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A Switched State Feedback Law for the Stabilization of LTI Systems

Keith R. Santarelli

Prepared by
Sandia National Laboratories
P.O. Box 5800
Albuquerque, New Mexico 87185 and Livermore, California 94550

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A Switched State Feedback Law for the Stabilization of LTI Systems

Keith R. Santarelli
Discrete Math & Complex Systems
Sandia National Laboratories

Abstract

Inspired by prior work in the design of switched feedback controllers for second order systems, we develop a switched state feedback control law for the stabilization of LTI systems of arbitrary dimension. The control law operates by switching between two static gain vectors in such a way that the state trajectory is driven onto a stable $n-1$ dimensional hyperplane (where n represents the system dimension). We begin by brief y examining relevant geometric properties of the phase portraits in the case of two-dimensional systems to develop intuition, and we then show how these geometric properties can be expressed as algebraic constraints on the switched vector f elds that are applicable to LTI systems of arbitrary dimension. We then derive necessary and suff cient conditions to ensure stabilizability of the resulting switched system (characterized primarily by simple conditions on eigenvalues), and describe an explicit procedure for designing stabilizing controllers. We then show how the newly developed control law can be applied to the problem of minimizing the maximal Lyapunov exponent of the corresponding closed-loop state trajectories, and we illustrate the closed-loop transient performance of these switched state feedback controllers via multiple examples.

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1 Introduction

The study of switched linear systems is a problem that has pervaded the systems and control literature for over f ve decades (see, e.g., [1]— [37]). With roots in relay feedback systems [30] and certain branches of optimal control [6], the primary perspective for the study of switched linear systems has evolved into the following basic question: can we artif cially introduce switching into systems design so as to increase performance? While this question is simply stated, the answer is not. Indeed, the idea of introducing switching into systems design has led to a tremendous amount of research over the past decade-and-a-half which attempts to address this issue from a variety of different technical perspectives. A brief survey of the existing literature on the topic leads to two immediate conclusions, one in that a number of diff cult problems have been already formulated and solved, and another in that there are still quite a number of open questions which need to be addressed in order to make switched system design a mature engineering f eld.

While there are many varied technical approaches to designing switched linear systems, a basic theme which is followed by most is encapsulated in the following problem, f rst described by Liberzon and Morse in [15]. We consider a switching system of the form

$$\dot{x} = A_{\sigma(t)}x(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the continuous state, $\sigma(t) \in \{1, 2, \ldots, k\}$ is a piecewise constant function of time (referred to as the *switching signal*), and A_i, $1 \leq i \leq k$ are given $n \times n$ linear transformations. A generic design problem that can be posed for such a setup is the following: construct a switching signal $\sigma(t)$ which makes the switching system of Eqn. 1 asymptotically stable.

The above problem, and certain generalizations of it, have led to a number of problems/techniques that have been studied in the literature: quadratic stabilizability techniques attempt to f nd a (piecewise) quadratic Lyapunov function which can be used to produce a switching law that minimizes a piecewise quadratic cost function at every time instant ([14, 15, 28, 29]); techniques have been developed for low order systems (via phase portraits and/or algebraic techniques) which can effectively utilize unstable behavior of linear subsystems to create stable switched interconnections ([2, 12, 16],[23]—[27],[32]—[35]); [11] utilizes the Youla parameterization to devise a method of switching between stabilizing controllers for arbitrary switching signals; extensions from asymptotic stability to L2 gain stability have been considered in [10, 22, 26, 27, 31, 36, 37]; some recent work considers switched system design over polyhedra/polyhedral Lyapunov functions ([9],[18]—[20]).

1.1 Tradeoffs: "General" Methods vs. Low Order Methods

A qualitative examination of the literature indicates that methods for designing switching controllers typically fall into one of two categories: methods that focus on low order systems (typically no higher than two to three states) which exploit algebraic and geometric properties of the corresponding state-space descriptions to cleverly achieve stability through switching, and methods which apply to general (arbitrary order) state-space descriptions which are typically less reliant on system structure. The latter of these two families of problems appears to have a larger following for a good reason: methods which do not depend on order can be applied to a larger class of problems. Moreover, while low order methods often involve nonlinear/nonconvex constraints on the corresponding decision parameters (see, e.g., [12, 34]), general order methods are often formulated in a manner such that the resulting constraints have a linear structure (e.g., the Youla parameterization-based method of [11] or linear matrix inequalities that result from quadratic stabilizability methods). Hence, the resulting constraints can be solved eff ciently in high dimension.

Nevertheless, while general order methods provide obvious benef ts, they are not without their detriments. First, general order methods tend to focus on asymptotic behavior without paying explicit attention to transient characteristics. While methods which focus on asymptotic behavior in linear systems design often produce good transient behavior, the same cannot typically be said in switching systems design. For instance, methods which rely upon quadratic stabilization techniques many times produce closed-loop controllers which switch very frequently and which often produce "jagged" state trajectories (see, e.g., [28, 29] and the examples presented therein).[1] A more concerning issue, however, lies in that general order methods typically depend upon a restricted set of matrices A_i in order to operate properly. In some of the simplest methods, which are restricted to switch between stabilizing controllers, each matrix A_i, $1 \leq i \leq k$ is assumed to be Hurwitz (correspondingly Schur for discrete-time problems) [10, 11, 28, 36]. In less restrictive methods, a common assumption (typically used in quadratic stabilizability methods) is that some convex combination of the A_i's is Hurwitz (resp. Schur) [28], i.e., that there exist $\tau_i \geq 0$, $1 \leq i \leq k$ with $\sum \tau_i = 1$ such that $\sum \tau_i A_i$ is a Hurwitz (resp. Schur) matrix. While this second condition is clearly much less restrictive than the f rst, it does exclude certain "good" choices of switching laws, as is demonstrated by the following example.

Example 1.1. *This example is based off the author's prior work in [23]— [27]. Consider a double integrator in the controllability canonical form, i.e., a plant of the form*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{2}$$

$$y = x_1 \tag{3}$$

[1] To be fair, it should be noted that the same problem exists with some low order techniques as well (see, e.g., [12, 34]), though, in low order systems, this problem is easier to avoid, as is demonstrated by [23]—[27]).

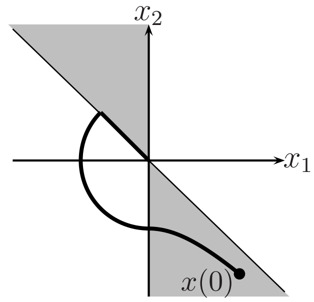

Figure 1. Sample phase portrait for plant of Eqn. 2 and 3 under the feedback law of Eqn. 4.

under the feedback law $u = v(x)y$*, where* $v(x)$ *is given by*

$$v(x) = \begin{cases} -1 & x_1(x_1 + x_2) > 0 \\ 1 & x_1(x_1 + x_2) \leq 0 \end{cases} . \tag{4}$$

The above control law corresponds to a switched state feedback law that switches between the matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{5}$$

A sample phase portrait of the resulting closed-loop interconnection is illustrated in Fig. 1.1, where the shaded region denotes where $v(x) = 1$ *and the non-shaded region denotes where* $v(x) = -1$*. While we shall not describe the detailed mechanisms of how/why this control law was selected nor its stability/robustness properties for a more generic class of second order systems, we shall highlight some of the geometric properties of the corresponding closed-loop control, along with some of the potential benefits of using such a controller for more advanced problems. First, noting that* $\begin{bmatrix} 1 & -1 \end{bmatrix}'$ *is an eigenvector of* A_1 *with eigenvalue* -1*, one can infer from the phase portrait that the controller implemented by the switching law of Eqn. 4 operates by driving the state trajectory onto this stable manifold. Indeed, along this stable eigenvector,* $x_1 + x_2 = 0$*, and, hence, once the state trajectory reaches this manifold at some time* t_0*,* $v(x(t)) = -1$ *for all* $t \geq t_0$*, and the state trajectory remains on this stable manifold for all future times.*

While the benefits of using a control law like Eqn. 4 are not immediately apparent, the author's previous work has focused on studying the performance enhancements of controllers with a structure similar to the one shown here, i.e., controllers of the form $u = v(x)y$ *with*

$$v(x) = \begin{cases} v_0 & x'F_1F_2'x \leq 0 \\ v_1 & x'F_1F_2'x > 0 \end{cases} \tag{6}$$

where $F_j \in \mathbb{R}^{2\times 1}$*,* $j = 1, 2$*. In Chapter 3 of [27] (a preliminary version of which can be found in [24]), it is shown that control laws of the form Eqn. 6 maximize the rate of convergence of the state trajectory* x *whenever the gain* $v(x)$ *is bounded (i.e.,* $|v(x)| \leq v_0$ *for some* $v_0 > 0$*). In [26], it is shown that a generalization of the above control law is finite L2 gain* *stable when exogenous inputs are introduced, and in [25], the use of these generalized controllers is shown to have performance benefits for a particular step tracking problem over other forms of LTI control.*

While the control laws of Eqn. 6 have provably "good" properties, it is important to point out that such control laws could never be found using standard quadratic stabilizability methods. For the particular example of the plant of Eqn. 2 and 3 with $v(x)$ given by Eqn. 4, Not only are the matrices A_1 and A_2 both individually unstable (A_1 has eigenvalues of ± 1, while A_2 has eigenvalues of $\pm j$), but no convex combination of these matrices is Hurwitz stable either. Indeed, any convex combination of A_1 and A_2 takes the form

$$\begin{bmatrix} 0 & 1 \\ w & 0 \end{bmatrix} \tag{7}$$

with $|w| \leq 1$. For $w \geq 0$, the above matrix has eigenvalues $\pm\sqrt{w}$, while for $w < 0$, it has eigenvalues of $\pm j\sqrt{|w|}$.

The previous example serves to illustrate a simple point: while general order methods may cover an overall broader class of systems to which they can be applied, they can "miss" certain forms of control that have good behavior because the corresponding conditions on the matrices A_i are too strict. On the other hand, the major criticism of a control law such as the one depicted in the example is that it is derived only for low order systems[2], and no immediate extensions to systems of general dimension have been apparent—until now.

The goal of this paper is to describe an extension of the control laws of Eqn. 6 that can be generalized to LTI systems of arbitrary f nite dimension. Deferring exact details of the problem description to the next section, we consider an extension where we switch between two static state feedback controllers to drive the state of the plant onto a stable manifold of dimension $n-1$ (for a system of dimension n) using a switching law very similar to the one of Eqn. 6, with the exception that the scalar gains are now vectors of gains, and the vectors F_j, $j = 1, 2$ are now $n-$length vectors. Assuming for the moment that such an extension is possible, such control laws possess certain benef ts, in addition to the obvious benef t of generalizing to arbitrary dimension:

- The switching laws have a very simple structure in that they correlate the signs of two linear measurements. That is, if $\mathrm{sgn}(F_1'x)\mathrm{sgn}(F_2'x) = 1$, one static gain is used, while if $\mathrm{sgn}(F_1'x)\mathrm{sgn}(F_2'x) = 0$ or -1, the other static gain is used. Hence, the switching law can be implemented using relatively coarse information about the system state.
- While perhaps not immediately obvious, by driving the state dynamics onto a linear manifold, certain aspects of performance will be easy to characterize in terms of

[2]This statement is not strictly true for the control laws of Eqn. 6, since it can be shown as in Chapter 6 of [27] that the techniques described therein can be applied to nonlinear/time-varying/higher order systems that have a good second order LTI approximant in an L2 gain sense.

eigenvalues associated with the manifold. Such a result is attractive because it allows for one to extend certain inferences from linear control to the switching realm.

1.2 Document Outline

We show how to construct an asymptotically stabilizing controller which switches between two static state feedback gains to drive the state of an LTI plant onto a stable manifold. We begin by brief y examining relevant geometric properties of the phase portraits in the case of two-dimensional systems to develop intuition and then show how these geometric properties can be expressed as algebraic constraints on the switched vector f elds. We then derive necessary and suff cient conditions to ensure stabilizability of the resulting switched system (characterized primarily by simple conditions on eigenvalues), and describe an explicit procedure for designing stabilizing controllers. We then show how the newly developed control law can be applied to the problem of minimizing the maximal Lyapunov exponent of the corresponding closed-loop state trajectories, and we illustrate the closed-loop transient performance of these switched state feedback controllers via multiple examples.

2 Geometric Considerations

The basic problem that we consider in this document is the following: given a reachable continuous-time LTI system $\dot{x} = Ax + Bu$, with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, find K_1, K_2, F_1, $F_2 \in \mathbb{R}^{1\times n}$ such that the switched system

$$\dot{x} = \begin{cases} (A + BK_1)x & x'F_1'F_2x \leq 0 \\ (A + BK_2)x & x'F_1'F_2x > 0 \end{cases} \tag{8}$$

is globally exponentially stable. For notational simplicity, we shall frequently refer to the matrices $A_1 \triangleq A + BK_1$ and $A_2 \triangleq A + BK_2$.

While there are many ways that the vectors K_1, K_2, F_1, and F_2 can be selected so as to achieve stability, here, we focus our efforts on designing control laws that, in a sense, mimic the geometric behavior of the second order control laws studied in [23]—[27], i.e., controllers that drive the state x of Eqn. 8 onto a stable hyperplane of dimension $n-1$ (we defer a demonstration of the utility of such control laws to a later section where we apply them to an application for minimizing the maximal Lyapunov exponent of a closed-loop system under gain constraints).

For second order systems, it is easy to design control laws with particular geometric properties by examining phase portraits. If, however, one desires to adapt such results to higher dimension, these geometric properties must somehow be translated into (relatively simple) algebraic constraints. In this section, we examine two second order examples to demonstrate the relevant geometric features of the control laws that we wish to design, and we show how to translate these geometric features into algebraic constraints that can be used to develop design algorithms for LTI systems of arbitrary dimension. We do not prove any formal statements until a later section; the purpose of this section is to provide geometric intuition for the algebraic constraints that we examine throughout the manuscript.

To begin, we shall start by examining the example of the last section in more detail. The matrices A_1 and A_2 are as in Eqn. 5, and corresponding values of K_i and F_i, $i = 1, 2$ can be determined by inspection: $K_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}$, $K_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $F_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and $F_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$. As we shall describe in a moment, for a given selection of K_1 and K_2, the choice of F_1 is unique to within a scaling factor, whereas the choice of F_2 is *not* unique. A more detailed diagram depicting the switching law of Example 1 (for an "arbitrary" choice of F_2) is provided in Fig. 2. Here v_2 represents the stable eigenvector of A_1 with eigenvalue $\lambda_2 = -1$, and N represents a vector normal to v_2 of clockwise orientation. The vector v_1 represents the unstable eigenvector of A_1 with eigenvalue $\lambda_1 = 1$, and w_1 represents a vector normal to v_1 of clockwise orientation. F_2 represents a normal vector of clockwise orientation to one side of the switching boundary between A_1 and A_2.

The first relevant geometric feature of Fig. 2 that we point out is that, since the object of the switched feedback law is to drive the state x onto the stable manifold of A_1, the stable

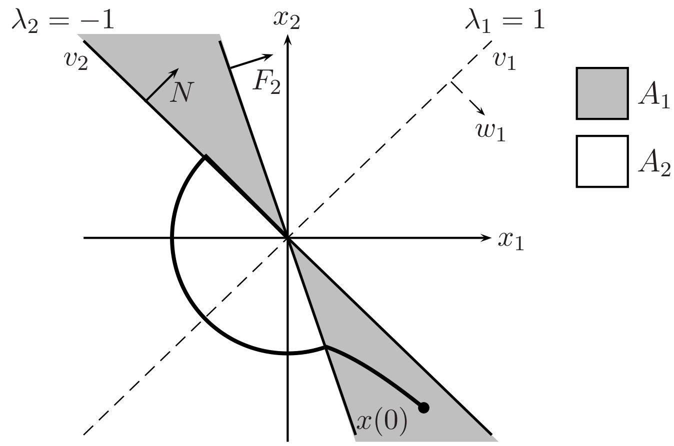

Figure 2. More detailed illustration of switching law for Example of Section 1.

eigenvector v_2 is always a switching boundary. That is, we can always choose $F_1 = N$, where N is some normal vector of appropriate orientation to the stable manifold, so that, overall, we consider switched output feedback laws of the form

$$\dot{x} = \begin{cases} (A + BK_1)x & x'N'F_2x \leq 0 \\ (A + BK_2)x & x'N'F_2x > 0 \end{cases} . \tag{9}$$

The second relevant feature we point out is that the region in the state space where A_1 is used *cannot* contain the unstable eigenvector v_1 of A_1. If such a situation were to occur, then any initial condition that were to lie along v_1 would grow exponentially for all time, and the resulting system would be unstable. Since A_1 is used in the region where $x'N'F_2'x \leq 0$, the prior constraint can be represented algebraically via

$$v_1'N'F_2v_1 > 0. \tag{10}$$

A third important feature of the switching law depicted in Fig. 2 relates to the "unidirectional" motion of the phase portraits. As can be seen from the sample phase portrait in the f gure, the state trajectory always rotates in a clockwise direction so that the angle $\theta(t)$ of the state trajectory is always non-decreasing. Such a condition guarantees lack of Zeno behavior along the switching boundaries and, hence, guarantees existence of solutions. Geometrically, this means that the vector f elds A_1x and A_2x have to "point" in the same direction across the boundaries $Nx = 0$ and $F_2x = 0$. To see the algebraic consequences of this along the boundary def ned by F_2, consider the set of x such that $F_2x = 0$ and $Nx \geq 0$. Geometrically, this set of x corresponds to all points lying along the ray in the second quadrant that are perpendicular to F_2. For this set of x, if the phase portrait is to rotate clockwise, then $F_2\dot{x} \geq 0$ for both $\dot{x} = A_1x$ and $\dot{x} = A_2x$. Note that, for the set of x given by $F_2x = 0$ and $Nx \leq 0$, the condition is reversed: $F_2\dot{x} \leq 0$ for both $\dot{x} = A_1x$ and $\dot{x} = A_2x$. Both of these sets of conditions can be combined to form a pair of quadratic constraints:

$$x'N'F_2A_1x \geq 0 \quad \forall x: \quad F_2x = 0 \tag{11}$$
$$x'N'F_2A_2x \geq 0 \quad \forall x: \quad F_2x = 0 \tag{12}$$

While perhaps not immediately obvious, satisfaction of the above quadratic constraints also guarantees that the phase portrait will reach the stable eigenvector v_s in f nite time, hence automatically ensuring that the switched system is well-behaved along the boundary def ned by $Nx = 0$.

