Dynamics and Control of Quadrotor UAVs Transporting a Rigid Body Connected via Flexible Cables

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***Abstract*— This paper is focused on the dynamics and control of arbitrary number of quadrotor UAVs transporting a rigid body payload. The rigid body payload is connected to quadrotors via flexible cables where each flexible cable is modeled as a system of serially-connected links. It is shown that a coordinate-free form of equations of motion can be derived for arbitrary numbers of quadrotors and links according to Lagrangian mechanics on a manifold. A geometric nonlinear controller is presented to transport the rigid body to a fixed desired position while aligning all of the links along the vertical direction. Numerical results are provided to illustrate the desirable features of the proposed control system.**

I. INTRODUCTION

There are various applications for aerial load transportation such as usage in construction, military operations, emergency response, or delivering packages. Load transportation with the cable-suspended load has been studied traditionally for a helicopter [1], [2] or for small unmanned aerial vehicles such as quadrotor UAVs [3], [4], [5].

In most of the prior works, the dynamics of aerial transportation has been simplified due to the inherent dynamic complexities. For example, it is assumed that the dynamics of the payload is considered completely decoupled from quadrotors, and the effects of the payload and the cable are regarded as arbitrary external forces and moments exerted to the quadrotors [6], [7], [8], thereby making it challenging to suppress the swinging motion of the payload actively, particularly for agile aerial transportations.

Recently, the coupled dynamics of the payload or cable has been explicitly incorporated into control system design [9]. In particular, a complete model of a quadrotor transporting a payload modeled as a point mass, connected via a flexible cable is presented, where the cable is modeled as serially connected links to represent the deformation of the cable [10]. In another distinct study, multiple quadrotors transporting a rigid body payload has been studied [11], but it is assume that the cables connecting the rigid body payload and quadrotors are always taut. These assumptions and simplifications in the dynamics of the system reduce the stability of the controlled system, particularly in rapid and aggressive load transportation where the motion of the cable and payload is excited nontrivially.

The first distinct contribution of this paper is presenting the complete dynamic model of an arbitrary number of

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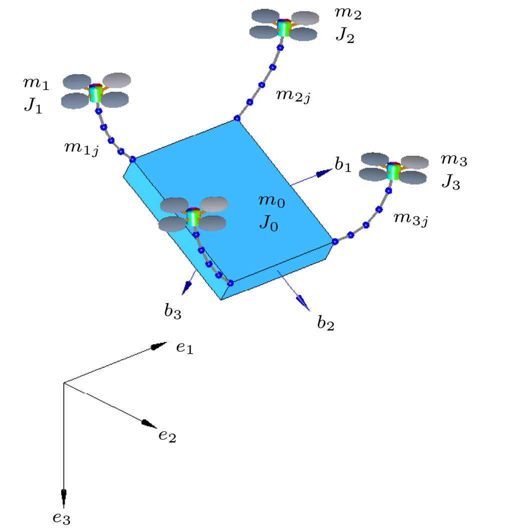


Fig. 1. Quadrotor UAVs with a rigid body payload. Cables are modeled as a serial connection of an arbitrary number of links (only 4 quadrotors with 5 links in each cable are illustrated).

quadrotors transporting a rigid body where each quadrotor is connected to the rigid body via a flexible cable. Each flexible cable is modeled as an arbitrary number of serially connected links, and it is valid for various masses and lengths. A coordinate free form of equations of motion is derived according to Lagrange mechanics on a nonlinear manifold for the full dynamic model. These sets of equations of motion are presented in a complete and organized manner without any simplification.

Another contribution of this study is designing a control system to stabilize the rigid body at desired position. Geometric nonlinear controllers presented in the author's previous study is utilized [12], [13], [14], and they are generalized for the presented model. More explicitly, we show that the rigid body payload is asymptotically transported into a desired location, while aligning all of the links along the vertical direction corresponding to a hanging equilibrium.

The unique property of the proposed control system is that the nontrivial coupling effects between the dynamics of rigid payload, flexible cables, and multiple quadrotors are explicitly incorporated into control system design, without any simplifying assumption. Another distinct feature is that the equations of motion and the control systems are developed directly on the nonlinear configuration manifold intrinsically. Therefore, singularities of local parameterization are completely avoided to generate agile maneuvers of the payload in a uniform way. In short, the proposed control system is particularly useful for rapid and safe payload transportation in complex terrain, where the position of the payload should be controlled concurrently while suppressing the deformation of the cables.

This paper is organized as follows. A dynamic model is presented and the problem is formulated at Section II. Control systems are constructed at Sections III and IV, which are followed by numerical examples in Section V. Due to the page limit, parts of proofs are relegated to [15].

II. Problem Formulation

Consider a rigid body with mass $m_0 \in \mathbb{R}$ and moment of inertia $J_0 \in \mathbb{R}^{3\times 3}$, being transported with arbitrary n numbers of quadrotors. The location of the mass center of the rigid body is denoted by $x_0 \in \mathbb{R}^3$, and its attitude is given by $R_0 \in \mathsf{SO(3)}$, where the special orthogonal group is given by $\mathsf{SO(3)} = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det(R) = 1\}$. Figure 1 illustrates the system with an inertial frame. We choose an inertial frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and body fixed frame $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ attached to the payload. We also consider a body fixed frame attached to the i-th quadrotor $\{\vec{b}_{1_i}, \vec{b}_{2_i}, \vec{b}_{3_i}\}$. In the inertial frame, the third axes $\vec{e}_3$ points downward with gravity and the other axes are chosen to form an orthonormal frame.

The mass and the moment of inertia of the i-th quadrotor are denoted by $m_i \in \mathbb{R}$ and $J_i \in \mathbb{R}^{3\times 3}$ respectively. The cable connecting each quadrotor to the rigid body is modeled as an arbitrary numbers of links for each quadrotor with varying masses and lengths. The direction of the j-th link of the i-th quadrotor, measured outward from the quadrotor toward the payload is defined by the unit vector $q_{ij} \in \mathsf{S}^2$, where $\mathsf{S}^2 = \{q \in \mathbb{R}^3 \mid \|q\| = 1\}$, where the mass and length of that link is denoted with m_{ij} and l_{ij} respectively. The number of links in the cable connected to the i-th quadrotor is defined as n_i.

The configuration manifold for this system is given by $\mathsf{SO(3)} \times \mathbb{R}^3 \times (\mathsf{SO(3)}^n) \times (\mathsf{S}^2)^{\sum_{i=1}^n n_i}$. The i-th quadrotor can generate a thrust force of $-f_i R_i e_3 \in \mathbb{R}^3$ with respect to the inertial frame, where $f_i \in \mathbb{R}$ is the total thrust magnitude of the i-th quadrotor. It also generates a moment $M_i \in \mathbb{R}^3$ with respect to its body-fixed frame. Throughout this paper, the two norm of a matrix A is denoted by $\|A\|$. The standard dot product is denoted by $x \cdot y = x^T y$ for any $x, y \in \mathbb{R}^3$.

A. Lagrangian

The kinematics equations for the links, payload, and quadrotors are given by

$$\dot{q}_{ij} = \omega_{ij} \times q_{ij} = \hat{\omega}_{ij} q_{ij}, \tag{1}$$

$$\dot{R}_0 = R_0 \hat{\Omega}_0, \tag{2}$$

$$\dot{R}_i = R_i \hat{\Omega}_i, \tag{3}$$

where $\omega_{ij} \in \mathbb{R}^3$ is the angular velocity of the j-th link in the i-th cable satisfying $q_{ij} \cdot \omega_{ij} = 0$. Also, $\Omega_0 \in \mathbb{R}^3$ is the angular velocity of the payload and $\Omega_i \in \mathbb{R}^3$ is the angular velocity of the i-th quadrotor, expressed with respect to the corresponding body fixed frame. The hat map $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by the condition that $\hat{x} y = x \times y$ for all $x, y \in \mathbb{R}^3$, and the inverse of the hat map is denoted by the vee map $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$.

The position of the i-th quadrotor is given by

$$x_i = x_0 + R_0 \rho_i - \sum_{a=1}^{n_i} l_{ia} q_{ia}, \tag{4}$$

where $\rho_i \in \mathbb{R}^3$ is the vector from the center of mass of the rigid body to the point that i-th quadrotor is connected to rigid body via the cable. Similarly the position of the j-th link in the cable connecting the i-th quadrotor to the rigid body is given by

$$x_{ij} = x_0 + R_0 \rho_i - \sum_{a=j+1}^{n_i} l_{ia} q_{ia}. \tag{5}$$

We derive equations of motion according to Lagrangian mechanics. Total kinetic energy of the system is given by

$$\begin{aligned} T =& \frac{1}{2} m_0 \|\dot{x}_0\|^2 + \sum_{i=1}^n \sum_{j=1}^{n_i} \frac{1}{2} m_{ij} \|\dot{x}_{ij}\|^2 + \frac{1}{2} \sum_{i=1}^n m_i \|\dot{x}_i\|^2 \\ &+ \frac{1}{2} \sum_{i=1}^n \Omega_i \cdot J_i \Omega_i + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0. \end{aligned} \tag{6}$$

The gravitational potential energy is given by

$$V = -m_0 g e_3 \cdot x_0 - \sum_{i=1}^n m_i g e_3 \cdot x_i - \sum_{i=1}^n \sum_{j=1}^{n_i} m_{ij} g e_3 \cdot x_{ij}, \tag{7}$$

where it is assumed that the unit-vector e_3 points downward along the gravitational acceleration as shown at Figure 1. The corresponding Lagrangian of the system is $L = T - V$.

B. Euler-Lagrange equations

Coordinate-free form of Lagrangian mechanics on the two-sphere S^2 and the special orthogonal group $\mathsf{SO(3)}$ for various multibody systems has been studied in [16], [17]. The key idea is representing the infinitesimal variation of $R_i \in \mathsf{SO(3)}$ in terms of the exponential map

$$\delta R_i = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} R_i \exp(\epsilon \hat{\eta}_i) = R_i \hat{\eta}_i, \tag{8}$$

for $\eta_i \in \mathbb{R}^3$. The corresponding variation of the angular velocity is given by $\delta\Omega_i = \dot{\eta}_i + \Omega_i \times \eta_i$. Similarly, the infinitesimal variation of $q_{ij} \in \mathsf{S}^2$ is given by

$$\delta q_{ij} = \xi_{ij} \times q_{ij}, \tag{9}$$

for $\xi_{ij} \in \mathbb{R}^3$ satisfying $\xi_{ij} \cdot q_{ij} = 0$. This lies in the tangent space as it is perpendicular to q_i. Using these, we obtain the following Euler-Lagrange equations.

