

Robust Stability of Uncertain Quantum Systems

Ian R. Petersen, Valery Ugrinovskii and Matthew R James

Abstract— This paper considers the problem of robust stability for a class of uncertain quantum systems subject to unknown perturbations in the system Hamiltonian. Some general stability results are given for different classes of perturbations to the system Hamiltonian. Then, the special case of a nominal linear quantum system is considered with either quadratic or non-quadratic perturbations to the system Hamiltonian. In this case, robust stability conditions are given in terms of strict bounded real conditions.

I. INTRODUCTION

An important concept in modern control theory is the notion of robust or absolute stability for uncertain nonlinear systems in the form of a Lur'e system with an uncertain nonlinear block which satisfies a sector bound condition; e.g., see [1]. This enables a frequency domain condition for robust stability to be given. This characterization of robust stability enables robust feedback controller synthesis to be carried out using H^∞ control theory; e.g., see [2]. The aim of this paper is to extend classical results on robust stability to the case of quantum systems. This is motivated by a desire to apply quantum H^∞ control such as presented in [3], [4] to nonlinear and uncertain quantum systems.

In recent years, a number of papers have considered the feedback control of systems whose dynamics are governed by the laws of quantum mechanics rather than classical mechanics; e.g., see [3]–[15]. In particular, the papers [12], [16] consider a framework of quantum systems defined in terms of a triple (S, L, H) where S is a scattering matrix, L is a vector of coupling operators and H is a Hamiltonian operator. The paper [16] then introduces notions of dissipativity and stability for this class of quantum systems. In this paper, we build on the results of [16] to obtain robust stability results for uncertain quantum systems in which the quantum system Hamiltonian is decomposed as $H = H_1 + H_2$ where H_1 is a known nominal Hamiltonian and H_2 is a perturbation Hamiltonian, which is contained in a specified

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set of Hamiltonians \mathcal{W} . The set of perturbation Hamiltonians \mathcal{W} corresponds to the set of exosystems considered in [16].

For this general class of uncertain quantum systems, a number of stability results are obtained. The paper then considers the case in which the nominal Hamiltonian H_1 is a quadratic function of annihilation and creation operators and the coupling operator vector is a linear function of annihilation and creation operators. This case corresponds to a nominal linear quantum system; e.g., see [3], [4], [8], [10], [15]. In this special case, robust stability results are obtained in terms of a frequency domain condition.

The remainder of the paper proceeds as follows. In Section II, we define the general class of uncertain quantum systems under consideration. In Section III, we consider a special class of quadratic perturbation Hamiltonians and obtain a robust stability result for this case. In Section IV, we consider a general class of non-quadratic perturbation Hamiltonians. In Section V, we specialize to the case of a linear nominal quantum systems and obtain a number of robust stability results for this case in which stability conditions are given in terms of a strict bounded real condition. In Section VI, we present an illustrative example and in Section VII, we present some conclusions.

II. QUANTUM SYSTEMS

We consider open quantum systems defined by parameters (S, L, H) where $H = H_1 + H_2$; e.g., see [12], [16]. The corresponding generator for this quantum system is given by

$$\mathcal{G}(X) = -i[X, H] + \mathcal{L}(X) \quad (1)$$

where $\mathcal{L}(X) = \frac{1}{2}L^\dagger[X, L] + \frac{1}{2}[L^\dagger, X]L$. Here, $[X, H] = XH - HX$ denotes the commutator between two operators and the notation † denotes the adjoint transpose of a vector of operators. Also, H_1 is a self-adjoint operator on the underlying Hilbert space referred to as the nominal Hamiltonian and H_2 is a self-adjoint operator on the underlying Hilbert space referred to as the perturbation Hamiltonian. The triple (S, L, H) , along with the corresponding generators define the Heisenberg evolution $X(t)$ of an operator X according to a quantum stochastic differential equation; e.g., see [16].

The problem under consideration involves establishing robust stability properties for an uncertain open quantum system for the case in which the perturbation Hamiltonian is contained in a given set \mathcal{W} . Using the notation of [16], the set \mathcal{W} defines a set of exosystems. This situation is illustrated in the block diagram shown in Figure 1. The main robust stability results presented in this paper will build on the following result from [16].

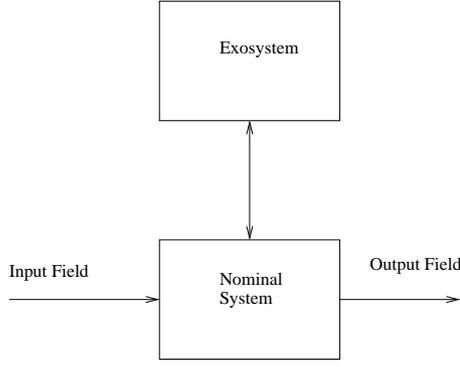


Fig. 1. Block diagram representation of an open quantum system interacting with an exosystem.