While simple, the conditions of Eqn. 10—12 represent the essential geometric properties for switching laws of the form Eqn. 9, and we shall exploit these properties to determine algorithms for selecting vectors K_1, K_2, and F_2 which guarantee exponential stability of the system of Eqn. 9 in a later section. There is, however, one small additional caveat related to the matrix A_2 that needs to be explored. In this example, A_2 is designed to have complex eigenvalues so as to induce rotation in the corresponding phase portraits. In general, it is *not* necessary for A_2 to have complex eigenvalues in order to induce rotation (and, hence, proper operation) of the switching law of Eqn. 9, as we now illustrate. Consider the problem of switching between matrices A_1 and A_2 where A_1 is as in Eqn. 5 but where A_2 is now given by

$$A_2 = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}. \tag{13}$$

The matrix A_2 has real eigenvalues $\tilde{\lambda}_2 = 2$ and $\tilde{\lambda}_1 = 3$ as is depicted in the diagram of Fig. 3 (the dotted lines represent the corresponding eigenvectors of A_2). A heuristic description of how the state evolves under the switching law depicted in the f gure for the depicted initial condition $x(0)$ is as follows. First, since $x(0)$ initially evolves according to the matrix A_1, the state begins to move "closer" (in an angular sense) to the unstable eigenvector v_1 of A_1. In doing so, the state trajectory "passes over" the unstable eigenvectors of the matrix A_2. When the state trajectory crosses the boundary $F_2x = 0$, the state begins to evolve according to $\dot{x} = A_2x$. Since, in the absence of switching, the state trajectory should tend toward the eigenvector with maximal eigenvalue (in this case $\tilde{\lambda}_1 = 3$, one should expect that the state trajectory should begin to move toward the region in the f rst quadrant bound by the x_2 axis and the left-most dotted line. Before such a phenomenon actually occurs, however, the state trajectory lands on the stable eigenvector v_2, and the state trajectory moves in toward the origin exponentially.

In this second example, all of the conditions given by Eqn. 10—12, must hold as before. The only additional constraint that must be imposed is that the two real eigenvectors of the matrix A_2 do *not* lie in the cone where A_2 is used. If we denote these two eigenvectors by $\tilde{v}_1$ and $\tilde{v}_2$, this amounts to the algebraic conditions:

$$\tilde{v}_1N'F_2\tilde{v}_1 \quad < \quad 0 \tag{14}$$
$$\tilde{v}_2N'F_2\tilde{v}_2 \quad < \quad 0 \tag{15}$$

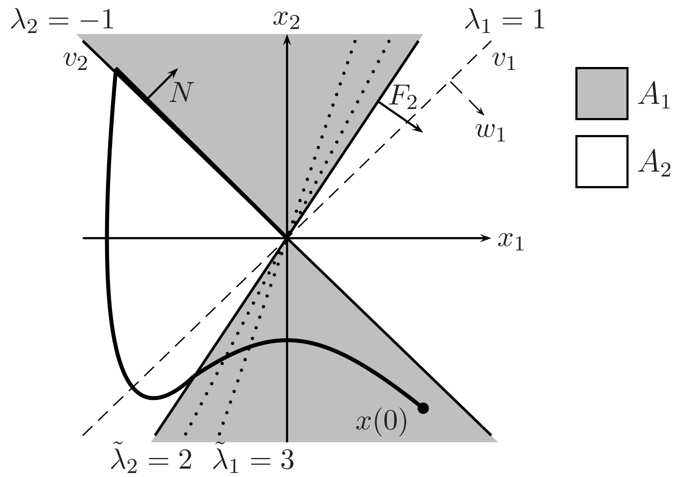

Figure 3. Example of switching law where rotation is induced via real eigenmodes.

By considering separate cases, we shall show that the conditions of Eqn. 10—12 when A_2 induces rotation via complex eigenvalues and Eqn. 10—15 when A_2 induces rotation via real eigenvalues can be used to f nd a control law of the form Eqn. 9 for appropriate choices of K_1, K_2, and F_2. Moreover, we shall show these algebraic conditions impose *constraints* on the allowable choices of K_1, K_2, and F_2 which will allow us to formulate a simple set of conditions to characterize the set of stabilizing controllers of the form Eqn. 9. This, in turn, will allow us to derive a simple method of designing exponentially stabilizing switching controllers.

3 Mathematical Preliminaries

In this section, we provide a number of mathematical statements which will be useful in establishing our main results. Proofs for all of these statements can be found in the appendix.

3.1 Rank One Quadratic Cones Containing Hyperplanes

The f rst statement we examine is the following:

Proposition 3.1. *Consider two linearly independent vectors M_1, $M_2 \in \mathbb{R}^{1\times n}$, and $C \in \mathbb{R}^{1\times n}$ such that the following condition holds:*

$$x' M_1' M_2 x \geq 0 \qquad \forall x : Cx = 0. \tag{16}$$

Then there exist constants α_1, $\alpha_2 \in \mathbb{R}$ with $\alpha_1\alpha_2 \geq 0$ such that

$$C = \alpha_1 M_1 - \alpha_2 M_2 \tag{17}$$

In layman's terms, Prop. 3.1 states that a quadratic cone def ned via a rank one matrix $M_1' M_2$ can contain a hyperplane *only* if the hyperplane is of a very restricted form. This statement will be useful in determining necessary conditions for the relative form of the vectors K_1, K_2, and F_2 that can be used to achieve stability.

3.2 Left Eigenvectors of Companion Matrices

For convenience, the majority of the main results will be proved for a particular state space description in which the closed-loop system matrices A_1 and A_2 are in *companion form*[3]:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_2 & -a_2 & \dots & -a_{n-1} \end{bmatrix}. \tag{18}$$

Companion matrices of the form Eqn. 18 have many useful properties which are commonly known, the f rst being that the characteristic polynomial of A is given by

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

[3]Such an assumption is for convenience only; we shall show later that switching controllers for systems that are not represented in companion form can be found via an appropriate coordinate transformation.

where the coeff cients a_k are as in Eqn. 18. The proofs of many of our main results will rely heavily on certain orthogonality conditions between the left and right (generalized) eigenvectors of companion matrices. For a general matrix, certain information regarding left-right eigenvector orthogonality is known. For instance, it is known that left generalized eigenvectors are orthogonal to right generalized eigenvectors for distinct eigenvalue pairings (i.e., if v_k is a (generalized) right eigenvector of A with eigenvalue λ_k, and w_j is a (generalized) left eigenvector of A with eigenvalue λ_j, and $\lambda_k \neq \bar{\lambda}_j$, then $w_j' v_k = 0$) [21]. Note that when A is diagonalizable with distinct eigenvalues, this reduces to the familiar condition that each left eigenvector w_j, $j = 1, 2, \ldots, n$, satisf es the condition $w_j' v_k = 0$, $k \neq j$.

There is one situation that is of importance to us that is not covered by the standard theorems regarding orthogonality of left and right eigenvectors within a *single* Jordan block:

Proposition 3.2. *Consider a matrix $A \in \mathbb{R}^{n \times n}$, $n \geq 2$, that is similar to a matrix of the form*

$$\begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda \end{bmatrix} \tag{19}$$

for some $\lambda \in \mathbb{C}$. Let w_1 represent the (single) left eigenvector of A and let v_k, $k = 1, 2, \ldots, n$, represent the right generalized eigenvectors of A as generated via the standard recursion $Av_1 = \lambda v_1$, $Av_k = \lambda v_k + v_{k-1}$, $k = 2, 3, \ldots, n$. Then $w_1' v_k = 0$ for $k = 1, 2, \ldots, n-1$.

Note that the above results holds for a general matrix, not just matrices of the companion for Eqn. 18. Put simply, the result states that the left eigenvector of a single Jordan block is orthogonal to the f rst $n-1$ generalized right eigenvectors. In particular, the left eigenvector is orthogonal to the right eigenvector, a somewhat surprising result since $w_k' v_k \neq 0$ for non-defective matrices.

With the aid of Prop. 3.2, we are able to prove the following very useful result about companion matrices of the form Eqn. 18:

Proposition 3.3. *Consider $A \in \mathbb{R}^{n \times n}$ of the companion form of Eqn. 18, and let λ_k, $k = 1, 2, \ldots L$ represent the distinct eigenvalues of A with multiplicity m_k. Let w_k represent a left eigenvector corresponding to eigenvalue λ_k which takes the form*

$$w_k = \begin{bmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_{n-2} & 1 \end{bmatrix}. \tag{20}$$

Then the following equality holds for every $s \in \mathbb{C}$:

$$s^{n-1} + \alpha_{n-2} s^{n-2} + \ldots \alpha_1 s + \alpha_0 = (s - \lambda_k)^{m_k - 1} \prod_{j=1, j \neq k}^{L} (s - \lambda_j)^{m_j}. \tag{21}$$

Prop. 3.3 has multiple benefits. First, from a computational perspective, if the eigenvalues of a companion matrix A are known, then the left eigenvectors can be computed by multiplying out the polynomial on the right hand side of Eqn. 21 and reading coefficients. Second, from a theoretical standpoint, certain statements regarding left eigenvectors will be made easier to prove by performing operations on the corresponding polynomials of Eqn. 21 rather than on the left eigenvector w_k itself. In particular, linear combinations of left eigenvectors are equivalent to sums of polynomials with common factors, a fact which we shall exploit in proving certain stability results.

4 Sufficient Conditions for Stabilizability

We now turn our attention to the first main issue of the paper: establishing a set of sufficient conditions for stabilizability via switching. We consider single input LTI systems of the form $\dot{x} = Ax + Bu$ where the pair (A, B) is reachable, with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$. Our goal is to find row vectors $K_1, K_2, F_1, F_2 \in \mathbb{R}^{1\times n}$ such that the switched system

$$\dot{x} = \begin{cases} (A + BK_1)x & x'F_1'F_2x \leq 0 \\ (A + BK_2)x & x'F_1'F_2x > 0 \end{cases} \tag{22}$$

is globally exponentially stable. We assume that $F_1 \neq \gamma F_2$ for any $\gamma \in \mathbb{R}$ (vectors F_1 and F_2 which do not satisfy this constraint implement switching laws which use the matrix $A + BK_1$ *only* on the hyperplane $F_1x = 0$, a measure zero set in $\mathbb{R}^n$). We will prove that, under the following assumptions, the switched system of Eqn. 22 is globally exponentially stable:

1. $A + BK_1$ has $n - 1$ eigenvalues in the right half plane (at least one of which is purely real), along with a single, real dominant eigenvalue λ_1 with corresponding right eigenvector v_1.

2. $F_1 = N$, where N is the normal vector to the hyperplane containing the $n-1$ smallest eigenvalues of $A + BK_1$.

3. F_2 is neither a left eigenvector of $A + BK_1$ nor $A + BK_2$.

4. The dominant right eigenvector v_1 of $A + BK_1$ satisfies the condition

$$v_1'N'F_2v_1 > 0.$$

5. The following two conditions hold:

$$\begin{array}{ll} x'N'F_2(A + BK_1)x \geq 0 \quad \forall x: & F_2x = 0 \\ x'N'F_2(A + BK_2)x \geq 0 \quad \forall x: & F_2x = 0. \end{array}$$

6. One of the following conditions must hold:

   - The matrix A_2 has a pair of conjugate symmetric eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathbb{C}$, $\tilde{\lambda}_1 = \tilde{\lambda}_2'$, $\mathrm{Im}\{\tilde{\lambda}_1\} \neq 0$ such that the corresponding right eigenvector $\tilde{v}_1$ of $\tilde{\lambda}_1$ satisfies the condition $N\tilde{v}_1 \neq 0$.
   - The matrix A_2 has a pair of real eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathbb{R}$ such that the corresponding right eigenvectors $\tilde{v}_1$ and $\tilde{v}_2$ satisfy the conditions

$$\begin{array}{rcl} \tilde{v}_1'N'F_2\tilde{v}_1 & < & 0 \\ \tilde{v}_2'N'F_2\tilde{v}_2 & < & 0. \end{array}$$

We shall prove that any switching law of the form Eqn. 22 is globally exponentially stable whenever the above assumptions are satisf ed, and we shall develop relatively simple algorithms for designing stabilizing controllers which satisfy these assumptions. Initially, we focus on the special case where the LTI system to be controlled is written in the *controllability canonical form*, where the A and B matrices have the special structure

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_2 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{23}$$

It is well-known that whenever the pair (A, B) is controllable, there exists a coordinate transformation T which puts the matrices $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}B$ in the form of Eqn. 23 [4], so we lose no generality in making this assumption (we shall make a formal statement to this effect in a later section). It is also clear that, for any vector $K \in \mathbb{R}^{1\times n}$, the matrix $A+BK$ is a companion matrix of the form Eqn. 18. This allows us to prove the following important statement regarding item 5 above, whose proof can be found in the appendix:

Lemma 4.1. *Consider matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ in the companion form of Eqn. 18 where A_1 has $n-1$ eigenvalues $\lambda_k \in \mathbb{C}^-$ $k = 2, 3 \dots, n$ along with a single real eigenvalue λ_1 of multiplicity 1 with $\lambda_1 > \mathrm{Re}\{\lambda_k\}$ for $k \geq 2$. Let $N \in \mathbb{R}^{1\times n}$ denote a vector that is normal to the subspace of $\mathbb{R}^n$ generated by the corresponding eigenvectors v_k for $k \geq 2$, and consider $F \in \mathbb{R}^{1\times n}$, $F \neq \gamma N$ for any $\gamma \in \mathbb{R}$, that satisfies the following conditions:*

$$x'N'FA_1x \geq 0 \quad \forall x : \quad Fx = 0 \tag{24}$$

$$x'N'FA_2x \geq 0 \quad \forall x : \quad Fx = 0. \tag{25}$$

The following statements are true:

1. *$N' = w_1$, where w_1 represents a left eigenvector of A_1 corresponding to eigenvalue λ_1.*

2. *$F' = \mu_1 w_1 + \mu_j w_j$ for some $\mu_1, \mu_j \in \mathbb{R}$, where w_j is a left eigenvector of A_1, $j \neq 1$. Also, $F' = \tilde{\mu}_1 \tilde{w}_j + \tilde{\mu}_l \tilde{w}_l$ for some $\tilde{\mu}_1, \tilde{\mu}_l \in \mathbb{R}$, where either $\tilde{w}_j$ and $\tilde{w}_l$ are left eigenvectors of A_2, or $\tilde{w}_j$ and $\tilde{w}_l$ are a left eigenvector and corresponding first generalized left eigenvector corresponding to a repeated eigenvalue $\tilde{\lambda}_j$ of A_2.*

3. *A_1 and A_2 have at least $n-2$ eigenvalues in common.*

4. *There exists an invertible transformation $T \in \mathbb{R}^{2\times 2}$ such that*

$$\begin{bmatrix} w_1' \\ w_j' \end{bmatrix} = T \begin{bmatrix} \tilde{w}_j' \\ \tilde{w}_l' \end{bmatrix}.$$

Regarding item 2 of Lemma 4.1, by arranging left eigenvectors appropriately, we can always assume without loss of generality that F can be represented as both $F' = \mu_1 w_1 + \mu_2 w_2$ and $F' = \tilde{\mu}_1 \tilde{w}_1 + \tilde{\mu}_2 \tilde{w}_2$. We henceforth use this notation throughout the remainder of the document.