Proposition 1: By using the above expressions, the equations of motion can be obtained from Hamilton's principle:

$$M_T\ddot{x}_0-\sum_{i=1}^{n}\sum_{j=1}^{n_i}M_{0ij}l_{ij}\ddot{q}_{ij}-\sum_{i=1}^{n}M_{iT}R_0\hat{\rho}_i\dot{\Omega}_0$$
$$=M_Tge_3+\sum_{i=1}^{n}-f_iR_ie_3-\sum_{i=1}^{n}M_{iT}R_0\hat{\Omega}_0^2\rho_i, \tag{10}$$
$$\bar{J}_0\dot{\Omega}_0+\sum_{i=1}^{n}M_{iT}\hat{\rho}_iR_0^T\ddot{x}_0-\sum_{i=1}^{n}\sum_{j=1}^{n_i}M_{0ij}l_{ij}\hat{\rho}_iR_0^T\ddot{q}_{ij}$$
$$=\sum_{i=1}^{n}\hat{\rho}_iR_0^T(-f_iR_ie_3+M_{iT}ge_3)-\hat{\Omega}_0\bar{J}_0\Omega_0, \tag{11}$$
$$\sum_{k=1}^{n_i}M_{0ij}l_{ik}\hat{q}_{ij}^2\ddot{q}_{ik}-M_{0ij}\hat{q}_{ij}^2\ddot{x}_0+M_{0ij}\hat{q}_{ij}^2R_0\hat{\rho}_i\dot{\Omega}_0$$
$$=M_{0ij}\hat{q}_{ij}^2R_0\hat{\Omega}_0^2\rho_i-\hat{q}_{ij}^2(M_{0ij}ge_3-f_iR_ie_3), \tag{12}$$
$$J_i\dot{\Omega}_i+\Omega_i\times J_i\Omega_i=M_i. \tag{13}$$

Here the total mass M_T of the system and the mass of the i-th quadrotor and its flexible cable M_{iT} are defined as

$$M_T=m_0+\sum_{i=1}^{n}M_{iT},\ M_{iT}=\sum_{j=1}^{n_i}m_{ij}+m_i, \tag{14}$$

and the constants related to the mass of links are given as

$$M_{0ij}=m_i+\sum_{a=1}^{j-1}m_{ia}, \tag{15}$$

The equations of motion can be rearranged in a matrix form as follow

$$\mathbf{N}\ddot{X}=\mathbf{P} \tag{16}$$

where the state vector $X\in\mathbb{R}^{D_X}$ with $D_X=6+3\sum_{i=1}^{n}n_i$ is given by

$$X=[x_0,\ \Omega_0,\ q_{1j},\ q_{2j},\ \cdots,\ q_{nj}]^T, \tag{17}$$

and matrix $\mathbf{N}\in\mathbb{R}^{D_X\times D_X}$ is defined as

$$\mathbf{N}=\begin{bmatrix}M_TI_3 & \mathbf{N}_{x_0\Omega_0} & \mathbf{N}_{x_01} & \mathbf{N}_{x_02} & \cdots & \mathbf{N}_{x_0n}\\ \mathbf{N}_{\Omega_0x_0} & \bar{J}_0 & \mathbf{N}_{\Omega_01} & \mathbf{N}_{\Omega_02} & \cdots & \mathbf{N}_{\Omega_0n}\\ \mathbf{N}_{1x_0} & \mathbf{N}_{1\Omega_0} & \mathbf{N}_{qq1} & 0 & \cdots & 0\\ \mathbf{N}_{2x_0} & \mathbf{N}_{2\Omega_0} & 0 & \mathbf{N}_{qq2} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{N}_{nx_0} & \mathbf{N}_{n\Omega_0} & 0 & 0 & \cdots & \mathbf{N}_{qqn}\end{bmatrix}, \tag{18}$$

where the sub-matrices are defined as

$$\mathbf{N}_{x_0\Omega_0}=-\sum_{i=1}^{n}M_{iT}R_0\hat{\rho}_i;\ \mathbf{N}_{\Omega_0x_0}=\mathbf{M}_{x_0\Omega_0}^T,$$
$$\mathbf{N}_{x_0i}=-[M_{0i1}l_{i1}I_3,\ M_{0i2}l_{i2}I_3,\ \cdots,\ M_{0in_i}l_{in_i}I_3],$$
$$\mathbf{N}_{\Omega_0i}=-[M_{0i1}l_{i1}\hat{\rho}_iR_0^T,\ M_{0i2}l_{i2}\hat{\rho}_iR_0^T,\ \cdots,\ M_{0in_i}l_{in_i}\hat{\rho}_iR_0^T],$$
$$\mathbf{N}_{ix_0}=-[M_{0i1}\hat{q}_{i1}^2,\ M_{0i2}\hat{q}_{i2}^2,\ \cdots,\ M_{0in_i}\hat{q}_{in_i}^2]^T,$$
$$\mathbf{N}_{i\Omega_0}=[M_{0i1}\hat{q}_{i1}^2R_0\hat{\rho}_i,\ M_{0i2}\hat{q}_{i2}^2R_0\hat{\rho}_i,\ \cdots,\ M_{0in_i}\hat{q}_{in_i}^2R_0\hat{\rho}_i]^T, \tag{19}$$

and the sub-matrix $\mathbf{N}_{qqi}\in\mathbb{R}^{3n_i\times3n_i}$ is given by

$$\mathbf{N}_{qqi}=\begin{bmatrix}-M_{011}l_{i1}I_3 & M_{012}l_{i2}\hat{q}_{i2}^2 & \cdots & M_{01n_i}l_{in_i}\hat{q}_{in_i}^2\\ M_{021}l_{i1}\hat{q}_{i1}^2 & -M_{022}l_{i2}I_3 & \cdots & M_{02n_i}l_{in_i}\hat{q}_{in_i}^2\\ \vdots & \vdots & & \vdots\\ M_{0n_i1}l_{i1}\hat{q}_{i1}^2 & M_{0n_i2}l_{i2}\hat{q}_{i2}^2 & \cdots & -M_{0n_in_i}l_{in_i}I_3\end{bmatrix}. \tag{20}$$

The $\mathbf{P}\in\mathbb{R}^{D_X}$ matrix is

$$\mathbf{P}=[P_{x_0},\ P_{\Omega_0},\ P_{1j},\ P_{2j},\ \cdots,\ P_{nj}]^T, \tag{21}$$

and sub-matrices of $\mathbf{P}$ matrix are also defined as

$$P_{x_0}=M_Tge_3+\sum_{i=1}^{n}-f_iR_ie_3-\sum_{i=1}^{n}M_{iT}R_0\hat{\Omega}_0^2\rho_i,$$
$$P_{\Omega_0}=-\hat{\Omega}_0\bar{J}_0\Omega_0+\sum_{i=1}^{n}\hat{\rho}_iR_0^T(M_{iT}ge_3-f_iR_ie_3),$$
$$P_{ij}=-\hat{q}_{ij}^2(-f_iR_ie_3+M_{0ij}ge_3)+M_{0ij}\hat{q}_{ij}^2R_0\hat{\Omega}_0^2\rho_i$$
$$+M_{0ij}\|\dot{q}_{ij}\|^2q_{ij}.$$

Proof: See Appendix A ∎

These equations are derived directly on a nonlinear manifold without any simplification. The dynamics of the payload, flexible cables, and quadrotors are considered explicitly, and they avoid singularities and complexities associated to local coordinates.

III. CONTROL SYSTEM DESIGN FOR SIMPLIFIED DYNAMIC MODEL

A. Control Problem Formulation

Let $x_{0_d}\in\mathbb{R}^3$ be the desired position of the payload. The desired attitude of the payload is considered as $R_{0_d}=I_{3\times3}$, and the desired direction of links is aligned along the vertical direction. The corresponding location of the i-th quadrotor at this desired configuration is given by

$$x_{i_d}=x_{0_d}+\rho_i-\sum_{a=1}^{n_i}l_{ia}e_3. \tag{22}$$

We wish to design control forces f_i and control moments M_i of quadrotors such that this desired configuration becomes asymptotically stable.

B. Simplified Dynamic Model

Control forces for each quadrotor is given by $-f_iR_ie_3$ for the given equations of motion (10), (11), (12), (13). As such, the quadrotor dynamics is underactuated. The total thrust magnitude of each quadrotor can be arbitrary chosen, but the direction of the thrust vector is always along the third body fixed axis, represented by R_ie_3. But, the rotational attitude dynamics of the quadrotors are fully actuated, and they are not affected by the translational dynamics of the quadrotors or the dynamics of links.

Based on these observations, in this section, we simplify the model by replacing the $-f_iR_ie_3$ term by a fictitious control input $u_i\in\mathbb{R}^3$, and design an expression for u to asymptotically stabilize the desired equilibrium. In another words, we assume that the attitude of the quadrotor can be instantaneously changed. The effects of the attitude dynamics are studied at the next section.

C. Linear Control System

The control system for the simplified dynamic model is developed based on the linearized equations of motion. At the desired equilibrium, the position and the attitude of the payload are given by x_{0_d} and $R_0^* = I_3$, respectively, where the superscript $*$ denotes the value of a variable at the desired equilibrium throughout this paper. Also, we have $q_{ij}^* = e_3$ and $R_i^* = I_3$. In this equilibrium configuration, the control input for the i-th quadrotor is

$$u_i^* = -f_i^* R_i^* e_3, \tag{23}$$

where the total thrust is $f_i^* = (M_{iT} + \frac{m_0}{n})g$.

