2013 American Control Conference (ACC)
Washington, DC, USA, June 17-19, 2013

Output Synchronization for Heterogeneous Networks of Discrete-time Introspective Right-invertible Agents with Uniform Constant Communication Delay

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***Abstract*— In this paper, we consider synchronization problems for heterogeneous networks of introspective, right-invertible, discrete-time linear agents with uniform constant communication delay. We first design decentralized controllers for solving the output synchronization problem for a set of network topologies under arbitrary bounded delay. We then apply the proposed scheme to solve the formation problem with arbitrarily given formation vectors. Finally, we consider the output regulation problem, where the output of each agent has to track an *a prior* specified reference trajectory, generated by an exosystem. In this case, we assume that the common root agent has access to its own output relative to the reference trajectory. We then solve the problem for a set of network topologies with delay whose upper bound depends exosystem and some characteristic of network topologies.**

I. Introduction

The synchronization problem in a network has received substantial attention due to its wide application areas, e.g., some papers [1]–[13], and recent books [14]–[16].

The existing literature can be generally divided into two categories depending whether the agent models are identical or not, that is, *homogeneous* networks (i.e., networks where the agent models are identical) and *heterogeneous* networks (i.e., networks where the agent models are non-identical). All the aforementioned references focus on homogeneous networks, however, the recent focus in the literature is to study the synchronization problem for heterogeneous networks. The existing results for heterogeneous networks can be further divided into two categories: *introspective* case and *non-introspective* case. The agents are said to be introspective (see the definition in [13], [17]) if they possess self-knowledge about their own states. While most works [15], [17]–[19] focus the introspective case, few works [13], [20] consider the non-introspective case.

Although most works focused on the case where the agent models are continuous-time, synchronization in homogeneous networks of discrete-time agents has been studied in [4], [7], [12], [21]. In [4], the author consider a network of first-order agents with fixed topologies and switching topologies. In [7], the author considers a special case of neutrally stable agents with full actuation (i.e., $B = I$). A distributed observer-based synchronization controller was developed in [21], which makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates. All the aforementioned works only consider synchronization for homogeneous networks. In our previous work [22], we studied synchronization for heterogeneous networks of introspective, right-invertible discrete-time linear agents. This paper extends the above result to the case where the communication is tolerate the uniform constant delay.

Due to the ubiquity of communication delay during the transmission of information, the research has also been directed to synchronization in networks with time-delays. While most works in this direction focus on the continuous-time case, see for instance, [4], [23], [24], Xu *et al.* [25] consider state synchronization for homogeneous networks of non-introspective, non-right-invertible, discrete-time agents with uniform constant communication delay. This paper is different from [25] since the agent models are non-identical, although introspective and right-invertible.

This paper considers synchronization problems for heterogeneous networks of introspective, right-invertible, discrete-time linear agents with uniform constant communication delay for a set of network topologies. The underlying principle is to use pre-compensators and an observer-based pre-feedback within each agent to yield a network of almost identical agents by exploiting the self-knowledge and the right-invertibility property of the agents. Specifically, for the output synchronization problem, agent models are manipulated to a common model whose system matrix has all its eigenvalues at 1. We then show that the arbitrary bounded constant delay can be tolerated. We also apply the proposed scheme to solve the formation problem for arbitrarily given formation vectors. Finally, for the output regulation problem, we show that the upper bound on the uniform constant delay depends only on the system matrix of the exosystem and some characteristic of network topologies.

II. Notations and Preliminaries

In this paper, the following notations are used. $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{R}^+$ denote respectively the sets of all complex numbers, integers and positive real numbers. For any open set $\mathscr{D} \subset \mathbb{C}$, $\partial\mathscr{D}$ and $\overline{\mathscr{D}}$ denote its boundary and closure. For $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}^+$, $\mathscr{D}(z_0, r)$ denotes an open disc centered at z_0 with

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The work of Tao Yang has been supported in part by the Knut and Alice Wallenberg Foundation and the Swedish Research Council. The work of Tao Yang, Xu Wang and Ali Saberi is partially supported by National Science Foundation grant NSF-0901137 and NAVY grants ONR KKK777SB001 and ONR KKK760SB0012.

978-1-4799-0176-0/$31.00 ©2013 AACC

radius r. In particular we denote:

$$\mathbb{C}^{\odot} := \overline{\mathscr{D}(0,1)}, \quad \mathbb{C}^{\bigcirc} := \partial\mathscr{D}(0,1).$$

For any $k_1, k_2 \in \mathbb{Z}$, and $k_1 \le k_2$,

$$\overline{[k_1,k_2]} := \{k \in \mathbb{Z} | k_1 \le k \le k_2\}.$$

For column vectors $x_1, \ldots, x_n$, the stacking column vector of $x_1, \ldots, x_n$ is denoted the column vector by $[x_1; \ldots; x_n]$.

