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Semiglobal Stabilization of Sandwich Systems by Dynamic Output Feedback

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Abstract— **We consider the problem of stabilizing a class of sandwich systems, consisting of two linear subsystems connected in cascade by a saturated scalar signal, with partial-state measurement available from the second subsystem only. We present conditions for semiglobal stabilization and demonstrate their sufficiency by explicit construction of a stabilizing controller. This controller is a mathematical construction that is not intended for practical implementation in its current form. Central to the stabilization strategy is a detection scheme that determines whether the saturation is active or inactive within intervals of a freely chosen length.**

I. Introduction

Many physical systems can be modeled as interconnections of several distinct subsystems, some of which are linear and some of which are nonlinear. One common type of structure consists of a static nonlinear element *sandwiched* between two linear systems, as illustrated in Fig. 1. This type of structure can occur, for example, when an actuator with linear dynamics and an output nonlinearity is connected to a linear system. We refer to the system in Fig. 1 as a *sandwich system*.

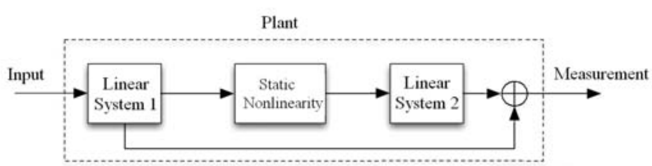

Fig. 1. Sandwich system

In this paper we focus on sandwich systems where the sandwiched nonlinearity is a saturation. Saturations can occur due to the limited capacity of an actuator, limited range of a sensor, or physical limitations within a system. Physical quantities such as speed, acceleration, pressure, flow, current, voltage, and so on, are always limited to a finite range, and saturations are therefore a ubiquitous feature of physical systems. Our primary goal is to investigate conditions for semiglobal stabilization by output feedback. Due to space constraints we limit ourselves to the case when the available output is a linear combination of the states of the second subsystem only, as illustrated in Fig. 2. This formulation captures the main challenge of the output-feedback stabilization problem, namely, that the states of the first subsystem can only be observed when the saturation is inactive. We refer to the two linear subsystems as the L_1 and L_2 subsystems.

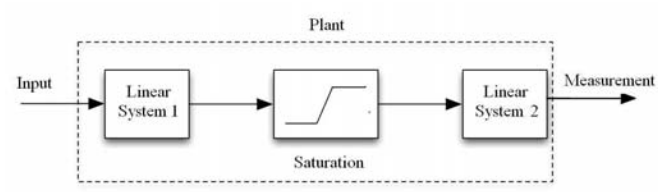

Fig. 2. Sandwich system with saturation nonlinearity and partial-state measurement from second subsystem only

Stabilization of sandwich systems has been studied previously, for example, in [1]–[4]. The main technique used in [1]–[4] is based on approximate inversion of the sandwiched nonlinearity. Inversion is a viable approach for some types of nonlinearities, a prominent example being the deadzone nonlinearity, which is right-invertible. Saturations, however, have a limited range and are therefore not amenable to inversion except in a small region; thus, a different approach is required. In [5], the authors considered full-state feedback stabilization of sandwich systems with saturation nonlinearities. The technique introduced in [5] is a generalization of the *low-gain* design methodologies developed in [6]–[8] for stabilization of linear systems subject to actuator saturation. Roughly, a pre-feedback is designed to make the L_1 subsystem exponentially stable, so that saturation is avoided after an initial transient. The pre-feedback is then augmented by a control law designed for the overall system with a sufficiently low gain to guarantee that saturation is avoided as the whole state is brought to the origin.

When full-state measurement is not available, it is natural to construct an observer to estimate the states. For the system in question, observer design is complicated by the saturation, which separates the L_1 subsystem from the output. In general, the saturation must therefore be deactivated before all the states of the system can be identified.

The problem considered in this paper is related to the problem of stabilizing a linear system with a saturated output. This problem has been considered for single-input single-output (SISO) systems in [9], [10], and the results in [9] have been extended to multiple-input multiple-output (MIMO) systems in [11]. In the approach from [9], the output is

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brought out of saturation by applying an input that grows sufficiently fast to catch up with any internal instabilities, based only on the sign of the output. When the output comes out of saturation, the state is identified and controlled to the origin in a deadbeat manner.

A. Stabilization Strategy

In this paper we combine the method from [5] for state-feedback stabilization of sandwich systems with the method from [9] for stabilization of systems with an output saturation. As in [9], our strategy is to deactivate the saturation without knowing the full state of the system. Once the saturation is deactivated, the states are identified and controlled to the origin using the method from [5]. A difficulty with this approach is the lack of direct knowledge of whether the saturation is active or inactive at any given time. Consequently, an integral part of the strategy is to detect whether the saturation is active or not based only on the available output.