The variation of x_0 is given by

$$\delta x_0 = x_0 - x_{0_d}, \tag{24}$$

and the variation of the attitude of the payload is defined as

$$\delta R_0 = R_0^* \hat{\eta}_0 = \hat{\eta}_0,$$

for $\eta_0 \in \mathbb{R}^3$. The variation of q_{ij} can be written as

$$\delta q_{ij} = \xi_{ij} \times e_3, \tag{25}$$

where $\xi_{ij} \in \mathbb{R}^3$ with $\xi_{ij} \cdot e_3 = 0$. The variation of ω_{ij} is given by $\delta\omega_{ij} \in \mathbb{R}^3$ with $\delta\omega_{ij} \cdot e_3 = 0$. Therefore, the third element of each of ξ_{ij} and $\delta\omega_{ij}$ for any equilibrium configuration is zero, and they are omitted in the following linearized equations. The state vector of the linearized equation is composed of $C^T\xi_{ij} \in \mathbb{R}^2$, where $C = [e_1,\ e_2] \in \mathbb{R}^{3\times 2}$. The variation of the control input $\delta u_i \in \mathbb{R}^{3\times 1}$, is given as $\delta u_i = u_i - u_i^*$.

Proposition 2: The linearized equations of the simplified dynamic model are given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\mathbf{x} = \mathbf{B}\delta u, \tag{26}$$

where the state vector $\mathbf{x} \in \mathbb{R}^{D_\mathbf{x}}$ with $D_\mathbf{x} = 6 + 2\sum_{i=1}^n n_i$ is given by

$$\mathbf{x} = \left[\delta x_0, \eta_0, C^T\xi_{1j}, C^T\xi_{2j}, \cdots, C^T\xi_{nj}\right],$$

and $\delta u = [\delta u_1^T,\ \delta u_2^T,\ \cdots,\ \delta u_n^T]^T \in \mathbb{R}^{3n\times 1}$. The matrix $\mathbf{M} \in \mathbb{R}^{D_\mathbf{x}\times D_\mathbf{x}}$ are defined as

$$\mathbf{M} = \begin{bmatrix} M_T I_3 & \mathbf{M}_{x_0\Omega_0} & \mathbf{M}_{x_0 1} & \mathbf{M}_{x_0 2} & \cdots & \mathbf{M}_{x_0 n} \\ \mathbf{M}_{\Omega_0 x_0} & \bar{J}_0 & \mathbf{M}_{\Omega_0 1} & \mathbf{M}_{\Omega_0 2} & \cdots & \mathbf{M}_{\Omega_0 n} \\ \mathbf{M}_{1x_0} & \mathbf{M}_{1\Omega_0} & \mathbf{M}_{qq1} & 0 & \cdots & 0 \\ \mathbf{M}_{2x_0} & \mathbf{M}_{2\Omega_0} & 0 & \mathbf{M}_{qq2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{M}_{nx_0} & \mathbf{M}_{n\Omega_0} & 0 & 0 & \cdots & \mathbf{M}_{qqn} \end{bmatrix},$$

where the sub-matrices are defined as

$$\mathbf{M}_{x_0\Omega_0} = -\sum_{i=1}^n M_{iT}\hat{\rho}_i;\ \mathbf{M}_{\Omega_0 x_0} = \mathbf{M}_{x_0\Omega_0}^T,$$

$$\begin{aligned} \mathbf{M}_{x_0 i} &= [M_{0i1} l_{i1}\hat{e}_3 C,\ M_{0i2} l_{i2}\hat{e}_3 C,\ \cdots,\ M_{0in_i} l_{in_i}\hat{e}_3 C], \\ \mathbf{M}_{\Omega_0 i} &= [M_{0i1} l_{i1}\hat{\rho}_i C,\ M_{0i2} l_{i2}\hat{\rho}_i C,\ \cdots,\ M_{0in_i} l_{in_i}\hat{\rho}_i C], \\ \mathbf{M}_{ix_0} &= -[M_{0i1} C^T\hat{e}_3,\ M_{0i2} C^T\hat{e}_3,\ \cdots,\ M_{0in_i} C^T\hat{e}_3], \end{aligned} \tag{27}$$

$$\mathbf{M}_{i\Omega_0} = [M_{0i1} C^T\hat{e}_3\hat{\rho}_i,\ M_{0i2} C^T\hat{e}_3\hat{\rho}_i,\ \cdots,\ M_{0in_i} C^T\hat{e}_3\hat{\rho}_i], \tag{28}$$

and the sub-matrix $\mathbf{M}_{qqi} \in \mathbb{R}^{2n_i\times 2n_i}$ is given by

$$\mathbf{M}_{qqi} = \begin{bmatrix} M_{i11} l_{i1} I_2 & M_{i12} l_{i2} I_2 & \cdots & M_{i1n_i} l_{in_i} I_2 \\ M_{i21} l_{i1} I_2 & M_{i22} l_{i2} I_2 & \cdots & M_{i2n_i} l_{in_i} I_2 \\ \vdots & \vdots & & \vdots \\ M_{in_i1} l_{i1} I_2 & M_{in_i2} l_{i2} I_2 & \cdots & M_{in_in_i} l_{in_i} I_2 \end{bmatrix}. \tag{29}$$

The matrix $\mathbf{G} \in \mathbb{R}^{D_\mathbf{x}\times D_\mathbf{x}}$ is defined as

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{G}_{\Omega_0\Omega_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{G}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{G}_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \mathbf{G}_n \end{bmatrix},$$

where $\mathbf{G}_{\Omega_0\Omega_0} = \sum_{i=1}^n \frac{m_0}{n} g\hat{\rho}_i\hat{e}_3$ and the sub-matrices $\mathbf{G}_i \in \mathbb{R}^{2n_i\times 2n_i}$ are

$$\mathbf{G}_i = \mathrm{diag}[(-M_{iT} - \frac{m_0}{n} + M_{0ij})ge_3 I_2].$$

The matrix $\mathbf{B} \in \mathbb{R}^{D_\mathbf{x}\times 3n}$ is given by

$$\mathbf{B} = \begin{bmatrix} I_3 & I_3 & \cdots & I_3 \\ \hat{\rho}_1 & \hat{\rho}_2 & \cdots & \hat{\rho}_n \\ \mathbf{B_B} & 0 & 0 & 0 \\ 0 & \mathbf{B_B} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \mathbf{B_B} \end{bmatrix},$$

where $\mathbf{B_B} = -[C^T\hat{e}_3,\ C^T\hat{e}_3,\ \cdots,\ C^T\hat{e}_3]^T$.

Proof: See Appendix B ■

We present the following PD-type control system for the linearized dynamics

$$\delta u_i = -K_{x_i}\mathbf{x} - K_{\dot{x}_i}\dot{\mathbf{x}}, \tag{30}$$

for controller gains $K_{x_i}, K_{\dot{x}_i} \in \mathbb{R}^{3\times D_\mathbf{x}}$. Provided that (26) is controllable, we can choose the combined controller gains $K_x = [K_{x_1}^T, \ldots\ K_{x_n}^T]^T$, $K_{\dot{x}} = [K_{\dot{x}_1}^T, \ldots K_{\dot{x}_n}^T]^T \in \mathbb{R}^{3n\times D_\mathbf{x}}$ such that the equilibrium is asymptotically stable for the linearized equation (26).

IV. CONTROL SYSTEM DESIGN FOR THE FULL DYNAMIC MODEL

The control system designed at the previous section is based on a simplifying assumption that each quadrotor can generates a thrust along any direction. In the full dynamic model, the direction of the thrust for each quadrotor is parallel to its third body-fixed axis always. In this section, the attitude of each quadrotor is controlled such that the third body-fixed axis becomes parallel to the direction of the ideal control force designed in the previous section. The central idea is that the attitude R_i of the quadrotor is controlled such that its total thrust direction $-R_ie_3$, corresponding to the third body-fixed axis, asymptotically follows the direction of the fictitious control input u_i. By choosing the total thrust magnitude properly, we can guarantee asymptotical stability for the full dynamic model.

Let $A_i \in \mathbb{R}^3$ be the ideal total thrust of the i-th quadrotor that asymptotically stabilize the desired equilibrium. Therefor, we have

$$A_i = u_i^* + \delta u_i = -K_{x_i}\mathbf{x} - K_{\dot{x}_i}\dot{\mathbf{x}} + u_i^*, \tag{31}$$

where f_i^* and u_i^* are the total thrust and control input of each quadrotor at its equilibrium respectively.

From the desired direction of the third body-fixed axis of the i-th quadrotor, namely $b_{3_i} \in \mathsf{S}^2$, is given by

$$b_{3_i} = -\frac{A_i}{\|A_i\|}. \tag{32}$$

This provides a two-dimensional constraint on the three dimensional desired attitude of each quadrotor, such that there remains one degree of freedom. To resolve it, the desired direction of the first body-fixed axis $b_{1_i}(t) \in \mathsf{S}^2$ is introduced as a smooth function of time. Due to the fact that the first body-fixed axis is normal to the third body-fixed axis, it is impossible to follow an arbitrary command $b_{1_i}(t)$ exactly. Instead, its projection onto the plane normal to b_{3_i} is followed, and the desired direction of the second body-fixed axis is chosen to constitute an orthonormal frame [13]. More explicitly, the desired attitude of the i-th quadrotor is given by

$$R_{i_c} = \left[-\frac{(\hat{b}_{3_i})^2 b_{1_i}}{\|(\hat{b}_{3_i})^2 b_{1_i}\|} \quad \frac{\hat{b}_{3_i} b_{1_i}}{\|\hat{b}_{3_i} b_{1_i}\|} \quad b_{3_i}\right], \tag{33}$$

which is guaranteed to be an element of $\mathfrak{so}(3)$. The desired angular velocity is obtained from the attitude kinematics equation, $\Omega_{i_c} = (R_{i_c}^T \dot{R}_{i_c})^\vee \in \mathbb{R}^3$. Define the tracking error vectors for the attitude and the angular velocity of the i-th quadrotor as

$$e_{R_i} = \frac{1}{2}(R_{i_c}^T R_i - R_i^T R_{i_c})^\vee,\; e_{\Omega_i} = \Omega_i - R_i^T R_{i_c}\Omega_{i_c}, \tag{34}$$