An important corollary to Lemma 4.1 is the following:

Corollary 4.1. *Let the assumptions of Lemma 4.1 hold, and let $v_k \in \mathbb{R}^n$, $k = 1, 2, \ldots, n$ represent a right eigenvector or a generalized right eigenvector of A_1. Then there exist at least $n - 2$ values of k for which the following holds: $Nv_k = 0$, and v_k is either a right eigenvector or a generalized right eigenvector of A_2.*

Proof. Because A_1 and A_2 are both companion matrices, the generalized eigenspace of each matrix is completely determined by the corresponding eigenvalues per Eqn. 78. Hence, if the two matrices have at least $n-2$ eigenvalues in common, they have at least $n-2$ (generalized) right eigenvectors in common, as well. More careful examination of the proof of Lemma 4.1 reveals that the $n-2$ eigenvalues which are guaranteed to be in common between A_1 and A_2 have corresponding (generalized) eigenvectors which satisfy $Nv_k = 0$. □

4.1 Proof of Stability

Under the six assumptions presented at the beginning of this section, along with the result of Lemma 4.1, we now prove that the switched system of Eqn. 22 is globally exponentially stable. Under the assumptions on the matrix $A + BK_1$, it is clear that any initial condition which lies in the stable hyperplane $Nx(0) = 0$ satisf es $Nx(t) = 0$ for all $t \geq 0$ and, hence, decays exponentially toward the origin. The main problem, then, is to show that the conditions presented at the beginning of the section guarantee that any initial condition which does *not* lie on the stable hyperplane decays exponentially toward 0, as well.

The essential manner in which stability is achieved by the switching law of Eqn. 22 is the following: the switching surface def ned by the vector F_2 and the feedback gains K_1 and K_2 are chosen in such a way that any initial condition which does not initially lie on the stable hyperplane $Nx(0) = 0$ is driven onto the stable hyperplane in some f nite time T such that $Nx(T) = 0$. Assuming this occurs, $Nx(t) = 0$ for all $t \geq T$, and exponential stability follows. We prove that all initial conditions are driven onto the stable hyperplane in f nite time by considering two separate cases:

Case 1: $x(0)'N'F_2x(0) > 0$. In this case, we show under the given assumptions that there exists T such that $Nx(T) = 0$.

Case 2: $x(0)'N'F_2x(0) \leq 0$, $Nx(0) \neq 0$. In this case, we show under the given assumptions that there exists some f nite time $\tilde{T}$ such that $x(\tilde{T})'N'F_2x(\tilde{T}) > 0$. Now, stability follows by considering case 1.

To relate these conditions to the second order examples that were provided in Section 2, case 1 corresponds to the case where the initial condition lies in the white region of Fig. 2 and 3, while case 2 corresponds to the case where the initial condition lies in the gray shaded region of these figures.

For all parts that follow, we assume that the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of the matrix $A + BK_2$ are distinct. The proofs for the case where the eigenvalues are repeated are similar.

4.1.1 Case 1

We consider the specific case where $Nx(0) > 0$ and $F_2x(0) > 0$; the case where both of these quantities are negative follows via a symmetry argument.

To prove that there exists some time T such that $Nx(T) = 0$, we will need the result of the following proposition:

Proposition 4.1. *Consider those $x \in \mathbb{R}^n$ for which $F_2x = 0$ and $F_2(A + BK_2)x = 0$. For all such x, $Nx = 0$.*

Proof. By item 3 of Lemma 4.1, F_2 can be represented as $F_2' = \tilde{\mu}_1\tilde{w}_1 + \tilde{\mu}_2\tilde{w}_2$. Moreover, by assumption 3 presented at the beginning of the section, both $\tilde{\mu}_1$ and $\tilde{\mu}_2$ must be nonzero. For the case where $\tilde{w}_1$ and $\tilde{w}_2$ represent left eigenvectors for distinct eigenvalues (the proof for a repeated eigenvalue is similar), $(A+BK_2)'F_2' = \tilde{\lambda}_1\tilde{\mu}_1\tilde{w}_1 + \tilde{\lambda}_2\tilde{\mu}_2\tilde{w}_2$. The constraint $F_2x = 0$ yields the relationship $\tilde{\mu}_1\tilde{w}_1'x = -\tilde{\mu}_2\tilde{w}_2'x$. Substitution of this expression into the expression for $F_2(A+BK_2)$ yields the constraint $\tilde{\mu}_1(\tilde{\lambda}_1 - \tilde{\lambda}_2)\tilde{w}_1'x = 0$. Since the eigenvalues are assumed distinct and $\tilde{\mu}_1 \neq 0$, we conclude $\tilde{w}_1'x = 0$. Substituting this back into the expression determined by $F_2x = 0$, we find that $\tilde{w}_2'x = 0$, as well.

Using item 4 of Lemma 4.1, there exists an invertible transformation T such that

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = T \begin{bmatrix} \tilde{w}_1' \\ \tilde{w}_2' \end{bmatrix}.$$

Multiplication of both sides by x yields $w_1'x = 0$ and $w_2'x = 0$ whenever both $\tilde{w}_1'x = 0$ and $\tilde{w}_2'x = 0$. Now, by item 1 of Lemma 4.1, $N' = w_1$, and hence $Nx = 0$. □

We first wish to show that if the state trajectory ever leaves the cone $x(t)'N'F_2x(t) > 0$, it must do so by passing through the hyperplane $Nx(t) = 0$. More formally, suppose there exists some time T for which $x(T)'N'F_2x(T) = 0$ and $x(t)'N'F_2x(t) > 0$ for all $0 \leq t < T$. It follows that either $Nx(T) = 0$ or $F_2x(T) = 0$. As we show now, the latter situation cannot occur unless *both* $Nx(T) = 0$ and $F_2x(T) = 0$.

Under the given assumptions, we find that $x(t)$ must satisfy the conditions $F_2x(t) > 0$ for $0 \leq t < T$ and $F_2x(T) = 0$. Hence, $F_2\dot{x}(T) \leq 0$. Note, however, by item 5 of the assumptions presented at the beginning of the section that $F_2\dot{x}(T) \geq 0$ and, therefore, $F_2\dot{x}(T) = 0$. Moreover, by Prop. 4.1, if both $F_2x(T)$ and $F_2\dot{x}(T)$ are zero, then $Nx(T) = 0$, i.e., the state trajectory has already crossed onto the other switching surface. Hence, for any $x(T)$ with $F_2x(T) = 0$ and $Nx(T) > 0$, we f nd that $F_2\dot{x}(T) > 0$, which contradicts the fact that $F_2\dot{x}(T) \leq 0$. Hence, the state trajectory can only ever leave the given cone by satisfying the constraint $Nx(T) = 0$.

Now, note that any initial condition $x(0)$ may be written in the form

$$x(0) = \sum_{k=1}^{n} \alpha_k \tilde{v}_k$$

where $\tilde{v}_k$, $k = 1, 2, \ldots, n$ form a basis for the (generalized) eigenspace of A_2. Using the fact that $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are both distinct, it follows that the corresponding left eigenvectors $\tilde{w}_1$ and $\tilde{w}_2$ can be normalized to satisfy $\tilde{w}_i'\tilde{v}_k = \delta_{ik}$ for $i = 1, 2$, $k = 1, 2, \ldots, n$. Now, using item 4 of Lemma 4.1, there exist constants β_1 and β_2 such that $w_1 = \beta_1\tilde{w}_1 + \beta_2\tilde{w}_2$. Since $N' = w_1$, we f nd that

$$Nx(0) = \alpha_1\beta_1 + \alpha_2\beta_2 > 0$$

and, moreover, that

$$Nx(t) = \alpha_1\beta_1 e^{\tilde{\lambda}_1 t} + \alpha_2\beta_2 e^{\tilde{\lambda}_2 t} \tag{26}$$

whenever $x(t)$ lies in the cone $x(t)'N'F_2x(t) > 0$. We now separately consider the cases where $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are complex-valued, $\tilde{\lambda}_1 = \tilde{\lambda}_2'$, and $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathbb{R}$, $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$. In the former case, we may rewrite Eqn. 26 in the form

$$Ae^{\sigma t}\cos(\omega t + \phi)$$

where $\tilde{\lambda}_1 = \sigma + j\omega$, $\omega > 0$, and where A is nonzero due to the assumption in the f rst part of item 6 that $N\tilde{v}_1 \neq 0$. Moreover, under the above assumption that $Nx(0) > 0$, we have that $A\cos(\phi) = Nx(0) > 0$. Because $Nx(\pi/\omega) = -A\cos(\phi)\exp(\sigma\pi/\omega) < 0$, we conclude that there exists $T < \pi/\omega$ such that $Nx(T) = 0$.

In the case where $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are both real and distinct, assume without loss of generality that $\tilde{\lambda}_1 > \tilde{\lambda}_2$. Item 6 at the beginning of the section requires existence of right eigenvectors $\tilde{v}_1$ and $\tilde{v}_2$ corresponding to $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ such that

$$\begin{aligned} -\gamma_1 &\triangleq N\tilde{v}_1 < 0 \\ -\gamma_2 &\triangleq N\tilde{v}_2 < 0 \\ \delta_1 &\triangleq F_2\tilde{v}_1 > 0 \\ \delta_2 &\triangleq F_2\tilde{v}_2 > 0 \end{aligned}$$

Because both F_2' and N' are linear combinations of $\tilde{w}_1$ and $\tilde{w}_2$, we f nd that

$$\begin{aligned} Nx(t) &= -\alpha_1\gamma_1 e^{\tilde{\lambda}_1 t} - \alpha_2\gamma_2 e^{\tilde{\lambda}_2 t} \\ F_2x(t) &= \alpha_1\delta_1 e^{\tilde{\lambda}_1 t} + \alpha_2\delta_2 e^{\tilde{\lambda}_2 t}. \end{aligned}$$

We now show that $\alpha_1 > 0$. First, assume that $\alpha_1 = 0$. Then $Nx(t) = -\alpha_2\gamma_2 e^{\tilde{\lambda}_2 t}$ which implies that $\alpha_2 < 0$ to satisfy the constraint $Nx(0) > 0$. However, since $F_2x(t) = \alpha_2\delta_2 e^{\tilde{\lambda}_2 t}$, this leads to the conclusion $F_2x(0) < 0$, which contradicts our assumption that $x(0)'N'F_2x(0) > 0$. Hence, $\alpha_1 \neq 0$.

Now assume that $\alpha_1 < 0$. Because $Nx(0) > 0$, $-\alpha_1\gamma_1 - \alpha_2\gamma_2 > 0$. Moreover, since $\tilde{\lambda}_1 > \tilde{\lambda}_2$,

$$Nx(t) = -\alpha_1\gamma_1 e^{\tilde{\lambda}_1 t} - \alpha_2\gamma_2 e^{\tilde{\lambda}_2 t} > (-\alpha_1\gamma_1 - \alpha_2\gamma_2)e^{\tilde{\lambda}_2 t} > 0$$

for all $t \geq 0$. However, because $\tilde{\lambda}_1 > \tilde{\lambda}_2$, $\operatorname{sgn}(F_2x(t)) = \operatorname{sgn}(\alpha_1\delta_1) = -1$ for suff ciently large t. This, consequently, implies that there exists T such that $F_2x(T) = 0$ which we have already shown cannot happen. Thus, $\alpha_1 > 0$.

Now, for suff ciently large t, $\operatorname{sgn}(Nx(T)) = \operatorname{sgn}(-\alpha_1\delta_1) = -1$, which implies that there exists T for which $Nx(T) = 0$.

4.1.2 Case 2

We now wish to show that for any initial condition with $x(0)'N'F_2x(0) \leq 0$, $Nx(0) \neq 0$, there exists $\tilde{T} > 0$ such that $x(\tilde{T})'N'F_2x(\tilde{T} > 0$. To begin, consider those $x(0)$ for which $Nx(0) > 0$, $F_2x(0) = 0$. Under the conditions of item 3 presented at the beginning of the section, $F_2\dot{x}$ is nondecreasing for such x under both the action $\dot{x} = (A + BK_1)x$ and $\dot{x} = (A + BK_2)x$. Hence, $x(t)$ will initially evolve according to $\dot{x} = (A + BK_2)x$. Moreover, as we showed in the proof of Case 1, $F_2(A + BK_2)x > 0$ for all x on the surface $F_2x = 0$ which do not lie in the hyperplane $Nx = 0$, so we conclude that for suff ciently small δ, $x(\delta)'N'F_2x(\delta) > 0$. A similar argument can be made for those x satisfying $Nx(0) < 0$, $F_2x(0) = 0$

From the above, we f nd that it is suff cient to show that any initial condition lying in the cone $x(0)'N'F_2x(0) < 0$ satisf es $F_2x(\tilde{T}) = 0$ for some $\tilde{T} > 0$ (because of time-invariance, the argument in the above paragraph can be applied to a new initial condition $\tilde{x}(0) = x(\tilde{T})$). We assume without loss of generality that $Nx(0) > 0$ and $F_2x(0) < 0$. Similar to case 1, we write $x(0)$ in the form

$$x(0) = \sum_{k=1}^{n} \alpha_k v_k$$

where v_k, $k = 1, 2, \ldots, n$ forms a basis for the generalized eigenspace of $A + BK_1$. Because $N' = w_1$, we f nd that

$$Nx(t) = \alpha_1\gamma_1 e^{\lambda_1 t}$$

where $\gamma_1 = Nv_1 > 0$ for an appropriate choice of v_1. Since $Nx(0) > 0$, we conclude that $\alpha_1 > 0$.

Now, from item 2 of Lemma 4.1, recall that $F_2' = \mu_1 w_1 + \mu_2 w_2$ for some $\mu_1, \mu_2 \in \mathbb{R}$. Simple computation shows that $v_1'N'F_2v_1 = \mu_1\gamma_1^2$. By the assumption of item 4 presented at the beginning of the section, $v_1' N' F_2 v_1 > 0$, and, hence, $\mu_1 > 0$. Without loss of generality, we henceforth take $\mu_1 = 1$.

Now, with $F_2' = w_1 + \mu_2 w_2$, we can write

$$F_2 x(t) = \alpha_1 \gamma_1 e^{\lambda_1 t} + \alpha_2 \gamma_2 e^{\lambda_2 t}$$

with $\gamma_2 = \mu_2 w_2' v_2$. At $t = 0$, $\alpha_1\gamma_1 + \alpha_2\gamma_2 < 0$ by assumption. However, for t suff ciently large $\mathrm{sgn}(F_2 x(t)) = \mathrm{sgn}(\alpha_1 \gamma_1) = 1$, since $\lambda_1 > \lambda_2$ by assumption. Hence, there exists $\tilde{T}$ such that $F_2 x(\tilde{T}) = 0$.

4.2 Remarks

While potentially complicated to read, the formal proofs which establish suff ciency of the conditions presented at the beginning of the section for stabilizability are conceptually simple; because $A + BK_1$ and $A + BK_2$ share $n - 2$ eigenvalues, the quantities $Nx(t)$ and $F_2 x(t)$ evolve in a “second order manner” and, therefore, have relatively simple behavior which allows us to show that the state trajectory can be driven onto the stable manifold in f nite time.

A very important question which shall not be addressed formally in this document is the issue of stability in the presence of *time delays*. Since any real system has f nite bandwidth, it is unreasonable to expect in real applications that the feedback gain can be switched instantaneously from K_2 to K_1 at the exact point in time the state trajectory crosses the stable manifold $Nx(t) = 0$. One should expect, however, that if the delay between the point in time that the state trajectory crosses the stable manifold and the time at which the gain is switched is small, then the state trajectory should still decay exponentially since the projection of the state on to the stable manifold will “dominate” the projection onto the unstable eigenvector λ_1. Indeed, such a statement can be proved formally and is the subject of a future manuscript. While we provide no formal proof of this important point in this document, we will show results of numerical simulations at the end of the paper which do inherently possess small time delays so as to provide a “sanity check” that the switching algorithms described herein maintain robustness with respect to this issue.

5 Necessary Conditions for Stabilizability

The previous section provides a set of conditions such that, if satisfied, the control law of Eqn. 22 is globally exponentially stable. In this section, we formulate a set of necessary conditions which must be satisfied for the assumptions of the last section to be valid. Assumption 1 regarding the placement of the eigenvalues of $A+BK_1$ can always be satisfied by the assumed reachability of the pair (A, B); assumption 2 simply picks a specific choice of the vector F_1 and, hence, is trivially satisfied. The remaining assumptions, however, are not completely trivial as certain conditions must hold for these assumptions to be valid. As before, we assume that the matrices $A+BK_1$ and $A+BK_2$ are in companion form, that the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $A+BK_2$ are distinct, and we further assume that neither $\tilde{\lambda}_1$ nor $\tilde{\lambda}_2$ is equal to any eigenvalue λ_k of $A+BK_1$. Also, in the process of proving item 2 of Lemma 4.1, we found that F_2 can be expressed as $F_2' = w_1 + \mu_2 w_2$ with $\mu_2 \neq 0$. For notational simplicity, we now express this condition as $F_2' = w_1 + \mu w_2$, $\mu \neq 0$.