A matrix $D = [d_{ij}] \in \mathbb{R}^{N\times N}$ is called a row stochastic matrix if $d_{ij} \ge 0$ for any $i, j \in \{1, \ldots, N\}$, and $\sum_{j=1}^{N} d_{ij} = 1$ for $i \in \{1, \ldots, N\}$. The matrix D can be associated with a directed graph (digraph) $G = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{1, \ldots, N\}$ and an arc $(j,i) \in \mathscr{E}$ if $d_{ij} > 0$. G is *undirected* if $d_{ij} = d_{ji}$ for any $i, j \in \{1, \ldots, N\}$. Otherwise, G is *directed*. A *directed path* from vertex i_1 to i_k is a sequence of vertices $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathscr{E}$ for $j = 1, \ldots, k-1$. A directed graph G contains a *directed spanning tree* if there is a node r such that there exists a directed path between r and every other node. For such a case, node r is often called a root.

III. Heterogeneous networks

Consider a heterogeneous network of N discrete-time introspective linear agents of the form

$$\begin{cases} x^i(k+1) = A^i x^i(k) + B^i u^i(k), \\ y^i(k) \quad = C_y^i x^i(k), \\ z^i(k) \quad = C_z^i x^i(k), \\ \zeta^i(k) \quad = \sum_{j=1}^{N} d_{ij}(y^i(k-\kappa) - y^j(k-\kappa)), i \in \mathscr{V}, \end{cases} \tag{1}$$

where $x^i \in \mathbb{R}^{n_i}$, $u^i \in \mathbb{R}^{m_i}$, $y^i, \zeta^i \in \mathbb{R}^p$, $z^i \in \mathbb{R}^{q_i}$ and κ is an unknown integer satisfying $\kappa \in \overline{[0,\bar{\kappa}]}$, with the integer $\bar{\kappa} \ge 0$.

The matrix $D = [d_{ij}] \in \mathbb{R}^{N\times N}$ is a row stochastic matrix, and moreover, $d_{ii} > 0$. This D matrix defines a communication topology that can be captured by a digraph $G = (\mathscr{V}, \mathscr{E})$. We make the following assumption regarding the digraph G.

Assumption 1 *The digraph G contains a directed spanning tree, and the matrix D is a row stochastic matrix with $d_{ii} > 0$ for all $i \in \mathscr{V}$.*

Under Assumption 1, it follows from [6, Corollary 3.5] that the matrix D has a simple eigenvalue at 1 with corresponding right eigenvector $\mathbf{1}$, and the remaining eigenvalues are strictly within the unit circle. Let $\lambda_1, ..., \lambda_N$ denote the eigenvalues of the matrix D such that $\lambda_1 = 1$ and $|\lambda_i| < 1$, $i \in \{2, ..., N\}$. We then define a set of communication topologies:

Definition 1 *For a given $\delta \in (0,1]$, let $\mathscr{G}_\delta$ denote a set of communication topologies, such that for each topology $G \in \mathscr{G}_\delta$, Assumption 1 holds and $|\lambda_i| < \delta$, $i \in \{2, ..., N\}$.*

The following assumption on the agent dynamics is also made throughout the paper.

Assumption 2 *For each agent $i \in \{1, \ldots, N\}$,*

1) *(A^i, B^i) is stabilizable;*
2) *(A^i, C_y^i) is detectable;*
3) *(A^i, B^i, C_y^i) is right-invertible;*
4) *(A^i, C_z^i) is detectable.*

Remark 1 *Right-invertibility of a triple (A^i, B^i, C_y^i) means that, given a reference output $y_r(k)$, there exist an initial condition $x^i(0)$ and an input $u^i(k)$ such that $y^i(k) = y_r(k)$ for all the non-negative integers k. For example, every single-input single-output system is right-invertible, unless its transfer function is identically zero.*

IV. Output synchronization

In this section, we consider the output synchronization problem for heterogeneous networks with unknown uniform constant communication delay. The definition of output synchronization is given as follows:

Definition 2 *A heterogeneous network* (1) *is said to achieve output synchronization if*

$$\lim_{k\to\infty} (y^i(k) - y^j(k)) = 0, \quad \forall i, j \in \mathscr{V}.$$

Next, we formulate the output synchronization problem.

