. \tag{22}$$

Proposition 2: ([8]) If the system in (19) is an affine parameter dependent model then it is quadratically stable iff there exists $X = X^\top > 0$ such that

$$A(\boldsymbol{\rho})^\top X + XA(\boldsymbol{\rho}) < 0, \tag{23}$$

for all $\boldsymbol{\rho} \in \mathcal{P}_0$.
The importance of this result lies in the fact that the quadratic stability can be concluded from a *finite set* of matrix inequalities.

IV. LMI Analysis

A. Affine Parametrization

Applying Proposition 2 requires that $\bar{A}(\boldsymbol{x})$ in (17) be written as an affine model. Towards this end, we set a compact domain of interest, Ω_x, such that $\boldsymbol{x}(t) \in \Omega_x$ for all $t \ge 0$ and $\boldsymbol{x}(0) \in \Omega_x$. Without loss of generality, Ω_x is defined as a hypercube whose corners are $\Omega_{x_0} := \{\boldsymbol{x} = (x_1, \ldots, x_n) \;:\; x_j \in \{\underline{x}_j, \overline{x}_j\}, j = 1, \ldots, n\}$. Note that Ω_{x_0} is a set of corners: $\boldsymbol{x}_i \in \Omega_{x_0}$, $i = 1, \ldots, 2^n$.

Let $\bar{A}(\boldsymbol{x})$ in (17) be decomposed as follows:

$$\bar{A}(\boldsymbol{x}) = \underbrace{\begin{bmatrix} A_m & \mathbf{0}_{n\times N} \\ \mathbf{0}_{N\times n} & -\sigma I_N \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{b}\boldsymbol{\phi}(\boldsymbol{x})^\top \\ -\gamma\phi(\boldsymbol{x})\boldsymbol{b}^\top P & \mathbf{0}_{N\times N} \end{bmatrix}}_{A_r(\boldsymbol{x})}. \tag{24}$$

Then, Figure 1 illustrates in the two-dimensional Euclidean space the complexity associated with parametrization when one attempts to directly obtain vertices $A_i = \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{b}\boldsymbol{\phi}(\boldsymbol{x}_i)^\top \\ -\gamma\phi(\boldsymbol{x}_i)\boldsymbol{b}^\top P & \mathbf{0}_{N\times N} \end{bmatrix}$ such that $A_r(\boldsymbol{x}) \in co\{A_i\}$ from Ω_{x_0}. Since $\boldsymbol{\phi}(\boldsymbol{x})$ may distort the rectangle in the transformed space, it is generally not guaranteed that $\boldsymbol{\phi}(\boldsymbol{x}) \in co\{\boldsymbol{\phi}(\boldsymbol{x}_i)\}$. Nevertheless, for a simple LMI technique to be applicable to the system in (17), it is desirable for the resulting parametrization to be convex and independent of the rate at which parameter variations occur. It has been shown that considering the time-rate of parameter variation leads to partial differential LMIs and makes the analysis problem complicated [7].

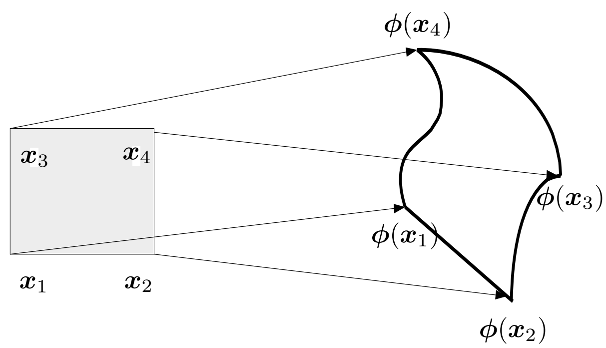

Fig. 1. Diagram for mapping $\boldsymbol{\phi}(\boldsymbol{x})$

In this paper, we note that the basis function $\boldsymbol{\phi}(\boldsymbol{x}) = [\phi_1(\boldsymbol{x}), \ldots, \phi_N(\boldsymbol{x})]^\top$ is known and the domain $\Omega_x = co\{\boldsymbol{x}_i\}$ is compact, and hence we can calculate the domain to which each basis function belongs, i.e., $\phi_j(\boldsymbol{x}) \in [\min(\phi_j(\boldsymbol{x})), \max(\phi_j(\boldsymbol{x}))] = [\underline{\phi}_j, \overline{\phi}_j]$. Moreover, this parametrization leads to:

$$A_r(\boldsymbol{x}) = \sum_{j=1}^{N} \phi_j(\boldsymbol{x}) A_j, \tag{25}$$

where $A_j \in \mathbb{R}^{(n+N)\times(n+N)}$ is a matrix such that $A_j(1:n, k) = \boldsymbol{b},\ A_j(k, 1:n) = -\gamma\boldsymbol{b}^\top P$ if $k = j$, and $A_j(k,l) = 0$ otherwise ($k \neq j$ nor $l \neq j$). The notation $1:n$ is used to represent indices from 1 to n. Then, it is immediately clear that the parametrization

$$\bar{A}(\boldsymbol{x}) = A_0 + \sum_{j=1}^{N} \phi_j(\boldsymbol{x}) A_j = \bar{A}(\boldsymbol{\rho}(t)) \tag{26}$$

is affine with respect to $\boldsymbol{\rho}(t) = \boldsymbol{\phi}(\boldsymbol{x}(t))$ and $\boldsymbol{\rho} \in \mathcal{P} := co(\mathcal{P}_0)$ where $\mathcal{P}_0 := \{\boldsymbol{\rho} = (\phi_1, \ldots, \phi_N) \ : \ \phi_j \in \{\underline{\phi}_j, \overline{\phi}_j\}, j = 1, \ldots, N\}$ (In regard to Section III, we can see that $k = N$ and $d = n + N$).

Remark 1: For a neural network (NN)-based adaptive approach, the basis function $\boldsymbol{\phi}(\boldsymbol{x})$ have a uniform structure, and obtaining min and max values for the basis vector on the compact domain collapses into finding those values with a single basis function.

B. LMI Analysis

The rationale in performing our LMI-based stability analysis is that the dynamics in (17) is viewed as an exponentially stable system perturbed by the unknown but constant disturbance $\sigma\boldsymbol{W}$. Exponential stability for the homogeneous system in (17)

$$\dot{\boldsymbol{\zeta}} = \bar{A}(\boldsymbol{x})\boldsymbol{\zeta} = \bar{A}(\boldsymbol{\rho})\boldsymbol{\zeta} \tag{27}$$

is checked as a feasibility problem of the following LMI by Proposition 2.

Lemma 1: The system in (27) is exponentially stable if there exists $X = X^\top > 0$ such that

$$\bar{A}(\boldsymbol{\rho})^\top X + X\bar{A}(\boldsymbol{\rho}) < 0,\ \forall \boldsymbol{\rho} \in \mathcal{P}_0. \tag{28}$$

Note that the analysis in (18) guarantees the feasibility of the LMI in (28) because X_0 in (13) satisfies (28). Once exponential stability is established, the UUBness of $\boldsymbol{\zeta}$ is ensured by the following lemma.

Lemma 2: Suppose that there exists $X = X^\top > 0,\ \mu > 0$ such that

$$\bar{A}(\boldsymbol{\rho})^\top X + X\bar{A}(\boldsymbol{\rho}) < -\mu I,\ \forall \boldsymbol{\rho} \in \mathcal{P}_0. \tag{29}$$

Then the system in (17) is UUB.

Proof: Consider $V(\boldsymbol{\zeta}) = \boldsymbol{\zeta}^\top X \boldsymbol{\zeta}$. For $\boldsymbol{\zeta} \in \mathbb{R}^{n+N}$, define $f_\zeta(\boldsymbol{\rho}) := \boldsymbol{\zeta}^\top[\bar{A}(\boldsymbol{\rho})^\top X + X\bar{A}(\boldsymbol{\rho}) + \mu I]\boldsymbol{\zeta}$ with $\boldsymbol{\rho} \in \mathcal{P}$. Since $f_\zeta(\boldsymbol{\rho})$ is an affine function of $\boldsymbol{\rho}$, $f_\zeta(\boldsymbol{\rho})$ is a convex function of $\boldsymbol{\rho}$. By Proposition 1, Eq. (29) is equivalent to $f_\zeta(\boldsymbol{\rho}) < 0$ for $\forall \boldsymbol{\rho} \in \mathcal{P}$. Therefore, we have $\dot{V}(\boldsymbol{\zeta}) = f_\zeta(\boldsymbol{\rho}) - \mu\|\boldsymbol{\zeta}\|^2 + 2\boldsymbol{\zeta}^\top X\bar{B}\sigma\boldsymbol{W} \leq -\mu\|\boldsymbol{\zeta}\|^2 + 2\|\boldsymbol{\zeta}\|\left\|X\bar{B}\right\|\sigma\|\boldsymbol{W}\| \leq -\mu\|\boldsymbol{\zeta}\|\left[\|\boldsymbol{\zeta}\| - d_\zeta\right]$, where $d_\zeta = 2\sigma/\mu\left\|X\bar{B}\right\|\|\boldsymbol{W}\|$. Therefore, $\dot{V} \leq 0$ whenever $\|\boldsymbol{\zeta}\| \geq d_\zeta$, and $\boldsymbol{\zeta}(t)$ is UUB. ■