Before deriving necessary conditions for each of the assumptions 3—6, we derive the following useful result.

Proposition 5.1. *The left eigenvectors w_1 and w_2 of $A+BK_1$ are related to the left eigenvectors $\tilde{w}_1$ and $\tilde{w}_2$ of $A+BK_2$ via the relationships*

$$\begin{aligned} w_1 &= \alpha_{11}\tilde{w}_1 + \alpha_{12}\tilde{w}_2 \quad (27) \\ w_2 &= \alpha_{21}\tilde{w}_1 + \alpha_{22}\tilde{w}_2 \quad (28) \end{aligned}$$

with

$$\alpha_{11} = \frac{\tilde{\lambda}_1 - \lambda_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2}, \quad \alpha_{12} = \frac{\lambda_2 - \tilde{\lambda}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2}, \quad \alpha_{21} = \frac{\tilde{\lambda}_1 - \lambda_1}{\tilde{\lambda}_1 - \tilde{\lambda}_2}, \quad \alpha_{22} = \frac{\lambda_1 - \tilde{\lambda}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2}. \quad (29)$$

Proof. From the proof of item 3 of Lemma 4.1, we find that the coefficients of the left eigenvectors w_1, w_2, $\tilde{w}_1$, and $\tilde{w}_2$ can be taken as the coefficients of the polynomials

$$\begin{aligned} w_1 &: (s-\lambda_2)p(s) \\ w_2 &: (s-\lambda_1)p(s) \\ \tilde{w}_1 &: (s-\tilde{\lambda}_2)p(s) \\ \tilde{w}_2 &: (s-\tilde{\lambda}_1)p(s) \end{aligned}$$

where $p(s)$ is a polynomial containing the $n-2$ common eigenvalues between $A+BK_1$ and $A+BK_2$. From item 4 of Lemma 4.1, there exists T of the form

$$T = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

such that

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = T \begin{bmatrix} \tilde{w}_1' \\ \tilde{w}_2' \end{bmatrix}.$$

Note that a linear relationship between the two sets of eigenvectors implies a linear relationship between the corresponding sets of polynomials, as well. In particular,

$$\begin{aligned} (s-\lambda_2)p(s) &= \left[\alpha_{11}(s-\tilde{\lambda}_2) + \alpha_{12}(s-\tilde{\lambda}_1)\right] p(s) \\ (s-\lambda_1)p(s) &= \left[\alpha_{21}(s-\tilde{\lambda}_2) + \alpha_{22}(s-\tilde{\lambda}_1)\right] p(s). \end{aligned}$$

Since the above relationships must hold for all $s \in \mathbb{C}$, we conclude

$$\begin{aligned} s-\lambda_2 &= (\alpha_{11}+\alpha_{12})s - \alpha_{11}\tilde{\lambda}_2 - \alpha_{12}\tilde{\lambda}_1 \\ s-\lambda_1 &= (\alpha_{21}+\alpha_{22})s - \alpha_{21}\tilde{\lambda}_2 - \alpha_{22}\tilde{\lambda}_1. \end{aligned}$$

Equating the coeff cients of s and equating the constants in each of the above relationships yields the values in Eqn. 29. $\square$

We now examine each of the remaining assumptions, starting with assumption 3.

5.0.1 Assumption 3

Since $F_2' = w_1 + \mu w_2$ with $\mu \neq 0$, F_2' is not a left eigenvector of $A+BK_1$. If we now express w_1 and w_2 in terms of $\tilde{w}_1$ and $\tilde{w}_2$, we f nd that

$$w_1 + \mu w_2 = (\alpha_{11} + \mu\alpha_{21})\tilde{w}_1 + (\alpha_{12} + \mu\alpha_{22})\tilde{w}_2.$$

It is clear from the above expression that F_2' is not a left eigenvector of $A+BK_2$ if and only if both the coeff cients of $\tilde{w}_1$ and $\tilde{w}_2$ are nonzero, which can be expressed as:

$$\tilde{\lambda}_1 - \lambda_2 + (\tilde{\lambda}_1 - \lambda_1)\mu \neq 0 \quad (30)$$

$$\tilde{\lambda}_2 - \lambda_2 + (\tilde{\lambda}_2 - \lambda_1)\mu \neq 0 \quad (31)$$

5.0.2 Assumption 4

We assume the condition $v_1' N' F_2 v_1 > 0$. Substituting the expressions $N' = w_1$, $F_2' = w_1 + \mu w_2$, and using the fact that $w_2' v_1 = 0$, we f nd that the given assumption is equivalent to $(w_1' v_1)^2 > 0$, which is trivially true.

5.0.3 Assumption 5

The first condition can be expressed as

$$x'w_1(w_1+\mu w_2)'(A+BK_1)x \geq 0 \qquad \forall x : (w_1+\mu w_2)'x = 0.$$

The inequality constraint can be rewritten as

$$x'w_1(\lambda_1 w_1 + \mu\lambda_2 w_2)'x \geq 0.$$

Upon substituting, $w_1'x = -\mu w_2'x$ into the above, we find

$$\mu^2(\lambda_1-\lambda_2)(w_2'x)^2 \geq 0.$$

Since $\lambda_1 > \lambda_2$ by assumption, the above relationship is automatically satisfied.

The second condition of assumption 5 yields a non-trivial constraint. To begin, note that the condition can be written as

$$x'w_1(w_1+\mu w_2)'(A+BK_2)x \geq 0 \qquad \forall x : (w_1+\mu w_2)'x = 0.$$

We can now express $w_1+\mu w_2$ in terms of $\tilde{w}_1$ and $\tilde{w}_2$ via Prop. 5.1:

$$\begin{aligned} x'w_1(w_1+\mu w_2)'(A+BK_2)x &= x'w_1\left[(\alpha_{11}+\mu\alpha_{21})\tilde{w}_1 + (\alpha_{12}+\mu\alpha_{22})\tilde{w}_2\right]'(A+BK_2)x \\ &= x'w_1\left[\tilde{\lambda}_1(\alpha_{11}+\mu\alpha_{21})\tilde{w}_1 + \tilde{\lambda}_2(\alpha_{12}+\mu\alpha_{22})\tilde{w}_2\right]' x. \end{aligned}$$

If we now express $\tilde{w}_1$ and $\tilde{w}_2$ in terms of w_1 and w_2, and then make the substitution $w_1'x = -\mu w_2'x$, algebraic manipulation yields that the above inequality constraint is equivalent to

$$-\mu\left[\tilde{\lambda}_2-\lambda_2+(\tilde{\lambda}_2-\lambda_1)\mu\right]\left[\tilde{\lambda}_1-\lambda_2+(\tilde{\lambda}_1-\lambda_1)\mu\right](w_2'x)^2 \geq 0.$$

A necessary and sufficient condition for the above constraint to hold for all x satisfying $(w_1+\mu w_2)'x = 0$ is

$$\mu\left[\tilde{\lambda}_2-\lambda_2+(\tilde{\lambda}_2-\lambda_1)\mu\right]\left[\tilde{\lambda}_1-\lambda_2+(\tilde{\lambda}_1-\lambda_1)\mu\right] \leq 0.$$

Under the assumption that $\mu \neq 0$, and the constraints of Eqn. 30 and 31, we arrive at the necessary and sufficient condition

$$\mu\left[\tilde{\lambda}_2-\lambda_2+(\tilde{\lambda}_2-\lambda_1)\mu\right]\left[\tilde{\lambda}_1-\lambda_2+(\tilde{\lambda}_1-\lambda_1)\mu\right] < 0. \tag{32}$$

5.0.4 Assumption 6

When the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are real, the corresponding eigenvectors must satisfy the constraints

$$\begin{aligned} \tilde{v}_1' N' F_2 \tilde{v}_1 &< 0 \\ \tilde{v}_2' N' F_2 \tilde{v}_2 &< 0. \end{aligned}$$

Expressing w_1 and w_2 in terms of $\tilde{w}_1$ and $\tilde{w}_2$, the above constraints are equivalent to

$$\begin{aligned} \tilde{v}_1'(\alpha_{11}\tilde{w}_1 + \alpha_{12}\tilde{w}_2)\left[(\alpha_{11} + \alpha_{21}\mu)\tilde{w}_1 + (\alpha_{21} + \alpha_{22}\mu)\tilde{w}_2\right]' \tilde{v}_1 &< 0 \\ \tilde{v}_2'(\alpha_{11}\tilde{w}_1 + \alpha_{12}\tilde{w}_2)\left[(\alpha_{11} + \alpha_{21}\mu)\tilde{w}_1 + (\alpha_{21} + \alpha_{22}\mu)\tilde{w}_2\right]' \tilde{v}_2 &< 0. \end{aligned}$$

Under the assumption that $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ is not equal to any eigenvalue λ_k of $A + BK_1$, we f nd that $\tilde{w}_i'\tilde{v}_j = \delta_{ij}$, $i, j = 1, 2$ via Cor. 4.1:

$$\begin{aligned} \alpha_{11}(\alpha_{11} + \alpha_{21}\mu)(\tilde{w}_1'\tilde{v}_1)^2 &< 0 \\ \alpha_{12}(\alpha_{12} + \alpha_{22}\mu)(\tilde{w}_2'\tilde{v}_2)^2 &< 0. \end{aligned}$$

Since neither $\tilde{w}_1'\tilde{v}_1$ nor $\tilde{w}_2'\tilde{v}_2$ are zero, the above conditions reduce to

$$(\tilde{\lambda}_1 - \lambda_2)\left[\tilde{\lambda}_1 - \lambda_2 + (\tilde{\lambda}_1 - \lambda_1)\mu\right] < 0 \tag{33}$$

$$(\tilde{\lambda}_2 - \lambda_2)\left[\tilde{\lambda}_2 - \lambda_2 + (\tilde{\lambda}_2 - \lambda_1)\mu\right] < 0. \tag{34}$$

5.1 Design Preliminaries: Conditions on Eigenvalues and μ Parameter

The necessary conditions derived in the last subsections can be condensed to form a very simple set of conditions on the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of the matrix $A + BK_2$, and the parameter μ to guarantee stability of the switched system of Eqn. 22. In this section, we develop conditions for two separate cases: when $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ form a complex conjugate pair ($\tilde{\lambda}_1 = \tilde{\lambda}_2'$, $\mathrm{Im}\{\tilde{\lambda}_1\} \neq 0$), and when both eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathbb{R}$, $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$. Also, while we do not formally derive the results here, we shall present conditions for the case where $\tilde{\lambda}_1 = \tilde{\lambda}_2$, $\tilde{\lambda}_1 \in \mathbb{R}$.

5.1.1 Complex Eigenvalues

When the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are complex-valued, the sum total of the conditions derived in the last section are described compactly as a constraint on μ given by Eqn. 32:

$$\mu\left[\tilde{\lambda}_2 - \lambda_2 + (\tilde{\lambda}_2 - \lambda_1)\mu\right]\left[\tilde{\lambda}_1 - \lambda_2 + (\tilde{\lambda}_1 - \lambda_1)\mu\right] < 0.$$

Because $\tilde{\lambda}_1 = \tilde{\lambda}_2'$, the roots (as a function of μ) of each of the above bracketed terms are complex-valued, and it is straightforward to verify that the product of these two bracketed terms is equal to a positive def nite quadratic polynomial in μ. Hence, the above constraint is satisf ed if and only if $\mu < 0$.

5.1.2 Real Non-repeated Eigenvalues

In the case where the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are real and non-repeated, a total of three conditions must be satisf ed, as dictated by Eqn. 32, 33, and 34:

$$\begin{aligned}
\mu\left[\tilde{\lambda}_2 - \lambda_2 + (\tilde{\lambda}_2 - \lambda_1)\mu\right]\left[\tilde{\lambda}_1 - \lambda_2 + (\tilde{\lambda}_1 - \lambda_1)\mu\right] &< 0\\
(\tilde{\lambda}_1 - \lambda_2)\left[\tilde{\lambda}_1 - \lambda_2 + (\tilde{\lambda}_1 - \lambda_1)\mu\right] &< 0\\
(\tilde{\lambda}_2 - \lambda_2)\left[\tilde{\lambda}_2 - \lambda_2 + (\tilde{\lambda}_2 - \lambda_1)\mu\right] &< 0.
\end{aligned}$$

It is clear that the above inequalities are equivalent to the following set of inequalities:

$$\mu(\tilde{\lambda}_1 - \lambda_2)(\tilde{\lambda}_2 - \lambda_2) < 0 \tag{35}$$

$$(\tilde{\lambda}_1 - \lambda_2)\left[\tilde{\lambda}_1 - \lambda_2 + (\tilde{\lambda}_1 - \lambda_1)\mu\right] < 0 \tag{36}$$

$$(\tilde{\lambda}_2 - \lambda_2)\left[\tilde{\lambda}_2 - \lambda_2 + (\tilde{\lambda}_2 - \lambda_1)\mu\right] < 0. \tag{37}$$

In order to derive conditions on μ and the associated eigenvalues λ_1, λ_2, $\tilde{\lambda}_1$, and $\tilde{\lambda}_2$, we must separately consider six separate cases[4]:

$$\begin{aligned}
&\tilde{\lambda}_1 > \tilde{\lambda}_2 > \lambda_1 > \lambda_2, \qquad && \tilde{\lambda}_1 > \lambda_1 > \tilde{\lambda}_2 > \lambda_2\\
&\tilde{\lambda}_1 > \lambda_1 > \lambda_2 > \tilde{\lambda}_2, && \lambda_1 > \tilde{\lambda}_1 > \tilde{\lambda}_2 > \lambda_2\\
&\lambda_1 > \tilde{\lambda}_1 > \lambda_2 > \tilde{\lambda}_2, && \lambda_1 > \lambda_2 > \tilde{\lambda}_1 > \tilde{\lambda}_2.
\end{aligned}$$

[4]Recall that *both* λ_1 and λ_2 are real by the f rst assumption presented at the beginning of Section 4, so that the inequalities written below make sense.

We shall consider two cases formally and will leave the remaining cases to the reader. In the case where $\tilde{\lambda}_1 > \tilde{\lambda}_2 > \lambda_1 > \lambda_2$, Eqn. 35—37 can be satisfied if and only if

$$\mu < \min\left\{-\left(\frac{\tilde{\lambda}_1 - \lambda_2}{\tilde{\lambda}_1 - \lambda_1}\right), -\left(\frac{\tilde{\lambda}_2 - \lambda_2}{\tilde{\lambda}_2 - \lambda_1}\right)\right\}.$$

In the case where $\tilde{\lambda}_1 > \lambda_1 > \tilde{\lambda}_2 > \lambda_2$, Eqn. 35 implies $\mu > 0$, whereas Eqn. 37 implies that

$$\mu < -\left(\frac{\tilde{\lambda}_1 - \lambda_2}{\tilde{\lambda}_1 - \lambda_1}\right) < 0.$$

Hence the conditions of Eqn. 35—37 cannot be satisfied in this case.