Because the saturation is separated from the output by a dynamical system, we generally cannot expect to detect activation or deactivation of the saturation instantly. Instead, we shall consider arbitrarily small time intervals and create a detection scheme to determine whether, on any such interval, the saturation is active or not. We shall furthermore determine the sign of the saturation if it is indeed active. When the saturation is detected as inactive for an entire interval, the state of the full system can be determined. This strategy requires that the output of the L_1 subsystem is driven out of saturation for at least one entire interval. To guarantee that this happens, we make the time intervals sufficiently small relative to the size of a bounded set of admissible initial conditions. Our result is therefore semiglobal rather than global; that is, the region of attraction is bounded but can be made arbitrarily large by decreasing the length of the time intervals.

We emphasize that the main purpose of this paper is to investigate solvability conditions for semiglobal stabilization of the sandwich system in Fig. 2. Although we do so by explicit construction of a stabilizing controller, we do not claim that this controller achieves good performance in most cases. Nevertheless, we introduce design ideas that will be used in future work with attention to performance.

II. System Description

The class of sandwich systems considered in this paper is described by the following equations:

$$L_1 : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ z(t) = Cx(t), \end{cases} \tag{1a}$$

$$L_2 : \begin{cases} \dot{\omega}(t) = M\omega(t) + N\sigma(z(t)), \\ y(t) = G\omega(t), \end{cases} \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $\omega(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}^p$, and $z(t) \in \mathbb{R}$. The function $\sigma(\cdot)$ is a standard saturation described by $\sigma(z(t)) = \operatorname{sign}(z(t)) \min\{1, |z(t)|\}$. The input $u(t)$ is assumed to be piecewise continuous. We assume without loss of generality that G has full row rank. For ease of notation, we define $\chi(t) := \operatorname{col}(x(t), \omega(t))$.

In the region where the saturation is inactive (that is, when $|z(t)| \leq 1$), the system equations can be merged in a single linear system:

$$\dot{\chi}(t) = \mathcal{A}\chi(t) + \mathcal{B}u(t) \tag{2a}$$

$$y(t) = \mathcal{C}\chi(t). \tag{2b}$$

where

$$\mathcal{A} := \begin{bmatrix} A & 0 \\ NC & M \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \mathcal{C} := \begin{bmatrix} 0 & G \end{bmatrix}.$$

The system is initialized at time $t = 0$.

Assumption 1: The pair $(\mathcal{A}, \mathcal{B})$ is controllable, and the pair $(\mathcal{C}, \mathcal{A})$ is observable.

It follows from Assumption 1 that the pairs (A, B) and (M, N) are controllable, and that the pairs (C, A) and (G, M) are observable.

Assumption 2: The eigenvalues of M are located in the closed left-half plane, and the triple (G, M, N) has no invariant zeros at the origin.

The assumption that the eigenvalues of M are located in the closed left-half plane is necessary to ensure stabilizability of the system, even in the case of full-state feedback, as explained in [5]. We use the assumption that the triple (G, M, N) has no zeros at the origin to facilitate detection of an active or inactive saturation. This can be intuitively understood by noting that a zero at the origin would block constant inputs to the L_2 subsystem from being visible at the output $y(t)$. It would therefore be impossible to use the output $y(t)$ to separate between different constant inputs to the L_2 subsystem, including a positive saturation ($\sigma(z(t)) = 1$), a negative saturation ($\sigma(z(t)) = -1$), and a zero signal ($\sigma(z(t)) = 0$).

III. Saturation Detection

We wish to design a detection scheme to determine whether the saturation in (1) is active or inactive, based only on knowledge of the output $y(t)$ and the input $u(t)$. To this end, we divide the time $t > 0$ into intervals $(kT - T, kT]$, $k = 1, 2, \ldots$, where the interval length $T > 0$ is a design parameter that can be made arbitrarily small. The detection scheme will determine at time kT whether on the preceding interval $(kT - T, kT]$, the saturation was active for the entire interval, inactive for the entire interval, or both active and inactive within the interval.