Lemma 1 (See Lemma 3.4 of [16].): Consider an open quantum system defined by (S, L, H) and suppose there exists a non-negative self-adjoint operator V on the underlying Hilbert space such that

$$\mathcal{G}(V) + cV \leq \lambda \quad (2)$$

where $c > 0$ and λ are real numbers. Then for any plant state, we have

$$\langle V(t) \rangle \leq e^{-ct} \langle V \rangle + \frac{\lambda}{c}, \quad \forall t \geq 0.$$

Here $V(t)$ denotes the Heisenberg evolution of the operator V and $\langle \cdot \rangle$ denotes quantum expectation; e.g., see [16].

A. Commutator Decomposition

Given set of non-negative self-adjoint operators \mathcal{P} and real parameters $\gamma > 0$, $\delta \geq 0$, we now define a particular set of perturbation Hamiltonians \mathcal{W}_1 . This set \mathcal{W}_1 is defined in terms of the commutator decomposition

$$[V, H_2] = [V, z^\dagger]w - w^\dagger[z, V] \quad (3)$$

for $V \in \mathcal{P}$ where w and z are given vectors of operators. Here, the notation $[z, V]$ for a vector of operators z and a scalar operator V denotes the corresponding vector of commutators. Also, this set will be defined in terms of the sector bound condition:

$$w^\dagger w \leq \frac{1}{\gamma^2} z^\dagger z + \delta. \quad (4)$$

Indeed, we define

$$\mathcal{W}_1 = \left\{ \begin{array}{l} H_2 : \exists w, z \text{ such that (4) is satisfied} \\ \text{and (3) is satisfied } \forall V \in \mathcal{P} \end{array} \right\}. \quad (5)$$

Using this definition, we obtain the following theorem.

Theorem 1: Consider a set of non-negative self-adjoint operators \mathcal{P} and an open quantum system (S, L, H) where $H = H_1 + H_2$ and $H_2 \in \mathcal{W}_1$ defined in (5). If there exists a $V \in \mathcal{P}$ and real constants $c > 0$, $\tilde{\lambda} \geq 0$ such that

$$-i[V, H_1] + \mathcal{L}(V) + [V, z^\dagger][z, V] + \frac{1}{\gamma^2} z^\dagger z + cV \leq \tilde{\lambda}, \quad (6)$$

then

$$\langle V(t) \rangle \leq e^{-ct} \langle V \rangle + \frac{\tilde{\lambda} + \delta}{c}, \quad \forall t \geq 0.$$

Proof: Let $V \in \mathcal{P}$ be given and consider $\mathcal{G}(V)$ defined in (1). Then

$$\mathcal{G}(V) = -i[V, H_1] + \mathcal{L}(V) - i[V, z^\dagger]w + iw^\dagger[z, V] \quad (7)$$

using (3). Now since V is self-adjoint $[V, z^\dagger]^\dagger = [z, V]$. Therefore,

$$\begin{aligned} 0 &\leq ([V, z^\dagger] - iw^\dagger) ([V, z^\dagger] - iw^\dagger)^\dagger \\ &= [V, z^\dagger][z, V] + i[V, z^\dagger]w - iw^\dagger[z, V] + w^\dagger w. \end{aligned}$$

Substituting this into (7), it follows that

$$\mathcal{G}(V) \leq -i[V, H_1] + \mathcal{L}(V) + [V, z^\dagger][z, V] + \frac{1}{\gamma^2} z^\dagger z + \delta \quad (8)$$

using (4). Hence, (6) implies (2) holds with $\lambda = \tilde{\lambda} + \delta$. Therefore, the result follows from Lemma 1. \square

B. Alternative Commutator Decomposition

We now consider an alternative set of perturbation Hamiltonians \mathcal{W}_2 . For a given set of non-negative self-adjoint operators \mathcal{P} and real parameters $\gamma > 0$, $\delta_1 \geq 0$, $\delta_2 \geq 0$, this set \mathcal{W}_2 is defined in terms of the commutator decomposition

$$\begin{aligned} [V, H_2] &= [V, z]w_1^* - w_1[z^*, V] \\ &\quad + \frac{1}{2}[z, [V, z]]w_2^* - \frac{1}{2}w_2[z, [V, z]]^* \end{aligned} \quad (9)$$

for $V \in \mathcal{P}$ where w_1 , w_2 and z are given scalar operators. Here, the notation $*$ denotes the adjoint of an operator. Also, the set \mathcal{W}_2 will be defined in terms of the sector bound condition

$$w_1 w_1^* \leq \frac{1}{\gamma^2} z z^* + \delta_1 \quad (10)$$

and the condition

$$w_2 w_2^* \leq \delta_2. \quad (11)$$

Then, we define

$$\mathcal{W}_2 = \left\{ \begin{array}{l} H_2 : \exists w_1, w_2, z \text{ such that (10) and (11)} \\ \text{are satisfied and (9) is satisfied } \forall V \in \mathcal{P} \end{array} \right\}. \quad (12)$$

Using this definition, we obtain the following theorem.

