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Invariance Principles for Switched Differential-Algebraic Equations Under Arbitrary and Dwell-Time Switching

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***Abstract*— We investigate the invariance properties of a class of switched systems in which the value of a switching signal determines the current mode of operation (among a finite number of them) and, for each fixed mode, its dynamics are described by a Differential-Algebraic Equation (DAE). Motivated by the lack of invariance principles for such systems, we develop such principles for switched DAE systems under arbitrary and dwell-time switching. By obtaining a hybrid system model that describes the switched DAE system, we build from invariance results for hybrid systems and generate invariance principles for such switched systems. Examples are included to illustrate the results.**

I. INTRODUCTION

In this paper, we consider a class of switched systems with a state evolving according to a linear DAE in-between switching instants and, due to the algebraic constraints of the DAE in each mode, potentially exhibiting jumps in the state at switching instants. These type of systems arise in several applications in engineering such as robot manipulators, vehicular traffic systems, power systems, biological systems, and mechanical systems. These type of systems can be modelled as switched Differential-Algebraic Equations (switched DAEs), where its switching behaviour obeys some switching signal.

Several authors have analyzed switched DAE systems from many perspectives, with most of the research being focused in establishing asymptotic stability of the origin. The consistency of initial conditions together with Lyapunov theory is used to study the stability properties of the origin in [1]. In [2], a chronological survey of Lyapunov stability of linear descriptor systems is presented. In [3], Lyapunov's direct method is used to analyse the asymptotic stability of the origin. Sufficient conditions for exponential stability of switched singular system with stable subsystems are presented in [4].

Surprisingly, there is a lack of results regarding the invariance properties of switched DAE systems, being perhaps the main reason the difficulty in guaranteeing a sequential compactness property of the solutions to such systems [5]. To fill this gap, we build from the concept of solution of hybrid systems in [6] (referred here as *hybrid inclusions*), and propose a model for switched DAE systems using the framework for *hybrid DAEs* introduced in [7]. Thus, the proposed model, when it satisfies certain mild conditions, has the structural properties required to build an invariance-like result. Starting from previous results on invariance properties of hybrid DAE systems in [7], invariance principles for hybrid systems in [8], and switched systems in [9], we develop invariance principles for switched DAE systems under arbitrary and dwell-time switching.

The remainder of this paper is organized as follows. Required background is presented in Section II. Models for switched DAE systems and the invariance principles for arbitrary and dwell-time switching are presented in Section III-A and Section III-B, respectively.

II. PRELIMINARIES

A. Modeling differential-algebraic systems

In this paper, we consider the class of linear switched DAE systems given by

$$E_\sigma \dot{\xi} = A_\sigma \xi, \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the state, $\sigma : [0,\infty) \to \Sigma$ is the switching signal, and Σ is a finite discrete set. Solutions to (1) are typically given by (right or left) continuous functions (see [10] and references therein). Definition 2.4 below introduces the notion of solution to (1) employed here.

Definition 2.1: (DAE regularity [11, Definition 1-2.1]) *The collection* (E_σ, A_σ) *is regular if for each* $\sigma \in \Sigma$ *the matrix pencil* $sE_\sigma - A_\sigma \in \mathbb{R}^{m\times n}$ $(s \in \mathbb{C})$ *is regular. The matrix pencil* $sE_\sigma - A_\sigma$ *is called regular if* $n = m$ *and there exists a constant* $s \in \mathbb{C}$ *such that* $det(sE_\sigma - A_\sigma) \neq 0$, *or* $det(sE_\sigma - A_\sigma)$ *is not the zero polynomial. The matrix pair* (E_σ, A_σ) *and the corresponding DAE is called regular whenever* (E_σ, A_σ) *is regular.*

In order to define a switched DAE system as in [3], we define first some concepts regarding the linear subspaces where solutions to (1) belong. Due to the algebraic constraints in (1), the solutions to (1) evolve within a set called the consistency space.

Definition 2.2: (Consistency space[1]) *Given* $\sigma' \in \Sigma$, *the consistency space for* (1) *is given by*

$$\mathfrak{C}_{\sigma'} := \{\xi_0 \in \mathbb{R}^n | \exists \text{ a solution } \xi : [0,\tau) \to \mathbb{R}^n \text{ to } (1), \\ \xi(0) = \xi_0,\ \tau > 0,\ \sigma \equiv \sigma'\}$$

[1]This definition is adapted from [3, Definition 2.1].

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For a linear switched DAE system as in (1) for each $\sigma \in \Sigma$ the consistency space is given by a linear subspace[2]. Moreover, this consistency space can be characterized by a set or by a basis. The consistency spaces can be computed using the quasi-Weierstrass form (qWf) and the Wong Sequences, which are introduced in [12]. The Wong sequences are used to calculate the consistency and inconsistency spaces, which are calculated from the basis of the linear subspaces (see more in [3]). Notice that, for each $\sigma \in \Sigma$, the consistency space $\mathfrak{C}_\sigma$ is computed directly from the system data (the matrix pair (E_σ, A_σ)). The consistency space can be described by a set in $\mathbb{R}^n$ as follows.

Definition 2.3: (Consistency set). *Given $\sigma \in \Sigma$, the consistency set for system (1) is*[3] $\mathfrak{O}_\sigma := \{\xi \mid \xi \in span(\mathfrak{C}_\sigma)\}$.

Given that $\mathfrak{C}_\sigma$ is a basis with finitely many column vectors, the operator $span$ over $\mathfrak{C}_\sigma$ leads to a closed set $\mathfrak{O}_\sigma$. Now we can define a solution to a switched DAE system. Below, the domain of definition of a function f is denoted $\operatorname{dom} f$.

