Alternative approach to anti-windup synthesis for double integrator systems

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***Abstract*— This paper presents a simple approach for anti-windup synthesis for double integrator systems. The parameters of the anti-windup compensator can be chosen using simple *linear-based* guidelines which, nevertheless, also provide *nonlinear* stability guarantees. The results are constructed on the basis of a Popov-like sufficient condition presented in [1]. The advantage of the method is that design and re-design of the anti-windup compensator is exceptionally simple, requires no optimisation and yet offers the engineer great design transparency.**

I. INTRODUCTION

Anti-windup (AW) compensators are designed to work with existing controllers to prevent performance degradation and maintain stability in systems during periods of saturation. An important feature of an anti-windup compensator is that it only becomes active whenever saturation occurs and the original control loop remains unchanged as long as saturation does not occur. In recent years, the study of anti-windup techniques has grown steadily and this has led to major developments in approaches that provide favourable stability and performance results for systems with input saturation. Examples of relevant papers are [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and recent books on the topic include [14], [15], [16], [17]

Many modern approaches to anti-windup design are formulated and solved using linear matrix inequalities (LMIs) to ensure that the anti-windup compensator bestows some sort of stability and performance guarantees on the system under consideration [18]. However, the use of LMIs may seem excessive in some situations, especially in the design of compensators for relatively simple systems. In addition, the $\mathcal{L}_2$ induced-norm used to measure performance in many LMI approaches is also a rather nebulous quantity and is not always a reliable indication of a nonlinear system's practical performance. Finally, although LMI-based approaches make anti-windup design systematic and tractable, typically one "optimal" solution is returned. This may not necessarily be the only solution yielding a "good" anti-windup compensator, but rather there may exist a family of AW compensators for which the designer can choose anyone of them.

In this paper, we examine saturation in systems containing double integrators within the anti-windup framework presented in [8], [19]. Double integrators describe, or approximately describe, many systems, including Euler-Lagrange systems [20], aircraft systems, and especially the single-axis dynamics of quadrotors which inspired the work presented here. The novelty here is that we provide a *direct* approach to AW design which circumvents the use of LMIs and $\mathcal{L}_2$ -type performance indices. The stability analysis we use originates from the results presented in [1] which provides a Popov-like sufficient condition for global asymptotic stability based upon a Lure-Postnikov Lyapunov function.

It transpires that, for the double integrator AW problems we consider here, the analysis of [1] provides a very large set of stabilising AW compensators. Instead of choosing amongst these compensators using $\mathcal{L}_2$ -type performance measures, we propose using standard linear system time domain performance criteria based on the compensator's natural frequency and damping ratio. This leads to simple, transparent formulae for choosing the AW parameters and there is clear correlation between these and the corresponding time-domain performance. Once stability has been guaranteed, subsequent designs and redesigns of the AW compensator only require the selection of suitable parameters based on the speed and damping criteria sought.

The paper is structured as follows. Section II briefly describes the AW framework considered and some limitations. Section III presents the direct synthesis conditions for double-integrator plants and the tuning guidelines. Examples are used in Section IV to illustrate the approach.

A. Notation

The saturation function is defined as $\mathrm{sat}(.) : \mathbb{R}^m \longmapsto \mathbb{R}^m$ for $u = [u_1, \ldots, u_m]$ and $u_i > 0$, $i \in I[1, m]$ such that

$$\mathrm{sat}(u) = [\mathrm{sat}(u_1), \ldots, \mathrm{sat}(u_m)]'$$
$$\mathrm{sat}(u_i) = \min\{|u_i|, \bar{u}_i\} \times \mathrm{sign}(u_i)$$

The deadzone function $\mathrm{Dz}(.) : \mathbb{R}^m \longmapsto \mathbb{R}^m$ is simply

$$\mathrm{Dz}(u) = [\mathrm{Dz}(u_1), \ldots, \mathrm{Dz}(u_m)]' \quad (1)$$
$$\mathrm{Dz}(u) = u - \mathrm{sat}(u) \quad (2)$$

For brevity, we denote $\tilde{u} = \mathrm{Dz}(u)$. The notation $\mathrm{He}\{A\} = A + A'$. $\mathbf{P}^m$ is the set of $m \times m$ symmetric positive-definite matrices. $\mathbf{N}^m$ is the set of $m \times m$ symmetric non-negative definite matrices and $\mathbf{D}$ is the set of diagonal matrices.