Theorem 2: Consider a set of non-negative self-adjoint operators \mathcal{P} and an open quantum system (S, L, H) where $H = H_1 + H_2$ and $H_2 \in \mathcal{W}_2$ defined in (12). If there exists a $V \in \mathcal{P}$ and real constants $c > 0$, $\tilde{\lambda} \geq 0$ such that $\mu = [z, [V, z]]$ is a constant and

$$-i[V, H_1] + \mathcal{L}(V) + [V, z][z^*, V] + \frac{1}{\gamma^2} z z^* + cV \leq \tilde{\lambda}. \quad (13)$$

Then

$$\langle V(t) \rangle \leq e^{-ct} \langle V \rangle + \frac{\tilde{\lambda} + \delta_1 + \mu\mu^*/4 + \delta_2}{c}, \quad \forall t \geq 0.$$

Proof: Let $V \in \mathcal{P}$ be given and consider $\mathcal{G}(V)$ defined in (1). Then

$$\begin{aligned} \mathcal{G}(V) &= -i[V, H_1] + \mathcal{L}(V) - i[V, z]w_1^* + iw_1[z^*, V] \\ &\quad - i\mu w_2^* + iw_2\mu^* \end{aligned} \quad (14)$$

using (9). Now $[V, z]^* = z^*V - Vz^* = [z^*, V]$ since V is self-adjoint. Therefore,

$$\begin{aligned} 0 &\leq ([V, z] - iw_1)([V, z] - iw_1)^* \\ &= [V, z][z^*, V] + i[V, z]w_1^* \\ &\quad - iw_1[z^*, V] + w_1w_1^* \end{aligned}$$

and hence

$$-i[V, z]w_1^* + iw_1[z^*, V] \leq [V, z][z^*, V] + w_1w_1^*. \quad (15)$$

Also,

$$\begin{aligned} 0 &\leq \left(\frac{1}{2}\mu - iw_2\right) \left(\frac{1}{2}\mu - iw_2\right)^* \\ &= \frac{1}{4}\mu\mu^* - \frac{i}{2}w_2\mu^* + \frac{i}{2}\mu w_2^* + w_2w_2^* \end{aligned}$$

and hence

$$\frac{i}{2}w_2\mu^* - \frac{i}{2}\mu w_2^* \leq \frac{1}{4}\mu\mu^* + w_2w_2^*. \quad (16)$$

Substituting (15) and (16) into (14), it follows that

$$\begin{aligned} \mathcal{G}(V) &\leq -i[V, H_1] + \mathcal{L}(V) + [V, z][z^*, V] \\ &\quad + \frac{1}{\gamma^2}zz^* + \delta_1 + \mu\mu^*/4 + \delta_2 \end{aligned} \quad (17)$$

using (10) and (11). Then it follows from (13) that

$$\mathcal{G}(V) + cV \leq \tilde{\lambda} + \delta_1 + \mu\mu^*/4 + \delta_2.$$

Then the result of the theorem follows from Lemma 1. \square

III. QUADRATIC PERTURBATIONS OF THE HAMILTONIAN

In this section, we consider a set \mathcal{W}_3 of quadratic perturbation Hamiltonians of the following form

$$H_2 = \frac{1}{2} \begin{bmatrix} \zeta^\dagger & \zeta^T \end{bmatrix} \Delta \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} \quad (18)$$

where $\Delta \in \mathbb{C}^{2m \times 2m}$ is a Hermitian matrix of the form

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2^\# & \Delta_1^\# \end{bmatrix} \quad (19)$$

and $\Delta_1 = \Delta_1^\dagger$, $\Delta_2 = \Delta_2^T$. Also, $\zeta = E_1a + E_2a^\#$. Here a is a vector of annihilation operators on the underlying Hilbert space and $a^\#$ is the corresponding vector of creation operators. Also, in the case of matrices, the notation \dagger refers to the complex conjugate transpose of a matrix. In the case vectors of operators, the notation $\#$ refers to the vector of adjoint operators and in the case of complex matrices, this notation refers to the complex conjugate matrix.

The annihilation and creation operators are assumed to satisfy the canonical commutation relations:

$$\begin{aligned} \left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \right] &= \begin{bmatrix} a \\ a^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \\ &\quad - \left(\begin{bmatrix} a \\ a^\# \end{bmatrix}^\# \begin{bmatrix} a \\ a^\# \end{bmatrix}^T \right)^T \\ &= J \end{aligned} \quad (20)$$

where $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$; e.g., see [9], [13], [15].

The matrix Δ is subject to the norm bound

$$\|\Delta\| \leq \frac{2}{\gamma} \quad (21)$$

where $\|\cdot\|$ denotes the matrix induced norm (maximum singular value). Then we define

$$\mathcal{W}_3 = \left\{ H_2 \text{ of the form (18) such that conditions (19 and (21) are satisfied} \right\}. \quad (22)$$

Using this definition, we obtain the following lemma.