and a configuration error function on $\mathsf{SO}(3)$ as follows

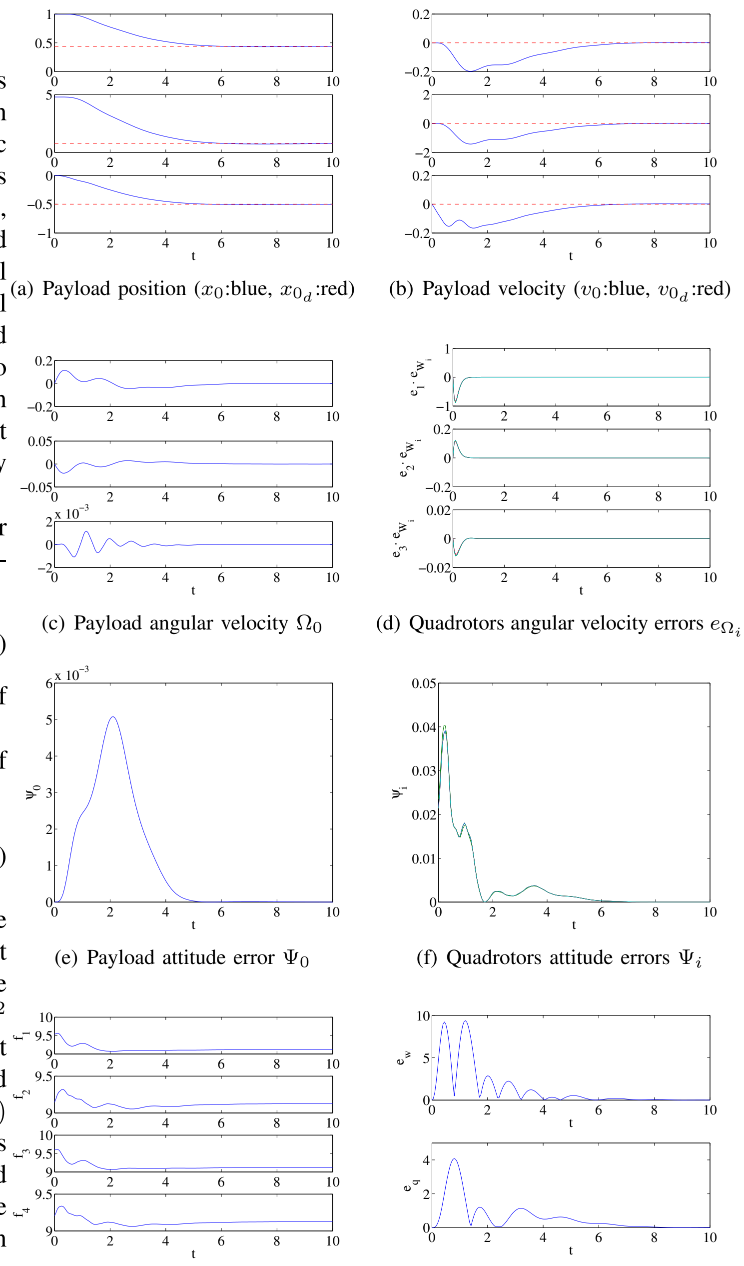
$$\Psi_i = \frac{1}{2}\operatorname{tr}[I - R_{i_c}^T R_i]. \tag{35}$$

The thrust magnitude is chosen as the length of u_i, projected on to $-R_ie_3$, and the control moment is chosen as a tracking controller on $\mathsf{SO}(3)$:

$$\begin{aligned} f_i &= -\, A_i \cdot R_i e_3, && (36)\\ M_i &= -\, k_R e_{R_i} - k_\Omega e_{\Omega_i} \\ &\quad + (R_i^T R_{c_i}\Omega_{c_i})^\wedge J_i R_i^T R_{c_i}\Omega_{c_i} + J_i R_i^T R_{c_i}\dot{\Omega}_{c_i}, && (37)\end{aligned}$$

where k_R and k_Ω are positive constants.

Stability of the corresponding controlled systems for the full dynamic model can be studied by showing the the error due to the discrepancy between the desired direction b_{3_i} and the actual direction R_ie_3. This stability is shown via a Lyapunov analysis.



(a) Payload position (x_0:blue, x_{0_d}:red)

(b) Payload velocity (v_0:blue, v_{0_d}:red)

(c) Payload angular velocity Ω_0

(d) Quadrotors angular velocity errors e_{Ω_i}

(e) Payload attitude error Ψ_0

(f) Quadrotors attitude errors Ψ_i

(g) Quadrotors total thrust inputs f_i

(h) Direction error e_q, and angular velocity error e_ω for the links

Fig. 2. Stabilization of a rigid-body connected to multiple quadrotors

Proposition 3: Consider the full dynamic model defined by (10), (11), (12), (13). For the command x_{0_d} and the desired direction of the first body-fixed axis b_{1_i}, control inputs for quadrotors are designed as (36) and (37). Then, the equilibrium of zero tracking errors for e_{x_0}, $\dot{e}_{x_0}$, e_{R_0}, e_{Ω_0}, $e_{q_{ij}}$, $e_{\omega_{ij}}$, e_{R_i}, e_{Ω_i}, is exponentially stable.

Proof: See Appendix C ■

V. NUMERICAL EXAMPLE

We demonstrate the desirable properties of the proposed control system with numerical examples. Two cases are presented. At the first case, a payload is transported to a desired position from the ground. The second case considers stabilization of a payload with large initial attitude errors.

A. Stabilization of the Rigid Body

Consider four quadrotors ($n = 4$) connected via flexible cables to a rigid body payload. Initial conditions are chosen as

$$x_0(0) = [1.0,\ 4.8,\ 0.0]^T\,\mathrm{m},\ v_0(0) = 0_{3\times1},$$
$$q_{ij}(0) = e_3,\ \omega_{ij}(0) = 0_{3\times1},\ R_i(0) = I_{3\times3},\ \Omega_i(0) = 0_{3\times1}$$
$$R_0(0) = I_{3\times3},\ \Omega_0 = 0_{3\times1}.$$

The desired position of the payload is chosen as

$$x_{0_d}(t) = [0.44,\ 0.78,\ -0.5]^T\,\mathrm{m}. \tag{38}$$

The mass properties of quadrotors are chosen as

$$m_i = 0.755\,\mathrm{kg},$$
$$J_i = \mathrm{diag}[0.557,\ 0.557,\ 1.05] \times 10^{-2}\mathrm{kgm}^2. \tag{39}$$

The payload is a box with mass $m_0 = 0.5\,\mathrm{kg}$, and its length, width, and height are 0.6, 0.8, and $0.2\,\mathrm{m}$, respectively. Each cable connecting the rigid body to the i-th quadrotor is considered to be $n_i = 5$ rigid links. All the links have the same mass of $m_{ij} = 0.01\,\mathrm{kg}$ and length of $l_{ij} = 0.15\,\mathrm{m}$. Each cable is attached to the following points of the payload

$$\rho_1 = [0.3,\ -0.4,\ -0.1]^T\,\mathrm{m},\ \rho_2 = [0.3,\ 0.4,\ -0.1]^T\,\mathrm{m},$$
$$\rho_3 = [-0.3,\ -0.4,\ -0.1]^T\,\mathrm{m},\ \rho_4 = [-0.3,\ 0.4,\ -0.1]^T\,\mathrm{m}.$$

Numerical simulation results are presented at Figure 2, which shows the position and velocity of the payload, and its tracking errors. We have also presented the link direction error defined as

$$e_q = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \|q_{ij} - e_3\|.$$

B. Payload Stabilization with Large Initial Attitude Errors

In the second case, we consider large initial errors for the attitude of the payload and quadrotors. Initially, the rigid body is tilted in its b_1 axis by 30 degrees, and the initial direction of the links are chosen such that two cables are curved along the horizontal direction. The initial conditions are given by

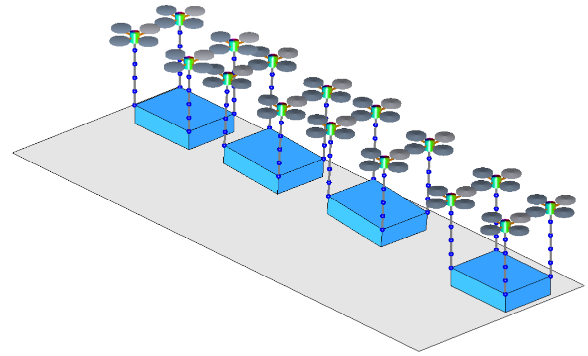
$$x_0(0) = [2.4,\ 0.8,\ -1.0]^T,\ v_0(0) = 0_{3\times1},$$
$$\omega_{ij}(0) = 0_{3\times1},\ \Omega_i(0) = 0_{3\times1}$$
$$R_0(0) = R_x(30^\circ),\ \Omega_0 = 0_{3\times1},$$

where $R_x(30^\circ)$ denotes the rotation about the first axis by 30°. The initial attitude of quadrotors are chosen as

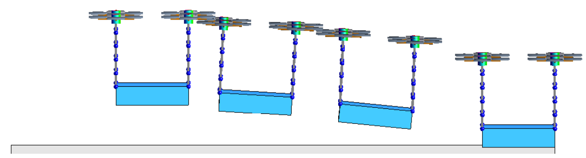
$$R_1(0) = R_y(-35^\circ),\ R_2(0) = I_{3\times3},$$
$$R_3(0) = R_y(-35^\circ),\ R_4(0) = I_{3\times3}.$$

The properties of quadrotors and cables are identical to the previous case. The payload mass is $m = 1.0\,\mathrm{kg}$, and its length, width, and height are 1.0, 1.2, and $0.2\,\mathrm{m}$, respectively.

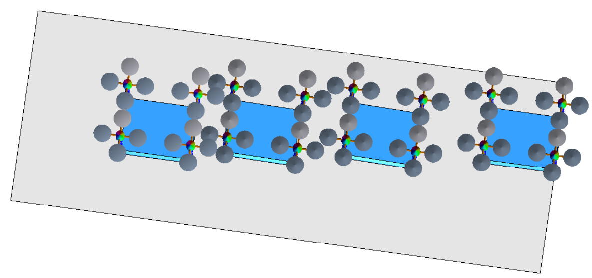
Figure 4 illustrates the tracking errors, and the total thrust of each quadrotor. Snapshots of the controlled maneuvers is also illustrated at Figure 5. It is shown that the proposed controller is able to stabilize the payload and cables at their desired configuration even from the large initial attitude errors.