Problem 1 *Consider a heterogenous network of N agents* (1). *For a given set $\mathscr{G}_\delta$ and a given integer $\bar{\kappa} \ge 0$, the output synchronization problem with a set of communication topologies $\mathscr{G}_\delta$ for any $\kappa \in \overline{[0,\bar{\kappa}]}$ is to find, if possible, a linear dynamical controller*

$$\begin{cases} \hat{x}^i(k+1) = A_c^i \hat{x}^i(k) + B_c^i \zeta^i(k) + E_c^i z^i(k), \\ u^i(k) \quad = C_c^i \hat{x}^i(k) + D_c^i \zeta^i(k) + M_c^i z^i(k) \end{cases} \tag{2}$$

for each agent $i \in \mathscr{V}$, such that the output synchronization can be achieved for the network with any communication topology $G \in \mathscr{G}_\delta$ and $\kappa \in \overline{[0,\bar{\kappa}]}$.

Remark 2 *Since (A^i, C_z^i) is detectable, one can always design a local stabilizing measurement feedback controller so that the network achieves output synchronization in the sense that $\lim_{k\to\infty} y^i(k) = 0$. Such a case is not interested in this paper. We are aiming to reach synchronization with a nontrivial and possibly desirable synchronization trajectory.*

The following theorem is concerned with the output synchronization problem as defined in Problem 1.

Theorem 1 *For a given set $\mathscr{G}_\delta$, and an arbitrarily given integer $\bar{\kappa} \ge 0$, Problem 1 is solvable via N decentralized controllers of the form* (2).

We shall prove Theorem 1 by explicitly constructing the synchronization controller of the form (2) via a two-step design procedure. First, we design a local pre-compensators and an local observer-based pre-feedback for each agent to make the agents almost identical to a new common model except for different geometrically decreasing sequences.We then show that such geometrically decreasing sequences are irrelevant and the output synchronization problem in the original heterogenous network of agents (1) can be reduced to the state/output synchronization problem in a homogeneous network with the same communication topology.

A. Towards Homogeneous Networks

For introspective agents, Xu *et al.* [22] show that their self-knowledge about their own states provide us additional freedom to manipulate their internal dynamics through the use of pre-feedbacks so as to disguise them as being almost identical to the rest of the network, which is recapped in the following Lemma.

Lemma 1 *[22] Consider a heterogeneous network of N agents of the form* (1). *Let n_d denote the maximum order of infinite zeros of (A^i, B^i, C^i). Suppose a triple (A, B, C) is given such that*

1) $\operatorname{rank}(C) = p$.
2) *(A, B, C) is invertible, of uniform rank $n_q \geq n_d$ and has no invariant zeros.*

Then for each agent $i \in \mathscr{V}$, there exists a compensator

$$\begin{cases} \xi^i(k+1) = A_h^i \xi^i(k) + B_h^i z^i(k) + E_h^i v^i(k), \\ u^i(k) \quad = C_h^i \xi^i(k) + D_h^i v^i(k), \end{cases} \tag{3}$$

such that the interconnection of (1) *and* (3) *can be written in the following form:*

$$\begin{cases} \bar{x}^i(k+1) = A\bar{x}^i(k) + B\left(v^i(k) + d^i(k)\right), \\ y^i(k) \quad = C\bar{x}^i(k), \\ \zeta^i(k) \quad = \sum_{j=1}^N d_{ij}(y^i(k-\kappa) - y^j(k-\kappa)), \end{cases} \tag{4}$$

where d^i are generated by

$$\begin{cases} e^i(k+1) = A_s^i e^i(k), \\ d^i(k) \quad = C_s^i e^i(k), \end{cases} \tag{5}$$

and A_s^i is Schur stable.

Remark 3 *We have the following observations*

1) *The properties that (A, B, C) is invertible and has no invariant zero implies that (A, B) is controllable and (A, C) is observable.*
2) *The triple (A, B, C) is arbitrarily assignable as long as the properties 1) and 2) in Lemma 1 are fulfilled. They play a role as design parameters. We shall use this freedom in various places in this paper.*

Remark 4 *Lemma 1 shows that we can design a pre-compensator of the form* (3) *to make the agent models identical to a new common model characterized by a priori given triple (A, B, C), except for different geometrically decreasing sequences injected in the range space of B. Moreover, we may freely choose the matrix A as long as the properties listed in Remark 3 are satisfied.*

B. Connection to Homogeneous Networks

The resulting network can be viewed as a homogenous network (4) affected by the different geometrically decreasing sequences d^i generated by (5). Next, we show that the injection of such sequences is irrelevant for solving the output synchronization problem, which in turn implies that the output synchronization problem in the original heterogenous network of agents (1) can be reduced to the state/output synchronization problem in a homogeneous network with the same communication topology.