If we repeat similar analyses for the remaining cases, the conditions of Eqn. 35—37 can be expressed in the following way: if we define the set Λ as

$$\Lambda = \left\{(\hat{\lambda}_1, \hat{\lambda}_2) \in \mathbb{R}^2 : \min\{\hat{\lambda}_1, \hat{\lambda}_2\} > \lambda_1\right\} \cup \left\{(\hat{\lambda}_1, \hat{\lambda}_2) \in \mathbb{R}^2 : \max\{\hat{\lambda}_1, \hat{\lambda}_2\} < \lambda_2\right\}, \tag{38}$$

then $(\tilde{\lambda}_1, \tilde{\lambda}_2) \in \Lambda$, and

$$\mu < \min\left\{-\left|\frac{\tilde{\lambda}_1 - \lambda_2}{\tilde{\lambda}_1 - \lambda_1}\right|, -\left|\frac{\tilde{\lambda}_2 - \lambda_2}{\tilde{\lambda}_2 - \lambda_1}\right|\right\}. \tag{39}$$

5.1.3 Real Repeated Eigenvalues

While we do not derive the result formally, the analysis of this section can be adjusted to account for the case when the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are real and equal. In this case, the corresponding set of constraints on $\tilde{\lambda}_1$ and μ are

$$\tilde{\lambda}_1 \in \Lambda, \qquad \Lambda = \{\lambda \in \mathbb{R} : \lambda < \lambda_2 \cup \lambda > \lambda_1\} \tag{40}$$

$$\mu < -\left|\frac{\tilde{\lambda}_1 - \lambda_2}{\tilde{\lambda}_1 - \lambda_1}\right|. \tag{41}$$

6 Stability Conditions for Arbitrary State Space Descriptions and Framework for Design

6.1 Coordinate Changes

While the conditions of the past two sections have been derived for a particular state space description, the same conditions apply to *arbitrary* state space descriptions under a simple assumption, as we explain now. First, note that if the system of Eqn. 22 is exponentially stable, then the under the transformation $z = Tx$ for some invertible transformation T, the system

$$\dot{z} = \begin{cases} (\hat{A} + \hat{B}\hat{K}_1)z & z'\hat{F}_1'\hat{F}_2 z \leq 0 \\ (\hat{A} + \hat{B}\hat{K}_2)z & z'\hat{F}_1'\hat{F}_2 z > 0 \end{cases} \tag{42}$$

with

$$\hat{A} = TAT^{-1}, \quad \hat{B} = TB, \quad \hat{K}_i = K_i T^{-1}, \quad \hat{F}_i = F_i T^{-1}, \quad i = 1, 2 \tag{43}$$

is globally exponentially stable, as well. It is straightforward to show that all of the sufficient conditions developed for the controllability canonical realization are invariant with respect to a coordinate change. As a demonstration, we show that item 4 is invariant with respect to a coordinate transformation. If we denote $\hat{v}_1$ as the eigenvector of $\hat{A} + \hat{B}\hat{K}_1$ with corresponding eigenvalue λ_1, then it is clear that $\hat{v}_1 = Tv_1$. Now,

$$v_1' N' F_2 v_1 = \hat{v}_1 (T^{-1})' N' F_2 T^{-1} \hat{v}_1 = \hat{v}_1' \hat{N}' \hat{F}_2 \hat{v}_1.$$

Hence, $v_1' N' F_2 v_1 < 0$ if and only if $\hat{v}_1' \hat{N}' \hat{F}_2 \hat{v}_1 < 0$. Similar analyses can be performed to show that the remainder of statements introduced in Section 4 are invariant to a coordinate change, as well.

Regarding the necessary conditions that have been derived, the one critical assumption that has been made is satisfaction of the equalities

$$w_1 = \alpha_{11}\tilde{w}_1 + \alpha_{12}\tilde{w}_2 \tag{44}$$

$$w_2 = \alpha_{21}\tilde{w}_1 + \alpha_{22}\tilde{w}_2 \tag{45}$$

with α_{ij} as in Eqn. 29. Such constraints implicitly impose conditions on the relative scaling factors of the left eigenvectors w_1 and w_2. For instance, consider an example where the state space description is in the controllability canonical form and where w_1 and w_2 have been selected according to the procedure outlined in the previous sections. Suppose the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are complex-valued. Then if we choose the vector F_2 in the form, $F_2' = w_1 + \mu w_2$, the conditions of the last section imply that $\mu < 0$ in order for the corresponding switching controller to be stabilizing. If, however, we were to have selected $\hat{w}_2 = -w_2$ as the corresponding left eigenvector and, consequently, parameterized F_2 as

$F_2' = w_1 + \mu\hat{w}_2$, it is clear that the corresponding necessary condition on μ is now $\mu > 0$. Clearly, then, one needs to proceed with caution when selecting left eigenvectors for the design of the switching vector F_2.

Fortunately, there is a simple way to alleviate the above issue. First, note that if we multiply Eqn. 44 and 45 by any right eigenvector $\tilde{v}_1$ with eigenvalue $\tilde{\lambda}_1$, we find that $w_1'\tilde{v}_1 = \alpha_{11}\tilde{w}_1'\tilde{v}_1$, and $w_2'\tilde{v}_1 = \alpha_{21}\tilde{w}_1'\tilde{v}_1$. Since $\tilde{w}_1'\tilde{v}_1 \neq 0$ by the assumption that $\tilde{\lambda}_1$ is not repeated, we find that

$$\frac{w_1'\tilde{v}_1}{w_2'\tilde{v}_1} = \frac{\alpha_{11}}{\alpha_{21}} \tag{46}$$

for the controllability canonical description examined in the prior sections. Moreover, since under a change of coordinates $z = Tx$, left eigenvectors $\hat{w}_k$ and right eigenvectors $\hat{v}_j$ can be expressed as $\hat{w}_k = (T^{-1})'w_k$ and $\hat{v}_j = Tv_j$, where w_k and v_j are left eigenvectors in the original $x-$space, we see that inner products are preserved: $\tilde{w}_k'\tilde{v}_j = w_k'T^{-1}Tv_j = w_k'v_j$. Hence, the necessary conditions developed to this point will hold for arbitrary state-space descriptions provided that the left eigenvectors w_1 and w_2 are chosen so as to satisfy Eqn. 46. Such a condition can be guaranteed in a very simple way. If we let $\hat{w}_1$ and w_2 be arbitrary left eigenvectors of $A + BK_1$ with corresponding eigenvalues λ_1 and λ_2, and we let $\tilde{v}_1$ be an arbitrary right eigenvector of $A + BK_2$, then the normalized left eigenvector

$$w_1 = \frac{\alpha_{11}}{\alpha_{21}}\frac{w_2'\tilde{v}_1}{\hat{w}_1'\tilde{v}_1}\hat{w}_1 \tag{47}$$

automatically satisfies the condition of Eqn. 46. Alternatively, some simple algebra shows that, for an arbitrary pair of left eigenvectors w_1 and w_2, the necessary conditions derived in the last section can be applied to $\hat{\mu}$ with

$$\hat{\mu} = \frac{\alpha_{21}}{\alpha_{11}}\frac{w_1'\tilde{v}_1}{\hat{w}_1'\tilde{v}_1}\mu. \tag{48}$$

While the above result is derived for the case where the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are distinct, a similar result can be derived for the case of repeated eigenvalues $\tilde{\lambda}_1 = \tilde{\lambda}_2$.

6.2 Basic Design Principles

We now go about describing a basic process which can be used to design stabilizing control laws of the form in Eqn. 8. We assume that K_1 has been chosen so that the matrix $A + BK_1$ has a real dominant eigenvalue λ_1, and $n - 1$ stable eigenvalues, at least one of which (λ_2) is real. Design of a stabilizing control law is equivalent to finding vectors F_1, F_2, and K_2. This can be achieved in multiple ways by carrying out the following steps:

Step 1: Pick one real negative eigenvalue of $A + BK_1$ and call this λ_2. Compute left eigenvectors w_1 and w_2 of $A+BK_1$ corresponding to eigenvalues λ_1 and λ_2. Choose $F_1' = w_1$.

Step 2, option 1: Select a gain vector K_2 such that

1. $A + BK_2$ has the same eigenvalues as $A + BK_1$, with the exception of the eigenvalues λ_1 and λ_2.
2. The remaining two eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $A + BK_2$ are not real and are not eigenvalues of $A + BK_1$.

Compute a right eigenvector $\tilde{v}_1$ corresponding to eigenvalue $\tilde{\lambda}_1$, and f nd some value of μ such that

$$\frac{\tilde{\lambda}_1 - \lambda_1}{\tilde{\lambda}_1 - \lambda_2} \frac{w_1' \tilde{v}_1}{w_2' \tilde{v}_1} \mu < 0.$$

For such a value of μ, set $F_2' = w_1 + \mu w_2$.

Step 2, option 2: Select a gain vector K_2 such that

1. $A + BK_2$ has the same eigenvalues as $A + BK_1$, with the exception of the eigenvalues λ_1 and λ_2.
2. The remaining two eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $A + BK_2$ are real and unequal, are not eigenvalues of $A + BK_1$, and satisfy the condition $(\tilde{\lambda}_1, \tilde{\lambda}_2) \in \Lambda$ where

$$\Lambda = \left\{ (\hat{\lambda}_1, \hat{\lambda}_2) \in \mathbb{R}^2 : \min\{\hat{\lambda}_1, \hat{\lambda}_2\} > \lambda_1 \right\} \cup \left\{ (\hat{\lambda}_1, \hat{\lambda}_2) \in \mathbb{R}^2 : \max\{\hat{\lambda}_1, \hat{\lambda}_2\} < \lambda_2 \right\}.$$

Compute a right eigenvector $\tilde{v}_1$ corresponding to eigenvalue $\tilde{\lambda}_1$, and f nd some value of μ such that

$$\frac{\tilde{\lambda}_1 - \lambda_1}{\tilde{\lambda}_1 - \lambda_2} \frac{w_1' \tilde{v}_1}{w_2' \tilde{v}_1} \mu < \min \left\{ - \left| \frac{\tilde{\lambda}_1 - \lambda_2}{\tilde{\lambda}_1 - \lambda_1} \right|, - \left| \frac{\tilde{\lambda}_2 - \lambda_2}{\tilde{\lambda}_2 - \lambda_1} \right| \right\}.$$

For such a value of μ, set $F_2' = w_1 + \mu w_2$.

Step 2, option 3: Select a gain vector K_2 such that

1. $A + BK_2$ has the same eigenvalues as $A + BK_1$, with the exception of the eigenvalues λ_1 and λ_2.
2. The remaining two eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $A + BK_2$ are real and *equal*, are not eigenvalues of $A + BK_1$, and satisfy the condition

$$\tilde{\lambda}_1 \in \Lambda, \qquad \Lambda = \{\lambda : \lambda > \lambda_1 \cup \lambda < \lambda_2\}.$$

Compute a *generalized* right eigenvector $\tilde{v}_2$ corresponding to eigenvalue $\tilde{\lambda}_1$, and f nd some value of μ such that

$$\frac{\tilde{\lambda}_1 - \lambda_1}{\tilde{\lambda}_1 - \lambda_2} \frac{w_1' \tilde{v}_2}{w_2' \tilde{v}_2} \mu < - \left| \frac{\tilde{\lambda}_1 - \lambda_2}{\tilde{\lambda}_1 - \lambda_1} \right|.$$

For such a value of μ, set $F_2' = w_1 + \mu w_2$.

Note that when the state space description is in the controllability canonical form, and we choose the left eigenvectors according to the procedure outlined in Prop. 3.3, the conditions on μ presented above simplify to the conditions on μ derived in Sections 5.1.1—5.1.3 (i.e., all of the coefficients multiplying μ in options 1 through 3 above are all automatically equal to 1).

The algorithm described above indicates a relatively simple way of designing stabilizing controllers: once we have determined some value λ_2 to "remove" from the spectrum of $A + BK_1$, this determines a left eigenvector w_2 used to define F_2. Now we choose new eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ (subject to mild conditions on placement) to replace λ_1 and λ_2 in $A + BK_2$, and compute a range of values for μ for which $F_2' = w_1 + \mu w_2$ yields a stabilizing controller. Nevertheless, there are two important questions that remain unanswered:

1. How does one go about choosing the matrix $A + BK_1$ in the first place?
2. It is possible that $A + BK_1$ has several real eigenvalues in the open left half-plane, and, hence, there are several choices for the parameter λ_2 and, correspondingly, w_2. How does one go about picking the value of λ_2, then, to achieve "good" performance?

It should be clear that the answers to the above questions are application specific and, in particular, a function of the way performance is measured. What we explore in the remainder of the document is a particular application where the above questions have a "natural" answer. In doing so, we shall also illustrate a potential performance benefit of using switching controllers of the form Eqn. 22 over standard LTI state feedback controllers.

7 Application: Minimization of Maximal Lyapunov Exponents

Recall that the *Lyapunov exponent* R of the autonomous dynamical system

$$\dot{x} = f(x), \qquad x(0) = x_0$$

is given by[5]

$$R = \lim_{T \to \infty} \frac{1}{T} \ln\left(||x(T)||\right). \tag{49}$$

The above quantity is, in general, highly specific to the initial condition $x(0)$, and the *maximal Lyapunov exponent* is the largest (assuming it exists) of the Lyapunov exponents over all initial conditions. For the linear system $\dot{x} = Ax$, it is clear the maximal Lyapunov exponent is equal to $\mathrm{Re}\{\lambda_1(A)\}$, where $\lambda_1(\cdot)$ denotes the eigenvalue with maximal real part.

For the dynamical systems that are achieved via the switched state feedback controllers we have been investigating in this document, we have shown that, in finite time, the state $x(t)$ is driven onto a subspace of the matrix $A + BK_1$ which is spanned by all of the right eigenvectors and generalized right eigenvectors *except* the right eigenvector corresponding to the maximal eigenvalue λ_1. Hence, it follows that the maximal Lyapunov exponent for this particular type of control law is given by $\mathrm{Re}(\lambda_2(A + BK_1)\}$, where $\lambda_2(\cdot)$ denotes the eigenvalue with *second* largest real part.

The advantage of the above observation from a design perspective is clear: if we wish to design a state feedback controller for the LTI plant $\dot{x} = Ax + Bu$, then it appears that the switched output feedback architecture that we consider in this document should outperform a linear feedback controller $u = Kx$; however, it is important to quantify by *how much* the switched feedback architecture can outperform a linear controller to ascertain whether use of a switched feedback controller in place of a linear one is warranted in practice.

In what follows, we examine a particular case study in which the plant to be controlled is an $n-$th order integrator in the controllability canonical form, i.e., we consider a plant of the form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u. \tag{50}$$

We consider the problem of trying to minimize the maximal Lyapunov exponent that can be achieved by using 1) a static state feedback controller and 2) the switched state feedback

[5]Typically, we require additional assumptions for this definition to be true, such as homogeneity of the vector field f, but we shall not focus on such issues here.

controller of Eqn. 8. In particular, we consider the case where the controller gains are *bounded*: for a gain vector $K = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix}$, we require that $|k_i| \leq A$ for some $A > 0$ and for $i = 0, 1, \ldots, n-1$. More specifically, by considering an asymptotic case where A is taken sufficiently large, we develop an upper bound on the performance gain of utilizing the given switching controller vs. a linear controller, and we explicitly parameterize a suboptimal switching controller which converges asymptotically to the upper bound for large n.

7.1 Linear Controller

We first consider the problem of designing a linear controller $u = Kx$ so as to minimize the maximal real part of eigenvalues of $A + BK_1$. In the case where the gain bound A on the coefficients of the gain vector K is taken sufficiently large, this problem has a simple solution:

Proposition 7.1. *Consider the $n-$th order integrator of Eqn. 50 under the feedback law $u = Kx$, $K = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix}$, with $|k_i| \leq A$ for some $A > 0$ and $i = 0, 1, \ldots, n-1$. Then, for A sufficiently large, the minimal value L of the real part of $\lambda_1(A + BK_1)$, where $\lambda_1(\cdot)$ is represents the eigenvalue with maximal real part, is given by*

$$L = -A^{1/n}. \tag{51}$$

Proof. To begin, note that the characteristic polynomial of $A + BK$ is given by

$$s^n - k_{n-1}s^{n-1} - \cdots - k_1 s - k_0.$$

Now, because $|\lambda_1\lambda_2\cdots\lambda_n| = |k_0|$, where λ_i represent the eigenvalues of $A+BK$, ordered such that $\mathrm{Re}\{\lambda_i\} \geq \mathrm{Re}\{\lambda_{i+1}\}$, we find that $|\lambda_1\lambda_2\cdots\lambda_n| \leq A$. This implies that $|\lambda_1|^n \leq A$, and, hence, that $\mathrm{Re}\{\lambda_1\} \geq -|\lambda_1| \geq -A^{\frac{1}{n}}$.