Lemma 2: For any set of self-adjoint operators \mathcal{P} ,

$$\mathcal{W}_3 \subset \mathcal{W}_1.$$

Proof: Given any $H_2 \in \mathcal{W}_3$, let

$$w = \frac{1}{2} \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2^\# & \Delta_1^\# \end{bmatrix} \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \Delta_1\zeta + \Delta_2\zeta^\# \\ \Delta_2^\#\zeta + \Delta_1^\#\zeta^\# \end{bmatrix}$$

and

$$z = \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} = \begin{bmatrix} E_1 & E_2 \\ E_2^\# & E_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = E \begin{bmatrix} a \\ a^\# \end{bmatrix}. \quad (23)$$

Hence,

$$H_2 = w^\dagger z = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} E^\dagger \Delta E \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

Then, for any $V \in \mathcal{P}$,

$$\begin{aligned} [V, z^\dagger]w &= \frac{1}{2} \left(\begin{array}{l} V\zeta^\dagger\Delta_1\zeta + V\zeta^\dagger\Delta_2\zeta^\# \\ +V\zeta^T\Delta_2^\#\zeta + V\zeta^T\Delta_1^\#\zeta^\# \end{array} \right) \\ &\quad - \frac{1}{2} \left(\begin{array}{l} \zeta^\dagger V\Delta_1\zeta + \zeta^\dagger V\Delta_2\zeta^\# \\ +\zeta^T V\Delta_2^\#\zeta + \zeta^T V\Delta_1^\#\zeta^\# \end{array} \right). \end{aligned}$$

Also,

$$\begin{aligned} w^\dagger[z, V] &= \frac{1}{2} \left(\begin{array}{l} \zeta^\dagger\Delta_1\zeta V + \zeta^T\Delta_2^\#\zeta V \\ +\zeta^\dagger\Delta_2\zeta^\# V + \zeta^T\Delta_1^\#\zeta^\# V \end{array} \right) \\ &\quad - \frac{1}{2} \left(\begin{array}{l} \zeta^\dagger V\Delta_1\zeta + \zeta^T V\Delta_2^\#\zeta \\ +\zeta^\dagger V\Delta_2\zeta^\# + \zeta^T V\Delta_1^\#\zeta^\# \end{array} \right). \end{aligned}$$

Hence,

$$\begin{aligned} [V, z^\dagger]w - w^\dagger[z, V] &= \frac{1}{2} \left(V\zeta^\dagger\Delta_1\zeta + V\zeta^\dagger\Delta_2\zeta^\# + V\zeta^T\Delta_2^\#\zeta + V\zeta^T\Delta_1^\#\zeta^\# \right) \\ &\quad - \frac{1}{2} \left(\begin{array}{l} \zeta^\dagger\Delta_1\zeta V + \zeta^T\Delta_2^\#\zeta V \\ +\zeta^\dagger\Delta_2\zeta^\# V + \zeta^T\Delta_1^\#\zeta^\# V \end{array} \right) \\ &= VH_2 - H_2V = [V, H_2] \end{aligned}$$

and thus (3) is satisfied. Also, condition (21) implies

$$\frac{1}{4} \begin{bmatrix} \zeta^\dagger & \zeta^T \end{bmatrix} \Delta \Delta \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} \leq \frac{1}{\gamma^2} \begin{bmatrix} \zeta^\dagger & \zeta^T \end{bmatrix} \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix}$$

which implies (4) for any $\delta \geq 0$. Hence, $H_2 \in \mathcal{W}_1$. Since, $H_2 \in \mathcal{W}_3$ was arbitrary, we must have $\mathcal{W}_3 \subset \mathcal{W}_1$. \square

IV. NON-QUADRATIC PERTURBATION HAMILTONIANS

In this section, we define a set of non-quadratic perturbation Hamiltonians denoted \mathcal{W}_4 . For a given set of non-negative self-adjoint operators \mathcal{P} and real parameters $\gamma > 0$, $\delta_1 \geq 0$, $\delta_2 \geq 0$, the set \mathcal{W}_4 is defined in terms of the following power series (which is assumed to converge in the sense of the induced operator norm on the underlying Hilbert space)

$$H_2 = f(\zeta, \zeta^*) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} S_{k\ell} \zeta^k (\zeta^*)^\ell = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} S_{k\ell} H_{k\ell}. \quad (24)$$

Here $S_{k\ell} = S_{\ell k}^*$, $H_{k\ell} = \zeta^k (\zeta^*)^\ell$, and ζ is a scalar operator on the underlying Hilbert space. It follows from this definition that

$$H_2^* = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} S_{k\ell}^* \zeta^\ell (\zeta^*)^k = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} S_{\ell k} \zeta^\ell (\zeta^*)^k = H_2$$

and thus H_2 is a self-adjoint operator. Note that it follows from the use of Wick ordering that the form (24) is the most general form for a perturbation Hamiltonian defined in terms of a single scalar operator ζ .