Definition 2.4: (Solution to a switched DAE system) *A solution $\phi = (\phi_\xi, \sigma)$ to the switched DAE system* (1) *consists of a piecewise continuous state function $\phi_\xi : \operatorname{dom} \phi_\xi \to \mathfrak{O}_{\sigma(t)}$ and a piecewise constant input function $\sigma : \operatorname{dom} \sigma \to \Sigma$, both right continuous, such that $E_{\sigma(t)} \dot{\phi}_\xi(t) = A_{\sigma(t)} \phi_\xi(t)$ for almost all $t \in \operatorname{dom} \phi_\xi$, with $\operatorname{dom} \phi = \operatorname{dom} \phi_\xi = \operatorname{dom} \sigma$.*

A solution ϕ to the switched DAE system in (1) is complete if its domain, denoted $\operatorname{dom} \phi_\xi$, is equal to $[0, \infty)$ and precompact if the solution itself is complete and bounded, where by bounded we mean that there exists a bounded set K such that $\phi_\xi(t) \in K$ for all $t \in \operatorname{dom} \phi_\xi$.

Next, the consistency and differential projectors, which are used to describe the explicit solution formula for switched DAE systems, are defined.

Definition 2.5: (Consistency and differential projectors [13, Definition 6.4.1]) *For the quasi-Weierstrass transformation in [7, Theorem 3.4] define*[4]

- *Consistency projector:* $\Pi_\sigma := T_\sigma \begin{bmatrix} I_{n_1^\sigma} & 0 \\ 0 & 0_{n_2^\sigma} \end{bmatrix} T_\sigma^{-1}$
- *Differential projector:* $\Pi_\sigma^{\text{diff}} := T_\sigma \begin{bmatrix} I_{n_1^\sigma} & 0 \\ 0 & 0_{n_2^\sigma} \end{bmatrix} S_\sigma$

B. Modeling hybrid systems as hybrid inclusions

The hybrid system modeling framework employed here follows [6]. A hybrid system is given by a hybrid inclusion of the form

$$\mathcal{H}: \quad x \in \mathbb{R}^m \quad \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x) \end{cases}$$

[2]For a linear DAE system, $\mathfrak{C}_\sigma$ is given by a linear subspace of $\mathbb{R}^n$ (see, e.g., [3, Remark 2.2]). Then, the basis $\mathfrak{C}_\sigma$ is given by a finite set of column vectors.

[3]The span of a set of vectors S is defined as the set of all finite linear combinations of elements of S, e.g., $span(S) = \left\{\sum_{i=1}^k \lambda_i v_i | k \in \mathbb{N},\ v_i \in S, \lambda_i \in \mathbb{R}\right\}$.

[4]For each $\sigma \in \Sigma$, n_1^σ and n_2^σ are given by n_1 and n_2 in [7, Theorem 3.4]. The matrix $0_{n_2^\sigma} \in \mathbb{R}^{n_2^\sigma \times n_2^\sigma}$ is the zero matrix, also $I_{n_1^\sigma} \in \mathbb{R}^{n_1^\sigma \times n_1^\sigma}$ is the identity matrix.

where its data is given by a set $C \subset \mathbb{R}^m$, called the *flow set*; a set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, called the *flow map*; a set $D \subset \mathbb{R}^m$, called the *jump set*; and a set-valued mapping $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, called the *jump map*. The flow map F defines the continuous dynamics on the flow set C, while the jump map G defines the discrete dynamics on the jump set D. These objects are referred to as the *data* of the hybrid system $\mathcal{H}$, which at times is explicitly denoted as $\mathcal{H} = (C, F, D, G)$.

C. Hybrid DAE systems with linear flow

Here, we introduce a class of hybrid systems for modelling DAE systems with jumps in the state that are triggered by state conditions. This class of hybrid system was introduced in [7][5]. We refer to these systems as hybrid DAE systems and denote them as $\mathcal{H}_{DAE}$.

The state vector is given by $x = (\xi, \sigma) \in \mathbb{R}^n \times \Sigma$, where Σ is a finite discrete set. The hybrid DAE system is given by

$$\mathcal{H}_{DAE} \begin{cases} \begin{bmatrix} E_\sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} f_\sigma(\xi) \\ 0 \end{bmatrix} =: F(x) & x \in C \\ \begin{bmatrix} \xi^+ \\ \sigma^+ \end{bmatrix} \in \bigcup_{\tilde{\sigma} \in \varphi_\sigma(\xi)} \begin{bmatrix} g(\xi, \sigma, \tilde{\sigma}) \\ \tilde{\sigma} \end{bmatrix} =: G(x) & x \in D \end{cases} \tag{2a}$$

where

$$C := \bigcup_{\sigma \in \Sigma} (C_\sigma \cap (\mathfrak{O}_\sigma \times \{\sigma\})) \tag{2b}$$

$$f_\sigma(\xi) := A_\sigma \xi \tag{2c}$$

$$D := \bigcup_{\sigma \in \Sigma} \big((D_\sigma \cap (\mathfrak{O}_\sigma \times \{\sigma\})) \cup ((\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \{\sigma\}) \big)$$

$g(\xi, \sigma, \tilde{\sigma}) := g_D(\xi, \sigma, \tilde{\sigma}) \cup g_{\mathfrak{O}}(\xi, \sigma, \tilde{\sigma})$ for all $(\xi, \sigma) \in D$, and for all $\tilde{\sigma} \in \varphi_\sigma(\xi)$, and

$$g_D(\xi, \sigma, \tilde{\sigma}) := \begin{cases} \Pi_{\tilde{\sigma}} g_\sigma(\xi) & \text{if } x \in D_\sigma \cap (\mathfrak{O}_\sigma \times \{\sigma\}) \\ \emptyset & \text{otherwise,} \end{cases} \tag{2d}$$

$$g_{\mathfrak{O}}(\xi, \sigma, \tilde{\sigma}) := \begin{cases} \Pi_{\tilde{\sigma}} \xi & \text{if } x \in (\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \{\sigma\} \\ \emptyset & \text{if } x \in \mathfrak{O}_\sigma \times \{\sigma\}, \end{cases} \tag{2e}$$