To illustrate the approach, suppose that the saturation is active in the positive direction for an entire interval; that is, for some arbitrary $k \in 1, 2, \ldots$, $z(t) \geq 1$ for all $t \in (kT - T, kT]$. On this interval the L_2 subsystem behaves like a linear system with $\sigma(z(t)) = 1$ as a constant input. Hence the output satisfies the following equation for all $\tau \in (0, T]$:

$$y(kT - T + \tau) = G\mathrm{e}^{M\tau}\omega(kT - T) + Gu_s^*(\tau), \tag{3}$$

where

$$u_s^*(\tau) := \int_0^\tau \mathrm{e}^{M(\tau-\gamma)} N \, \mathrm{d}\gamma. \tag{4}$$

If we premultiply (3) by $\mathrm{e}^{M^\top\tau}G^\top$ and integrate from 0 to τ, we obtain

$$\xi_s(kT-T;\tau) = D_s(\tau)\omega(kT-T) + S_s(\tau), \tag{5}$$

where

$$\xi_s(kT-T;\tau) = \int_0^\tau \mathrm{e}^{M^\top\gamma}G^\top y(kT-T+\gamma)\,\mathrm{d}\gamma, \tag{6a}$$

$$D_s(\tau) = \int_0^\tau \mathrm{e}^{M^\top\gamma}G^\top G\mathrm{e}^{M\gamma}\,\mathrm{d}\gamma, \tag{6b}$$

$$S_s(\tau) = \int_0^\tau \mathrm{e}^{M^\top\gamma}G^\top G u_s^*(\gamma)\,\mathrm{d}\gamma. \tag{6c}$$

Each of the quantities in (6) can be computed based on the available output data. Furthermore, we can use (5) to calculate $\omega(kT-T)$ as

$$\omega(kT-T) = D_s^{-1}(\tau)(\xi_s(kT-T;\tau) - S_s(\tau)). \tag{7}$$

Thus (7) represents a deadbeat observer for the L_2 subsystem when the saturation is active in the positive direction. Note that $D_s(\tau)$ is the observability Gramian of the observable pair (G, M), and thus it is invertible for all $\tau \in (0, T]$.

We wish to use (7) to detect whether the saturation really is active in the positive direction on the entire interval $(kT-T, kT]$. To do so we premultiply (7) by $G\mathrm{e}^{M\tau}$. Since $D_s(\tau)$ becomes singular as $\tau \to 0$, we also multiply the expression by $\det(D_s(\tau))$ to obtain $\det(D_s(\tau))D_s^{-1}(\tau) = \mathrm{adj}(D_s(\tau))$, where $\mathrm{adj}(D_s(\tau))$ is the adjugate of $D_s(\tau)$, which is bounded on $(0, T]$. Rearranging the resulting expression and using (3), we obtain

$$\begin{aligned}&\det(D_s(\tau))(y(kT-T+\tau) - Gu_s^*(\tau))\\ &\quad - G\mathrm{e}^{M\tau}\mathrm{adj}(D_s(\tau))(\xi_s(kT-T;\tau) - S_s(\tau)) = 0.\end{aligned} \tag{8}$$

Equation (8) can be checked using available output data, and it holds for each $\tau \in (0, T]$ if the saturation is active in the positive direction on the entire interval. If the saturation is not active in the positive direction on the entire interval, one might expect (8) not to hold, at least for some $\tau \in (0, T]$. Indeed, this expectation turns out to be true under the assumptions made in this paper. Our detection scheme is therefore based on checking the validity of (8). We create a similar test to check whether the saturation is active in the negative direction on the entire interval. Finally, we do the same based on the model (2) and input $u(t)$ to check whether the saturation is inactive on the entire interval.

A. Detectors

We define the following quantities:

$$\begin{aligned}e_{k+} = \int_0^T &\|\det(D_s(\tau))(y(kT-T+\tau) - Gu_s^*(\tau))\\ &- G\mathrm{e}^{M\tau}\mathrm{adj}(D_s(\tau))(\xi_s(kT-T;\tau) - S_s(\tau))\|\,\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}e_{k-} = \int_0^T &\|\det(D_s(\tau))(y(kT-T+\tau) + Gu_s^*(\tau))\\ &- G\mathrm{e}^{M\tau}\mathrm{adj}(D_s(\tau))(\xi_s(kT-T;\tau) + S_s(\tau))\|\,\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}e_{k0} = \int_0^T &\|\det(D_0(\tau))(y(kT-T+\tau)\\ &- Gu_0^*(kT-T;\tau))\\ -\mathcal{C}\mathrm{e}^{\mathcal{A}\tau}&\mathrm{adj}(D_0(\tau))(\xi_0(kT-T;\tau) - S_0(kT-T;\tau))\|\,\mathrm{d}\tau,\end{aligned}$$

where

$$u_0^*(kT-T;\tau) = \int_0^\tau \mathrm{e}^{\mathcal{A}(\tau-\gamma)}\mathcal{B}u(kT-T+\gamma)\,\mathrm{d}\gamma,$$

$$\xi_0(kT-T;\tau) = \int_0^\tau \mathrm{e}^{\mathcal{A}^\top\gamma}\mathcal{C}^\top y(kT-T+\gamma)\,\mathrm{d}\gamma,$$

$$D_0(\tau) = \int_0^\tau \mathrm{e}^{\mathcal{A}^\top\gamma}\mathcal{C}^\top\mathcal{C}\mathrm{e}^{\mathcal{A}\gamma}\,\mathrm{d}\gamma,$$

$$S_0(kT-T;\tau) = \int_0^\tau \mathrm{e}^{\mathcal{A}^\top\gamma}\mathcal{C}^\top\mathcal{C}u_0^*(kT-T;\gamma)\,\mathrm{d}\gamma.$$

The functions u_0^*, ξ_0, D_0, and S_0 correspond to the functions defined in (4), (6) but are based on the system matrices of the system (2) and the input $u(t)$, rather than the system matrices of the L_2 subsystem and the input 1.