Also, we let

$$f'(\zeta, \zeta^*) = \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} k S_{k\ell} \zeta^{k-1} (\zeta^*)^\ell, \quad (25)$$

$$f''(\zeta, \zeta^*) = \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} k(k-1) S_{k\ell} \zeta^{k-2} (\zeta^*)^\ell \quad (26)$$

and consider the sector bound condition

$$f'(\zeta, \zeta^*)^* f'(\zeta, \zeta^*) \leq \frac{1}{\gamma^2} \zeta \zeta^* + \delta_1 \quad (27)$$

and the condition

$$f''(\zeta, \zeta^*)^* f''(\zeta, \zeta^*) \leq \delta_2. \quad (28)$$

Then we define the set \mathcal{W}_4 as follows:

$$\mathcal{W}_4 = \left\{ H_2 \text{ of the form (24) such that conditions (27) and (28) are satisfied} \right\}. \quad (29)$$

Note that the condition (28) effectively amounts to a global Lipschitz condition on the quantum nonlinearity.

In this section, the set of non-negative self-adjoint operators \mathcal{P} will be assumed to satisfy the following assumption:

Assumption 1: Given any $V \in \mathcal{P}$, the quantity

$$\mu = [\zeta, [V, \zeta]] = \zeta[V, \zeta] - [V, \zeta]\zeta$$

is a constant.

Lemma 3: Suppose the set of self-adjoint operators \mathcal{P} satisfies Assumption 1. Then

$$\mathcal{W}_4 \subset \mathcal{W}_2.$$

Proof: First, we note that given any $V \in \mathcal{P}$ and $k \geq 1$,

$$\begin{aligned} V\zeta &= [V, \zeta] + \zeta V; \\ &\vdots \\ V\zeta^k &= \sum_{n=1}^k \zeta^{n-1} [V, \zeta] \zeta^{k-n} + \zeta^k V. \end{aligned} \quad (30)$$

Also for any $n \geq 1$,

$$\begin{aligned} \zeta[V, \zeta] &= [V, \zeta]\zeta + \mu; \\ &\vdots \\ \zeta^{n-1}[V, \zeta] &= [V, \zeta]\zeta^{n-1} + (n-1)\zeta^{n-2}\mu. \end{aligned} \quad (31)$$

Therefore using (30) and (31), it follows that

$$\begin{aligned} V\zeta^k &= \sum_{n=1}^k [V, \zeta] \zeta^{n-1} \zeta^{k-n} + (n-1)\zeta^{n-2} \zeta^{k-n} \mu \\ &\quad + \zeta^k V \\ &= \sum_{n=1}^k [V, \zeta] \zeta^{k-1} + (n-1)\zeta^{k-2} \mu + \zeta^k V \\ &= k[V, \zeta] \zeta^{k-1} + \frac{k(k-1)}{2} \zeta^{k-2} \mu + \zeta^k V \end{aligned}$$

which holds for any $k \geq 0$. Similarly

$$\begin{aligned} (\zeta^*)^k V &= k(\zeta^*)^{k-1} [\zeta^*, V] + \frac{k(k-1)}{2} \mu^* (\zeta^*)^{k-2} \\ &\quad + V(\zeta^*)^k. \end{aligned}$$

Now given any $H_2 \in \mathcal{W}_4$, $k \geq 0$, $\ell \geq 0$, we have

$$\begin{aligned} [V, H_{k\ell}] &= k[V, \zeta] \zeta^{k-1} (\zeta^*)^\ell + \frac{k(k-1)}{2} \mu \zeta^{k-2} (\zeta^*)^\ell \\ &\quad + \zeta^k V (\zeta^*)^\ell \\ &\quad - k\zeta^\ell (\zeta^*)^{k-1} [\zeta^*, V] - \frac{k(k-1)}{2} \mu^* \zeta^\ell (\zeta^*)^{k-2} \\ &\quad - \zeta^\ell V (\zeta^*)^k \\ &= k[V, \zeta] \zeta^{k-1} (\zeta^*)^\ell - k\zeta^\ell (\zeta^*)^{k-1} [\zeta^*, V] \\ &\quad + \frac{k(k-1)}{2} \mu \zeta^{k-2} (\zeta^*)^\ell \\ &\quad - \frac{k(k-1)}{2} \mu^* \zeta^\ell (\zeta^*)^{k-2}. \end{aligned} \quad (32)$$

Therefore,

$$\begin{aligned} [V, H_2] &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} S_{k\ell} [V, H_{k\ell}] \\ &= [V, \zeta] f'(\zeta, \zeta^*) - f'(\zeta, \zeta^*)^* [\zeta^*, V] \\ &\quad + \frac{1}{2} \mu f''(\zeta, \zeta^*) - \frac{1}{2} \mu^* f''(\zeta, \zeta^*)^*. \end{aligned} \quad (33)$$