where $\mathfrak{O}_\sigma$ are the consistency sets and Π_σ are the projectors as in Definition 2.3 and Definition 2.5. The sets C_σ and D_σ are subsets in $\mathbb{R}^{n+1}$ that define where the evolution of the system according to F and G are possible, respectively. At jumps, the map g defines the changes of ξ while φ_σ determines the changes of σ. The set $C_\sigma \cap (\mathfrak{O}_\sigma \times \{\sigma\})$ is the collection of points in $\mathbb{R}^{n+1}$ where the system is allowed to flow (since $\mathfrak{O}_\sigma$ is the set of points where flow is "consistent"). Also, the set $D_\sigma \cap (\mathfrak{O}_\sigma \times \{\sigma\})$ is where state jumps according to g_σ are allowed. Then, the data of $\mathcal{H}_{DAE}$ on the state space $\mathbb{R}^n \times \Sigma$ is given by $(E_\sigma, C_\sigma, f_\sigma, D_\sigma, g_\sigma, \varphi_\sigma)$. Note that $\mathfrak{O}_\sigma$ and Π_σ are generated using E_σ and A_σ. As in the hybrid systems description in [6], we define solutions to hybrid DAEs using hybrid time domains. Therefore, during flows, solutions are parametrized by $t \in \mathbb{R}_{\geq 0} := [0, \infty)$,

[5]The model in [7] considers single-valued functions φ_σ. Here, we generalize the hybrid DAE model to allow for set-valued maps φ_σ to make it suitable for the study of switched DAE systems.

while at jumps they are parametrized by $j \in \mathbb{N} := \{0, 1, \ldots\}$.

Definition 2.6: (Hybrid time domain [6, Definition 2.3].) *A subset $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if $\mathcal{T} = \bigcup_{j=0}^{\mathcal{I}-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_{\mathcal{I}}$. It is a hybrid time domain if for all $(\tau, \alpha) \in \mathcal{T}$, $\mathcal{T} \cap ([0, \tau] \times \{0, 1, \ldots, \alpha\})$ is a compact hybrid domain.*

To define the solution concept for hybrid DAE systems, first we define a hybrid arc.

Definition 2.7: (Hybrid arc [6, Definition 2.4].) *A function $\phi : \mathcal{T} \to \mathbb{R}^{n+1}$ is a hybrid arc if $\mathcal{T}$ is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j := \{t : (t, j) \in \mathcal{T}\}$.*

A hybrid arc is a solution to a hybrid DAE system if it satisfies the system dynamics.

Definition 2.8: (solution) *A hybrid arc $\phi = (\phi_\xi, \phi_\sigma)$ is a solution to $\mathcal{H}_{DAE}$ if*[6] *$\phi(0, 0) \in \overline{C} \cup D$ and*

(S1) *(Flow condition) for each $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \mathcal{T}\}$ has nonempty interior*

$$\begin{bmatrix} \phi_\xi(t,j) \\ \phi_\sigma(t,j) \end{bmatrix} \in C \text{ for all } t \in \text{int } I^j,$$

$$t \mapsto \begin{bmatrix} \phi_\xi(t,j) \\ \phi_\sigma(t,j) \end{bmatrix} \text{ satisfies for almost all } t \in I^j$$

$$\begin{bmatrix} E_{\phi_\sigma(t,j)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_\xi(t,j) \\ \dot{\phi}_\sigma(t,j) \end{bmatrix} = \begin{bmatrix} f_{\phi_\sigma(t,j)}(\phi_\xi(t,j)) \\ 0 \end{bmatrix}$$

(S2) *(Jump condition) for each $(t, j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$, $\phi(t, j) \in D$ and $\phi(t, j+1) \in G(\phi(t, j))$.*

A solution ϕ is maximal if there does not exist another solution ψ such that $\operatorname{dom} \phi$ is a proper subset of $\operatorname{dom} \psi$ and $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \operatorname{dom} \phi$. A solution ϕ is complete if $\operatorname{dom} \phi$ is unbounded and precompact if it is complete and bounded. We will employ the range of a solution ϕ, which is denoted as $\operatorname{rge} \phi$, i.e., $\operatorname{rge} \phi = \phi(\operatorname{dom} \phi)$.

1) Some concepts of invariance: The following notions are used to develop invariance principles for switched DAE systems.

Definition 2.9: (Weak Invariance) *For the hybrid DAE system $\mathcal{H}_{DAE}$, the set M is said to be:*

- *weakly forward invariant if for each $x_0 \in M$, there exists at least one complete solution ϕ to $\mathcal{H}_{DAE}$ from x_0 with $\operatorname{rge} \phi \subset M$.*
- *weakly backward invariant if for each $x^* \in M$, $N > 0$, there exist $x_0 \in M$ and at least one solution ϕ to $\mathcal{H}_{DAE}$ from x_0 such that for some $(t^*, j^*) \in \operatorname{dom} \phi$, $t^* + j^* \geq N$, we have $\phi(t^*, j^*) = x^*$ and $\phi(t^*, j^*) \in M$ for all $(t, j) \preceq (t^*, j^*)$, $(t, j) \in \operatorname{dom} \phi$;*
- *weakly invariant if it is both weakly forward invariant and weakly backward invariant.*

[6]The notation $\overline{K}$ stands for the closure of K.

III. Invariance Principles for Switched DAEs

In this section, we propose invariance principles for switched DAE systems given by (1) under certain families of switching signals. In particular, we consider arbitrarily fast switching signals and dwell-time switching signals. Particularly, σ is a *dwell-time switching signal* with dwell-time $\tau_D > 0$ if $t_{j+1} - t_j \geq \tau_D$ for $j = 1, 2, \ldots$; see [8] for similar constructions. To develop the invariance principles, we model the switched linear DAE system under the said family of switching signals as a hybrid DAE system as in (2).