(a) 3D perspective

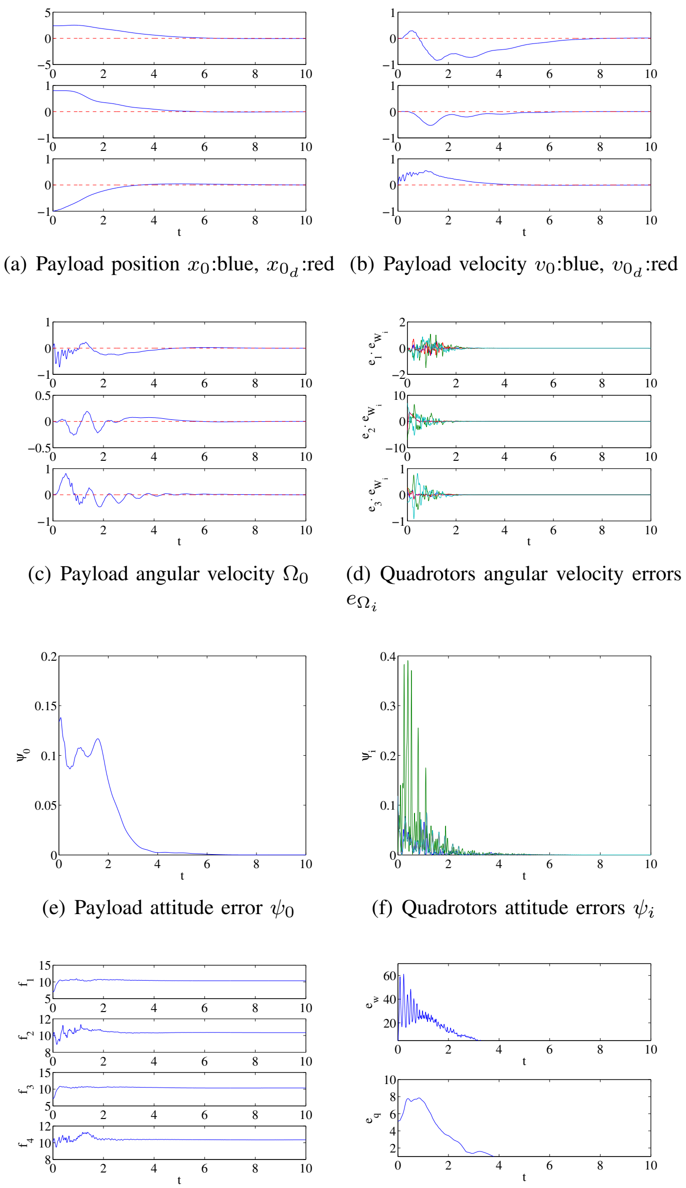


(b) Side view



(c) Top view

Fig. 3. Snapshots of controlled maneuver



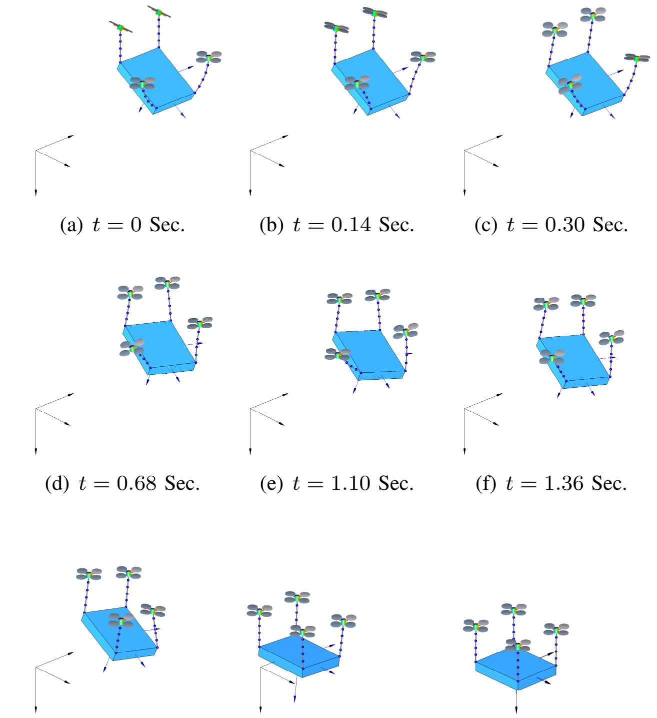
(a) Payload position x_0:blue, x_{0_d}:red (b) Payload velocity v_0:blue, v_{0_d}:red

(c) Payload angular velocity Ω_0 (d) Quadrotors angular velocity errors e_{Ω_i}

(e) Payload attitude error ψ_0 (f) Quadrotors attitude errors ψ_i

(g) Quadrotors total thrust inputs f_i (h) Direction error e_q, and angular velocity error e_ω for the links

Fig. 4. Stabilization of a payload with multiple quadrotors connected with flexible cables.



(a) $t = 0$ Sec. (b) $t = 0.14$ Sec. (c) $t = 0.30$ Sec.

(d) $t = 0.68$ Sec. (e) $t = 1.10$ Sec. (f) $t = 1.36$ Sec.

(g) $t = 1.98$ Sec. (h) $t = 3.48$ Sec. (i) $t = 10$ Sec.

Fig. 5. Snapshots of the controlled maneuver. A short animation is also available at http://youtu.be/Mp4Riw6xBl4

Appendix

A. Proof for Proposition 1

1) Kinetic Energy: The kinetic energy of the whole system is composed of the kinetic energy of quadrotors, cables and the rigid body, as

$$T =\frac{1}{2}m_0\|\dot{x}_0\|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\frac{1}{2}m_{ij}\|\dot{x}_{ij}\|^2 + \frac{1}{2}\sum_{i=1}^{n} m_i\|\dot{x}_i\|^2 + \frac{1}{2}\sum_{i=1}^{n}\Omega_i\cdot J_i\Omega_i + \frac{1}{2}\Omega_0\cdot J_0\Omega_0. \tag{40}$$

Substituting the derivatives of (4) and (5) into the above expression we have

$$T =\frac{1}{2}m_0\|\dot{x}_0\|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\frac{1}{2}m_{ij}\|\dot{x}_0 + \dot{R}_0\rho_i - \sum_{a=j+1}^{n_i} l_{ia}\dot{q}_{ia}\|^2 + \frac{1}{2}\sum_{i=1}^{n} m_i\|\dot{x}_0 + \dot{R}_0\rho_i - \sum_{a=1}^{n_i} l_{ia}\dot{q}_{ia}\|^2 + \frac{1}{2}\sum_{i=1}^{n}\Omega_i\cdot J_i\Omega_i + \frac{1}{2}\Omega_0\cdot J_0\Omega_0. \tag{41}$$

We expand the above expression as follow

$$T = \frac{1}{2}(m_0\|\dot{x}_0\|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n_i} m_{ij}\|\dot{x}_0\|^2 + \sum_{i=1}^{n} m_i\|\dot{x}_0\|^2)$$
$$+ \frac{1}{2}\sum_{i=1}^{n}(\sum_{j=1}^{n_i} m_{ij}\|\dot{R}_0\rho_i\|^2 + m_i\|\dot{R}_0\rho_i\|^2)$$
$$+ \sum_{i=1}^{n}(\sum_{j=1}^{n_i} m_{ij}\dot{x}_0\cdot\dot{R}_0\rho_i + m_i\dot{x}_0\cdot\dot{R}_0\rho_i)$$
$$+ \frac{1}{2}\sum_{i=1}^{n}(\sum_{j=1}^{n_i} m_{ij}\|\sum_{a=j+1}^{n_i} l_{ia}\dot{q}_{ia}\|^2 + m_i\|\sum_{a=1}^{n_i} l_{ia}\dot{q}_{ia}\|^2)$$
$$- \sum_{i=1}^{n}(\sum_{j=1}^{n_i} m_{ij}\dot{x}_0\cdot\sum_{a=j+1}^{n_i} l_{ia}\dot{q}_{ia} + \dot{x}_0\cdot\sum_{a=1}^{n_i} l_{ia}\dot{q}_{ia})$$
$$- \sum_{i=1}^{n}(\sum_{j=1}^{n_i} m_{ij}\dot{R}_0\rho_i\cdot\sum_{a=j+1}^{n_i} l_{ia}\dot{q}_{ia} + m_i\dot{R}_0\rho_i\cdot\sum_{a=1}^{n_i} l_{ia}\dot{q}_{ia})$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\Omega_i\cdot J_i\Omega_i + \frac{1}{2}\Omega_0\cdot J_0\Omega_0, \tag{42}$$

and substituting (14), (15), it is rewritten as

$$T = \frac{1}{2}M_T\|\dot{x}_0\|^2 + \frac{1}{2}\sum_{i=1}^{n} M_{iT}\|\dot{R}_0\rho_i\|^2 + \sum_{i=1}^{n}(M_{iT}\dot{x}_0\cdot\dot{R}_0\rho_i)$$
$$+ \sum_{i=1}^{n}\sum_{j,k=1}^{n_i} M_{0ij}l_{ik}\dot{q}_{ij}\cdot\dot{q}_{ik} - \sum_{i=1}^{n}(\dot{x}_0\cdot\sum_{j=1}^{n_i} M_{0ij}l_{ij}\dot{q}_{ij})$$
$$- \sum_{i=1}^{n}(\dot{R}_0\rho_i\cdot\sum_{j=1}^{n_i} M_{0ij}l_{ij}\dot{q}_{ij})$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\Omega_i\cdot J_i\Omega_i + \frac{1}{2}\Omega_0\cdot J_0\Omega_0. \tag{43}$$

2) Potential Energy: We can derive the potential energy expression by considering the gravitational forces on each part of system as given

$$V = -m_0 g e_3\cdot x_0 - \sum_{i=1}^{n} m_i g e_3\cdot x_i - \sum_{i=1}^{n}\sum_{j=1}^{n_i} m_{ij} g e_3\cdot x_{ij}. \tag{44}$$