We now show that, when A is sufficiently large, the polynomial $(s + A^{\frac{1}{n}})^n$ has coefficients which are bounded by A. The coefficient a_m of the term s^m for this polynomial is given by

$$a_m = \begin{pmatrix} n \\ m \end{pmatrix} A^{\frac{m}{n}}.$$

It is clear that, by taking A sufficiently large, $|a_m| \leq A$ for each $m = 0, 1, \ldots, n-1$. Hence, there exists a polynomial satisfying the gain constraints which achieves the lower bound on the minimal value of $\mathrm{Re}\{\lambda_1\}$, and we conclude that $L = -A^{\frac{1}{n}}$. □

7.2 Switched State Feedback Controller

7.2.1 Lower bound on Second Largest Eigenvalue

Since the switched state feedback laws we examine here have maximal Lyapunov exponent $\mathrm{Re}\{\lambda_2(A+BK_1)\}$, we must f rst develop a lower bound on this quantity as a function of the gain bound A. To do this will require several steps. We f rst need to show the following, the proof of which can be found in the appendix:

Proposition 7.2. *Suppose that the polynomial $s^n + a^*_{n-1}s^{n-1} + \cdots a^*_1 s + a^*_0$ is such that the root with second largest real part λ_2 is the smallest that can be achieved over all polynomials of the form $s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0$ with $|a_i| \leq A$ for some $A > 0$ sufficiently large. Then the root λ_1 of $s^n + a^*_{n-1}s^{n-1} + \cdots a^*_1 s + a^*_0$ with maximal real part is purely real and nonnegative.*

Using Prop. 7.2, we can establish the following lower bound on $\mathrm{Re}\{\lambda_2\}$:

Proposition 7.3. *For any $\lambda_1 > 0$,*

$$\mathrm{Re}\{\lambda_2\} \geq \max\left\{-\sqrt[n-1]{\lambda_1^{n-1} + A\lambda_1^{n-2} + \cdots + A\lambda_1 + A}, -\sqrt[n-1]{\frac{A}{\lambda_1}}\right\}. \tag{52}$$

Proof. We f rst prove the following intermediate result:

$$\prod_{k=2}^{n}(-\lambda_k) \leq \lambda_1^{n-1} + A\lambda_1^{n-2} + \cdots + A\lambda_1 + A. \tag{53}$$

The proof follows a repeated pattern, of which we show the f rst two steps. First, by virtue of the fact that the coeff cient of the term s^{n-1} is bounded above by A,

$$\sum_{j=1}^{n}(-\lambda_j) \leq A$$

which can be rewritten as

$$\sum_{j=2}^{n}(-\lambda_j) \leq A + \lambda_1. \tag{54}$$

Now, by virtue of the fact that the coeff cient of the term s^{n-2} is also bounded above by A, we have

$$\sum_{k=1}^{n}(-\lambda_k)\sum_{j>k}^{n}(-\lambda_j) \leq A$$

which can be rewritten as

$$\sum_{k=2}^{n}(-\lambda_k)\sum_{j>k}^{n}(-\lambda_j) \leq A + \lambda_1 \sum_{j=2}^{n}(-\lambda_j).$$

Using the inequality in Eqn. 54 to substitute into the summation on the righthand side above (valid because $\lambda_1 \geq 0$ by Prop. 7.2), we f nd

$$\sum_{k=2}^{n}(-\lambda_k)\sum_{j>k}^{n}(-\lambda_j) \leq A + \lambda_1(A + \lambda_1) = \lambda_1^2 + A\lambda_1 + A.$$

We can keep repeating this process—writing out the expression for the coeff cient of s^k in terms of the roots of the polynomial, bringing the terms multiplying λ_1 over to the righthand side, and using the last inequality we developed to get a new inequality—we eventually arrive at the inequality of Eqn. 53. From this inequality, it is clear that $\mathrm{Re}\{\lambda_2\}$ is larger than the f rst quantity in the maximum of Eqn. 52. Moreover, by virtue of the fact that

$$\left|\prod_{k=1}^{n}(-\lambda_k)\right| \leq A,$$

we f nd that $\mathrm{Re}\{\lambda_2\}$ is larger than the second expression in the maximum of Eqn. 52. □

Because the bound of Eqn. 52 holds for any $\lambda_1 > 0$, one can optimize the value of λ_1 to minimize this lower bound. It is clear that the minimizing value of λ_1 occurs where both of the expressions in braces are equal. Setting these two expressions equal yields the equation

$$\lambda_1^n + A\lambda_1^{n-1} + \cdots + A\lambda_1 - A = 0. \tag{55}$$

By Descartes' rule of signs, Eqn. 55 has at most one positive root λ_1. As we show now, there exists a positive real root satisfying $0.5 < \lambda_1 < 1$. First, note that when $\lambda_1 = 0.5$, the left hand side of Eqn. 55 evaluates to

$$\left(\frac{1}{2}\right)^n (1 - A) < 0$$

for large A. On the other hand, when $\lambda_1 = 1$, the left hand side evaluates to

$$1 + (n-2)A > 0$$

for $n \geq 2$. Thus, the polynomial of Eqn. 55 changes sign between 0.5 and 1, and, hence, a root exists between these two values. This observation allows us to obtain the following lower bounds whose proofs are immediate and are left to the reader:

Corollary 7.1. *For $A > 0$ sufficiently large,*

$$\mathrm{Re}\{\lambda_2\} \geq -\sqrt[n-1]{\frac{A}{\lambda_1^*}} \tag{56}$$

where λ_1^* is the unique positive solution of Eqn. 55. In particular, since $\lambda_1^* > 0.5$ for all values of n,

$$\mathrm{Re}\{\lambda_2\} \geq -\sqrt[n-1]{2A}. \tag{57}$$

7.2.2 Upper bound on Performance Increase, and Suboptimal Switching Controller Design

A natural way to measure the increase in performance we obtain by using a switched feedback controller vs. a linear one is to compute the quotient of the minimal value of the maximum Lyapunov exponents in each case. For the linear case, we are able to compute this exactly. For the switched feedback controller case, we are able to obtain an upper bound. Hence, by dividing the upper bound for the switching controller by the exact minimum value obtained for the linear controller, we obtain an upper bound on the performance increase obtainable via controller switching. If we denote this upper bound by G, simple division of the upper bound Eqn. 57 by the value L in Prop. 7.1 yields

$$G = \frac{1}{\sqrt[n-1]{\lambda_1^*}} \sqrt[n(n-1)]{A} \leq \sqrt[n-1]{2} \sqrt[n(n-1)]{A}. \tag{58}$$

Several comments are in order. First, because $\sqrt[n-1]{2} \to 1$ as $n \to \infty$, we see that the upper bound G converges asymptotically to $\sqrt[n(n-1)]{A}$, a result which will be useful in a moment when we consider the task of designing a switched feedback controller. Second, we see that, whenever A is taken large enough, the upper bound in performance gain can be made arbitrarily large (and, as we show in a moment, the *actual* performance gain can be made arbitrarily large as well). In practice, when the dimension n is large, the values of A needed to obtain a given performance increase become prohibitively large; for instance, the upper bound suggests that, in order to gain roughly a factor of 10 gain increase in performance, one needs to consider values of A on the order of $10^{n(n-1)}$. This suggests that, for high dimension, a more sophisticated switching architecture (the topic of future work) may be necessary to obtain reasonable performance increases, but note for now that, when the dimension n is of moderate size, the upper bound indicates promise for practical gains.

Unfortunately, it is currently unknown whether the upper bound G is obtainable, because it is unknown whether a polynomial which achieves the lower bound on the real part of the second largest eigenvalue, and which simultaneously has coeff cients which lie between A and $-A$, actually exists.[6] One could try, for instance, the polynomial

$$\left(s + \sqrt[n-1]{\frac{A}{\lambda_1^*}}\right)^{n-1} (s - \lambda_1 *),$$

but it is easy to verify that the coeff cient of the s term in the above polynomial exceeds A when A is suff ciently large.

Fortunately, a suboptimal polynomial whose second largest real-part eigenvalue is close to the lower bound in Eqn. 57 exists, and is given by

$$s\left(s + \sqrt[n-1]{A}\right)^{n-1}. \tag{59}$$

[6]For a given order n and value of $A > 0$, this problem can clearly be solved numerically, but here we seek a polynomial of form that can be generalized to arbitrary values of n and A.

By appropriately switching between a matrix whose characteristic polynomial is given by Eqn. 59, and a matrix whose characteristic polynomial is

$$(s - \sqrt[2(n-1)]{A})^2 \left(s + \sqrt[n-1]{A}\right)^{n-2}, \tag{60}$$

one can drive the state trajectory of the switched system onto a stable manifold with $n-1$ repeated eigenvalues of value $-\sqrt[n-1]{A}$. It is easy to verify that all of the coeff cients of the polynomials in Eqn. 59 and 60 are bounded in magnitude by A when A is large, and, hence, for $n-$th order integrators written in the controllability canonical form Eqn. 50, one can f nd a controller subject to the given gain bounds which has performance increase G_s given by

$$G_s = \sqrt[n(n-1)]{A}. \tag{61}$$

As $n \to \infty$, it is clear that G_s asymptotically converges to the upper bound G, and it can be verif ed numerically that $G_s \geq 0.786G$ for all values of $n \geq 2$, so that the performance obtained by the suboptimal switching controller is never more than 22% away from the upper bound G.

It should be noted that the polynomials of Eqn. 59 and 60 are in *no way* unique; one can choose different characteristic polynomials subject to the gain bound A which, when switched appropriately, also achieve the performance increase G_s of Eqn. 61. Such non-uniqueness should be viewed as advantageous since it allows for additional optimization metrics to be considered in conjunction with the objective of minimizing the maximal Lyapunov exponent.

The process of f nding a switching controller of the form Eqn. 8 now reduces to f nding switching boundaries F_1 and F_2, and this can be performed by following the algorithm described in Section 6. We shall illustrate the process of designing a suboptimal controller in the next section where we present multiple examples.

8 Examples

In this section, we provide multiple examples to illustrate the design techniques described in the previous sections. We f rst provide an example whereby we design multiple switching controllers to illustrate the basic design techniques described in Section 6. We then present an example where we design a controller which minimizes the maximal Lyapunov exponent for a triple integrator in the controllability canonical form and provide some insight for future application areas.

8.1 Example 1: Switched Feedback Controller Design for a Sixth Order System

In this f rst example, we consider a sixth-order integrator in the controllability canonical form of Eqn. 50 where the gain vector K_1 is selected such that the matrix $A + BK_1$ is given by

$$A + BK_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 54 & 81 & -9 & -70 & -44 & -11 \end{bmatrix}. \tag{62}$$

The characteristic polynomial of $A + BK_1$ is given by

$$(s + 3)^3(s + 2)(s + 1)(s - 1). \tag{63}$$

Our objective is to f nd a switched feedback controller of the form Eqn. 8 such that the state trajectory is driven onto the stable invariant subspace of the matrix $A + BK_1$, spanned by the (generalized) eigenvectors corresponding to the eigenvalues $-3, -2$ and -1. To do so requires three objects: selection of a gain vector K_2, and computation of the switching boundary vectors F_1 and F_2. We shall actually design two separate switching controllers—one where the matrix $A+BK_2$ has complex eigenvalues, and one where it has purely real eigenvalues—to illustrate the design procedure for the f rst two options listed in Section 6.

8.1.1 Controller 1: Complex Eigenvalues

Recall that the matrices $A + BK_1$ and $A + BK_2$ must have $n - 2$ eigenvalues in common. Therefore, only two eigenvalues are allowed to differ between the characteristic polynomials of $A+BK_1$ and $A+BK_2$. For this example, we (arbitrarily) will “move” the eigenvalues located at 1 and -1 of the matrix $A + BK_1$ to eigenvalues of $\pm j$ for the matrix $A + BK_2$, so that the matrix $A + BK_2$ has characteristic polynomial

$$(s+3)^3(s+2)(s^2+1). \tag{64}$$

The gain vector K_2 corresponding to the above characteristic polynomial is given by

$$K_2 = \begin{bmatrix} -54 & -81 & -99 & -92 & -46 & -11 \end{bmatrix}. \tag{65}$$

The switching boundary vector F_1 is a normal vector to the stable invariant subspace of the matrix $A + BK_1$. Because we are dealing with an example in the controllability canonical form, using the result of Prop. 3.3, F_1 may be found by multiplying out the polynomial

$$(s+3)^3(s+2)(s+1)$$

and stacking the coeff cients of the resulting expanded polynomial into the vector F_1 in ascending powers of s:

$$F_1 = \begin{bmatrix} 54 & 135 & 126 & 56 & 12 & 1 \end{bmatrix}. \tag{66}$$

Recall, now, that $F_2 = w_1' + \mu w_2'$, where $w_1' = F_1$ is the left eigenvector of $A + BK_1$ corresponding to the eigenvalue 1, and w_2 is some other left eigenvector of $A + BK_1$. Specif cally, w_2 corresponds to the left eigenvector with the eigenvalue that is "removed" from $A + BK_1$ to form the characteristic polynomial of $A + BK_2$ which, in this case, is the left eigenvector corresponding to the eigenvalue -1:

$$w_2 = \begin{bmatrix} -54 & -27 & 36 & 34 & 10 & 1 \end{bmatrix}. \tag{67}$$

According to the procedure outlined in Section 6, we may set $F_2 = w_1' + \mu w_2'$ for any value of $\mu < 0$ to achieve a stable closed-loop interconnection. If we choose $\mu = -1$, the switching boundary vector F_2 is given via

$$F_2 = \begin{bmatrix} 108 & 162 & 90 & 22 & 2 & 0 \end{bmatrix}. \tag{68}$$

Fig. 8.1.1 shows a sample phase portrait for the resulting closed-loop system for the initial condition $x(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$. The trajectory was computed in MATLAB by approximating the continuous-time system via the sampled-data system

$$x((k+1)T) = \begin{cases} \tilde{A}_1 x(kT) & x'F_1'F_2 x \leq 0 \\ \tilde{A}_2 x(kT) & x'F_1'F_2 x > 0 \end{cases} \tag{69}$$

where $\tilde{A}_i = \exp((A+BK_i)T)$, with $T = 0.001$. The waveforms $x_1(t)$ through $x_6(t)$ represent the individual components of the state vector $x(t)$, and the waveform denoted "Switching signal" is given by

$$\sigma(t) = \operatorname{sgn}\left(x(t)'F_2'F_1 x(t)\right). \tag{70}$$

The switching signal $\sigma(t)$ indicates which of the matrices $\tilde{A}_1$ or $\tilde{A}_2$ is being used at any given time; when $\sigma(t) = 1$, the state evolves according to $\dot{x} = (A + BK_2)x$, while when $\sigma(t) = -1$, the state evolves according to $\dot{x} = (A + BK_1)x$.

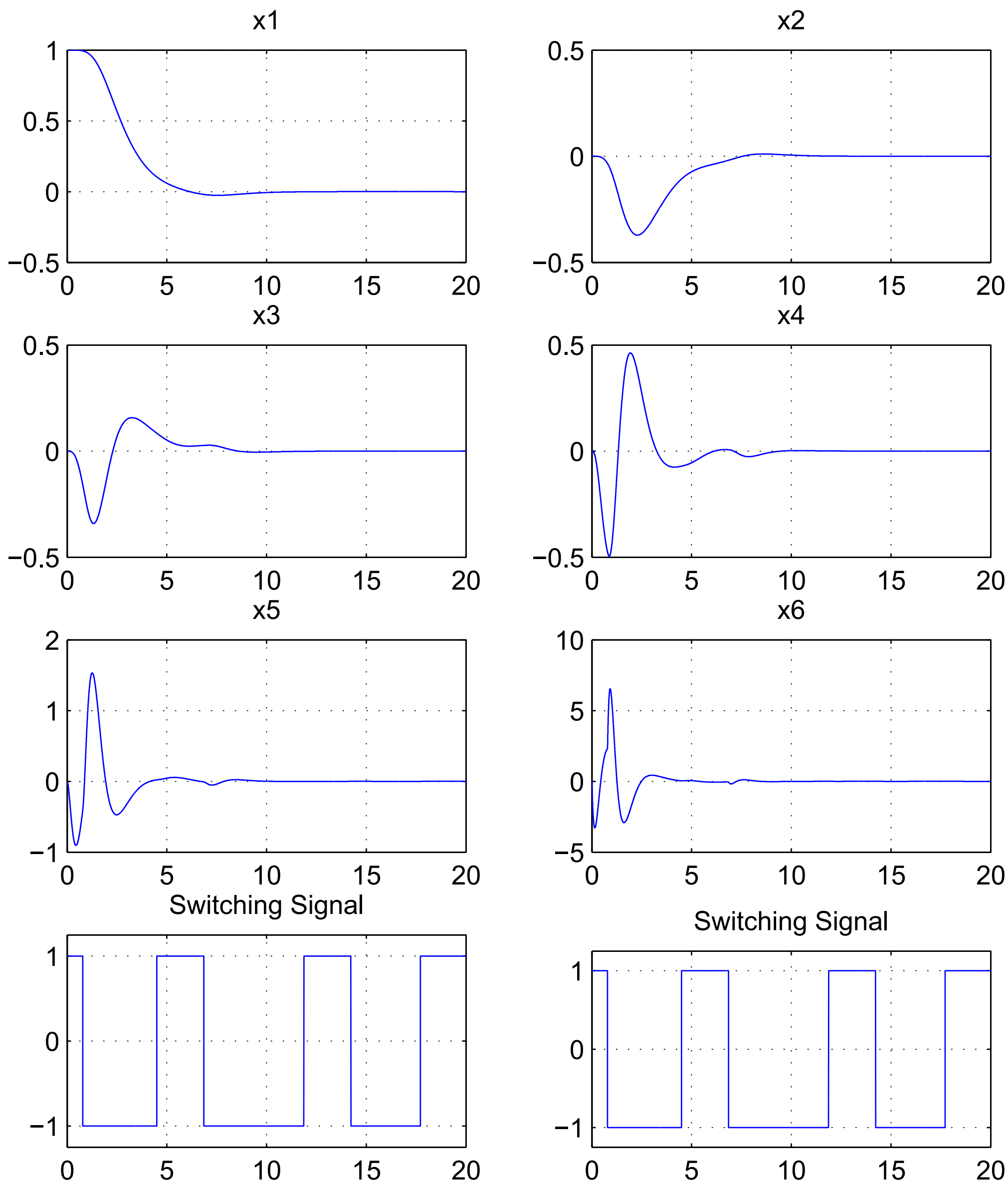

Figure 4. Sample phase portrait for the switching controller of Section 8.1.1. The variables x_1 through x_6 represent the individual components of the state vector x, and the value of the switching signal as a function of time is plotted in both columns for convenience.