A. Invariance principle under arbitrary switching

Given $E_\sigma, A_\sigma \in \mathbb{R}^{n \times n}$ for each $\sigma \in \Sigma$, a switched DAE system under arbitrary switching signals can be captured by the following hybrid DAE system:

$$\mathcal{H}_{DAE}^{\mathcal{SW}} \begin{cases} \begin{bmatrix} E_\sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} f_\sigma(\xi) \\ 0 \end{bmatrix} =: F(x) & x \in C \\ \begin{bmatrix} \xi^+ \\ \sigma^+ \end{bmatrix} \in \bigcup_{\tilde{\sigma} \in \Sigma \setminus \{\sigma\}} \begin{bmatrix} \Pi_{\tilde{\sigma}} \xi \\ \tilde{\sigma} \end{bmatrix} =: G(x) & x \in D \end{cases} \tag{3}$$

where $f_\sigma(\xi) := A_\sigma \xi$, $C := \bigcup_{\sigma \in \Sigma} \big(\mathfrak{O}_\sigma \times \{\sigma\}\big)$, and $D := \bigcup_{\sigma \in \Sigma} \big(\mathbb{R}^n \times \{\sigma\}\big)$. This hybrid DAE system has data $(E_\sigma, C_\sigma, f_\sigma, D_\sigma, g_\sigma, \varphi_\sigma)$ where $C_\sigma = \mathbb{R}^n \times \{\sigma\}$, $f_\sigma(\xi) = A_\sigma \xi$, $D_\sigma = \mathbb{R}^n \times \{\sigma\}$, $g_\sigma(\xi) = \xi$, and $\varphi_\sigma(\xi) = \Sigma \setminus \{\sigma\}$. For simplicity, we write it as $\mathcal{H}_{DAE}^{\mathcal{SW}} = ((E_\sigma, A_\sigma), \Sigma)$.

The data of the switched DAE system in (1) is assumed to satisfy the following regularity property.

Assumption 3.1: (Basic conditions for switched DAE systems). *Given the switched DAE system in* (1)*, we have that*

(C1) *For each $\sigma \in \Sigma$, the DAE with data (E_σ, A_σ) is regular (see Definition 2.1).*

Following [8, Theorem 4.7], we consider locally Lipschitz functions V. The generalized directional gradient (in the sense of Clarke) of V at x in the direction v is given by $V^\circ(x, v) = \max_{\zeta \in \partial V(x)} \langle \zeta, v \rangle$, where, $\partial V(x)$ is a closed, convex, and nonempty set equal to the convex hull of all limits of sequences $\nabla V(x_i)$, where x_i is any sequence converging to x.

Theorem 3.2: (Invariance principle for switched DAE systems under arbitrary switching). *Consider a switched DAE system given by* (1)*. Suppose that, for each $\sigma \in \Sigma$, the matrix pair (E_σ, A_σ) satisfies Assumption 3.1. Let Π_σ^{diff} be given by Definition 2.5. Furthermore, suppose there exist a function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ that is continuous on $\mathbb{R}^{n+1}$ and locally Lipschitz on an open set containing C, and functions $u_C : C \to \mathbb{R}$ and $u_D : \hat{D} \to \mathbb{R}$ such that, for all $(\xi, \sigma) \in C$, we have*

$$V^\circ\left((\xi, \sigma), \begin{bmatrix} \Pi_\sigma^{\text{diff}} A_\sigma \xi \\ 0 \end{bmatrix}\right) \leq u_C(\xi, \sigma) \tag{4a}$$

$$V(\eta) - V(\xi, \sigma) \leq u_D(\xi, \sigma) \quad \forall \eta \in G(\xi, \sigma) \tag{4b}$$

where $\hat{D} := C = \bigcup_{\sigma \in \Sigma} \big(\mathfrak{O}_\sigma \times \{\sigma\}\big)$ and G is given in (3)*. Suppose that ϕ is a precompact solution to the switched DAE system in* (1)*. Moreover, suppose that $K \subset \mathbb{R}^{n+1}$ is nonempty and $\overline{\operatorname{rge} \phi} \subset K$. If $u_C(\xi, \sigma) \leq 0$ and $u_D(\xi, \sigma) \leq 0$*

for all $(\xi,\sigma) \in K$, then ϕ approaches the largest weakly invariant set M for $\mathcal{H}_{DAE}^{\mathcal{SW}}$ which is a subset of[7]

$$V^{-1}(r) \cap K \cap \left(u_C^{-1}(0) \cup \left(u_D^{-1}(0) \cap G\left(u_D^{-1}(0)\right)\right)\right) \quad (5)$$

for some constant $r \in V(K)$. △

Remark 3.3: *Notice that given that σ is a component of the state vector it is possible to consider different functions V, u_C, and u_D for each σ. In particular, the results in Section III-B use functions V explicitly indexed by σ.*

Next, we use Theorem 3.2 to describe the largest weakly invariant set for a switched DAE system under arbitrary switching.

Example 3.4: ($2D$ switched DAE system) Consider the switched DAE system (as presented in [10, Example 1b]) with the data given by

$$E_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $\Sigma := \{1,2\}$. The consistency spaces and projectors are given by

$$\mathfrak{C}_1 = im\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),\ \Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ \Pi_1^{\text{diff}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$
$$\mathfrak{C}_2 = im\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right),\ \Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \Pi_2^{\text{diff}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

Thus, the consistency sets are given by $\mathfrak{D}_1 := \{\xi \in \mathbb{R}^2 \mid \xi_1 = \xi_2\}$, $\mathfrak{D}_2 := \{\xi \in \mathbb{R}^2 \mid \xi_1 = 0\}$. Now, consider the Lyapunov-like function $V : \mathbb{R}^{2+1} \to \mathbb{R}$ given by $V(\xi,\sigma) = \sigma\left(\frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2\right)$ Also, consider the function $u_D : \hat{D} \to \mathbb{R}$, which for points $x \in \hat{D}$, is given by $u_D(x) := 0$ and the function $u_C : C \to \mathbb{R}$, which for points in $x \in C$, is given by $u_C(x) := -\xi_1^2 - \xi_2^2$ where $\hat{D} := \cup_{\sigma\in\Sigma}\left(\mathfrak{D}_\sigma \times \{\sigma\}\right)$ and $C := \cup_{\sigma\in\Sigma}\left(\mathfrak{D}_\sigma \times \{\sigma\}\right)$. Also, notice that the matrix pairs (E_σ, A_σ) satisfy Assumption 3.1. Observe that the assumptions in Theorem 3.2 hold for V, u_C, and u_D. Computing the sets involved in (5), we have that from Theorem 3.2 every precompact solution to (1) converges to the largest weakly invariant subset in M, where $M = \left\{x \in \mathbb{R}^2 \times \{1,2\} \middle| \ \xi_1^2/2 + \xi_2^2/2 = r,\ \sigma = 1\right\} \cup \left\{x \in \mathbb{R}^2 \times \{1,2\} \middle| \ \xi_1 = 0, \xi_2^2 = r,\ \sigma = 2\right\}$.