II. ANTI-WINDUP FRAMEWORK

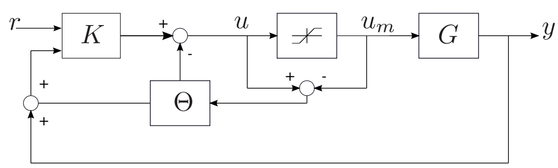

Fig. 1. A full-order anti-windup structure

In this section, the anti-windup design technique in [21] is revisited. Consider the control structure depicted in Figure 1 where $r(t) \in \mathbb{R}^{n_r}$ is the reference, $y \in \mathbb{R}^p$ is the output, $u \in$

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$\mathbb{R}^m$ is the plant input, $G(s)$ is the plant, $K(s)$ is the controller and $\Theta(s)$ is the anti-windup compensator. The state-space realization of the plant $G(s)$ is given as,

$$G(s) \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] \quad A_p \in \mathbb{R}^{n \times n} \tag{3}$$

The approach in [8], [21] interprets the anti-windup design problem as s search for a transfer function matrix $M(s)$ such that the anti-windup compensator $\Theta(s)$ has the structure:

$$\Theta = \begin{bmatrix} M(s) - I \\ G(s)M(s) \end{bmatrix} \tag{4}$$

$M(s)$ is chosen as part of a right coprime factorisation of the plant; $G(s) = N(s)M^{-1}(s)$. If the order of the coprime factorisation is the same as that of the plant, a state space realization of the anti-windup compensator $\Theta(s)$ is

$$\Theta(s) = \begin{bmatrix} M(s) - I \\ N(s) \end{bmatrix} \sim \left[\begin{array}{c|c} A_p + B_pF & B_p \\ \hline F & 0 \\ C_p + D_pF & D_p \end{array}\right] \tag{5}$$

where F is chosen such that $A_p + B_pF$ is Hurwitz. With this formulation, Figure 1 can be redrawn as Figure 2 which makes the analysis of the system with saturation and anti-windup more convenient because it decouples the system into three distinct subsystems. From Figure 2, observe that the mapping $\mathcal{T}_p : u_{lin} \mapsto y_d$ determines the deviation of the nonlinear system from the nominal linear system. Assuming the nominal plant-controller interconnection is asymptotically stable, stability and performance of the saturated system may be assessed by considering the stability of the nonlinear loop represented by the mapping $\mathcal{T}_p$:

$$\mathcal{T}_p \sim \begin{cases} \dot{x} &= (A_p + B_pF)x + B_p\mathrm{Dz}(u_{lin} - u_d) \\ u_d &= Fx \\ y_d &= (C_p + D_pF)x + D_p\mathrm{Dz}(u_{lin} - u_d) \end{cases} \tag{6}$$

If the plant $G(s) \in \mathcal{RH}_\infty$ (A_p is Hurwitz), a matrix F guaranteeing global exponential stability, and finite $\mathcal{L}_2$ gain of the map $\mathcal{T}_p : u_{lin} \mapsto y_d$, always exists. Furthermore, such an F can be computed by solving a simple set of LMI's [21]. However, if the plant contains a double integrator, then $G(s) \notin \mathcal{RH}_\infty$, which makes the LMI's in [21] infeasible. To overcome this, a small adjustment to these LMI's can be made: if there exist matrices $Q \in \mathbf{P}^n$, $U \in \mathbf{DP}^m$ and $L \in \mathbb{R}^{m \times n}$ such that the following LMI is satisfied

$$\mathrm{He}\left\{\begin{bmatrix} A_pQ + B_pL & B_pU & 0 & 0 \\ -\epsilon L & -U & \epsilon I & 0 \\ 0 & 0 & -\frac{\gamma}{2} & 0 \\ C_pQ + D_pL & D_pU & 0 & -\frac{\gamma}{2} \end{bmatrix}\right\} < 0 \tag{7}$$