Now letting

$$z = \zeta, \quad w_1 = f'(\zeta, \zeta^*)^*, \quad \text{and} \quad w_2 = f''(\zeta, \zeta^*)^*, \quad (34)$$

it follows that condition (9) is satisfied. Furthermore, conditions (10), (11) follow from conditions (27), (28) respectively. Hence, $H_2 \in \mathcal{W}_2$. Since, $H_2 \in \mathcal{W}_4$ was arbitrary, we must have $\mathcal{W}_4 \subset \mathcal{W}_2$. \square

V. THE LINEAR CASE

We now consider the case in which the nominal quantum system corresponds to a linear quantum system; e.g., see [3], [4], [8], [10], [15]. In this case, we assume that H_1 is of the form

$$H_1 = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (35)$$

where $M \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix of the form

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^\# & M_1^\# \end{bmatrix}$$

and $M_1 = M_1^\dagger$, $M_2 = M_2^T$. In addition, we assume L is of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (36)$$

where $N_1 \in \mathbb{C}^{m \times n}$ and $N_2 \in \mathbb{C}^{m \times n}$. Also, we write

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = N \begin{bmatrix} a \\ a^\# \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_2^\# & N_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

In addition we assume that V is of the form

$$V = \begin{bmatrix} a^\dagger & a^T \end{bmatrix} P \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (37)$$

where $P \in \mathbb{C}^{2n \times 2n}$ is a positive-definite Hermitian matrix of the form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\# & P_1^\# \end{bmatrix}. \quad (38)$$

Hence, we consider the set of non-negative self-adjoint operators \mathcal{P}_1 defined as

$$\mathcal{P}_1 = \left\{ \begin{array}{l} V \text{ of the form (37) such that } P > 0 \text{ is a} \\ \text{Hermitian matrix of the form (38)} \end{array} \right\}. \quad (39)$$

In the linear case, we will consider a specific notion of robust mean square stability.

Definition 1: An uncertain open quantum system defined by (S, L, H) where $H = H_1 + H_2$ with H_1 of the form (35), $H_2 \in \mathcal{W}$, and L of the form (36) is said to be *robustly mean square stable* if for any $H_2 \in \mathcal{W}$, there exist constants $c_1 > 0$, $c_2 > 0$ and $c_3 \geq 0$ such that

$$\begin{aligned} & \left\langle \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}^\dagger \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \right\rangle \\ & \leq c_1 e^{-c_2 t} \left\langle \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \begin{bmatrix} a \\ a^\# \end{bmatrix} \right\rangle + c_3 \quad \forall t \geq 0. \end{aligned} \quad (40)$$

Here $\begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}$ denotes the Heisenberg evolution of the vector of operators $\begin{bmatrix} a \\ a^\# \end{bmatrix}$; e.g., see [16].

In order to address the issue of robust mean square stability for the uncertain linear quantum systems under consideration, we first require some algebraic identities.

Lemma 4: Given $V \in \mathcal{P}_1$, H_1 defined as in (35) and L defined as in (36), then

$$\begin{aligned} [V, H_1] &= \begin{bmatrix} [a^\dagger & a^T] P \begin{bmatrix} a \\ a^\# \end{bmatrix}, \frac{1}{2} [a^\dagger & a^T] M \begin{bmatrix} a \\ a^\# \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger [PJM - MJ P] \begin{bmatrix} a \\ a^\# \end{bmatrix}. \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{L}(V) &= \frac{1}{2} L^\dagger [V, L] + \frac{1}{2} [L^\dagger, V] L \\ &= \text{Tr} \left(P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \\ &\quad - \frac{1}{2} \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger (N^\dagger J N J P + P J N^\dagger J N) \begin{bmatrix} a \\ a^\# \end{bmatrix}. \end{aligned}$$

Furthermore,

$$\left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, [a^\dagger & a^T] P \begin{bmatrix} a \\ a^\# \end{bmatrix} \right] = 2JP \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

Proof: The proof of these identities follows via straightforward but tedious calculations using (20). \square

A. Quadratic Hamiltonian Perturbations

We now specialize the results of Section III to the case of a linear nominal system in order to obtain concrete conditions for robust mean square stability. In this case, we use the relationship (23):

$$z = \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} = \begin{bmatrix} E_1 & E_2 \\ E_2^\# & E_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = E \begin{bmatrix} a \\ a^\# \end{bmatrix}, \quad (41)$$

to show that the following following strict bounded real conditions provides a sufficient condition for robust mean square stability when $H_2 \in \mathcal{W}_3$:

1) The matrix

$$F = -iJM - \frac{1}{2} JN^\dagger JN \text{ is Hurwitz}; \quad (42)$$

2)

$$\left\| E(sI - F)^{-1} D \right\|_\infty < \frac{\gamma}{2} \quad (43)$$

where $D = iJE^\dagger$.

This leads to the following theorem.

Theorem 3: Consider an uncertain open quantum system defined by (S, L, H) such that $H = H_1 + H_2$ where H_1 is of the form (35), L is of the form (36) and $H_2 \in \mathcal{W}_3$. Furthermore, assume that the strict bounded real conditions (42), (43) are satisfied. Then the uncertain quantum system is robustly mean square stable.