From Theorem 3.2, every precompact solution to the switched DAE system under arbitrary switching converges to the largest weakly invariant set inside M for some $r \in V(K)$. This is a tight result as M is weakly forward invariant. In fact, there are precompact solutions ϕ from M that have constant ϕ_σ, in which case, ϕ_ξ approaches the origin. Also, for every solution ϕ with $\operatorname{dom}\phi = \overset{\infty}{\underset{j=0}{\cup}}([t_j, t_{j+1}], j)$ with infinitely many intervals $[t_j, t_{j+1}]$ such that $t_{j+1} - t_j > 0$, due to the strictly decaying rate of V during flows, the largest invariant set is $\{0\} \times \Sigma$. Notice that for the hybrid DAE system $\mathcal{H}_{DAE}^{\mathcal{SW}}$ there are Zeno solutions. In particular, from every point in M, there is a discrete solution that stays in M (in backward and forward time), making the set M weakly forward invariant for some $r > 0$. However, notice that the notion of solution to (1) in Definition 2.4, does not allow Zeno solutions. More precisely, Zeno solutions to $\mathcal{H}_{DAE}^{\mathcal{SW}}$ are not solutions to the switched DAE system in (1). □

B. Invariance principle under dwell-time switching

As a motivational example, we present a variation of Example 3.4 for which fast switching would lead to an empty omega limit set, but when adding a constraint on the switching signal, particularly a dwell-time constraint with large enough τ_D, each maximal solution is bounded and its omega limit set is nonempty. Later, we propose a hybrid DAE model for such switching systems under dwell-time switching as well as an invariance principle.

Example 3.5: ($2D$ switched DAE system revisited) Consider the same data as in Example 3.4, but with a variation on the second mode given by $E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$. The consistency spaces and projectors for the second mode are given by

$$\mathfrak{C}_2 = im\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right),\ \Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\ \Pi_2^{\text{diff}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Notice that, even though it is possible to craft strictly decreasing Lyapunov-like functions for each system mode, given the consistency projectors an arbitrary switching signal may lead to unbounded solutions. For example, consider a solution ϕ to (1) such that for the given switching signal, $\operatorname{dom}\phi = \overset{\infty}{\underset{j=0}{\cup}}([t_j, t_{j+1}], j)$ where $t_{j+1} - t_j < \ln(2)\frac{1}{2}$ for all $j \geq 0$, such a solution does not converge to any weakly invariant set, in fact, any solution starting away from the origin diverges to infinity as is shown in Figure 1(b) (right). Then, one may want to restrict the switching signal to have solutions that do not diverge. For example, one may want to restrict the switching signal to belong to the family of dwell-time switching signals. □

Given matrices $E_\sigma, A_\sigma \in \mathbb{R}^{n\times n}$ for each $\sigma \in \Sigma$, a switched DAE system under dwell-time switching signals can be captured by the following hybrid DAE system[8]:

$$\mathcal{H}_{DAE}^{\tau_D}\begin{cases} \begin{bmatrix} \begin{bmatrix} E_\sigma & 0 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\tau} \end{bmatrix} \\ \dot{\sigma} \end{bmatrix} \in \begin{bmatrix} f_\sigma(\xi) \\ \kappa_{\tau_D}(\tau) \\ 0 \end{bmatrix} =: F(x) & x \in C \\ \begin{bmatrix} \xi^+ \\ \tau^+ \\ \sigma^+ \end{bmatrix} \in \bigcup_{\tilde{\sigma}\in\Sigma\setminus\{\sigma\}} \begin{bmatrix} \Pi_{\tilde{\sigma}}\xi \\ 0 \\ \tilde{\sigma} \end{bmatrix} =: G(x) & x \in D \end{cases} \quad \text{(7a)}$$

where

$$f_\sigma(\xi) := A_\sigma \xi \qquad \forall \sigma \in \Sigma,\ \xi \in \mathfrak{D}_\sigma \quad \text{(7b)}$$

$$\kappa_{\tau_D}(\tau) := \begin{cases} 1 & \text{if } \tau < \tau_D \\ [0,1] & \text{if } \tau = \tau_D \\ 0 & \text{if } \tau > \tau_D \end{cases} \qquad \forall \tau \in [0,\infty) \quad \text{(7c)}$$

$$C := \bigcup_{\sigma\in\Sigma} \left(C_\sigma \cap (\mathfrak{D}_\sigma \times [0,\infty) \times \{\sigma\})\right) \quad \text{(7d)}$$

$$D := \bigcup_{\sigma\in\Sigma} \big((D_\sigma \cap (\mathfrak{D}_\sigma \times [0,\infty) \times \{\sigma\})) \cup ((\mathbb{R}^n \setminus \mathfrak{D}_\sigma) \times [0,\infty) \times \{\sigma\})\big) \quad \text{(7e)}$$

[7] The notation $f^{-1}(r)$ stands for the $r-$level set of f on $\operatorname{dom} f$, i.e., $f^{-1}(r) := \{z \in \operatorname{dom} f | f(z) = r\}$.