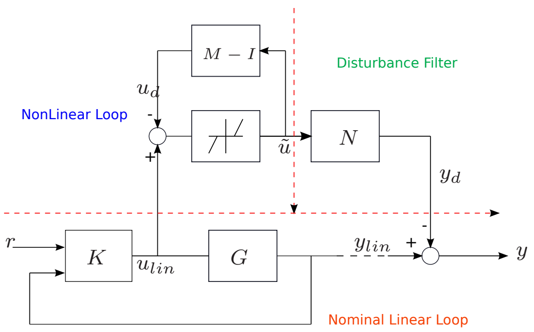


Fig. 2. Equivalent representation of structure

then $F = LQ^{-1}$ can be used to construct the anti-windup compensator (5). In this case, it is assumed that the standard deadzone no longer occupies the Sector$[0, I]$, but is restricted to some narrower sector, Sector$[0, \epsilon I]$ where $0 < \epsilon < 1$, so stability is only guaranteed locally. Note however that, as ϵ approaches one, stability is closer to being administered globally. This approach, or variants thereof, has been successfully used in a number of applications, e.g. [22], [23]. Other approaches for handling systems with imaginary axis eigenvalues can be found, for example, in [24], [25].

In general, these LMI approaches are flexible and able to provide local stability for plants $G(s) \notin \mathcal{RH}_\infty$. However, they are not able to provide *global* stability without further development and the performance it provides is focused on an $\mathcal{L}_2$ measure of this. For plants with simple and/or apparent structures, one would naturally expect a simpler and more transparent approach to be obtained. The next section describes such an approach for double integrator plants.

III. AW Synthesis for Double Integrator Systems

A. Stability Analysis

Consider a saturated linear system described by the following state-space equations

$$\dot{x}(t) = Ax(t) + B\mathrm{sat}(u(t)) \tag{8}$$
$$u(t) = Kx(t) \tag{9}$$

After appropriate similarity transformations, the state-space matrices are assumed to be structured as

$$A = \begin{bmatrix} A_z & 0 \\ 0 & A_s \end{bmatrix} \quad B = \begin{bmatrix} B_z \\ B_s \end{bmatrix} \tag{10}$$

where $A_z \in \mathbb{R}^{n_z \times n_z}$ and has eigenvalues on the imaginary axis, and $A_s \in \mathbb{R}^{n_s \times n_s}$ is Hurwitz. $B_z \in \mathbb{R}^{n_z \times m}$, $B_s \in \mathbb{R}^{n_s \times m}$ and $n = n_z + n_s$.

Sufficient conditions for global stability of the above system were given in [1] using a Popov-like Lyapunov function. The Lyapunov function is novel since it comprises a positive *semi*-definite quadratic term and an additional integral term.

Theorem 1: If there exist matrices $R_z \in \mathbf{N}^{n_z}$, $R_s \in \mathbf{N}^{n_s}$, $R_2 \in \mathbf{DN}^m$ $N \in \mathbf{DN}^m$, $P \in \mathbf{N}^{(n_z+ns)}$ such that

$$R = \begin{bmatrix} R_z & 0 \\ 0 & R_s \end{bmatrix} \tag{11}$$

and the following matrix equations and inequalities are satisfied:

$$0 = A^TP + PA + R \tag{12}$$
$$0 = B^TP + NKA + R_2K \tag{13}$$
$$0 < 2R_2 - (NKB + B^TK^TN) \tag{14}$$
$$0 < P + K^TNK \tag{15}$$

then the origin of the system (8)-(9) is globally asymptotically stable if, either (i) (A, K) is observable or (ii) (A, K) is detectable and (A, R) is observable. Furthermore a Lyapunov function proving global asymptotic stability is given by

$$V(x) = x^TPx + 2\sum_{i=1}^{m} \int_0^{u_i = K_ix} N_i\mathrm{sat}_i(u_i)du_i$$

The proof of this theorem and the process of realizing these conditions follows that in [1]. Theorem 1 can be used either to guarantee global stability of a given saturated controller or to construct a stabilizing controller for the system in (8)-(9).