Several comments are in order. First, note that $\sigma(t)$ switches between the values $+1$ and -1 multiple times. This contradicts the fact that the *exact* solution to the nonlinear differential equation should have $\sigma(t) = -1$ for all $t \geq t_0$ for some $t_0 > 0$ (corresponding to the fact that, once the state is driven onto the stable hyperplane, it never leaves). Note in general, however, for the sampled data system of Eqn. 69, the gain vector will not switch from K_2 to K_1 at the *exact* point in time that the state trajectory crosses the stable hyperplane but will, rather, switch some small amount of time after this has happened. Therefore, as a general rule, $\sigma(t)$ will vary between $+1$ and -1.

Nevertheless, as the plots in the f gure seem to suggest, the state trajectory remains well-behaved despite this issue. In fact, one can formally prove that the switching control laws derived here are globally exponentially stable even in the presence of suff ciently small *time delays*. We omit this proof, both due to its length and because it is relevant to an extension of the work we present here which considers the problem of input-to-state stability. As an additional numerical verif cation of stability for this particular example, we uniformly grid the unit box $||x||_\infty = 1$ with grid size $\Delta = 0.2$ and simulate the closed-loop differential equation for every initial condition $x(0)$ on this grid. For each resulting state trajectory, we compute the value $||x(20)||_2/||x(0)||_2$ and f nd that the maximum value of this quantity over all initial conditions on the grid is 6.88×10^{-4}(corresponding to the initial condition $x(0) = \begin{bmatrix} -0.4 & -1 & -1 & -0.4 & 0 & 0 \end{bmatrix}'$). While this does not *formally* prove anything, it does provide a “sanity check” that the state trajectories are decaying for arbitrary initial conditions and that the phase portraits of Fig. 8.1.1 were not carefully crafted by choosing an initial condition with special properties.

Also, as a qualitative aside, note that the state variables $x_1(t)$ through $x_5(t)$ vary smoothly and do not possess any “jagged” behavior. Such behavior should be expected since $\dot{x}_j = x_{j+1}$ for *both* the matrices $A + BK_1$ and $A + BK_2$ for $j = 1, \ldots, 5$. The state variable $x_6(t)$ has discontinuous derivatives at the switching instants as is apparent from the f gure, but the state varies continuously between switching instants.

8.1.2 Controller 2: Real, Non-repeated Eigenvalues

For this example, we choose to move the eigenvalue 1 and one of the eigenvalues located at -3 from the matrix $A+BK_1$ to eigenvalues 2 and 3 for the matrix $A+BK_2$, corresponding to a characteristic polynomial

$$(s+3)^2(s+2)(s+1)(s-2)(s-3). \tag{71}$$

The gain vector K_2 which achieves this characteristic polynomial is

$$K_2 = \begin{bmatrix} -108 & -144 & 3 & 52 & 10 & -4 \end{bmatrix}. \tag{72}$$

Since the value of F_1 depends only upon the matrix $A + BK_1$, F_1 is the same as for the previous controller and is given by Eqn. 66. To compute a choice of F_2, we f rst compute the left eigenvector w_2 corresponding to the eigenvalue -3:

$$w_2 = \begin{bmatrix} -18 & -21 & 10 & 20 & 8 & 1 \end{bmatrix}. \tag{73}$$

We can choose $F_2 = w_1' + \mu w_2'$ for any value of μ which satisfies the condition of Eqn. 39 (corresponding to the condition on μ in step 2, option 2 of Section 6). Using the values $\lambda_1 = 1$, $\lambda_2 = -3$, $\tilde{\lambda}_1 = 2$ and $\tilde{\lambda}_2 = 3$, we find that this condition reduces to $\mu < \min\{-5, -3\} = -5$. Hence, by choosing $\mu = -6$, we arrive at

$$F_2 = \begin{bmatrix} 162 & 261 & 66 & -64 & -36 & -5 \end{bmatrix}. \tag{74}$$

The coordinates of the state trajectory along with the value of the switching signal $\sigma(t)$ are plotted in Fig. 8.1.2 for the initial condition $x(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$.

8.2 Example 2: Minimization of Maximal Lyapunov Exponent for a Triple Integrator

As an example of the problem described in Section 7, we consider the task of designing a switching controller which minimizes the maximal Lyapunov exponent for a triple integrator in the controllability canonical form subject to a gain bound of $A = 10^6$. According to the results of that section, the gain G_s which can be achieved by using a switched feedback controller in place of a linear controller is 10 (see Eqn. 61 for the value $A = 10^6$ and $n = 3$). In particular, by switching between companion matrices with characteristic polynomials

$$s(s + 1000)^2$$

and

$$(s + 1000)(s - 10\sqrt{10})^2,$$

one can achieve a maximal Lyapunov exponent of -1000. One can calculate the following parameters of the switching controller following the methods outlined in Section 6:

$$\begin{aligned}
K_1 &= \begin{bmatrix} 0 & -10^6 & -2000 \end{bmatrix} \\
K_2 &= \begin{bmatrix} -10^6 & 1000 - 20000\sqrt{10} & 1000 - 20\sqrt{10} \end{bmatrix} \\
F_1 &= \begin{bmatrix} 10^6 & 2000 & 1 \end{bmatrix} \\
w_2' &= \begin{bmatrix} 0 & 1000 & 1 \end{bmatrix}.
\end{aligned}$$

Following the procedure outlined in the third option of the algorithm outlined in Section 6, one finds that choosing $F_2 = F_1 + \mu w_2'$ will achieve a stable closed-loop system whenever $\mu < -1 - 10\sqrt{10}$.

Fig. 8.2 shows two trajectories. The top trajectory corresponds to the state variable $x_1(t)$ for a switching controller with the above parameters with $F_2 = F_1 + \mu w_2'$ with $\mu = -100$

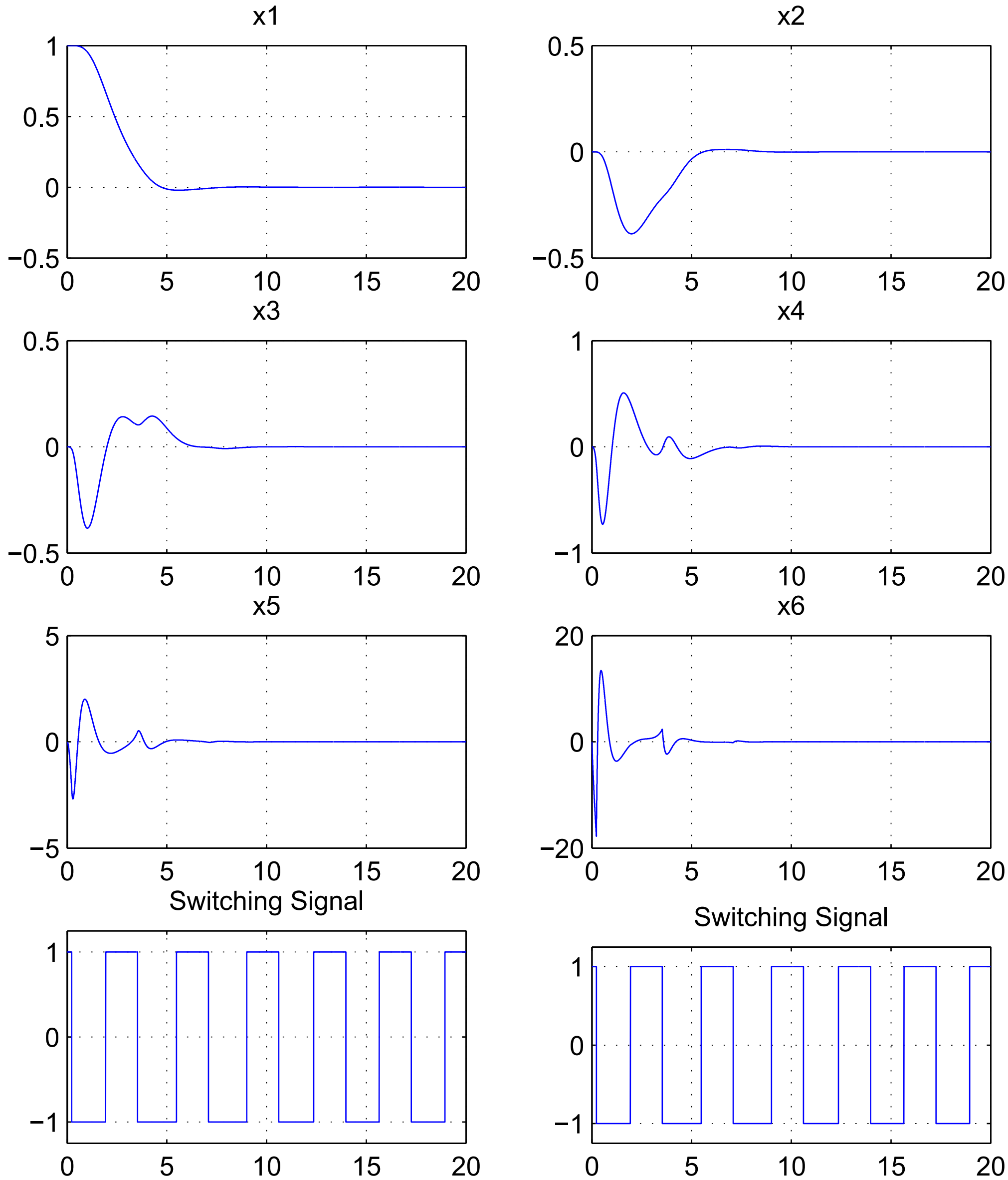

Figure 5. Sample phase portrait for the switching controller of Section 8.1.2. The variables x_1 through x_6 represent the individual components of the state vector x, and the value of the switching signal as a function of time is plotted in both columns for convenience.

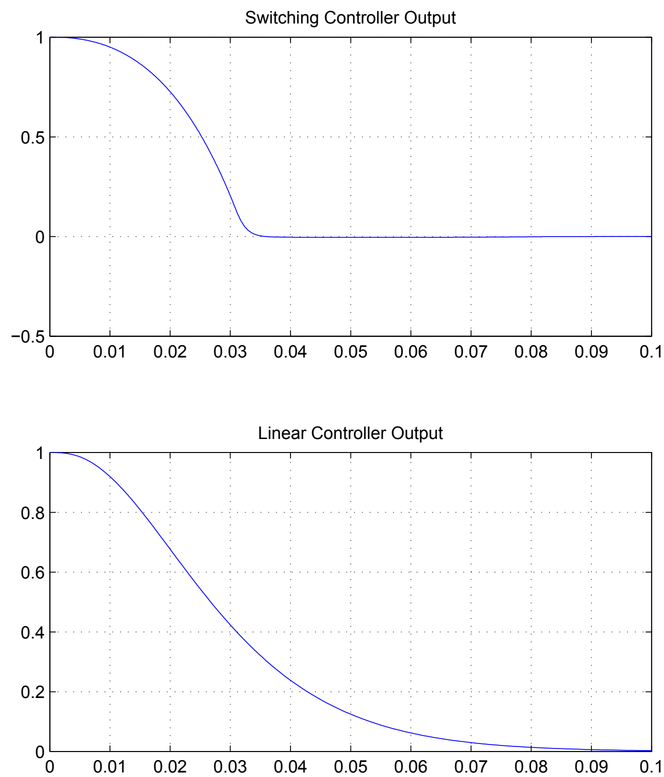

Figure 6. Comparison of $x_1(t)$ for switched feedback controller (top) and linear feedback controller (bottom).

when $x(0) = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array} \right]'$. The lower trajectory is the state variable $x_1(t)$ for a *linear* controller which satisfies the given gain bound of 10^6 with characteristic polynomial $(s + 100)^3$ for the same initial condition. Note that this characteristic polynomial minimizes the maximal Lyapunov exponent over all linear feedback controllers subject to the gain bound. The plots hint at an additional application for the switched feedback control laws described in this document. If one considers the process of designing some sort of stabilizing controller for the plant $P(s) = 1/s^3$, for which the output $y(t)$ is the first state variable $x_1(t)$ when the remainder of the plant dynamics are in the controllability canonical form, then the figure suggests that a switching controller may lead to performance increases over linear control when performance is measured jointly by the maximal Lyapunov exponent and by *settling time*. Indeed, the 1% settling time of the top waveform is 0.034, while the bottom waveform takes 0.084 to settle, a factor of nearly 2.5 improvement (in addition to the factor of 10 improvement in the Lyapunov exponent). A formal investigation of this topic is the subject of future work.

9 Conclusion

We have derived a new switched state feedback control architecture based off of our previous work in designing switched output feedback control laws for second order systems. In addition to formally proving that control laws of the prescribed form achieve global exponential stability, we have provided methods for designing switched state feedback control laws and, in essence, have characterized the set of switched state feedback control laws that achieve stability. We have presented an application which benefits from using the control laws described here, and we have provided multiple numerical examples to illustrate the design techniques and some of the practical benefits of using these switched state feedback control laws.

The work presented here is hardly an end. First, the problem of examining switched *output* feedback controllers is an important problem which must be investigated since the full state vector is not always immediately available and must be estimated. Also, we hope to extend the results related to finite L2 gain stability for the second order counterpart of this problem [26] to problems of general dimension, so as to formally quantify the effects of disturbances on performance for the class of switched feedback systems considered here. The investigation of applications where these switched feedback control laws can have significant impact in performance over linear control design is an important area of future research, as well.

The main goal of the work presented here is to characterize a *set* of switched feedback controllers that achieve stability. One important future direction is to consider the problem of *optimal* switched feedback design whereby one searches for a switched feedback controller to optimize a (or multiple) performance objective(s). Such methods not only seek out “good” controllers among the set of stabilizing controllers, but they also provide methods for automatically finding a controller rather than having to make certain arbitrary choices as we did in the examples presented here.

Appendix: Proofs of Technical Statements

Proof of Prop. 3.1

To begin, note that the statement is trivial when $n = 2$ since any two linearly independent vectors in $\mathbb{R}^2$ form a basis for $\mathbb{R}^2$. Hence, we need only consider the more general case where $n \geq 3$. Also, note that, by appropriately scaling the vector C, we may consider the two cases where α_1 is either 1 or 0 without loss of generality. We prove the statement for the case where $\alpha = 1$ and leave the (simpler) case of $\alpha = 0$ to the reader.

We prove the statement via contradiction. Assume that the statement holds for C of the form

$$C = M_1 - \alpha_2 M_2 + \tilde{M}$$

where $\tilde{M} \notin \mathrm{span}\{M_1, M_2\}$. Now, consider $\tilde{x} = x_0 + \delta_1 y_1 + \delta_2 y_2$, where x_0, y_1, and y_2 are all nonzero and satisfy the following constraints

$$\begin{array}{ll} M_2 x_0 > 0 & \\ M_2 y_1 = 0 & M_2 y_2 = 0 \\ M_1 y_1 \neq 0 & M_1 y_2 = 0 \\ \tilde{M} y_1 = 0 & \tilde{M} y_2 \neq 0 \end{array}$$

(Note that since $n \geq 3$, such nonzero choices of y_1 and y_2 are guaranteed to exist). The constraint $C\tilde{x} = 0$ can be written in the form

$$M_1(x_0 + \delta_1 y_1) - \alpha_2 M_2 x_0 + \tilde{M}(x_0 + \delta_2 y_2) = 0.$$

Observe that for any values of x_0 and α_2, there exist δ_1 and δ_2 such that the above constraint is satisf ed, since $M_1 y_1 \neq 0$ and $\tilde{M} y_2 \neq 0$. Moreover, for any choice of x_0 and α_2, there always exists a choice of δ_2 such that $\alpha_2 M_2 x_0 - \tilde{M}(x_0 + \delta_2 y_2) < 0$. Hence, there exists $\tilde{x}$ such that

$$C\tilde{x} = 0, \qquad M_2\tilde{x} > 0, \qquad \alpha_2 M_2 \tilde{x} - \tilde{M}\tilde{x} < 0. \tag{75}$$

However, since $C\tilde{x} = 0$,

$$M_1\tilde{x} = \alpha_2 M_2 \tilde{x} - \tilde{M}\tilde{x}$$

from which it follows that

$$\tilde{x}' M_1' M_2 \tilde{x} = (M_2\tilde{x})(\alpha_2 M_2 \tilde{x} - \tilde{M}\tilde{x}).$$

By virtue of the inequalities in Eqn. 75, the right hand side of the above expression is negative. Therefore, there exists $\tilde{x}$ with $C\tilde{x} = 0$ such that $\tilde{x} M_1' M_2 \tilde{x} < 0$, which contradicts our original assumption that $x' M_1' M_2 x \geq 0$ for all x such that $Cx = 0$.