Using (4) and (5), we obtain

$$V = -m_0 g e_3\cdot x_0 - \sum_{i=1}^{n} m_i g e_3\cdot(x_0 + R_0\rho_i - \sum_{a=1}^{n_i} l_{ia}q_{ia})$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{n_i} m_{ij} g e_3\cdot(x_0 + R_0\rho_i - \sum_{a=j+1}^{n_i} l_{ia}q_{ia}), \tag{45}$$

and utilizing (15), we can simplify the potential energy as

$$V = -M_T g e_3\cdot x_0 - \sum_{i=1}^{n} M_{iT} g e_3\cdot R_0\rho_i + \sum_{i=1}^{n}\sum_{j=1}^{n_i} M_{0ij}l_{ij}q_{ij}\cdot e_3. \tag{46}$$

3) Derivatives of Lagrangian: We develop the equation of motion for the Lagrangian $L = T - V$. The derivatives of the Lagrangian are given by

$$D_{\dot{x}_0}L = M_T\dot{x}_0 + \sum_{i=1}^{n} M_{iT}\dot{R}_0\rho_i - \sum_{i=1}^{n}\sum_{j=1}^{n_i} M_{0ij}l_{ij}\dot{q}_{ij}, \tag{47}$$

$$D_{x_0}L = M_T g e_3, \tag{48}$$

$$D_{\dot{q}_{ij}}L = \sum_{i=1}^{n}\sum_{j=1}^{n_i} M_{0ij}l_{ik}\dot{q}_{ik} - \sum_{i=1}^{n} M_{0ij}l_{ij}(\dot{x}_0 + \dot{R}_0\rho_i), \tag{49}$$

$$D_{q_{ij}}L = -\sum_{i=1}^{n} M_{0ij}l_{ij}e_3, \tag{50}$$

where $D_{\dot{x}_0}$ denote the derivative with respect to $\dot{x}_0$, and other derivatives are defined similarly. We also have

$$D_{\Omega_0}L = J_0\Omega_0 + \sum_{i=1}^{n} M_{iT}\hat{\rho}_i R_0^T\dot{x}_0,$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{n_i} M_{0ij}l_{ij}\hat{\rho}_i R_0^T\dot{q}_{ij} - \sum_{i=1}^{n} M_{iT}\hat{\rho}_i^2\Omega_0, \tag{51}$$

$$D_{\Omega_0}L = \bar{J}_0\Omega_0 + \sum_{i=1}^{n}\hat{\rho}_i R_0^T(M_{iT}\dot{x}_0 - \sum_{j=1}^{n_i} M_{0ij}l_{ij}\dot{q}_{ij}), \tag{52}$$

$$D_{\Omega_i}L = \sum_{i=1}^{n} J_i\Omega_i, \tag{53}$$

where $\bar{J}_0$ is defined as

$$\bar{J}_0 = J_0 - \sum_{i=1}^{n} M_{iT}\hat{\rho}_i^2. \tag{54}$$

The derivation of the Lagrangian with respect to R_0 is given by

$$D_{R_0}L\cdot\delta R_0 = \sum_{i=1}^{n} M_{iT}R_0\hat{\eta}_0\hat{\Omega}_0\rho_i\cdot\dot{x}_0$$
$$- \sum_{i=1}^{n} R_0\hat{\eta}_0\hat{\Omega}_0\rho_i\cdot\sum_{j=1}^{n_i} M_{0ij}l_{ij}\dot{q}_{ij}$$
$$+ \sum_{i=1}^{n} M_{iT} g e_3\cdot R_0\hat{\eta}_0\rho_i, \tag{55}$$

which can be rewritten as

$$D_{R_0}L\cdot\delta R_0 = d_{R_0}\cdot\eta_0, \tag{56}$$

where

$$d_{R_0} = \sum_{i=1}^{n}(((\widehat{\hat{\Omega}_0\rho_i}R_0^T(M_{iT}\dot{x}_0) - \sum_{j=1}^{n_i} M_{0ij}l_{ij}\dot{q}_{ij})$$
$$+ M_{iT}g\hat{\rho}_i R_0^T e_3)). \tag{57}$$

4) Lagrange-d'Alembert Principle: Consider $\mathfrak{G} = \int_{t_0}^{t_f} L$ be the action integral. Using the equations derived in previous section, the infinitesimal variation of the action integral can be written as

$$\begin{aligned}\delta\mathfrak{G} = \int_{t_0}^{t_f} & D_{\dot{x}_0}L \cdot \delta\dot{x}_0 + D_{x_0} \cdot \delta x_0 \\ & + D_{\Omega_0}L(\dot{\eta}_0 + \Omega_0 \times \eta_0) + d_{R_0}L \cdot \eta_0 \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n_i} D_{\dot{q}_{ij}}L(\dot{\xi}_{ij} \times q_{ij} + \xi_{ij} \times \dot{q}_{ij}) \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n_i} D_{q_{ij}}L \cdot (\xi_{ij} \times q_{ij}) \\ & + \sum_{i=1}^{n} D_{\Omega_i}L \cdot (\dot{\eta}_i + \Omega_i \times \eta_i). \end{aligned} \tag{58}$$

The total thrust at the i-th quadrotor with respect to the inertial frame is denoted by $u_i = -f_i R_i e_3 \in \mathbb{R}^3$ and the total moment at the i-th quadrotor is defined as $M_i \in \mathbb{R}^3$. The corresponding virtual work is given by

$$\begin{aligned}\delta W = \int_{t_0}^{t_f} & \sum_{i=1}^{n} u_i \cdot \{\delta x_0 + R_0\hat{\eta}_0\rho_i - \sum_{j=1}^{n_i} l_{ij}\dot{\xi}_{ij} \times q_{ij}\} \\ & + M_i \cdot \eta_i \, dt. \end{aligned} \tag{59}$$

According to Lagrange-d Alembert principle, we have $\delta\mathfrak{G} = -\delta W$ for any variation of trajectories with fixes end points. By using integration by parts and rearranging, we obtain the following Euler-Lagrange equations

$$\frac{d}{dt}D_{\dot{x}_i}L - D_{x_0}L = \sum_{i=1}^{n} u_i, \tag{60}$$

$$\frac{d}{dt}D_{\Omega_0} + \Omega_0 \times D_{\Omega_0} - d_{R_0} = \sum_{i=1}^{n} \hat{\rho}_i R_0^T u_i, \tag{61}$$

$$\hat{q}_{ij}\frac{d}{dt}D_{\dot{q}_{ij}}L - \hat{q}_{ij}D_{q_i}L = -l_{ij}\hat{q}_{ij}u_i, \tag{62}$$

$$\frac{d}{dt}D_{\Omega_i}L + \Omega_i \times D_{\Omega_i}L = M_i. \tag{63}$$

Substituting the derivatives of Lagrangians into the above expression and rearranging, the equations of motion are given by (10), (11), (12), (13).

B. Proof for Proposition 2

The variations of x and q are given by (24) and (25). From the kinematics equation $\dot{q}_{ij} = \omega_{ij} \times q_{ij}$ and

$$\delta\dot{q}_{ij} = \dot{\xi}_{ij} \times e_3 = \delta\omega_{ij} \times e_3 + 0 \times (\xi_{ij} \times e_3) = \delta\omega_{ij} \times e_3.$$

Since both sides of the above equation is perpendicular to e_3, this is equivalent to $e_3 \times (\dot{\xi}_{ij} \times e_3) = e_3 \times (\delta\omega_{ij} \times e_3)$, which yields

$$\dot{\xi}_{ij} - (e_3 \cdot \dot{\xi}_{ij})e_3 = \delta\omega_{ij} - (e_3 \cdot \delta\omega_{ij})e_3.$$

Since $\xi_{ij} \cdot e_3 = 0$, we have $\dot{\xi}_{ij} \cdot e_3 = 0$. As $e_3 \cdot \delta\omega_{ij} = 0$ from the constraint, we obtain the linearized equation for the kinematics equation of the link

$$\dot{\xi}_{ij} = \delta\omega_{ij}. \tag{64}$$

The infinitesimal variation of $R_0 \in \mathsf{SO(3)}$ in terms of the exponential map

$$\delta R_0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} R_0 \exp(\epsilon\hat{\eta}_0) = R_0\hat{\eta}_0, \tag{65}$$

for $\eta_0 \in \mathbb{R}^3$. Substituting these into (10), (11), and (12), and ignoring the higher order terms, we obtain the following sets of linearized equations of motion

$$\begin{aligned} & M_T\delta\ddot{x}_0 - \sum_{i=1}^{n} M_{iT}\hat{\rho}_i\delta\dot{\Omega}_0 \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n_i} M_{0ij}l_{ij}\hat{e}_3C(C^T\ddot{\xi}_{ij}) = \sum_{i=1}^{n}\delta u_i \end{aligned} \tag{66}$$

$$\begin{aligned} & \sum_{i=1}^{n} M_{iT}\hat{\rho}_i\delta\ddot{x}_0 + \bar{J}_0\delta\dot{\Omega}_0 + \sum_{i=1}^{n}\sum_{j=1}^{n_i} M_{0ij}l_{ij}\hat{\rho}_i\hat{e}_3C(C^T\ddot{\xi}_{ij}) \\ & + \sum_{i=1}^{n}\frac{m_0}{n}g\hat{\rho}_i\hat{e}_3\eta_0 = \sum_{i=1}^{n}\hat{\rho}_i\delta u_i \end{aligned} \tag{67}$$

$$\begin{aligned} & -M_{0ij}C^T\hat{e}_3\delta\ddot{x}_0 + M_{0ij}C^T\hat{e}_3\hat{\rho}_i\delta\dot{\Omega}_0 + \sum_{k=1}^{n_i} M_{0ij}l_{ik}I_2(C^T\ddot{\xi}_{ij}) \\ & = -C^T\hat{e}_3\delta u_i + (-M_{iT} - \frac{m_0}{n} + M_{0ij})ge_3I_2(C^T\xi_{ij}) \end{aligned} \tag{68}$$

$$\dot{\eta}_i = \delta\Omega_i, \ \dot{\eta}_0 = \delta\Omega_0, \ J_i\delta\Omega_i = \delta M_i, \tag{69}$$

which can be written in a matrix form as presented in (26). See [18] for detaied derivations for a similar dynamic system. We used $C^T\hat{e}_3^2C = -I_2$ to simplify these derivations.