B. Application to anti-windup design

In this work we are interested in examining stability of the nonlinear loop in Figure 2 when the plant is a double integrator, that is when the state-space matrices are:

$$A_p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_p = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \quad C_p = [1 \quad 0] \quad D_p = 0 \tag{16}$$

where $\beta \neq 0$ is an indefinite scalar. The unforced dynamics of the nonlinear loop ($\mathcal{T}_p$) are given by

$$\dot{x} = (A_p + B_p F)x + B_p \mathrm{Dz}(-u_d) \tag{17}$$
$$u_d = Fx \tag{18}$$

where F is the state-feedback matrix which determines the anti-windup compensator. The following result can be established as a corollary of Theorem 1.

Corollary 1: Assume $F = [F_a \quad F_b]$ is chosen such that $\mathrm{sign}(F_a) = \mathrm{sign}(F_b) = -\mathrm{sign}(\beta)$. Then the origin of the system (17)-(18) is globally asymptotically stable.

Proof: The proof uses the identity (2) and a simple application of Theorem 1; it closely follows Example 4.4 in [1]. First note that the dynamics (17)-(18) can be re-written as

$$\dot{x} = A_p x + B_p \mathrm{sat}(u_d) \tag{19}$$
$$u_d = Fx \tag{20}$$

The system is now in the form of (8)-(9) and, because the system is simply a double integrator $A_z = A_p$, $B_z = B_p$ and $K = F$, with $n_z = 2$, $n_s = 0$ and $m = 1$. Therefore equations (12)-(15) become

$$0 = A_p^T P + P A_p + R \tag{21}$$
$$0 = {B_p}^T P + NFA_p + R_2 F \tag{22}$$
$$0 < 2R_2 - (NFB_p + {B_p}^T F^T N) \tag{23}$$
$$0 < P + F^T NF \tag{24}$$

Theorem 1 allows the choices $R = 0$, $R_2 = 0$ and $N = 1$. In this case, equation (21) becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}' \begin{bmatrix} P_a & P_{b/c} \\ P_{b/c} & P_d \end{bmatrix} + \begin{bmatrix} P_a & P_{b/c} \\ P_{b/c} & P_d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} 0 & 0 \\ P_a & P_{b/c} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ P_a & P_{b/c} \end{bmatrix}' \tag{26}$$

Therefore $P_a = P_{b/c} = 0$. Equation (22) then becomes

$$[0 \quad 0] = [\beta P_{b/c} \quad \beta P_d + F_a] \tag{27}$$

Hence $P_d = -F_a/\beta$ and because, P_d must be positive semi-definite, it is necessary and sufficient to choose $\mathrm{sign}(F_a) = -\mathrm{sign}(\beta)$ or $P_d = 0$ and $F_a = 0$. Next, note that inequality (23) becomes

$$0 < -[F_a \quad F_b] \begin{bmatrix} 0 \\ \beta \end{bmatrix} + [0 \quad \beta] \begin{bmatrix} F_a \\ F_b \end{bmatrix} = -2F_b\beta \tag{28}$$

Thus for this inequality to hold, we must have $\mathrm{sign}(F_b) = -\mathrm{sign}(\beta)$. Finally, noting that $P_a = P_{b/c} = 0$ then, inequality (24) can be written as

$$0 < \begin{bmatrix} 0 & 0 \\ 0 & P_d \end{bmatrix} + \begin{bmatrix} F_a \\ F_b \end{bmatrix} [F_a \quad F_b] = \begin{bmatrix} F_a^2 & F_a F_b \\ F_a F_b & P_d + F_b^2 \end{bmatrix} \tag{29}$$

Therefore, for this inequality to hold we must strengthen our conclusion to $\mathrm{sign}(F_a) = -\mathrm{sign}(\beta)$; it cannot be zero or only positive semi-definiteness would be proven. Hence in this case, the conditions of Theorem 1 are fulfilled and the system will be globally asymptotically stable. □

C. Determination of suitable F for better performance

Corollary 1 implies that, for a double integrator plant *any* state-feedback matrix F with elements having the opposite sign to β will provide an anti-windup compensator ensuring global asymptotic stability. Typically, however, only a subset of this range will provide acceptable performance. In this section we propose choices of F based on simple approximations of the anti-windup compensator dynamics.