For an LMI implementation (for example MATLAB [9]), the feasibility test in Lemma 1 can be solved by obtaining the maximal μ in Lemma 2. Moreover, affine parametrization allows for maximizing the quadratic stability region. That is, even if the study of stability characteristics for the system in (17) is initiated with setting up the compact domain Ω_x, the resulting affine parametrization in (26) in turn enables us to find how far the domain can be expanded while still guaranteeing quadratic stability. The expanded domain can be derived as follows. For $\boldsymbol{\rho} \in \mathcal{P}$, each parameter $\rho_i(t) \in [\underline{\rho}_i, \overline{\rho}_i]$. Let $\rho_{c_i} = \frac{1}{2}(\underline{\rho}_i + \overline{\rho}_i)$ and $r_i = \frac{1}{2}(\overline{\rho}_i - \underline{\rho}_i)$ be the center and radius of each interval. Then, an LMI tool can solve the largest dilation factor θ such that the system remains quadratically stable whenever $\rho_i \in [\rho_{c_i} - \theta r_i, \rho_{c_i} + \theta r_i]$ by formulating it as a general eigenvalue problem [9], [10]. Let $\mathcal{P}_e := \{\boldsymbol{\rho} \ : \ \rho_i \in [\rho_{c_i} - \theta r_i, \rho_{c_i} + \theta r_i], \ i = 1, \ldots, N\}$, then $\mathcal{P}_e \supset \mathcal{P}$ if $\theta \geq 1$. The domain of attraction for $\boldsymbol{x}$ can then be determined as the pre-image of $\mathcal{P}_e$, $\boldsymbol{\phi}^{-1}(\mathcal{P}_e)$.

Lemma 3: Suppose that there exists $X = X^\top > 0, \ \mu > 0$ such that

$$\bar{A}(\boldsymbol{\rho})^\top X + X\bar{A}(\boldsymbol{\rho}) < -\mu X, \ \forall \boldsymbol{\rho} \in \mathcal{P}_0. \tag{30}$$

Then $\boldsymbol{\zeta}(t)$ in (27) is exponentially bounded by:

$$\begin{aligned} \|\boldsymbol{\zeta}(t)\| \leq \sqrt{\kappa(X)}\, \|\boldsymbol{\zeta}(0)\|\, e^{-\frac{\mu}{2}t} \\ + 2\sqrt{\kappa(X)}\sigma \|\boldsymbol{W}\| [1 - e^{-\frac{\mu}{2}t}], \end{aligned} \tag{31}$$

where $\kappa(X) = \lambda_{\max}(X)/\lambda_{\min}(X)$.

Proof: Let $\boldsymbol{\zeta}(t) = \Phi(t,0)\boldsymbol{\zeta}(0)$ be a solution to the system in (27), where $\Phi(t,0)$ is the transition matrix [11]. Consider $V(\boldsymbol{\zeta}) = \boldsymbol{\zeta}^\top X \boldsymbol{\zeta}$. Then following similar lines as in Lemma 2, we have: $\dot{V}(t) \leq -\mu V$, which leads to $V(t) \leq V(0)e^{-\mu t}$ for all $\boldsymbol{\rho} \in \mathcal{P}$. From the fact that $\lambda_{\min}(X) \|\boldsymbol{\zeta}\|^2 \leq V(\boldsymbol{\zeta}) \leq \lambda_{\max}(X) \|\boldsymbol{\zeta}\|^2$ it follows that $\|\boldsymbol{\zeta}(t)\|^2 \leq \kappa(X) e^{-\mu t} \|\boldsymbol{\zeta}(0)\|^2$. Since $\boldsymbol{\zeta}(0)$ is arbitrary,

$$\|\Phi(t,0)\| \leq \sqrt{\kappa(X)} e^{-\frac{\mu}{2}t}. \tag{32}$$

From this, the solution for the system in (17) is derived as $\boldsymbol{\zeta}(t) = \Phi(t,0)\boldsymbol{\zeta}(0) + \int_0^t \Phi(t,s)\bar{B}\sigma W ds$. Considering that $\|\bar{B}\| \leq 1$ together with (32) leads to (31). ■

Lemma 3 implies that the constant external disturbance $\sigma\boldsymbol{W}$ in (17) does not change the convergence rate that is determined by the homogeneous system in (27) even though the state $\boldsymbol{\zeta}$ does not converge to the origin. In fact, it is only guaranteed that $\lim_{t\to\infty} \|\boldsymbol{\zeta}(t)\| \leq 2\sqrt{\kappa(X)}\sigma \|\boldsymbol{W}\|$.