C. Proof for Proposition 3

We first show stability of the rotational dynamics of each quadrotor, and later it is combined with the stability analysis for the remaining parts.

1) Attitude Error Dynamics: Here, attitude error dynamics for e_{R_i}, e_{Ω_i} are derived and we find conditions on control parameters to guarantee the boundedness of attitude tracking errors. The time-derivative of $J_ie_{\Omega_i}$ can be written as

$$J_i\dot{e}_{\Omega_i} = \{J_ie_{\Omega_i} + d_i\}^{\wedge}e_{\Omega_i} - k_Re_{R_i} - k_\Omega e_{\Omega_i}, \tag{70}$$

where $d_i = (2J_i - \mathrm{tr}[J_i]I)R_i^TR_{i_d}\Omega_{i_d} \in \mathbb{R}^3$ [14]. The important property is that the first term of the right hand side is normal to e_{Ω_i}, and it simplifies the subsequent Lyapunov analysis.

2) Stability for Attitude Dynamics: Define a configuration error function on $\mathsf{SO(3)}$ as follows

$$\Psi_i = \frac{1}{2}\mathrm{tr}[I - R_{i_c}^TR_i]. \tag{71}$$

We introduce the following Lyapunov function

$$\mathcal{V}_2 = \sum_{i=1}^{n}\mathcal{V}_{2_i}, \tag{72}$$

where

$$\mathcal{V}_{2_i} = \frac{1}{2}e_{\Omega_i} \cdot J_i\dot{e}_{\Omega_i} + k_R\Psi_i(R_i, R_{d_i}) + c_{2_i}e_{R_i} \cdot e_{\Omega_i}. \tag{73}$$

Consider a domain D_2 given by

$$D_2 = \{(R_i, \Omega_i) \in \mathsf{SO(3)} \times \mathbb{R}^3 \mid \Psi_i(R_i, R_{d_i}) < \psi_{2_i} < 2\}. \tag{74}$$

In this domain we can show that $\mathcal{V}_2$ is bounded as follows [14]

$$z_{2_i}^T M_{i_{21}} z_{2_i} \leq \mathcal{V}_{2_i} \leq z_{2_i}^T M_{i_{22}} z_{2_i}, \tag{75}$$

and $z_{2_i} = [\|e_{R_i}\|, \|e_{\Omega_i}\|]^T \in \mathbb{R}^2$. Matrices $M_{i_{21}}$, $M_{i_{22}}$ are given by

$$\begin{aligned} M_{i_{21}} =& \frac{1}{2}\begin{bmatrix} k_R & -c_{2_i}\lambda_{M_i} \\ -c_{2_i}\lambda_{M_i} & \lambda_{m_i} \end{bmatrix}, \\ M_{i_{22}} =& \frac{1}{2}\begin{bmatrix} \frac{2k_R}{2-\psi_{2_i}} & c_{2_i}\lambda_{M_i} \\ c_{2_i}\lambda_{M_i} & \lambda_{M_i} \end{bmatrix}, \end{aligned}$$

The time derivative of $\mathcal{V}_2$ along the solution of the controlled system is given by

$$\dot{\mathcal{V}}_2 = \sum_{i=1}^{n} -k_\Omega \|e_{\Omega_i}\|^2 + c_{2_i}\dot{e}_{R_i} \cdot J_i e_{\Omega_i} + c_{2_i} e_{R_i} \cdot J_i \dot{e}_{\Omega_i}.$$

Substituting (70), the above equation becomes

$$\begin{aligned} \dot{\mathcal{V}}_2 =& \sum_{i=1}^{n} -k_\Omega \|e_{\Omega_i}\|^2 + c_{2_i}\dot{e}_{R_i} \cdot J_i e_{\Omega_i} - c_{2_i} k_R \|e_{R_i}\|^2 \\ &+ c_{2_i} e_{R_i} \cdot ((J_i e_{\Omega_i} + d_i)^\wedge e_{\Omega_i} - k_\Omega e_{\Omega_i}). \end{aligned}$$

We have $\|e_{R_i}\| \leq 1$, $\|\dot{e}_{R_i}\| \leq \|e_{\Omega_i}\|$ [19], and choose a constant B_{2_i} such that $\|d_i\| \leq B_{i_2}$. Then we have

$$\dot{\mathcal{V}}_2 \leq -\sum_{i=1}^{n} z_{2_i}^T W_{2_i} z_{2_i}, \tag{76}$$

where the matrix $W_{2_i} \in \mathbb{R}^{2\times 2}$ is given by

$$W_{2_i} = \begin{bmatrix} c_{2_i} k_R & -\frac{c_{2_i}}{2}(k_\Omega + B_{2_i}) \\ -\frac{c_{2_i}}{2}(k_\Omega + B_{2_i}) & k_\Omega - 2c_{2_i}\lambda_{M_i} \end{bmatrix}.$$

The matrix W_{2_i} is a positive definite matrix if

$$c_{2_i} < \min\{\frac{\sqrt{k_R \lambda_{m_i}}}{\lambda_{M_i}}, \frac{4k_\Omega}{8k_R\lambda_{M_i} + (k_\Omega + B_{i_2})^2}\}. \tag{77}$$

This implies that

$$\dot{\mathcal{V}}_2 \leq -\sum_{i=1}^{n} \lambda_m(W_{2_i}) \|z_{2_i}\|^2, \tag{78}$$

which shows stability of the attitude dynamics of quadrotors.

3) Error Dynamics of the Payload and Links: We derive the tracking error dynamics and a Lyapunov function for the translational dynamics of a payload and the dynamics of links. Later it is combined with the stability analyses of the rotational dynamics. From (10), (26), (31), and (36), the equation of motion for the controlled dynamic model is given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\mathbf{x} = \mathbf{B}(u - u^*) + \mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}}), \tag{79}$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \; u^* = \begin{bmatrix} -(M_{1T} + \frac{m_0}{n})ge_3 \\ -(M_{2T} + \frac{m_0}{n})ge_3 \\ \vdots \\ -(M_{nT} + \frac{m_0}{n})ge_3 \end{bmatrix}, \tag{80}$$

and $\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}})$ corresponds to the higher order terms. As $u_i = -f_i R_i e_3$ for the full dynamic model, $\delta u = u - u^*$ is given by

$$\delta u = \begin{bmatrix} -f_1 R_1 e_3 + (M_{1T} + \frac{m_0}{n})ge_3 \\ -f_2 R_2 e_3 + (M_{2T} + \frac{m_0}{n})ge_3 \\ \vdots \\ -f_n R_n e_3 + (M_{nT} + \frac{m_0}{n})ge_3 \end{bmatrix}. \tag{81}$$

The subsequent analyses are developed in the domain D_1

$$\begin{aligned} D_1 = \{(\mathbf{x}, \dot{\mathbf{x}}, R_i, e_{\Omega_i}) \in \mathbb{R}^{D_\mathbf{x}} \times \mathbb{R}^{D_\mathbf{x}} \times \mathsf{SO(3)} \times \mathbb{R}^3 \mid \\ \Psi_i < \psi_{1_i} < 1\}. \end{aligned} \tag{82}$$

In the domain D_1, we can show that

$$\frac{1}{2}\|e_{R_i}\|^2 \leq \Psi_i(R_i, R_{c_i}) \leq \frac{1}{2-\psi_{1_i}}\|e_{R_i}\|^2. \tag{83}$$

Consider the quantity $e_3^T R_{c_i}^T R_i e_3$, which represents the cosine of the angle between $b_{3_i} = R_i e_3$ and $b_{3_{c_i}} = R_{c_i} e_3$. Since $1 - \Psi_i(R_i, R_{c_i})$ represents the cosine of the eigen-axis rotation angle between R_{c_i} and R_i, we have $e_3^T R_{c_i}^T R e_3 \geq 1 - \Psi_i(R_i, R_{c_i}) > 0$ in D_1. Therefore, the quantity $\frac{1}{e_3^T R_{c_i}^T R_i e_3}$ is well-defined. We add and subtract $\frac{f_i}{e_3^T R_{c_i}^T R_i e_3} R_{c_i} e_3$ to the right hand side of (81) to obtain

$$\delta u = \begin{bmatrix} \frac{-f_1}{e_3^T R_{c_1}^T R_1 e_3} R_{c_1} e_3 - X_1 + (M_{1T} + \frac{m_0}{n})ge_3 \\ \frac{-f_2}{e_3^T R_{c_2}^T R_2 e_3} R_{c_2} e_3 - X_2 + (M_{2T} + \frac{m_0}{n})ge_3 \\ \vdots \\ \frac{-f_n}{e_3^T R_{c_n}^T R_n e_3} R_{c_n} e_3 - X_n + (M_{nT} + \frac{m_0}{n})ge_3 \end{bmatrix}. \tag{84}$$

where $X_i \in \mathbb{R}^3$ is defined by

$$X_i = \frac{f_i}{e_3^T R_{c_i}^T R_i e_3}((e_3^T R_{c_i}^T R_i e_3) R_i e_3 - R_{c_i} e_3). \tag{85}$$

Using

$$-\frac{f_i}{e_3^T R_{c_i}^T R_i e_3} R_{c_i} e_3 = -\frac{(\|A_i\| R_{c_i} e_3) \cdot R_i e_3}{e_3^T R_{c_i}^T R_i e_3} \cdot -\frac{A_i}{\|A_i\|} = A_i, \tag{86}$$

the equation (84) becomes

$$\delta u = \begin{bmatrix} A_1 - X_1 + (M_{1T} + \frac{m_0}{n})ge_3 \\ A_2 - X_2 + (M_{2T} + \frac{m_0}{n})ge_3 \\ \vdots \\ A_n - X_n + (M_{nT} + \frac{m_0}{n})ge_3 \end{bmatrix}. \tag{87}$$

Substituting (31) into the above equation, (79) becomes

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\mathbf{x} = \mathbf{B}(-K_\mathbf{x}\mathbf{x} - K_{\dot{\mathbf{x}}}\dot{\mathbf{x}} - X) + \mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}}), \tag{88}$$

where $X = [X_1^T,\ X_2^T,\ \cdots,\ X_n^T]^T \in \mathbb{R}^{3n}$. It is rewritten in the following matrix form

$$\dot{z}_1 = \mathbb{A}z_1 + \mathbb{B}(\mathbf{B}X + \mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}})), \tag{89}$$

where $z_1 = [\mathbf{x}, \dot{\mathbf{x}}]^T \in \mathbb{R}^{2D_\mathbf{x}}$ and

$$\mathbb{A} = \begin{bmatrix} 0 & I \\ -\mathbf{M}^{-1}(\mathbf{G} + \mathbf{B}K_\mathbf{x}) & -\mathbf{M}^{-1}\mathbf{B}K_{\dot{\mathbf{x}}} \end{bmatrix}, \mathbb{B} = \begin{bmatrix} 0 \\ \mathbf{M}^{-1} \end{bmatrix}. \tag{90}$$

We can also choose $K_\mathbf{x}$ and $K_{\dot{\mathbf{x}}}$ such that $\mathbb{A} \in \mathbb{R}^{2D_x \times 2D_x}$ is Hurwitz. Then for any positive definite matrix $Q \in \mathbb{R}^{2D_\mathbf{x} \times 2D_\mathbf{x}}$, there exist a positive definite and symmetric matrix $P \in \mathbb{R}^{2D_\mathbf{x} \times 2D_\mathbf{x}}$ such that $\mathbb{A}^T P + P\mathbb{A} = -Q$ according to [20, Thm 3.6].