The dynamics of the anti-windup compensator are governed by the equations (17)-(18), or, equivalently (19)-(20). Note that the saturation function can be replaced by a time-varying gain, which for $m = 1$ takes the form

$$\mathrm{sat}(u) = \sigma(u)u \qquad \sigma(.) : \mathbb{R} \mapsto [0, 1] \tag{30}$$

Using this in equations (19)-(20) yields

$$\dot{x} = (A_p + B_p \sigma(u) F)x =: A_{\sigma(u)} x \tag{31}$$

and the time-varying A-matrix has the explicit form

$$A_{\sigma(u)} = \begin{bmatrix} 0 & 1 \\ \beta\sigma(u)F_a & \beta\sigma(u)F_b \end{bmatrix} \tag{32}$$

Note that any F satisfying Corollary 1 will ensure global stability, but it is possible to use simple linear analysis to estimate the performance of the AW compensator. In particular, replacing $\sigma(u)$ by a constant $\sigma_0 \in [0, 1]$ means that

$$A_{\sigma_0} = \begin{bmatrix} 0 & 1 \\ \beta\sigma_0 F_a & \beta\sigma_0 F_b \end{bmatrix} \tag{33}$$

The eigenvalues of the nonlinear loop dynamics are therefore given by the roots of the characteristic equation

$$s^2 - \beta\sigma_0 F_b s - \beta\sigma_0 F_a = 0 \tag{34}$$

This can be compared to a standard second order characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{35}$$

where ω_n is the undamped natural frequency and ζ is the damping ratio. Comparing coefficients yields

$$\omega_n = \sqrt{-\beta\sigma_0 F_a} \quad \zeta = -\frac{F_b}{2}\sqrt{-\frac{\beta\sigma_0}{F_a}} \tag{36}$$

This implies that the speed of the nonlinear loop dynamics (ω_n), for a fixed saturation value σ_0 is purely a function of F_a, whereas the damping ratio is a function of both F_a and