[8] Note that the hybrid DAE in (7) slightly differs from the one in (3) due to the addition of the timer τ, which leads to the state $((\xi,\tau),\sigma)$ and a set-valued flow map.

where Π_σ and $\mathfrak{D}_\sigma$ are computed using the data (E_σ, A_σ) for each $\sigma \in \Sigma$ and $x = (\xi, \tau, \sigma)$. This hybrid DAE system has data $(E_\sigma, C_\sigma, f_\sigma, D_\sigma, g_\sigma, \varphi_\sigma)$ where $C_\sigma = \mathbb{R}^n \times [0, \tau_D] \times \{\sigma\}$, $D_\sigma = \mathbb{R}^n \times [\tau_D, \infty) \times \{\sigma\}$, $g_\sigma(\xi) = \xi$, and $\varphi_\sigma(\xi) = \Sigma \setminus \{\sigma\}$. For simplicity, we will write it as $\mathcal{H}^{\tau_D}_{DAE} = ((E_\sigma, A_\sigma), \Sigma, \tau_D)$. Notice that for each dwell-time solution to $\mathcal{H}^{\mathcal{SW}}_{DAE} = ((E_\sigma, A_\sigma), \Sigma)$ in (3) with dwell-time $\tau_D > 0$ there corresponds a solution to the hybrid DAE system $\mathcal{H}^{\tau_D}_{DAE}$ in (7).

Next, to locate the omega limit set of precompact solutions to switched DAE systems, such as the one in Example 3.5, an invariance principle for switched DAE systems under dwell-time switching signals is presented following the ideas in [9]. It requires Proposition 3.6, Assumptions 3.7, and Assumption 3.8, which are presented first.

Proposition 3.6: (auxiliary system $\mathcal{H}^1_{DAE}$) *Given $\sigma^* \in \Sigma$, consider the DAE system $E_{\sigma^*}\dot{\xi} = A_{\sigma^*}\xi$, where $(E_{\sigma^*}, A_{\sigma^*}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ satisfies Assumption 3.1. Let κ_{τ_D} be as in (7c), $\mathfrak{D}_{\sigma^*}$ as in Definition 2.3, and Π_{σ^*} and $\Pi^{\text{diff}}_{\sigma^*}$ as in Definition 2.5 from $(E_{\sigma^*}, A_{\sigma^*})$. Furthermore, suppose there exist a function $V_{\sigma^*} : \mathbb{R}^n \to \mathbb{R}$ that is continuous on $\mathbb{R}^n$ and locally Lipschitz on an open set $O_{\sigma^*} \subset \mathfrak{D}_{\sigma^*}$, and a continuous function $W_{\sigma^*} : O_{\sigma^*} \to \mathbb{R}_{\geq 0}$ such that $V^\circ_{\sigma^*}(\xi, \Pi^{\text{diff}}_{\sigma^*} A_{\sigma^*}\xi) \leq -W_{\sigma^*}(\xi)$ for all $\xi \in O_{\sigma^*}$ Define the following hybrid DAE system, denoted by $\mathcal{H}^1_{DAE}$, as*

$$\mathcal{H}^1_{DAE} \begin{cases} \begin{bmatrix} \begin{bmatrix} E_{\sigma^*} & 0 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\tau} \end{bmatrix} \\ \dot{\sigma} \end{bmatrix} \in \begin{bmatrix} f_{\sigma^*}(\xi) \\ \kappa_{\tau_D}(\tau) \\ 0 \end{bmatrix} & (\xi, \tau, \sigma^*) \in C^* \\ \begin{bmatrix} \xi^+ \\ \tau^+ \\ \sigma^+ \end{bmatrix} \in \begin{bmatrix} K \\ 0 \\ \sigma^* \end{bmatrix} & (\xi, \tau, \sigma^*) \in D^* \end{cases}$$

where $C^* = C_{\sigma^*} \cap (\mathfrak{D}_{\sigma^*} \times [0, \infty) \times \{\sigma^*\})$, $D^* = D_{\sigma^*} \cap (\mathfrak{D}_{\sigma^*} \times [0, \infty) \times \{\sigma^*\})$, C_{σ^*}, and D_{σ^*} are given as in (7), and $K \subset O_{\sigma^*}$ is a nonempty and compact set. Let $(\xi, \tau, \sigma^*) : \operatorname{dom}(\xi, \tau, \sigma^*) \to O_{\sigma^*} \times [0, \infty) \times \{\sigma^*\}$ be a complete solution to $\mathcal{H}^1_{DAE}$ such that $\operatorname{rge}\xi \subset K$ and such that $V_{\sigma^*}(\xi(t, j+1)) - V_{\sigma^*}(\xi(t, j)) \leq 0$ for all $(t, j) \in \operatorname{dom}(\xi, \tau, \sigma^*)$ such that $(t, j+1) \in \operatorname{dom}(\xi, \tau, \sigma^*)$. Then, for some constant $r \in \mathbb{R}$, the solution component ξ approaches the largest subset M of

$$V^{-1}_{\sigma^*}(r) \cap K \cap W^{-1}_{\sigma^*}(0) \tag{8}$$

that is invariant in the following sense: for each $\xi_0 \in M$ there exists a solution χ (given by the component ϕ_ξ in Definition 2.4 when σ is constant) to $E_{\sigma^*}\dot{\xi} = A_{\sigma^*}\xi$ on $[0, \tau_D/2]$ such that $\chi(t) \in M$ for all $t \in [0, \tau_D/2]$ and either $\chi(0) = \xi_0$ or $\chi(\tau_D/2) = \xi_0$.