In order to tolerate the arbitrarily given upper bound $\bar{\kappa}$ on the uniform unknown communication delay κ, we need to choose the triple (A, B, C) in Lemma 1 such that all the eigenvalues of the matrix A are located at 1. Such a triple (A, B, C) always exists and takes the following form:

$$A = A_0 + B_0 F, \quad B = B_0, \quad C = C_0, \tag{6}$$

where

$$A_0 = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_0 = \begin{bmatrix} I_p & 0 \end{bmatrix}, \tag{7}$$

and F is such that $A_0 + B_0 F$ has all the eigenvalues at 1. Such an F exists due to the fact that (A_0, B_0) is controllable.

For solving the output synchronization problem for a network of N agents (4) with the above triple (A, B, C) and (5) with a set of possible communication topologies $\mathscr{G}_\delta$, we consider N following decentralized controllers

$$\begin{cases} \chi^i(k+1) = A_c \chi^i(k) + B_c \zeta^i(k), \\ v^i(k) \quad = C_c \chi^i(k), \end{cases} \tag{8}$$

for each agent $i \in \mathscr{V}$, where $\chi^i \in \mathbb{R}^{n_c}$. Note that A_c, B_c and C_c are designed parameters, independent of the specific topology $G \in \mathscr{G}_\delta$, and will be determined later.

Define $\tilde{x}^i = [\bar{x}^i; \chi^i]$. Then the closed-loop of the system (4), (5), and the controller (8) can be written as

$$\begin{cases} \tilde{x}^i(k+1) = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \tilde{x}^i(k) + \begin{bmatrix} 0 \\ B_c \end{bmatrix} \zeta^i(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} d^i(k), \\ y^i(k) \quad = \begin{bmatrix} C & 0 \end{bmatrix} \tilde{x}^i(k), \\ \zeta^i(k) \quad = y^i(k-\kappa) - \sum_{j=1}^N d_{ij} y^j(k-\kappa). \end{cases}$$

Next, define $\tilde{x} = [\tilde{x}^1; \ldots; \tilde{x}^N]$, $d = [d^1; \ldots; d^N]$,

$$\bar{A} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ B_c \end{bmatrix}, \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \bar{E} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{9}$$

Then the overall dynamics of the N agents can be written as

$$\begin{aligned} \tilde{x}(k+1) = (I_N \otimes \bar{A})\tilde{x}(k) + \left((I_N - D) \otimes \bar{B}\bar{C}\right)\tilde{x}(k-\kappa) \\ + (I_N \otimes \bar{E})d(k). \end{aligned}$$

Define $\eta = [\eta^1; \ldots; \eta^N] = (T \otimes I_{pn_q+n_c})\tilde{x}$, where $\eta^i \in \mathbb{C}^{pn_q+n_c}$ and T is such that $J_L = T(I_N - D)T^{-1}$ is in the Jordan canonical form with $(1,1)$-th element of J_L, $J_L(1,1) = 0$. We then obtain the dynamics of η as follows:

$$\eta(k+1) = (I_N \otimes \bar{A})\eta(k) + (J_L \otimes \bar{B}\bar{C})\eta(k-\kappa) + (T \otimes \bar{E})d(k).$$

Let us recalled the following lemma from [22],

Lemma 2 *[22] Consider the network of N agents* (4) *with d^i generated by* (5), *and controllers* (8). *If $\lim_{k\to\infty} \eta^i(k) = 0$, where $i \in \{2, \ldots, N\}$, then $\lim_{k\to\infty}(y^i(k) - y^j(k)) = 0$, for any $i, j \in \mathscr{V}$.*

Next, we define $\bar{\eta} = [\eta^2; \ldots; \eta^N]$. Taking the dynamics of d into account yields

$$\begin{bmatrix} \bar{\eta}(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \bar{A} & (\bar{I}T \otimes \bar{E})\bar{C}_s \\ 0 & \bar{A}_s \end{bmatrix} \begin{bmatrix} \bar{\eta}(k) \\ e(k) \end{bmatrix} + \begin{bmatrix} \bar{J}_L \otimes \bar{B}\bar{C} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\eta}(k-\kappa) \\ e(k-\kappa) \end{bmatrix}, \tag{10}$$

where $e = [e^1; \ldots; e^N]$, $\bar{C}_s = \text{blkdiag}\{C_s^i\}_{i=1}^N$, $\bar{I} = [0, I_{N-1}]$, $\bar{A}_s = \text{blkdiag}\{A_s^i\}_{i=1}^N$, which is Schur stable due to its block diagonal structure and the fact that all the A_s^i for $i \in \mathscr{V}$ are Schur stable. Also $\bar{J}_L$ is such that $J_L = \text{blkdiag}(0, \bar{J}_L)$.