To facilitate detection, we also need an assumption regarding the control input $u(t)$.

Assumption 3: If for any $k \in 1, 2, \ldots$, the function $\tau \mapsto u(kT-T+\tau)$ is a Bohl function on $(0, T)$, then its spectrum does not contain any invariant zeros of the triple (G, M, N).[1]

Remark 1: Assumption 3 specifies a mild restriction on the allowable input signals $u(t)$ on any interval. The reason for this restriction is that some signals may create an output from the L_1 subsystem that is blocked by the zeros of the L_2 subsystem.

We can now state our result on saturation detection.

Theorem 1: For each $k \in 1, 2, \ldots$

1) $e_{k+} = 0$ if and only if for all $t \in (kT-T, kT]$, $z(t) \geq 1$
2) $e_{k-} = 0$ if and only if for all $t \in (kT-T, kT]$, $z(t) \leq -1$
3) $e_{k0} = 0$ and $e_{k+}, e_{k-} > 0$ if and only if for all $t \in (kT-T, kT]$, $-1 \leq z(t) \leq 1$ and for some $t \in (kT-T, kT]$, $|z(t)| < 1$

Proof: See Appendix. ∎

IV. Semiglobal Stabilization

In this section we use the detection scheme from the previous section to create a stabilizing control law for the system (1). Because the approach is semiglobal, we make the following assumption:

Assumption 4: The state $\chi(t)$ is initialized from some ***a priori*** known compact set K_0.

The control strategy can be divided into three consecutive stages, described in the following sections. In Stage 2, we apply a control on the form $-B^\top\mathrm{e}^{-A^\top\tau}\kappa$ on intervals $(0, \bar{T}]$, where κ is a constant. This approach is borrowed from [9] and is used to deactivate the saturation. To ensure that the control law satisfies the assumption about $u(t)$ in Theorem 1, we therefore replace it with the following assumption:

[1]A function $f(t)$ is a Bohl function if it is a linear combination of signals of the form $t^\alpha\mathrm{e}^{\lambda t}$, where the α's are nonnegative integers and the λ's are complex numbers. The set of λ's is called the spectrum of $f(t)$ [12].

Assumption 3′: No eigenvalues of $-A$ coincide with any invariant zeros of the triple (G, M, N).

We remark that this is not a necessary condition, as the control law in Stage 2 can easily be modified to ensure that Assumption 3 holds, even when Assumption 3′ does not hold.

Stage 1

In Stage 1 we do not apply any control, but wait until the saturation is either active or inactive for an entire interval $(k_1T - T, k_1T]$, as indicated by the condition $e_{k_1+}e_{k_1-}e_{k_10} = 0$. This is guaranteed to occur for some finite $k_1 \geq 1$ if T is chosen sufficiently small, because the unforced system cannot oscillate arbitrarily fast. In Stage 1, the control is therefore specified by

$$u(t) = 0, \quad t \in [0, k_1T]. \tag{9}$$

At time $t = k_1T$ we move to Stage 2.

Stage 2

In Stage 2 we apply a control to ensure that $z(t)$ is brought out of saturation for an entire interval $(k_2T - T, k_2T]$, as indicated by the condition $e_{k_20} = 0$ and $e_{k_2+}e_{k_2-} > 0$. Define

$$\delta = \begin{cases} 1, & e_{k_1+} = 0, \\ -1, & e_{k_1-} = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{T} > 0$ be some arbitrary fixed constant. We divide the time $t \geq k_1T$ into intervals $(k_1T + j\bar{T} - \bar{T}, k_1T + j\bar{T}]$, $j = 0, 1, \ldots$ The control in Stage 2 is defined based on [9] by

$$u(k_1T + j\bar{T} + \tau) = -B^{\mathsf{T}}\mathrm{e}^{-A^{\mathsf{T}}\tau}U_j, \ \forall t \in (k_1T, k_2T], \tag{10}$$

where $j = 0, 1, \ldots$ and $\tau \in (0, \bar{T}]$. The quantities U_j and $\beta(\bar{T})$ are defined by

$$U_j = \alpha^j \beta^{-1}(\bar{T})(\alpha h - \mathrm{e}^{A\bar{T}}h)\delta,$$
$$\beta(\bar{T}) = \int_0^{\bar{T}} \mathrm{e}^{A(\bar{T}-\gamma)}BB^{\mathsf{T}}\mathrm{e}^{-A^{\mathsf{T}}\gamma}\mathrm{d}\gamma,$$

Finally, α is defined such that $\alpha > \mathrm{e}^{2\bar{T}\|A\|}$ and h is any vector such that $Ch > 0$. At time k_2T, we move to Stage 3.