Proof: If the conditions of the theorem are satisfied, then it follows from the strict bounded real lemma that the matrix inequality

$$F^\dagger P + PF + 4PJE^\dagger EJP + \frac{E^\dagger E}{\gamma^2} < 0. \quad (44)$$

will have a solution $P > 0$ of the form (38); e.g., see [2], [4]. This matrix P defines a corresponding operator $V \in \mathcal{P}_1$ as in (37). Now it follows from Lemma 4 that

$$[z, V] = 2EJP \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (45)$$

where z is defined as in (23) and (41). Hence,

$$[V, z^\dagger][z, V] = 4 \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger PJE^\dagger EJP \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

Also,

$$z^\dagger z = \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger E^\dagger E \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

Hence using Lemma 4, we obtain

$$\begin{aligned} & -i[V, H_1] + \mathcal{L}(V) + [V, z^\dagger][z, V] + \frac{z^\dagger z}{\gamma^2} + cV \\ &= \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \begin{pmatrix} F^\dagger P + PF \\ +4PJE^\dagger EJP + \frac{E^\dagger E}{\gamma^2} \end{pmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} \\ &+ \text{Tr} \left(P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \end{aligned} \quad (46)$$

where $F = -iJM - \frac{1}{2}JN^\dagger JN$. Therefore, it follows from (44) that there exists a constant $c > 0$ such that condition (6) will be satisfied with

$$\tilde{\lambda} = \text{Tr} \left(P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \geq 0.$$

Hence choosing $\delta = 0$, it follows from Lemma 2, Theorem 1 and $P > 0$ that

$$\begin{aligned} & \left\langle \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}^\dagger \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \right\rangle \\ & \leq \frac{\langle V(t) \rangle}{\lambda_{\min}[P]} \\ & \leq e^{-ct} \frac{\langle V \rangle}{\lambda_{\min}[P]} + \frac{\tilde{\lambda}}{c\lambda_{\min}[P]} \\ & \leq e^{-ct} \left\langle \begin{bmatrix} a(0) \\ a^\#(0) \end{bmatrix}^\dagger \begin{bmatrix} a(0) \\ a^\#(0) \end{bmatrix} \right\rangle \frac{\lambda_{\max}[P]}{\lambda_{\min}[P]} \\ & \quad + \frac{\tilde{\lambda}}{c\lambda_{\min}[P]} \quad \forall t \geq 0. \end{aligned}$$

Hence, the condition (40) is satisfied with $c_1 = \frac{\lambda_{\max}[P]}{\lambda_{\min}[P]} > 0$, $c_2 = c > 0$ and $c_3 = \frac{\tilde{\lambda}}{c\lambda_{\min}[P]} \geq 0$. \square

B. Non-quadratic Hamiltonian Perturbations

We now specialize the results of Section IV to the case of a linear nominal system. In this case, we define

$$\begin{aligned} z &= \zeta = E_1 a + E_2 a^\# \\ &= \begin{bmatrix} E_1 & E_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = \tilde{E} \begin{bmatrix} a \\ a^\# \end{bmatrix} \end{aligned} \quad (47)$$

where z is assumed to be a scalar operator. Then, we show that the following strict bounded real condition

provides a sufficient condition for robust mean square stability when $H_2 \in \mathcal{W}_4$:

1) The matrix

$$F = -iJM - \frac{1}{2}JN^\dagger JN \text{ is Hurwitz;} \quad (48)$$

2)

$$\left\| \tilde{E}^\# \Sigma (sI - F)^{-1} \tilde{D} \right\|_\infty < \frac{\gamma}{2} \quad (49)$$

$$\text{where } \tilde{D} = J\Sigma\tilde{E}^T \text{ and } \Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

This leads to the following theorem.

Theorem 4: Consider an uncertain open quantum system defined by (S, L, H) such that $H = H_1 + H_2$ where H_1 is of the form (35), L is of the form (36) and $H_2 \in \mathcal{W}_4$. Furthermore, assume that the strict bounded real condition (48), (49) is satisfied. Then the uncertain quantum system is robustly mean square stable.

In order to prove this theorem, we require the following lemma.

Lemma 5: Given any $V \in \mathcal{P}_1$, then

$$\mu = [z, [z, V]] = [z^*, [z^*, V]]^* = -\tilde{E}\Sigma J P J \tilde{E}^T.$$

which is a constant. Hence, the set of operators \mathcal{P}_1 satisfies Assumption 1.

Proof: The proof of this result follows via a straightforward but tedious calculation using (20). \square

Proof of Theorem 4. If the conditions of the theorem are satisfied, then it follows from the strict bounded real lemma that the matrix inequality

$$F^\dagger P + PF + 4P J \Sigma \tilde{E}^T \tilde{E}^\# \Sigma J P + \frac{1}{\gamma^2} \Sigma \tilde{E}^T \tilde{E}^\# \Sigma < 0. \quad (50)$$

will have a solution $P > 0$ of the form (38); e.g., see [2], [4]. This matrix P defines a corresponding operator $V \in \mathcal{P}_1$ as in (37).