The guaranteed convergence rate can be obtained by finding the maximal μ that makes the LMI in (30) feasible, which is a standard problem in LMI methods [9]. Moreover, notice that the standard analysis in the literature [12]–[14] given in Eq. (18), which is based on $\lambda_{\min}(X_0) \|\boldsymbol{\zeta}\|^2 \leq V_0 \leq \lambda_{\max}(X_0) \|\boldsymbol{\zeta}\|^2$, also leads to the convergence rate ,$\mu_0 := \frac{c_1}{c_2}$, where $c_2 = \max\{\lambda_{\max}(P), 1/\gamma\}$. In other words, the Lyapunov candidate X_0 and μ_0 in the standard analysis are a solution for the LMI in (30). This implies that the maximal convergence rate μ_{opt} by solving the optimization problem for (30) is guaranteed to be larger than the standard convergence rate, μ_0.

Remark 2: The convergence rate μ can also indicate the degree to which norm-bounded unmatched uncertainty can be tolerated because the LMI in (30) guarantees $\bar{A}(\boldsymbol{\rho}(t) + \frac{\mu}{2}I < 0$. This in turn implies that the LMI-based approach has a potential for a less conservative estimate for the size of unmatched uncertainty compared to the standard analysis. The rigorous treatment of unmatched uncertainties will be presented in a future publication.

In the case of adaptive control, the weight estimation error $\widetilde{\boldsymbol{W}}$ can only be shown to be bounded, unless a persistency of excitation condition [6] is met. Therefore, Eq. (31) is not as meaningful in studying the performance of an adaptive controller, hence performance has been associated only with the tracking error, $\boldsymbol{e}$. The following lemma sheds light on how the tracking error can be analyzed.

Lemma 4: Suppose that there exists $X = X^\top > 0$, $\mu > 0$, and $\beta_1, \beta_2, \nu > 0$ such that

$$\begin{aligned} &\begin{bmatrix} A(\boldsymbol{\rho})^\top X + XA(\boldsymbol{\rho}) + \mu X & X\bar{B} \\ \bar{B}^\top X & -\nu I_N \end{bmatrix} < 0, \\ &\begin{bmatrix} \mu X & 0 & \bar{C}^\top \\ 0 & (\beta_2 - \nu)I & 0 \\ \bar{C} & 0 & \beta_1 I_N \end{bmatrix} > 0, \ \forall \boldsymbol{\rho} \in \mathcal{P}_0, \end{aligned} \tag{33}$$

Then the tracking error is upper bounded by:

$$\|\boldsymbol{e}(t)\| \leq \sqrt{\beta_1 \mu \lambda_{\max}(X)}\, \|\boldsymbol{\zeta}(0)\|\, e^{-\frac{\mu}{2}t} + \sqrt{\beta_1\beta_2}\sigma \|\boldsymbol{W}\|. \tag{34}$$

Proof: The LMIs in (33) are slightly modified from those in [8] that provide an upper bound for the $\mathcal{L}_1$-norm (the peak-to-peak norm) when the initial condition is set as zero. The proof given here is tailored from that in [8] in order to account for the fact that the external disturbance is constant. Consider $V(\boldsymbol{\zeta}) = \boldsymbol{\zeta}^\top X \boldsymbol{\zeta}$. By following the same lines from Lemma 2, from the first inequality, we have $\dot{V} + \mu V - \nu\sigma^2 \boldsymbol{W}^\top \boldsymbol{W} < 0$. This leads to:

$$\begin{aligned} V(t) &\leq V(0)e^{-\mu t} + \nu\sigma^2 \|\boldsymbol{W}\|^2 \int_0^t e^{-\mu(t-s)} ds \\ &\leq V(0)e^{-\mu t} + \nu/\mu\sigma^2 \|\boldsymbol{W}\|^2. \end{aligned} \tag{35}$$