Assumption 3.7: *For each $\sigma \in \Sigma$, $O_\sigma \subset \mathfrak{D}_\sigma$ is an open set, $V_\sigma : O_\sigma \to \mathbb{R}$ is locally Lipschitz on O_σ, $W_\sigma : O_\sigma \to \mathbb{R}_{\geq 0}$ is a continuous function, and $V^\circ_\sigma(\xi, f_\sigma(\xi)) \leq -W_\sigma(\xi)$ for all $\xi \in O_\sigma$ where $f_\sigma : \mathfrak{D}_\sigma \to \mathbb{R}^n$ is given by $f_\sigma(\xi) = \Pi^{\text{diff}}_\sigma A_\sigma \xi$ for all $\xi \in \mathfrak{D}_\sigma$, where $\mathfrak{D}_\sigma$ as in Definition 2.3. Furthermore, for each $\sigma \in \Sigma$, (E_σ, A_σ) satisfies Assumption 3.1.*

Assumption 3.8: *Given a dwell-time switching signal $t \to \sigma(t)$, the solution $t \to \xi(t)$ to (1) is such that for each $\sigma^* \in \Sigma$, for any two consecutive intervals (t_j, t_{j+1}), (t_k, t_{k+1}) such that $\sigma(t) = \sigma^*$ for all $t \in (t_j, t_{j+1})$ and all $t \in (t_k, t_{k+1})$, we have $V_{\sigma^*}(\xi(t_{j+1})) - V_{\sigma^*}(\xi(t_k)) \geq 0$.*

Now, we are ready to present the invariance principle that involves only properties of switched DAE systems (1) under dwell-time switching signals. Its proof uses an equivalence between the solutions to such systems and those to $\mathcal{H}^{\tau_D}_{DAE}$, and, for each $\sigma \in \Sigma$, exploits the property of $\mathcal{H}^1_{DAE}$ in Proposition 3.6

Theorem 3.9: (Invariance principle for switched DAE systems under dwell-time switching) *Let Assumption 3.7 hold, and let ϕ be a precompact* dwell-time solution, *with dwell-time $\tau_D > 0$, to the switched DAE system in* (1) *satisfying Assumption 3.8. Then, there exist $r_1, \ldots, r_{\sigma_{max}} \in \mathbb{R}$ such that ϕ_ξ approaches $M = \bigcup_{\sigma\in\Sigma} M_\sigma(r_\sigma, \tau_D)$, where $M_\sigma(r_\sigma, \tau_D)$ is the largest subset of $V^{-1}_\sigma(r_\sigma) \cap W^{-1}_\sigma(0)$ that is invariant in the following sense: for each constant $\sigma \in \Sigma$ and for each $\xi_0 \in M_\sigma(r_\sigma, \tau_D)$ there exists a solution χ to the DAE[9] $E_\sigma\dot{\xi} = A_\sigma\xi$ on $[0, \tau_D/2]$ such that $\chi(t) \in M_\sigma(r_\sigma, \tau_D)$ for all $t \in [0, \tau_D/2]$ and either $\chi(0) = \xi_0$ or $\chi(\tau_D/2) = \xi_0$.*

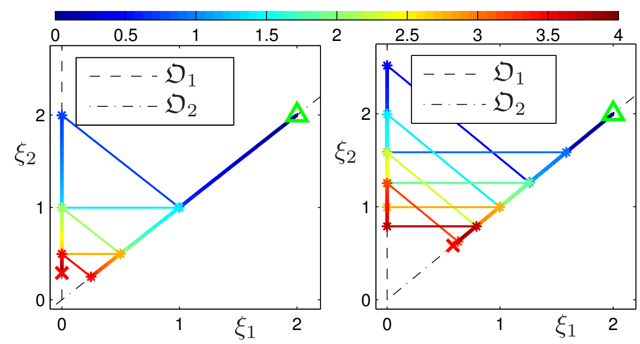
Proof: For each $\sigma^* \in \Sigma$ for which there is infinitely many disjoint intervals $(t_j, t_j + \Delta t_j)$, $j = 0, 1, \ldots$, $\Delta t_j \geq \tau_D$ on which $\sigma(t) = \sigma^*$, consider a hybrid arc z with $\operatorname{dom} z = \cup_{j=0}^{\infty} \left[\sum_{i=0}^{j-1} \Delta t_i, \sum_{i=0}^{j} \Delta t_i\right] \times \{j\}$ (with the convention that $\sum_{i=0}^{-1} \Delta t_i = 0$) defined by $z(t, j) = \xi\left(t_j + t - \sum_{i=0}^{j-1} \Delta t_i\right)$ for $t \in \left[\sum_{i=0}^{j-1} \Delta t_i, \sum_{i=0}^{j} \Delta t_i\right]$. Such a hybrid arc is a component of the solution to $\mathcal{H}^1_{DAE}$ of Proposition 3.6, and meets the assumptions of that proposition, with E_{σ^*}, A_{σ^*}, V_{σ^*}, W_{σ^*} for all $\sigma^* \in \Sigma$, and with $K \subset O_{\sigma^*}$ being any compact set such that $\xi(t, j) \in K$ whenever $\sigma(t) = \sigma^*$ (with K large enough to hold for jumps and flows of the solution component ξ). Propostion 3.6 implies the claim. ■

Example 3.10: ($2D$ switched DAE system under dwell-time switching) Consider a switched DAE system under a dwell-time switching signal with the data in Example 3.5. Also, for each $\sigma \in \{1, 2\}$ consider the functions $V_\sigma : \mathfrak{D}_\sigma \to \mathbb{R}$ and $W_\sigma : \mathfrak{D}_\sigma \to \mathbb{R}$ given by $V_\sigma(\xi) = \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2$, $W_\sigma(\xi) = \xi_1^2 + \xi_2^2$. Notice that Assumption 3.7 holds for V_σ and W_σ. To satisfy Assumption 3.8, we need to evaluate V along the solutions. The decrement rate of V_σ during flows is $\lambda = 2$ since $V^\circ_\sigma(\xi, \Pi^{\text{diff}}_\sigma A_\sigma\xi) = -2V_\sigma(\xi)$. Assumption 3.8 holds when $\tau_D \geq \ln(4)\frac{1}{2}$.