It follows from (47) that we can write

$$\begin{aligned} z^* &= E_1^\# a^\# + E_2^\# a = \begin{bmatrix} E_2^\# & E_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} \\ &= \tilde{E}^\# \Sigma \begin{bmatrix} a \\ a^\# \end{bmatrix}. \end{aligned}$$

Also, it follows from Lemma 4 that

$$[z^*, V] = 2\tilde{E}^\# \Sigma J P \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

Hence,

$$[V, z][z^*, V] = 4 \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger P J \Sigma \tilde{E}^T \tilde{E}^\# \Sigma J P \begin{bmatrix} a \\ a^\# \end{bmatrix}. \quad (51)$$

Also, we can write

$$z z^* = \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \Sigma \tilde{E}^T \tilde{E}^\# \Sigma \begin{bmatrix} a \\ a^\# \end{bmatrix}. \quad (52)$$

Hence using Lemma 4, we obtain

$$\begin{aligned}
& -i[V, H_1] + \mathcal{L}(V) + [V, z][z^*, V] + \frac{zz^*}{\gamma^2} \\
& = \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \left(\begin{array}{c} F^\dagger P + PF \\ +4PJ\Sigma\tilde{E}^T\tilde{E}^\#\Sigma JP \\ +\frac{1}{\gamma^2}\Sigma\tilde{E}^T\tilde{E}^\#\Sigma \end{array} \right) \begin{bmatrix} a \\ a^\# \end{bmatrix} \\
& + \text{Tr} \left(P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \quad (53)
\end{aligned}$$

where $F = -iJM - \frac{1}{2}JN^\dagger JN$.

From this, it follows using (50) that there exists a constant $c > 0$ such that condition (13) is satisfied with

$$\tilde{\lambda} = \text{Tr} \left(P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \geq 0.$$

Hence, it follows from Lemma 5, Lemma 3, Theorem 2 and $P > 0$ that

$$\begin{aligned}
& \left\langle \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}^\dagger \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \right\rangle \\
& \leq e^{-ct} \left\langle \begin{bmatrix} a(0) \\ a^\#(0) \end{bmatrix}^\dagger \begin{bmatrix} a(0) \\ a^\#(0) \end{bmatrix} \right\rangle \frac{\lambda_{\max}[P]}{\lambda_{\min}[P]} \\
& + \frac{\lambda}{c\lambda_{\min}[P]} \quad \forall t \geq 0 \quad (54)
\end{aligned}$$

where $\lambda = \tilde{\lambda} + \delta_1 + \mu\mu^*/4 + \delta_2$. Hence, the condition (40) is satisfied with $c_1 = \frac{\lambda_{\max}[P]}{\lambda_{\min}[P]} > 0$, $c_2 = c > 0$ and $c_3 = \frac{\lambda}{c\lambda_{\min}[P]} \geq 0$. \square

VI. ILLUSTRATIVE EXAMPLE

We consider an example of an open quantum system with

$$S = I, \quad H_1 = 0, \quad H_2 = \frac{1}{2}i \left((a^\dagger)^2 - a^2 \right), \quad L = \sqrt{\kappa}a,$$

which corresponds an optical parametric amplifier; see [17]. This defines a linear quantum system of the form considered in Theorem 3 with $M_1 = 0$, $M_2 = 0$, $N_1 = \sqrt{\kappa}$, $N_2 = 0$, $E_1 = \frac{1}{2}$, $E_2 = 0$, $\Delta_1 = 0$, $\Delta_2 = i$. Hence, $M = 0$, $N = \begin{bmatrix} \sqrt{\kappa} & 0 \\ 0 & \sqrt{\kappa} \end{bmatrix}$, $F = \begin{bmatrix} -\frac{\kappa}{2} & 0 \\ 0 & -\frac{\kappa}{2} \end{bmatrix}$ which is Hurwitz, $E = I$, and $D = iJ$. In this case,

$$\Delta\Delta = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, we can choose $\gamma = 1$ to ensure that (21) is satisfied and $H_2 \in \mathcal{W}_3$. Also,

$$\left\| E(sI - F)^{-1} D \right\|_\infty = \left\| \begin{bmatrix} \frac{1}{s+\kappa/2} & 0 \\ 0 & -\frac{1}{s+\kappa/2} \end{bmatrix} \right\|_\infty = \frac{2}{\kappa}.$$

Hence, it follows from Theorem 3 that this system will be mean square stable if $\frac{2}{\kappa} < \frac{1}{2}$; i.e., $\kappa > 4$.

VII. CONCLUSIONS

In this paper, we have considered the problem of robust stability for uncertain quantum systems with either quadratic and non-quadratic perturbations to the system Hamiltonian. The final stability results obtained are expressed in terms of strict bounded real conditions. Future research will be directed towards analyzing the stability of specific nonlinear quantum systems using the given robust stability results for the case of non-quadratic perturbations to the system Hamiltonian.

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