From the second inequality, we have $\|\boldsymbol{e}(t)\|^2 < \beta_1[\mu V(t) + (\beta_2 - \nu)\sigma^2 \|\boldsymbol{W}\|^2]$. By substituting (35), we have $\|\boldsymbol{e}(t)\|^2 < \beta_1[\mu V(0)e^{-\mu t} + \beta_2\sigma^2 \|\boldsymbol{W}\|^2] \leq$

$\beta_1\mu\lambda_{\max}(X)\|\boldsymbol{\zeta}(0)\|^2 e^{-\mu t}+\beta_1\beta_2\sigma^2\|\boldsymbol{W}\|^2$. This leads to (34). ■

Since the full state is available for feedback, by setting $\boldsymbol{x}_m(0)=\boldsymbol{x}(0)$, we can set $\boldsymbol{e}(0)=0$. Then, Eq.(34) leads to:

$$\lim_{t\to\infty}\|\boldsymbol{e}\|\le\sqrt{\beta_1\beta_2}\sigma\|\boldsymbol{W}\|,$$
$$\|\boldsymbol{e}\|_{\mathcal{L}_\infty}\le\sqrt{\beta_1\mu\lambda_{\max}(X)}\left\|\widetilde{\boldsymbol{W}}(0)\right\|+\sqrt{\beta_1\beta_2}\sigma\|\boldsymbol{W}\|, \tag{36}$$

where $\|\boldsymbol{e}\|_{\mathcal{L}_\infty}:=\sup_{t\ge 0}\|\boldsymbol{e}(t)\|$.

The guaranteed size of the ultimate bound is obtained by solving the optimization problem $\min\beta_1\beta_2$ such that the LMIs in (33) holds. In case of standard analysis in the literature [12]–[14], based on X_0 and μ_0, an upper bound for the tracking error is estimated by $V_0(t)\le V_0(0)e^{-\mu_0 t}+\frac{\nu_0}{\mu_0}\|\sigma\boldsymbol{W}\|^2$ and $\|\boldsymbol{e}(t)\|^2\le\frac{1}{\lambda_{\min}(P)}V_0(t)$. This leads to:

$$\|\boldsymbol{e}(t)\|^2\le\frac{1}{\lambda_{\min}(P)}[c_2\|\boldsymbol{\zeta}(0)\|^2 e^{-\mu_0 t}+\frac{\nu_0}{\mu_0}\|\sigma\boldsymbol{W}\|^2] \tag{37}$$

where $\nu_0=\frac{1}{\gamma\sigma}$. Therefore, the standard stability analysis in the literature is a single solution for the LMIs in Lemma 4 in which $X=X_0$, $\mu=\mu_0$, $\nu=\nu_0$, $\beta_1=\frac{1}{\mu_0\lambda_{\min}(P)}$, and $\beta_2=\nu_0$. This implies that the resulting ultimate bound from optimizing $\beta_1\beta_2$ in Lemma 4 is guaranteed to be smaller than the one in (37).

Remark 3: The LMIs in Lemma 4 is guaranteed to be feasible for any fixed μ that is obtained from Lemma 3. Let $\|\boldsymbol{e}(t)\|_{est_1}$ be the estimate for the tracking error with $\mu=\mu_{opt}$ in (33) and $\|\boldsymbol{e}(t)\|_{est_2}$ be the estimate with μ as a decision variable, for both of which $\beta_1\beta_2$ is minimized. Then, $\|\boldsymbol{e}(t)\|_{est}=\min\{\|\boldsymbol{e}(t)\|_{est_1},\|\boldsymbol{e}(t)\|_{est_2}\}$ denotes the best estimate for the maximal convergence rate with the smallest ultimate bound possible in the LMI analysis proposed in this paper.

V. Conclusions and Future Directions

In this paper we show that by casting the error dynamics of the tracking error and the weight estimation error into a proper form LMI-based analysis tools can be developed for stability and performance analysis of adaptive controllers with $\sigma-$modification. The analysis establishes guaranteed convergence rate, domain of attraction, and the transient and steady-state upper bound of the tracking error for adaptive systems. These tools provide less conservative estimates for the convergence rate and the ultimate bound than the standard analysis in the literature.

An additional benefit of employing LMI methods is that stability margins and robustness to uncertainties that cannot be parameterized can also be carried out under the framework of linear parameter varying systems. LMI-based analysis tools for guaranteed gain margin, phase margin, and time-delay margins, and robustness measures for unmatched uncertainties will be presented in forthcoming publications.

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