The following lemma shows that we will indeed move to Stage 3 within finite time.

Lemma 1: If $T > 0$ is chosen sufficiently small, then there exists a $k_2 \geq k_1$ such that $e_{k_20} = 0$ and $e_{k_2+}e_{k_2-} > 0$.

Proof: See Appendix. ■

Stage 3

In Stage 3 we bring the states to the origin by using the state-feedback approach from [5]. To do so, we need access to the full state $\chi(t)$. Since the saturation is inactive on the interval $(k_2T - T, k_2T]$, the full state $\chi(t)$ at time $k_2T - T$ can be calculated precisely by using a deadbeat approach, as indicated by the discussion in Section III. After this point $\chi(t)$ can be calculated by integrating (1). Thus, we define a state estimate $\hat{\chi}(t)$ for $t \geq k_2T$ by

$$\dot{\hat{\chi}}(t) = f(\hat{\chi}(t), u(t)), \tag{11a}$$

$$\hat{\chi}(k_2T) = \mathrm{e}^{\mathcal{A}T}D_0^{-1}(T)(\xi_0(k_2T - T; T) - S_0(k_2T - T; T)) + u_0^*(k_2T - T; T), \tag{11b}$$

where $f(\hat{\chi}(t), u(t))$ represents the right-hand side of (1).

Lemma 2: For all $t \geq k_2T$, $\hat{\chi}(t) = \chi(t)$.

Proof: See Appendix. ■

We use the precise knowledge of $\chi(t)$ for all $t \geq k_2T$ to implement a linear state-feedback control law according to [5]. We start by selecting F such that $A + BF$ is Hurwitz. We then find the unique solution $P_\varepsilon = P_\varepsilon^{\mathsf{T}} > 0$ of the algebraic Riccati equation

$$\begin{bmatrix} A + BF & 0 \\ NC & M \end{bmatrix}^{\mathsf{T}} P_\varepsilon + P_\varepsilon \begin{bmatrix} A + BF & 0 \\ NC & M \end{bmatrix} - P_\varepsilon \begin{bmatrix} BB^{\mathsf{T}} & 0 \\ 0 & 0 \end{bmatrix} P_\varepsilon + \varepsilon I = 0,$$

where $\varepsilon > 0$ is a low-gain parameter that must be chosen sufficiently small. The control in Stage 3 is now defined by

$$u(t) = \left(\begin{bmatrix} F & 0 \end{bmatrix} - \mathcal{B}^{\mathsf{T}}P_\varepsilon\right)\hat{\chi}(t), \quad \forall t > k_2T. \tag{12}$$

Since $\hat{\chi}(t) = \chi(t)$, we may use the state-feedback theory from [5]. From the proof of Lemma 1, we know that $\chi(k_2T)$ belongs to a compact set $K_2 \subset \mathbb{R}^{n+m}$, the size of which is bounded as a function of the set K_0 of admissible initial conditions. By [5, Theorem 3], the control law therefore ensures that $\chi(t) \to 0$ as $t \to \infty$, provided the low-gain parameter $\varepsilon > 0$ is chosen sufficiently small depending on K_0.

A. Asymptotic Stability

Based on the discussion in the previous section, we can now state the main result on semiglobal stabilization of the sandwich system.

Theorem 2: For any compact set $K_0 \subset \mathbb{R}^{n+m}$, there exist $T^* > 0$ and $\varepsilon^* > 0$ such that for all $0 < \varepsilon \leq \varepsilon^*$ and $0 < T \leq T^*$, the control law described in Stages 1–3 asymptotically stabilizes the system (1) with K_0 contained in the region of attraction.

Proof: See Appendix. ■

V. Discussion

As emphasized in the introduction, the primary purpose of this paper is to investigate solvability conditions for the semiglobal stabilization problem, not to construct a control law to ensure good performance. Although the control law presented in Section IV is theoretically stabilizing, there are several obvious drawbacks that must be addressed in a practical implementation.

First, the input applied to the system to deactivate the saturation grows exponentially larger for each interval $(k_1T + j\bar{T} - \bar{T}, k_1T + j\bar{T}]$, even when this is not necessary. If, for example, L_1 is asymptotically stable with $u(t) = 0$, then no input needs to be applied to deactivate the saturation, and if L_1 is marginally stable, only a small input needs to be applied. The exponentially growing input may cause $z(t)$ to pass quickly through the saturation, thus requiring T to be chosen small. As a consequence, the deadbeat observation of $\chi(t)$ may become poorly conditioned.