Suppose that the system (10) is globally asymptotically stable, then $\lim_{k\to\infty} \bar{\eta}(k) = 0$. It then follows from Lemma 2 below that $\lim_{k\to\infty}(y^i(k) - y^j(k)) = 0$ for any $i, j \in \mathscr{V}$. We then note that the system (10) is globally asymptotically stable if and only if

$$\det\left(zI - \begin{bmatrix} I_{N-1} \otimes \bar{A} & (\bar{I}T \otimes \bar{E})\bar{C}_s \\ 0 & \bar{A}_s \end{bmatrix} - z^{-\kappa} \begin{bmatrix} \bar{J}_L \otimes \bar{B}\bar{C} & 0 \\ 0 & 0 \end{bmatrix}\right) \neq 0 \tag{11}$$

for any $z \notin \mathbb{C}^{\odot}$.

Due to the upper block-triangular structures of both matrices in (11) and the fact that $\bar{A}_s$ is Schur stable, it is easy to see that (11) holds if and only if

$$\det\left(zI - (I_{N-1} \otimes \bar{A}) - z^{-\kappa}(\bar{J}_L \otimes \bar{B}\bar{C})\right) \neq 0, \quad \forall z \notin \mathbb{C}^{\odot}. \tag{12}$$

Note that $I_{N-1} \otimes \bar{A}$ and $\bar{J}_L \otimes \bar{B}\bar{C}$ are of upper block-triangular structure. We then have the following lemma.

Lemma 3 *Consider the network of N agents* (4) *with d^i generated by* (5)*, and controllers* (8)*. Let $\bar{A}, \bar{B}$ and $\bar{C}$ be defined by* (9)*. If the following systems*

$$\tilde{\eta}(k+1) = \bar{A}\tilde{\eta}(k) + (1-\lambda_i)\bar{B}\bar{C}\tilde{\eta}(k-\kappa) \tag{13}$$

are globally asymptotically stable for all λ_i, $i \in \{2, \ldots, N\}$, which are the eigenvalues of D that are not equal to 1*, and for any $\kappa \in \overline{[0, \bar{\kappa}]}$, then $\lim_{k\to\infty}(y^i(k) - y^j(k)) = 0$, $\forall i, j \in \mathscr{V}$.*

Note that the system given by (13) can be viewed as the closed-loop system of

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ z(k) \quad = (1-\lambda_i)Cx(k-\kappa), \end{cases} \tag{14}$$

and a compensator

$$\begin{cases} \chi(k+1) = A_c\chi(k) + B_c z(k), \\ u(k) \quad = C_c\chi(k). \end{cases} \tag{15}$$

Remark 5 *Lemma 3 shows that if the compensator of the form* (15) *simultaneously stabilizes the $N-1$ systems of the form* (14)*, then $\lim_{k\to\infty}(y^i(k) - y^j(k)) = 0$ for any $i, j \in \mathscr{V}$.*

We now need to design the parameters A_c, B_c and C_c to solve the simultaneous stabilization problem. Let us choose

$$A_c = A + KC, \quad B_c = -K, \quad C_c = \beta F_\varepsilon, \tag{16}$$

where K is such that $A + KC$ is Schur stable, $\beta > \frac{1}{1-\delta}$, and

$$F_\varepsilon = -(B'P_\varepsilon B + I)^{-1}B'P_\varepsilon A,$$

where $\varepsilon > 0$ is a low-gain parameter and $P_\varepsilon = P_\varepsilon' > 0$ is the unique solution of the following discrete-time algebraic Riccati equation

$$P_\varepsilon = A'P_\varepsilon A + \varepsilon I - A'P_\varepsilon B(B'P_\varepsilon B + I)^{-1}B'P_\varepsilon A. \tag{17}$$

We then have the following lemma, which is a slight different version of [25, Theorem 1].

Lemma 4 *For a given set $\mathscr{G}_\delta$, and an arbitrarily given integer $\bar{\kappa} \geq 0$, the compensator* (15) *with parameters A_c, B_c and C_c given by* (16) *simultaneously stabilizes the $N-1$ systems* (14)*.*

We are now ready to prove Theorem 1.

Proof: [Proof of Theorem 1] For a given set $\mathscr{G}_\delta$, and an arbitrarily given integer $\bar{\kappa} \geq 0$, it follows from Lemmas 1, 3, and 4 that there exists an $\varepsilon^* \in (0, 1]$, such that for $\varepsilon \in (0, \varepsilon^*]$, the composition of (3) and (8) with A_c, B_c and C_k given by (16), solves Problem 1. ■

V. Formation

In this section, we consider the formation problem. The definition of formation is given as follows.