4) Lyapunov Candidate for Simplified Dynamics: From the linearized control system developed at section 3, we use matrix P to introduce the following Lyapunov candidate for translational dynamics

$$\mathcal{V}_1 = z_1^T P z_1. \tag{91}$$

The time derivative of the Lyapunov function using the Leibniz integral rule is given by

$$\dot{\mathcal{V}}_1 = \dot{z}_1^T P z_1 + z_1^T P \dot{z}_1. \tag{92}$$

Substituting (89) into above expression

$$\dot{\mathcal{V}}_1 = z_1^T(\mathbb{A}^T P + P\mathbb{A})z_1 + 2z_1^T P\mathbb{B}(\mathbf{B}X + \mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}})). \tag{93}$$

Let $c_3 = 2\|P\mathbb{B}\mathbf{B}\|_2 \in \mathbb{R}$ and using $\mathbb{A}^T P + P\mathbb{A} = -Q$, we have

$$\dot{\mathcal{V}}_1 \le -z_1^T Q z_1 + c_3\|z_1\|\|X\| + 2z_1^T P\mathbb{B}\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}}). \tag{94}$$

The second term on the right hand side of the above equation corresponds to the effects of the attitude tracking error on the translational dynamics. We find a bound of X_i, defined at (85), to show stability of the coupled translational dynamics and rotational dynamics in the subsequent Lyapunov analysis. Since

$$f_i = \|A_i\|(e_3^T R_{c_i}^T R_i e_3), \tag{95}$$

we have

$$\|X_i\| \le \|A_i\|\|(e_3^T R_{c_i}^T R_i e_3)R_i e_3 - R_{c_i} e_3\|. \tag{96}$$

The last term $\|(e_3^T R_{c_i}^T R_i e_3)R_i e_3 - R_{c_i} e_3\|$ represents the sine of the angle between $b_{3_i} = R_i e_3$ and $b_{3_{c_i}} = R_{c_i} e_3$, since $(b_{3_{c_i}} \cdot b_{3_i})b_{3_i} - b_{3_{c_i}} = b_{3_i} \times (b_{3_i} \times b_{3_{c_i}})$. The magnitude of the attitude error vector, $\|e_{R_i}\|$ represents the sine of the eigen-axis rotation angle between R_{c_i} and R_i. Therefore, $\|(e_3^T R_{c_i}^T R_i e_3)R_i e_3 - R_{c_i} e_3\| \le \|e_{R_i}\|$ in D_1. It follows that

$$\begin{aligned} \|(e_3^T R_{c_i}^T R_i e_3)R_i e_3 - R_{c_i} e_3\| &\le \|e_{R_i}\| = \sqrt{\Psi_i(2 - \Psi_i)} \\ &\le \{\sqrt{\psi_{1_i}(2 - \psi_{1_i})} \triangleq \alpha_i\} < 1, \end{aligned} \tag{97}$$

therefore

$$\begin{aligned} \|X_i\| &\le \|A_i\|\|e_{R_i}\| \\ &\le \|A_i\|\alpha_i. \end{aligned} \tag{98}$$

We find an upper boundary for

$$A_i = -K_\mathbf{x}\mathbf{x} - K_{\dot{\mathbf{x}}}\dot{\mathbf{x}} + u_i^*, \tag{99}$$

by defining

$$\|u_i^*\| \le B_{1_i}, \tag{100}$$

for a given positive constant B_1. defining $K_{max} \in \mathbb{R}$

$$K_{\max} = \max\{\|K_\mathbf{x}\|, \|K_{\dot{\mathbf{x}}}\|\},$$

and then the upper bound of A is given by

$$\begin{aligned} \|A_i\| &\le K_{\max}(\|\mathbf{x}\| + \|\dot{\mathbf{x}}\|) + B_1 \\ &\le 2K_{\max}\|z_1\| + B_1. \end{aligned} \tag{101}$$

Using the above steps we can show that

$$\begin{aligned} \|X\| &\le \sum_{i=1}^n ((2K_{\max}\|z_1\| + B_1)\|e_{R_i}\|) \\ &\le (2K_{\max}\|z_1\| + B_1)\alpha, \end{aligned} \tag{102}$$

where $\alpha = \sum_{i=1}^n \alpha_i$. Then, we can simplify (94) as

$$\begin{aligned} \dot{\mathcal{V}}_1 \le &-(\lambda_{\min}(Q) - 2c_3 K_{\max}\alpha)\|z_1\|^2 \\ &+ \sum_{i=1}^n c_3 B_1\|z_1\|\|e_{R_i}\| + 2z_1^T P\mathbb{B}\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}}). \end{aligned} \tag{103}$$

5) Lyapunov Candidate for the Complete System: Let $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ be the Lyapunov function for the complete system. The time derivative of $\mathcal{V}$ is given by

$$\dot{\mathcal{V}} = \dot{\mathcal{V}}_1 + \dot{\mathcal{V}}_2. \tag{104}$$

Substituting (103) and (78) into the above equation

$$\begin{aligned} \dot{\mathcal{V}} \le &-(\lambda_{\min}(Q) - 2c_3 K_{\max}\alpha)\|z_1\|^2 + 2z_1^T P\mathbb{B}\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}}) \\ &+ \sum_{i=1}^n c_3 B_1\|z_1\|\|e_{R_i}\| - \sum_{i=1}^n \lambda_m(W_{2_i})\|z_{2_i}\|^2, \end{aligned} \tag{105}$$

and using $\|e_{R_i}\| \le \|z_{2_i}\|$, it can be written as

$$\begin{aligned} \dot{\mathcal{V}} \le &-(\lambda_{\min}(Q) - 2c_3 K_{\max}\alpha)\|z_1\|^2 + 2z_1^T P\mathbb{B}\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}}) \\ &+ \sum_{i=1}^n c_3 B_1\|z_1\|\|z_{2_i}\| - \sum_{i=1}^n \lambda_m(W_{2_i})\|z_{2_i}\|^2. \end{aligned} \tag{106}$$

The $2z_1^T P\mathbb{B}\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}})$ term in the above equation is indefinite. The function $\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}})$ satisfies

$$\frac{\|\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}})\|}{\|z_1\|} \to 0 \quad \text{as} \quad \|z_1\| \to 0.$$

Then, for any $\gamma > 0$ there exists $r > 0$ such that

$$\|\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}})\| < \gamma\|z_1\| \quad \forall \|z_1\| < r.$$

Therfore

$$2z_1^T P\mathbb{B}\mathfrak{g}(\mathbf{x}, \dot{\mathbf{x}}) \le 2\gamma\|P\|_2\|z_1\|^2. \tag{107}$$

Substituting the above inequality into (106)

$$\begin{aligned} \dot{\mathcal{V}} \le &-(\lambda_{\min}(Q) - 2c_3 K_{\max}\alpha)\|z_1\|^2 + 2\gamma\|P\|_2\|z_1\|^2 \\ &+ \sum_{i=1}^n c_3 B_1\|z_1\|\|z_{2_i}\| - \sum_{i=1}^n \lambda_m(W_{2_i})\|z_{2_i}\|^2, \end{aligned} \tag{108}$$

and rearranging

$$\dot{\mathcal{V}} \leq -\sum_{i=1}^{n}\Big(\frac{\lambda_{\min}(Q)-2c_3K_{\max}\alpha}{n}\|z_1\|^2 - c_3B_1\|z_1\|\|z_{2_i}\| + \lambda_m(W_{2_i})\|z_{2_i}\|^2\Big) + 2\gamma\|P\|_2\|z_1\|^2, \tag{109}$$

we obtain

$$\dot{\mathcal{V}} \leq -\sum_{i=1}^{n}(\mathbf{z}_i^T W_i \mathbf{z}_i) + 2\gamma\|P\|_2\|z_1\|^2, \tag{110}$$

where $\mathbf{z}_i = [\|z_1\|, \|z_{2_i}\|]^T \in \mathbb{R}^2$ and

$$W_i = \begin{bmatrix} \frac{\lambda_{\min}(Q)-2c_3K_{\max}\alpha}{n} & -\frac{c_3B_{1_i}}{2} \\ -\frac{c_3B_{1_i}}{2} & \lambda_m(W_{2_i}) \end{bmatrix}. \tag{111}$$

By using $\|z_1\| \leq \|\mathbf{z}_i\|$, we obtain

$$\dot{\mathcal{V}} \leq -\sum_{i=1}^{n}\Big(\lambda_{\min}(W_i) - \frac{2\gamma\|P\|_2}{n}\Big)\|\mathbf{z}_i\|^2. \tag{112}$$

Choosing $\gamma < n(\lambda_{\min}(W_i))/2\|P\|_2$, and

$$\lambda_m(W_{2_i}) > \frac{n\|\frac{c_3B_{1_i}}{2}\|^2}{\lambda_{\min}(Q)-2c_3K_{\max}\alpha}, \tag{113}$$

ensures that $\dot{\mathcal{V}}$ is negative definite. Then, the zero equilibrium is exponentially stable.

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