Second, the state estimation of $\chi(t)$ is based on deadbeat observation at $t = k_2 T$ and integration of the system equations from that point on. Clearly this is not a robust approach; any disturbance or modeling inaccuracy may cause the state estimate to diverge as $t \to \infty$. An obvious improvement would be to update $\hat{\chi}(t)$ using the deadbeat approach every time the saturation is inactive for an entire interval. Indeed, after some finite amount of time the saturation becomes inactive in every time interval.

Third, the algorithm passes through the three stages in a linear manner. A more robust approach would include a path back to Stage 2 from Stage 3, in case the control in Stage 3 fails to make the state converge and the saturation remains active. This can occur, for example, if an unknown disturbance to (2) causes $\hat{\chi}(t)$ to become inaccurate.

VI. Concluding Remarks

We have presented conditions for semiglobal stabilization of systems consisting of two linear systems connected in a saturated cascade connection. Sufficiency of the conditions is demonstrated through constructive design of a semiglobally stabilizing controller. Current research is focused on further development of a controller with emphasis on performance.

Appendix

Proof of Theorem 1: We first prove that if for all $\tau \in (0, T]$, $\sigma(z(kT - T + \tau)) = 1$, then $e_{k+} = 0$. Taking the norm on both sides of (8), we obtain

$$\begin{aligned}&\|\det(D_s(\tau))(y(kT - T + \tau) - Gu_s^*(\tau))\\&\quad - Ge^{M\tau}\mathrm{adj}(D_s(\tau))(\xi_s(kT - T; \tau) - S_s(\tau))\| = 0, \quad (13)\end{aligned}$$

for all $\tau \in (0, T]$. To form e_{k+} we integrate the left-hand side of (13) from 0 to T, and it follows that $e_{k+} = 0$. The same argument can be applied for e_{k-} and e_{k0}. This proves the *if* part of statements 1 and 2 of the theorem.

We now prove that if $e_{k+} = 0$, then for all $\tau \in (0, T]$, $z(kT - T + \tau) \geq 1$. We shall need the following lemma, which specifies that the output must correspond to a solution of L_2 with input 1, for some set of initial conditions.

Lemma 3: For each $k \in 1, 2, \ldots$, if $e_{k+} = 0$, then there exists a vector $\omega_0 \in \mathbb{R}^m$ such that for all $\tau \in (0, T]$, $y(kT - T + \tau)$ corresponds to the output of L_2 with input 1, initialized at time $t = kT - T$ with initial condition ω_0.

Proof: Suppose that $e_{k+} = 0$. Then from (8), we have

$$\begin{aligned}&y(kT - T + \tau) - Gu_s^*(\tau)\\&\qquad - Ge^{M\tau}D_s^{-1}(\tau)(\xi_s(kT - T; \tau) - S_s(\tau)) = 0.\end{aligned}$$

Premultiplying by $D_s^{-1}(\tau)e^{M^\top\tau}G^\top$, it is easily verified that we obtain

$$\begin{aligned}&D_s^{-1}(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}(\xi(kT - T; \tau) - S_s(\tau))\\&\qquad + \frac{\mathrm{d}}{\mathrm{d}\tau}(D_s^{-1}(\tau))(\xi_s(kT - T; \tau) - S_s(\tau)) = 0.\end{aligned}$$

Using integration by parts from 0 to τ therefore yields $D_s^{-1}(\tau)(\xi_s(kT - T; \tau) - S_s(\tau)) = \kappa$, where κ is a constant vector. Premultiplying by $D_s(\tau)$ and differentiating on $(0, T)$ yields

$$e^{M^\top\tau}G^\top y(kT - T + \tau) - e^{M^\top\tau}G^\top Gu_s^*(\tau) = e^{M^\top\tau}G^\top Ge^{M\tau}\kappa.$$

Because $G^\top$ has full column rank and $e^{M^\top\tau}$ is nonsingular, the above expression implies that

$$y(kT - T + \tau) - Gu_s^*(\tau) = Ge^{M\tau}\kappa.$$

Comparison with (3) shows that $y(kT - T + \tau)$ corresponds to the output of L_2 with input 1, initialized at time $t = kT - T$ with initial condition κ on $(0, T)$, which by continuity extends to $(0, T]$. ■