Definition 3 *Given a family of vectors $\{h_1, \ldots, h_N\}$, where $h_i \in \mathbb{R}^p$ for $i \in \mathscr{V}$. The heterogeneous network* (1) *is said to achieve formation if $\lim_{k\to\infty}[(y_i(k) - h_i) - (y_j(k) - h_j)] = 0$ for any $i, j \in \mathscr{V}$.*

We assume that the network infrastructure provides each agent with the following information

$$\hat{\zeta}^i(k) = \sum_{j=1}^N d_{ij}\left[(y_i(k-\kappa) - h_i) - (y_j(k-\kappa) - h_j)\right]. \tag{18}$$

Then the agent $i \in \mathscr{V}$ has the following dynamical equations:

$$\begin{cases} x^i(k+1) = A^i x^i(k) + B^i(k)u^i(k), \\ y^i(k) \quad = C_y^i x^i(k), \\ z^i(k) \quad = C_z^i x^i(k), \\ \hat{\zeta}^i(k) \quad = \sum_{j=1}^N d_{ij}\left[(y_i(k-\kappa) - h_i) - (y_j(k-\kappa) - h_j)\right]. \end{cases}$$

Let us formulate the formation problem to be solved.

Problem 2 *Consider a heterogeneous network of N agents* (1)*. For a given set $\mathscr{G}_\delta$, a given integer $\bar{\kappa} \geq 0$, and a given family of formation vectors $\{h_1, \cdots, h_N\}$, where $h_i \in \mathbb{R}^p$ for $i \in \mathscr{V}$, the formation problem with a set of communication topologies $\mathscr{G}_\delta$ for any $\kappa \in \overline{[0, \bar{\kappa}]}$ is to find, if possible, a linear dynamical controller*

$$\begin{cases} \hat{x}^i(k+1) = A_c^i\hat{x}^i(k) + B_c^i\zeta^i(k) + E_c^i z^i(k), \\ u^i(k) \quad = C_c^i\hat{x}^i(k) + D_c^i\zeta^i(k) + M_c^i z^i(k) \end{cases} \tag{19}$$

for each agent $i \in \mathscr{V}$, such that the formation can be achieved for the network with any network communication topology $G \in \mathscr{G}_\delta$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

Theorem 2 *For a given set $\mathscr{G}_\delta$, an arbitrarily given integer $\bar{\kappa} \geq 0$ and an arbitrarily given family of formation vectors*

$\{h_1,\cdots,h_N\}$, *where* $h_i \in \mathbb{R}^p$ *for* $i \in \mathscr{V}$, *Problem 2 is solvable via N decentralized controllers of the form* (19).

Proof: The proof follows from the proofs of [22, Theorem 3] and Theorem 1. Due to the limitation of the space, we have omitted the proof. ■

VI. Output regulation

In Section IV, we consider the output synchronization problem. Note that for this problem, we do not impose any restriction on the synchronization trajectories. The focus is to solve this problem for a large set of communication topologies and a large delay. On the other hand, it is important to consider the related problem of regulating the outputs toward a desired reference trajectory, generated by an autonomous exosystem

$$\left\{\begin{array}{ll} x_r(k+1) = A_r x_r(k), & x_r(0) = x_r^0, \\ y_r(k) = C_r x_r(k), & \end{array}\right. \tag{20}$$

where $x_r \in \mathbb{R}^r$ and $y_r \in \mathbb{R}^p$.

Assumption 3 *For the exosystem* (20),

1) (A_r, C_r) *is observable, and*
2) A_r *has all its eigenvalues in the closed unit circle.*

We introduce the following definition.

Definition 4 *A heterogeneous network of N agents* (1) *is said to achieve output regulation if* $\lim_{k\to\infty}(y^i(k)-y_r(k))=0$ *for any* $i \in \mathscr{V}$.

For solving the output regulation problem, we consider a subset $\mathscr{G}_s$ of $\mathscr{G}_1$, where $\mathscr{G}_1$ is the set of the network topologies, each of which satisfies Assumption 1. We assume that all the topologies in the given set $\mathscr{G}_s$ have a common root. Without loss of generality, we assume that the common root is agent 1. This (root) agent 1 measures its own output relative to output y_r of the exosystem, that is, agent 1 has access to the quantity $\psi^1 = d(y^1 - y_r)$, where $d = \frac{d_{11}}{2} > 0$, while $\psi^i = 0$ for $i \in \{2,\ldots,N\}$. Therefore, the agent $i \in \mathscr{V}$ has the following dynamical equations:

$$\left\{\begin{array}{ll} x^i(k+1) & = A^i x^i(k) + B^i u^i(k), \\ z^i(k) & = C_z^i x^i(k), \\ y^i(k) & = C_y^i x^i(k), \\ \bar{\zeta}^i(k) & = \sum_{j=1}^N d_{ij}(y^i(k-\kappa) - y^j(k-\kappa)) + \psi^i(k-\kappa). \end{array}\right.$$