To prove the result that $\alpha_1\alpha_2 \geq 0$, it is suff cient to prove that $\alpha_2 \geq 0$ for the case $\alpha = 1$. For all x such that $Cx = 0$, we have

$$M_1x = \alpha_2 M_2 x$$

from which it follows that

$$x_1' M_1' M_2 x = \alpha_2 x' M_2' M_2 x = \alpha_2 ||M_2x||^2.$$

Since $||M_2x||^2 \geq 0$ and $x' M_1' M_2 x \geq 0$ by assumption, it follows that $\alpha_2 \geq 0$.

Proof of Prop. 3.2

The left eigenvector w_1 satisf es the relationship $w_1' A = \lambda w_1'$, from which it follows that $w_1' A v_k = \lambda w_1' v_k$ for $k = 1, 2, \ldots, n$. From the recursive relationship for the right generalized eigenvectors, we also have $w_1' A v_k = \lambda w_1' v_k + w_1' v_{k-1}$ for $k = 2, 3, \ldots, n$. Hence, $w_1' v_{k-1} = 0$ for $k = 2, 3, \ldots, n$, or, equivalently, $w_1' v_k = 0$ for $k = 1, 2, \ldots, n-1$.

Proof of Prop. 3.3

We f rst prove the statement for the case where all of the eigenvalues of A are distinct, and then show how to augment the proof for the case of repeated eigenvalues. When all eigenvalues are distinct, the equality of Eqn. 21 can be written as

$$s^{n-1} + \alpha_{n-2}s^{n-2} + \ldots \alpha_1 s + \alpha_0 = (s - \lambda_1)\ldots(s - \lambda_{k-1})(s - \lambda_{k+1})\ldots(s - \lambda_n). \tag{76}$$

To begin, we recall a few facts about right eigenvectors of companion matrices, borrowed from [7]. The f rst is that distinct eigenvalues have only one linearly independent eigenvector (i.e., there is only a single Jordan block for each distinct eigenvalue). The second is that one such right eigenvector (corresponding to eigenvalue λ_j) is given by $v_j = \begin{bmatrix} 1 & \lambda_j & \lambda_j^2 & \ldots & \lambda_j^{n-1} \end{bmatrix}'$. Since $w_k' v_j = 0$ for $j \neq k$, then choosing w_k of the form Eqn. 20, we f nd

$$\lambda_j^{n-1} + \alpha_{n-2}\lambda_j^{n-2} + \ldots + \alpha_1 \lambda_j + \alpha_0 = 0. \tag{77}$$

Eqn. 77 shows that λ_j is a root of the polynomial on the lefthand side of Eqn. 76 for *every* $j = 1, 2, \ldots, n$, $j \neq k$. Since an $n-1$st degree polynomial is uniquely determined via $n-1$ roots to within a scaling factor, we conclude that the polynomial on the righthand side of Eqn. 76 is a scalar multiple of the polynomial on the lefthand side of Eqn. 76. Since, however, the leading coeff cients are both equal to 1, we conclude that the polynomials are exactly equal.

To prove the more general statement in which repeated eigenvalues may exist, we f rst recall that a basis for the generalized eigenspace corresponding to eigenvalue λ_j

with multiplicity m_j can be taken as ([7]):

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \lambda_j & 1 & 0 & \dots & 0 \\ \lambda_j^2 & 2\lambda_j & 1 & \dots & 0 \\ \lambda_j^3 & 3\lambda_j^2 & 3\lambda_j & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \lambda_j^{n-2} & (n-2)\lambda_j^{n-3} & \begin{pmatrix} n-2 \\ 2 \end{pmatrix}\lambda_j^{n-4} & \dots & \begin{pmatrix} n-2 \\ m_j-1 \end{pmatrix}\lambda_j^{n-m_j-1} \\ \lambda_j^{n-1} & (n-1)\lambda_j^{n-2} & \begin{pmatrix} n-1 \\ 2 \end{pmatrix}\lambda_j^{n-3} & \dots & \begin{pmatrix} n-1 \\ m_j-1 \end{pmatrix}\lambda_j^{n-m_j} \end{bmatrix} \tag{78}$$

If we let r_m, $m = 0, 1, \dots, m_j - 1$ represent the columns of the matrix in Eqn. 78, then we see that

$$r_m = \frac{1}{m!}\frac{d^m}{d\lambda_j^m} r_0, \qquad m = 0, 1, \dots, m_j - 1. \tag{79}$$

Denote now by $\beta_j(l, m)$

$$\beta_j(l,m) = \begin{cases} \begin{pmatrix} l \\ m \end{pmatrix} & l \geq m \\ 0 & l < m \end{cases} \tag{80}$$

for $l = 0, 1, \dots, n-1$, $m = 0, 1, \dots, m_j-1$. By [21], w_k is orthogonal to the entire generalized eigenspace corresponding to eigenvalue λ_j whenever $j \neq k$. This implies that

$$\beta_j(n-1,m)\lambda_j^{n-1-m} + \alpha_{n-2}\beta_j(n-2,m)\lambda_j^{n-2-m} + \dots + \alpha_1\beta_j(1,m)\lambda_j + \alpha_0\beta_j(0,m) = 0 \tag{81}$$

for each $m = 0, 1, \dots, m_j - 1$. By virtue of Eqn. 79, this condition is equivalent to

$$\frac{d^m}{d\lambda_j^m}\left(\lambda_j^{n-1} + \alpha_{n-2}\lambda_j^{n-2} + \dots + \alpha_1\lambda_j + \alpha_0\right) = 0 \tag{82}$$

for each $m = 0, 1, \dots, m_j - 1$, which implies that the lefthand side of Eqn. 21 has a root λ_j of order at least m_j for each $j \neq k$.

When $j = k$, Prop. 3.2 tells us that w_k is orthogonal to all but the *last* generalized eigenvector, corresponding to the last column of Eqn. 78. This implies that λ_k is a root of the polynomial on the lefthand side of Eqn. 21 of order at least $m_k - 1$. Noting that $\sum_{j=1}^{L} m_j = n$, we see that the above orthogonality constraints determine $n-1$ roots of the polynomial on the lefthand side of Eqn. 21. Again, since $n-1$ roots of an $n-1$st degree polynomial determine a polynomial within a scaling factor, and the leading coeff cients are the same on the left and the right, we determine the lefthand side and righthand side are equal.

Proof of Lemma 4.1

Item 1 is an immediate consequence of Prop. 3.2 and does not require A_1 to be in the companion form of Eqn. 18 to hold. To prove item 2, f rst note that the quadratic constraints of Eqn. 24 and 25 yield the following constraints, courtesy of Prop. 3.1:

$$F = \alpha_1 N - \alpha_2 A_1' F \tag{83}$$
$$F = \tilde{\alpha}_1 N - \tilde{\alpha}_2 A_2' F \tag{84}$$

for some $\alpha_1, \alpha_2, \tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathbb{R}$. Note that if either α_2 or $\tilde{\alpha}_2$ is zero, F violates the constraint that $F \neq \gamma N$ for all $\gamma \in \mathbb{R}$. Hence, we may rewrite the above in the form

$$A_1' F = \beta_1 F + \beta_2 N \tag{85}$$
$$A_2' F = \tilde{\beta}_1 F + \tilde{\beta}_2 N \tag{86}$$

for some $\beta_1, \beta_2, \tilde{\beta}_1, \tilde{\beta}_2 \in \mathbb{R}$. Examining Eqn. 85, we see that when the linear transformation A_1 is applied to F, the resulting vector is some combination of two linearly independent vectors. Hence, F must lie in a two-dimensional invariant subspace of A_1, which means that $F' = \mu_j w_j + \mu_l w_l$ where either w_j and w_l are left eigenvectors of A_1 corresponding to separate eigenvalues, or w_j and w_l are a left eigenvector and first generalized left eigenvector corresponding to a repeated eigenvalue of A_1. The parallel conclusion holds for the constraint of Eqn. 86, as well. The only difference between the two constraints is that N' is a left eigenvector of A_1 corresponding to an *isolated* eigenvalue (of multiplicity 1). Hence, we conclude that F' must be of the form $\mu_1 w_1 + \mu_k w_j$ where w_j is a left eigenvector of A_1, $j \neq 1$.

To prove item 3, we first introduce some notation. Let A_1 have L distinct eigenvalues, and let m_k denote the multiplicity of eigenvalue λ_k for $k = 1, 2, \ldots, L$. Similarly, let A_2 have $\tilde{L}$ distinct eigenvalues, and let $\tilde{m}_k$ denote the multiplicity of eigenvalue $\tilde{\lambda}_k$ for $k = 1, 2, \ldots, \tilde{L}$. Without loss of generality, assume that $F' = \mu_1 w_1 + \mu_2 w_2$, where w_2 represents the left eigenvector corresponding to the second distinct eigenvalue λ_2. For the representation of F in terms of the left eigenspace of A_2, we consider two separate cases, both of which we shall write in the form $F' = \tilde{\mu}_1 \tilde{w}_1 + \tilde{\mu}_2 \tilde{w}_2$. In one situation, $\tilde{w}_1$ and $\tilde{w}_2$ will represent left eigenvectors of distinct eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, while in a second situation, $\tilde{w}_1$ will represent a left eigenvector and $\tilde{w}_2$ will represent the first generalized left eigenvector corresponding to the same repeated eigenvalue $\tilde{\lambda}_1$. We investigate the former situation first.

To begin, by the result of Prop. 3.3, the left eigenvectors w_1 and w_2 of A_1 can be taken as the coefficients of the following polynomials:

$$\begin{aligned} w_1 : &\qquad (s-\lambda_2)^{m_2}(s-\lambda_3)^{m_3}\cdots(s-\lambda_L)^{m_L} \\ w_2 : &\qquad (s-\lambda_1)(s-\lambda_2)^{m_2-1}(s-\lambda_3)^{m_3}\cdots(s-\lambda_L)^{m_L} \end{aligned}$$

where we have explicitly used the fact that λ_1 occurs with multiplicity 1. Hence, it follows that $\mu_1 w_1 + \mu_2 w_2$ can be taken as the coefficients of the polynomial

$$\left[\mu_1(s-\lambda_2) + \mu_2(s-\lambda_1)\right](s-\lambda_2)^{m_2-1}(s-\lambda_3)^{m_3}\cdots(s-\lambda_L)^{m_L}. \tag{87}$$

Similarly, under the assumption that $\tilde{w}_1$ and $\tilde{w}_2$ are both left eigenvectors, we may take $\tilde{\mu}_1\tilde{w}_1 + \tilde{\mu}_2\tilde{w}_2$ as the coeff cients of the polynomial

$$\left[\tilde{\mu}_1(s-\tilde{\lambda}_2) + \tilde{\mu}_2(s-\tilde{\lambda}_1)\right](s-\tilde{\lambda}_1)^{\tilde{m}_1-1}(s-\tilde{\lambda}_2)^{\tilde{m}_2-1}(s-\tilde{\lambda}_3)^{\tilde{m}_3}\cdots(s-\tilde{\lambda}_L)^{\tilde{m}_L}. \quad (88)$$

Because $\mu_1 w_1 + \mu_2 w_2 = \tilde{\mu}_1\tilde{w}_1 + \tilde{\mu}_2\tilde{w}_2$, we conclude that the polynomials of Eqn. 87 and 88 are the same and that, in particular, their *roots* are the same. The polynomial of Eqn. 87 has roots λ_k of multiplicity $m_k - 1$ for $k = 1, 2$, and roots λ_k of multiplicity m_k for $k \geq 3$. Similarly, the polynomial of Eqn. 88 has roots $\tilde{\lambda}_k$ of multiplicity $\tilde{m}_k - 1$ for $k = 1, 2$, and roots $\tilde{\lambda}_k$ of multiplicity $\tilde{m}_k$ for $k \geq 3$. We conclude these two sets of values must be the same. Moreover, since the total number of elements in common is equal to $\sum_{k=1}^{L} m_k - 2 = n - 2$, we conclude that A_1 and A_2 have at least $n-2$ eigenvalues in common.

To establish the result for the case where $\tilde{w}_1$ and $\tilde{w}_2$ represent the left eigenvector and f rst generalized left eigenvector corresponding to a single eigenvalue, we rely on the following moderate generalization of Prop. 3.3 that we state without proof: a f rst generalized left eigenvector of the form $w_2 = \left[\begin{array}{cccccc} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-3} & 1 & 0 \end{array}\right]'$ with corresponding eigenvalue λ_k may be computed as the coeff cients of the polynomial

$$s^{n-2} + \alpha_{n-3}s^{n-3} + \cdots + \alpha_1 s + \alpha_0 = (s-\lambda_k)^{m_k-2}\prod_{j=1, j\neq k}^{L}(s-\lambda_j)^{m_j}. \quad (89)$$

Using this fact, we f nd that $\tilde{\mu}_1\tilde{w}_1 + \tilde{\mu}_2\tilde{w}_2$ can be represented as the coeff cients of the polynomial

$$\left[\tilde{\mu}_1(s-\tilde{\lambda}_1) + \tilde{\mu}_2\right](s-\tilde{\lambda}_1)^{\tilde{m}_1-2}(s-\tilde{\lambda}_2)^{\tilde{m}_2}\cdots(s-\tilde{\lambda}_L)^{\tilde{m}_L} \quad (90)$$

again assuming that $\tilde{w}_1$ and $\tilde{w}_2$ correspond to eigenvalue $\tilde{\lambda}_1$. The analysis from this point onward is the same as for the case of separate left eigenvectors, and the conclusion that A_1 and A_2 have at least $n-2$ eigenvalues in common still holds.

To prove item 4, Let $V \in \mathbb{R}^{n\times(n-2)}$ represent a basis for the eigenspace generated by the $n-2$ eigenvalues that are common between A_1 and A_2. It is clear that all of these eigenvalues are roots of the polynomials associated with the (generalized) left eigenvectors w_1, w_2, $\tilde{w}_1$, $\tilde{w}_2$ and, hence,

$$V'\left[\begin{array}{cc} w_1 & w_2 \end{array}\right] = 0, \qquad V'\left[\begin{array}{cc} \tilde{w}_1 & \tilde{w}_2 \end{array}\right] = 0.$$

Thus, $\mathrm{span}\{w_1, w_2\} = \mathrm{span}\{\tilde{w}_1, \tilde{w}_2\}$, which implies that there is an invertible linear transformation T which relates these two bases.

Proof of Prop. 7.2

To see that λ_1 must be purely real, assume that it is not. Then by construction, $\mathrm{Re}\{\lambda_1\} = \mathrm{Re}\{\lambda_2\}$ since all coeff cients are assumed real, and, hence, minimizing the real part of λ_2

would be equivalent to minimizing the real part of λ_1 subject to the gain constraint for A sufficiently large, which by Prop. 7.1 is equal to $-A^{\frac{1}{n}}$.

Now, consider the polynomial

$$s(s + A^{\frac{1}{n-1}})^n. \tag{91}$$

It is straightforward to verify that the coefficients of the above polynomial are bounded in magnitude by A whenever A is sufficiently large. Moreover, $\lambda_2 = -A^{\frac{1}{n-1}} < -A^{\frac{1}{n}}$. Hence, λ_1 must be real.

To prove that λ_1 must be nonnegative, since $|a_1| \leq A$, we have that

$$\left| \sum_{j=1}^{n} \prod_{k=1, k\neq j} (-\lambda_k) \right| \leq A.$$

We can rewrite one of the inequalities induced by the absolute value in the above expression as

$$\prod_{k=2}^{n} (-\lambda_k) \leq A + \lambda_1 \sum_{j=2}^{n} \prod_{k=2, k\neq j}^{n} (-\lambda_k).$$

Note that since $\mathrm{Re}\{\lambda_k\} < 0$ for $k \geq 2$, each of the products within the summation on the righthand side is positive, and hence a positive quantity multiplies λ_1. Hence, if $\lambda_1 < 0$, we find that

$$(-\lambda_2)(-\lambda_3)\cdots(-\lambda_n) < A$$

which implies that $\mathrm{Re}\{\lambda_2\} > -A^{\frac{1}{n-1}}$ where the inequality is strict. But the polynomial of Eqn. 91 indicates $\lambda_2 = -A^{\frac{1}{n-1}}$ is achievable without violating the bounds on the coefficients, and, hence, we conclude that $\lambda_1 \geq 0$.

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