Based on Lemma 3, suppose that for all $\tau \in (0, T]$, the output $y(kT - T + \tau)$ corresponds to the response of the L_2 subsystem with input 1, initialized at time $t = kT - T$ with initial condition ω_0. Then we may write $y(kT - T + \tau) = G\hat{\omega}(kT - T + \tau)$, where $\dot{\hat{\omega}}(kT - T + \tau) = M\omega(kT - T + \tau) + N$ and $\hat{\omega}(kT - T) = \omega_0$. Defining $\tilde{\omega}(kT - T + \tau) = \hat{\omega}(kT - T + \tau) - \omega(kT - T + \tau)$, we obtain the system

$$\dot{\tilde{\omega}}(kT - T + \tau) = M\tilde{\omega}(kT - T + \tau) + N\mu(kT - T + \tau),$$

where $\mu(kT - T + \tau) := 1 - \sigma(z(kT - T + \tau))$. From [13] it is easy to show that the $\tilde{\omega}$ system with output $\tilde{y}(kT - T + \tau) := G\tilde{\omega}(kT - T + \tau)$ is left-invertible with respect to the input $\mu(kT - T + \tau)$, because it is observable and the input is scalar. Since $\tilde{y}(kT - T + \tau) = 0$, it follows that $\mu(kT - T + \tau)$ must either be zero, or it must be blocked by the invariant zeros of the triple (G, M, N). If $\mu(kT - T + \tau) = 0$, then we have $\sigma(z(kT - T + \tau)) = 1$, as desired. If $\mu(kT - T + \tau)$ is a nonzero signal blocked by the invariant zeros of the triple (G, M, N), then it must be a Bohl function on the interval $(0, T)$ with a spectrum that contains only invariant zeros of (G, M, N). Furthermore, this signal must be non-constant, since (G, M, N) has no invariant zeros at the origin. This implies that $z(kT - T + \tau)$ must be a nonzero Bohl function with a spectrum containing an invariant zero of (G, M, N). Since the L_1 subsystem is a controllable and observable SISO system, this can only occur if either A has an eigenvalue that coincides with an invariant zero of (G, M, N) or if the input $u(kT - T + \tau)$ is a Bohl function on $(0, T)$ with a spectrum containing an invariant zero of (G, M, N). By Assumption 3, $u(kT - T + \tau)$ cannot be a Bohl function with a spectrum containing an invariant zero of (G, M, N), and thus A must have an eigenvalue that coincides with an invariant zero of (G, M, N). However, since the L_1 and L_2 subsystems are connected in cascade by a scalar signal, it is easy to show that this would lead to a pole-zero cancellation in the linear system (2), with a resulting loss of observability. This contradicts Assumption 1, and hence we must have $z(kT - T + \tau) \geq 1$.

The same argument holds for e_{k-} as for e_{k+}. We have therefore proven the ***only if*** part of statements 1 and 2 of the theorem, as well as the ***if*** part of statement 3. For statement 3, we still have to prove that $e_{k0} = 0$ and $e_{k+}, e_{k-} > 0$ cannot occur unless for all $\tau \in (0, T]$, $-1 \leq z(kT - T + \tau) \leq 1$ and for some $\tau \in (0, T]$, $|z(kT - T + \tau)| < 1$. We can use the same argument as in Lemma 3 to prove that $e_{k0} = 0$ implies that for all $\tau \in (0, T]$, $y(kT - T + \tau)$ corresponds exactly to the response of the χ system with input $u(kT - T + \tau)$, initialized at time $t = kT - T$ with some initial condition χ_0. Let $\hat{x}(kT - T + \tau)$ and $\hat{\omega}(kT - T + \tau)$ represent the corresponding trajectories. Following the same argument as above, we define $\tilde{\omega}(kT - T + \tau) = \hat{\omega}(kT - T + \tau) - \omega(kT - T + \tau)$ and obtain the system

$$\dot{\tilde{\omega}}(kT - T + \tau) = M\tilde{\omega}(kT - T + \tau) + N\mu(kT - T + \tau),$$

where $\mu(kT - T + \tau) := C\hat{x}(kT - T + \tau) - \sigma(z(kT - T + \tau))$. This can be rewritten as $\mu(kT - T + \tau) = z(kT - T + \tau) - \sigma(z(kT - T + \tau)) + C\tilde{x}(\tau)$, where $\dot{\tilde{x}}(\tau) = A\tilde{x}(\tau)$. As before, $\mu(kT - T + \tau)$ must be a Bohl function to ensure $\tilde{y}(kT - T + \tau) = 0$. Since $\tilde{x}(\tau)$ is a Bohl function, $z(kT - T + \tau) - \sigma(z(kT - T + \tau))$ must also be a Bohl function, which shows that either $|z(kT - T + \tau)| \geq 1$ or $|z(kT - T + \tau)| \leq 1$ holds for all $\tau \in (0, T]$ (otherwise $z(kT - T + \tau) - \sigma(z(kT - T + \tau))$ would be zero on a subinterval in $(0, T]$ and nonzero on another subinterval).