Let us formulate the output regulation problem as follows:

Problem 3 *Consider a heterogeneous network of N agents* (1) *and an exosystem* (20). *For a given set* $\mathscr{G}_s \subset \mathscr{G}$ *and a given integer* $\bar{\kappa} \geq 0$, *the output regulation problem with exosystem* (20) *and a set of communication topologies* $\mathscr{G}_s$ *for any* $\kappa \in \overline{[0,\bar{\kappa}]}$ *is to find, if possible, a linear dynamical controllers linear dynamical controller*

$$\left\{\begin{array}{ll} \hat{x}^i(k+1) & = A_c^i \hat{x}^i(k) + B_c^i \bar{\zeta}^i(k) + E_c^i z^i(k), \\ u^i(k) & = C_c^i \hat{x}^i(k) + D_c^i \bar{\zeta}^i(k) + M_c^i z^i(k) \end{array}\right. \tag{21}$$

for each agent $i \in \mathscr{V}$, *such that the output regulation can be achieved for the network with any communication topology* $G \in \mathscr{G}_s$ *and* $\kappa \in \overline{[0,\bar{\kappa}]}$.

Some preliminary work is needed before we present the main result. Let $\bar{G}$ denote an expanded network constructed from $G \in \mathscr{G}_s$ by adding the exosystem as node 0 and the edge from exosystem to agent 1 with weight $d = \frac{d_{11}}{2}$. Therefore, $\bar{\zeta}^i(k)$ for $i \in \mathscr{V}$ can be written as

$$\bar{\zeta}^i(k) = \sum_{j=0}^N \bar{d}_{ij}(y^i(k-\kappa) - y^j(k-\kappa)),$$

where $y^0 := y_r$, and

$$\bar{D} = [\bar{d}_{ij}] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{d_{11}}{2} & \frac{d_{11}}{2} & d_{12} & \cdots & d_{1N} \\ 0 & d_{21} & d_{22} & \cdots & d_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & d_{N1} & d_{N2} & \cdots & d_{NN} \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}.$$

This matrix $\bar{D}$ is also a row stochastic matrix and defines an expanded topology $\bar{G}$. Also note that the digraph $\bar{G}$ has a directed spanning tree rooted at agent 0. From [6, Corollary 3.5], it is easy to see that the matrix $\bar{D}$ has a simple eigenvalue at 1 while the remaining eigenvalues are strictly within the unit circle. Let $\bar{\lambda}_1,\ldots,\bar{\lambda}_{N+1}$ denote the eigenvalues of $\bar{G}$, such that $\bar{\lambda}_1 = 1$ and $|\bar{\lambda}_i| < 1$ for all $i \in \{2,\ldots,N+1\}$.

Assumption 4 *There exists a* $\bar{\delta} < 1$, *such that for each expanded network, we have* $|\bar{\lambda}_i| < \bar{\delta}$, $i \in \{2,\ldots,N+1\}$.

We are now ready to present our main result.

Theorem 3 *Consider a heterogeneous network* (1) *and an exosystem* (20). *For a given* $\mathscr{G}_s \subset \mathscr{G}$ *and a given integer* $\bar{\kappa} \geq 0$ *such that* $\bar{\kappa}\omega_{\max} < \arccos(\bar{\delta})$, *where* $\omega_{\max} = \max\{\omega \in [-\pi,\pi] \,|\, \det(e^{j\omega}I - A_r) = 0\}$, *Problem 3 is solvable via N decentralized controllers of the form* (21).

Proof: For the given exosystem (20), from [22, Appendix B], we know that there exists another exosystem

$$\left\{\begin{array}{ll} \tilde{x}_r(k+1) = \tilde{A}_r \tilde{x}_r(k), & \tilde{x}_r(0) = \tilde{x}_r^0, \\ y_r(k) = \tilde{C}_r \tilde{x}_r(k), & \end{array}\right. \tag{22}$$

such that for all $x_r^0 \in \mathbb{R}^r$, there exists $\tilde{x}_r^0 \in \mathbb{R}^{\tilde{r}}$ for which (22) produces the same output as the original exosystem (20). Furthermore, we can find a matrix $\tilde{B}_r$ such that the triple $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r)$ is invertible, of uniform rank n_q, where n_q is an integer greater than or equal to the maximal order of infinite zeros of (A^i, B^i, C^i), $i \in \mathscr{V}$ and all the observability index (see [26, Theorem 4.3.1]) of (A_r, C_r), and has no invariant zeros.