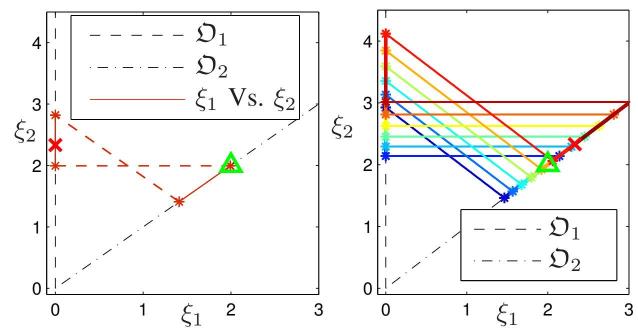
Consequently, following Theorem 3.9, ξ being a precompact dwell-time solution with dwell-time $\tau_D \geq \ln(4)\frac{1}{2}$, ξ approaches the largest invariant subset of $M = \bigcup_{\sigma\in\Sigma} M_\sigma(r_\sigma, \tau_D)$. Computing the sets involved in (8) with $W^{-1}_\sigma(0) = \{0\}$ for $\sigma \in \{1, 2\}$, we have that the only set that is invariant is the origin. Thus, any precompact dwell-time solution ξ with dwell-time $\tau_D \geq \ln(4)\frac{1}{2}$ approaches the origin. In Figure 1(a) (left), a numerical approximation of the solution to the system for a given initial condition and dwell-time $\tau_D = \ln(4)\frac{1}{2}$ is shown. □

Notice that the bound for τ_D in Example 3.10 may be not tight enough; actually, the bound on τ_D depends on the

[9]The notion of solution of a DAE is given by the component ϕ_ξ in Definition 2.4 when σ is constant.



(a) With $\tau_D = \ln(4)\frac{1}{2}$ (left) and $\tau_D = \ln(2)\frac{2}{3}$ (right).



(b) With $\tau_D = \ln(2)\frac{1}{2}$ (left) and $\tau_D = \ln(2)\frac{9}{20}$ (right).

Fig. 1. Solutions of the switched DAE system in Examples 3.5, 3.10, and 3.11 for different dwell-time switching signals using the same initial condition $\xi_1(0) = \xi_2(0) = 2$ and $\sigma(0) = 1$. Color gradient represents time, the green triangle represents the initial condition, and the red $\times$ symbol represents the final condition at $t = 4$.

selection of V_σ as is shown in the following example.

Example 3.11: (2D switched DAE system under dwell-time switching) Consider the switched DAE system in Example 3.10 under dwell-time $\tau_D > 0$ and the functions $V_\sigma : \mathfrak{D}_\sigma \to \mathbb{R}$ and $W_\sigma : \mathfrak{D}_\sigma \to \mathbb{R}$ for $\sigma \in \{1,2\}$ given by $V_\sigma(\xi) = 0$ and $W_\sigma(\xi) = 0$ for all $(\xi, \sigma) \in \mathfrak{D}_\sigma \times \{\sigma\}$.

To satisfy the conditions in Theorem 3.9, we need to ensure that solutions are precompact as in Proposition 3.6. Thus, we choose an arbitrary compact set K given by a ball of radius r centered in the origin for some $r \in \mathbb{R}$. We use an auxiliary function $\kappa : \mathbb{R}^2 \to \mathbb{R}$ to define K as $K := \{\xi \in \mathbb{R}^2 | \kappa(\xi) \leq r\}$, where $\kappa(\xi) := \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2}$. As in the previous example, it is possible to show that the function κ decays through flowing solutions by the decay rate $\lambda = 2$. Also, it is possible to show that at switching instants the function κ is given by $\kappa(\xi(t_k)) = 2\kappa(\xi(t_{j+1}))$ for all consecutive intervals (t_j, t_{j+1}) and $(t_k, t_{k+1}) \subset \operatorname{dom} \xi$. Then, to assure that the solution ξ belongs the compact set K (for some $r > 0$), after each jump $2e^{-2(t_{j+1}-t_j)}\kappa(\xi(t_j)) \leq r$ must hold. This condition is satisfied taking $t_{j+1} - t_j = \tau_D$, with $\tau_D \geq \ln\left(2\frac{\kappa(\xi(t_j))}{r}\right)\frac{1}{2}$. Thus, $\kappa(\xi(t_j)) = r$ give us the bound $\tau_D \geq \ln(2)\frac{1}{2}$ which makes ξ precompact for K. With the bound on τ_D it is possible to apply Theorem 3.9, and using auxiliary function κ we deduce that if $\tau_D = \ln(2)\frac{1}{2}$ then ξ approaches the largest invariant subset $\bigcup_{\sigma\in\{1,2\}} M_\sigma(r_\sigma, \tau_D)$ which, since $\operatorname{rge} \xi \subset K$, is a subset of $\bigcup_{\sigma\in\{1,2\}} V_\sigma^{-1}(r_\sigma) \cap K \cap W_\sigma^{-1}(0) = K \cap (\mathfrak{D}_1 \cup \mathfrak{D}_2)$. In Figure 1(b) (left), the solution to the system for a given initial condition and dwell-time $\tau_D = \ln(2)\frac{1}{2}$ is shown. Additionally, given the strict decrease of $\kappa(\xi(t))$ through flows given by $\langle \nabla\kappa(\xi), \Pi_\sigma^{\text{diff}} A_\sigma \xi \rangle < 0$ for $\sigma \in \{1,2\}$ and for all $\xi \in \mathbb{R}^2 \setminus \{0\}$, it is easy to see that if $\tau_D > \ln(2)\frac{1}{2}$ the largest invariant subset M is the origin. In Figure 1(a) (right), for $\tau_D = \ln(2)\frac{2}{3} > \ln(2)\frac{1}{2}$ it is shown how a solution for a given initial condition approaches the origin. Furthermore, if the dwell-time $\tau_D < \ln(2)\frac{1}{2}$, then that implies that $\kappa(\xi(t_j)) < r$ and the sequence $\kappa(\xi(t_k))$ for $k = \{1, 2, \ldots\}$ grows unbounded (see also Example 3.5). In Figure 1(b) (right), it is shown how the solution does not converge when $\tau_D = \ln(2)\frac{9}{20} < \ln(2)\frac{1}{2}$. □

IV. Conclusion

In this paper, we consider switched DAE systems, which are dynamical systems with multiple modes of operation and each mode of operation is described by a DAE. The analysis presented here borrows concepts from hybrid systems theory, as well as concepts from switched DAE systems to introduce an invariance principle for switched DAE systems under arbitrary and dwell-time switching signals, which were not available in the literature. We present examples where the definitions and results of this paper are exercised.

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