If $|z(kT - T + \tau)| \geq 1$, then e_{k+} or $e_{k-} = 0$. Hence, $e_{k0} = 0$ and $e_{k+}, e_{k-} > 0$ can only occur if for all $\tau \in (0, T]$, $-1 \leq z(kT - T + \tau) \leq 1$ and for some $\tau \in (0, T]$, $|z(kT - T + \tau)| < 1$. ■

Proof of Lemma 1: We start by noting that from the dynamics of the system, there is an upper bound on the maximum time before the saturation is active or inactive for an entire interval when $u(t) = 0$, provided T is sufficiently small. Using the fact that $\chi(t)$ is initialized from a compact set K_0, we therefore know that for small T there is a T-independent bound on $\|x(k_1 T)\|$.

We now prove that there is a finite $k_2 \geq k_1$ so that $e_{k_2 0} = 0$ and $e_{k_2 +} e_{k_2 -} > 0$. If $\delta = 0$, then $e_{k_1 0} = 0$ and $e_{k_1 +} e_{k_1 -} > 0$, and hence $k_2 = k_1$. Suppose instead that $\delta = 1$ and, for the purpose of establishing a contradiction, that $e_{k0} = 0$ and $e_{k+} e_{k-} > 0$ does not take place for any $k \geq k_1$, no matter how small $T > 0$ is chosen. Noting that $\delta = \mathrm{sign}(z(k_1 T))$, it follows directly from [9] that the sign of $z(t)$ switches before time $k_1 T + j'\bar{T}$, where j' is the smallest integer j that satisfies $\alpha^{j/2} > (\|C\|(\|x(k_1 T)\| + \|h\|))/(Ch)$. Since there is a T-independent bound on $\|x(k_1 T)\|$, it follows that j' is independent of T. From (10), we therefore see that there is a T-independent bound on $u(t)$ for all $t \in (k_1 T, k_1 T + j'\bar{T}]$. It follows that there is a T-independent bound on $x(t)$ for all $t \leq k_1 T + j'\bar{T}$. Based on this we know that there is a lower bound on the time that $z(t)$ is out of saturation before switching sign. However, if T is chosen smaller than half the length of that minimum interval, it is guaranteed that there is an entire interval $(k_2 T - T, k_2 T]$ in which $z(t)$ is out of saturation before switching sign, and hence $e_{k_2 0} = 0$ and $e_{k_2 +} e_{k_2 -} > 0$. The same argument holds if $\delta = -1$. ■

Proof of Lemma 2: Since the saturation is inactive on the interval $(k_2 - T, k_2 T]$, (2) is valid on this interval. Just as we may use (7) for the saturated system, we may therefore calculate $\chi(k_2 T - T)$ by $\chi(k_2 T - T) = D_0^{-1}(T)(\xi_0(k_2 T - T; T) - S_0(k_2 T - T; T))$. It therefore follows from the solution of the linear system (2) on $(k_2 T - T, k_2 T]$ that $\hat{\chi}(k_2 T)$ as defined in (11) satisfies $\hat{\chi}(k_2 T) = \chi(k_2 T)$. For $t \geq k_2 T$, $\hat{\chi}(t)$ evolves according to the same differential equation as $\chi(t)$, with the same initial condition and the same input. Hence for all $t \geq k_2 T$, $\hat{\chi}(t) = \chi(t)$. ■

Proof of Theorem 2: Through our discussion of the various stages, we have already proven that $\chi(t) \to 0$ as $t \to \infty$, provided $\varepsilon > 0$ and $T > 0$ are chosen sufficiently small. It remains to be shown that the origin of (1) is a stable equilibrium point.

On the interval $[0, T]$, we have $u(t) = 0$. Hence, if $\|\chi(0)\| \leq c$ for some sufficiently small constant $c > 0$, then the state evolves according to $\chi(t) = \mathrm{e}^{\mathcal{A}t}\chi(0)$ and the saturation remains strictly inactive for all $t \in [0, T]$. This implies that $e_{10} = 0$ and $e_{1+}, e_{1-} > 0$, and hence we move directly past Stage 2 to Stage 3. For sufficiently small $\chi(T) = \mathrm{e}^{\mathcal{A}t}\chi(0)$, the controller in Stage 3 ensures that for all $t \geq T$, $\|\chi(t)\| \leq \gamma\|\chi(T)\|$ for some $\gamma \geq 1$. It follows that for all sufficiently small $\|\chi(0)\|$, $\|\chi(t)\| \leq \gamma\|\mathrm{e}^{\mathcal{A}T}\|\|\chi(0)\|$, which shows that the origin is a stable equilibrium point. ■

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