The new exosystem can be rewritten as:

$$\left\{\begin{array}{ll} \tilde{x}_r(k+1) = \tilde{A}_r \tilde{x}_r(k) + \tilde{B}_r(v_r(k) + d_r(k)), \tilde{x}_r(0) = \tilde{x}_r^0 \\ y_r(k) = \tilde{C}_r \tilde{x}_r(k), \end{array}\right. \tag{23}$$

where $v_r(k) = 0$ and $d_r(k) = 0$ for all $k \geq 0$.

Following from the constructive proof of Lemma 1 in [22], we design a pre-compensator (3) for each agent $i \in \mathscr{V}$ such that the interconnection of (1) and (3) are almost identical to the exosystem system (23), that is,

$$\begin{cases} \bar{x}^i(k+1) = \tilde{A}_r\bar{x}^i(k) + \tilde{B}_r(v^i(k)+d^i(k)), \\ y^i(k) \quad = \tilde{C}_r\bar{x}^i(k), \\ \bar{\zeta}^i(k) \quad = \sum_{j=0}^{N}\bar{d}_{ij}(y^i(k-\kappa)-y^j(k-\kappa)), \end{cases} \tag{24}$$

where d^i is given by (5).

It is then easy to see that the output regulation for a heterogeneous network of N agents is converted to the output synchronization problem for an expanded network of $N+1$ agents by adding the exosystem system as agent 0 and the edge from agent 0 to agent 1 with weight d. Let us define $\bar{x}^0 := \tilde{x}_r$, $v^0 := v_r$, and $d^0 := d_r$, then the agent i, where $i \in \{0,1,\ldots,N\}$ has the following dynamics:

$$\begin{cases} \bar{x}^i(k+1) = \tilde{A}_r\bar{x}^i(k) + \tilde{B}_r(v^i(k)+d^i(k)), \\ y^i(k) \quad = \tilde{C}_r\bar{x}^i(k), \\ \bar{\zeta}^i(k) \quad = \sum_{j=0}^{N}\bar{d}_{ij}(y^i(k-\kappa)-y^j(k-\kappa)). \end{cases} \tag{25}$$

We then design the following controller

$$\begin{cases} \chi^i(k+1) = (\tilde{A}_r+\tilde{K}_r\tilde{C}_r)\chi_i - \tilde{K}_r\bar{\zeta}^i, \\ v^i(k) \quad = \tilde{\beta}\tilde{F}_\varepsilon\chi^i, \quad i \in \{0,1,\ldots,N\}, \end{cases} \tag{26}$$

where $\tilde{\beta} > \frac{1}{1-\bar{\delta}}$, the matrix $\tilde{K}_r$ is such that $\tilde{A}_r+\tilde{K}_r\tilde{C}_r$ is Hurwitz stable, and

$$\tilde{F}_\varepsilon = -(\tilde{B}_r'\tilde{P}_\varepsilon\tilde{B}+I)^{-1}\tilde{B}_r'\tilde{P}_\varepsilon\tilde{A}_r,$$

where $\varepsilon > 0$ is a low-gain parameter and $\tilde{P}_\varepsilon = \tilde{P}_\varepsilon' > 0$ is the unique solution of the following discrete-time algebraic Riccati equation

$$\tilde{P}_\varepsilon = \tilde{A}_r'\tilde{P}_\varepsilon\tilde{A}_r + \varepsilon I - \tilde{A}_r'\tilde{P}_\varepsilon\tilde{B}_r(\tilde{B}_r'\tilde{P}_\varepsilon\tilde{B}_r+I)^{-1}\tilde{B}_r'\tilde{P}_\varepsilon\tilde{A}_r.$$

For agent 0, we choose $\chi^0(0) = 0$ in (26). Therefore, $v^0(k) = 0$ for all $k \geq 0$ as desired since $\bar{\zeta}^0(k) = 0$ for all $k \geq 0$. It then follows from Theorem 1 and [25, Theorem 2] that there exist $\tilde{\beta}$ and ε^* such that for all $\varepsilon \in (0,\varepsilon^*]$, the composition of (3) and (26), solves the output synchronization for a set of the expanded network topologies. Hence, $\lim_{k\to\infty}(y^i(k) - y_r(k)) = 0$ for all $i \in \{1,\ldots,N\}$. ∎

VII. Conclusion

In this paper, we consider heterogeneous networks of introspective, right-invertible, discrete-time linear agents. We propose a decentralized control scheme for solving the output synchronization problem for a set of network topologies under arbitrary bounded communication delay. We then apply the proposed scheme to solve the formation problem with arbitrary formation vectors. Finally, we solve the output regulation problem for a set of network topologies with uniform constant delay whose upper bound depends exosystem and some characteristic of network topologies.

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