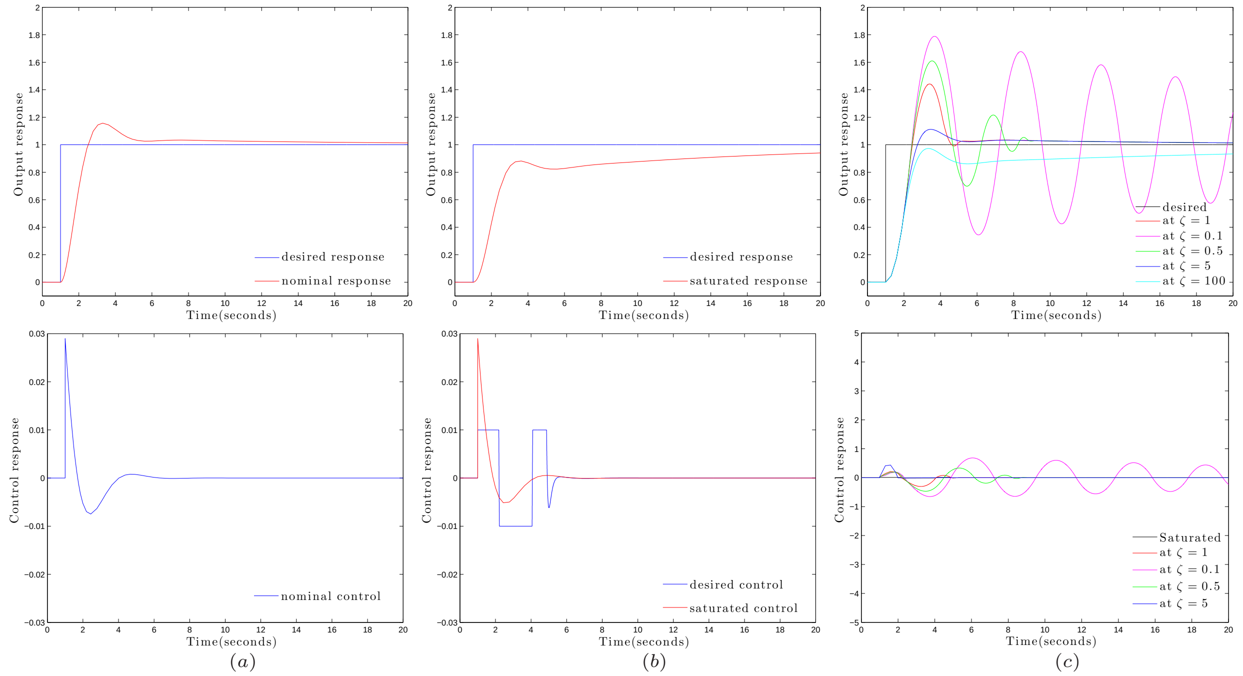


Fig. 3. Output and control response plots for textbook example. (a) Nominal; (b) Saturation, no AW; (c) Saturation, with AW at different ζ

F_b. Therefore F_a is used to set ω_n and then F_b to provides an appropriate damping ratio, thus:

$$F_a = -\omega_n^2/\beta\sigma_0 \quad F_b = -2\zeta\sqrt{-\frac{F_a}{\beta\sigma_0}} \tag{37}$$

However, note that in reality σ_0 is not constant, but varies within an interval $[0, 1]$. One therefore might expect that AW designs corresponding to compensators which are sufficiently well-damped and sufficiently fast for all σ_0 within a sub-interval of $[0, 1]$ to yield better responses for small enough saturation violations. Note that setting $\sigma_0 = 1$ provides the compensator dynamics when no control signal saturation occurs so one might expect that $A_{\sigma_0=1} = A_p + B_pF$ should be at least critically damped to enable a return to linear behaviour with no unwanted oscillations. However, a damping ratio greater than this would be required to ensure good damping when *saturation occurs* (i.e. when $\sigma_0 < 1$).

IV. Example

A. Textbook Example

Consider the double integrator plant $G_p(s) \sim (A_p, B_p, C_p, D_p)$ described by the state space matrices:

$$A_p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_p = \begin{bmatrix} 0 \\ 100 \end{bmatrix} \quad C_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A linear PD controller $K(s)$ with proportional gain $K_p = 0.001$ and derivative gain $K_d = 0.014$ was designed for the plant $G_p(s)$.

The saturation limits are fixed at ± 0.01 and according to Corollary 1, the elements of F must be negative since $\beta = 100$. This will ensure that global stability of the AW compensator for this plant system is guaranteed.

Several AW compensators were constructed using the different values of F listed in Table I. Using equation (37), the elements of F were chosen so that F_a corresponds to undamped natural frequencies of $\omega_n = 10 rad/s$ when $\sigma_0 = 1$. F_b corresponds to different damping ratios, again when $\sigma_0 = 1$.

Damping Ratio	F_a	F_b	Remark
$\zeta = 0.1$	-1	-0.02	$\zeta << 1$
$\zeta = 0.5$	-1	-0.1	$\zeta < 1$
$\zeta = 1$	-1	-0.2	$\zeta = 1$
$\zeta = 5$	-1	-1	$\zeta > 1$
$\zeta = 100$	-1	-20	$\zeta >> 1$

TABLE I
ANTI-WINDUP GAINS AND APPROXIMATE NONLINEAR LOOP CHARACTERISTICS

Figure 3 shows the output response and the corresponding control response for a step demand. Figure 3a shows the un-saturated response and Figure 3b shows the response degraded by saturation. Figure 3c shows the system response with AW, synthesized using different F values, engaged. When F is selected such that $\zeta = 1$ and $\zeta = 2$, the response of the system is significantly improved. When F corresponding to $\zeta = 0.1$ and $\zeta = 0.5$ is used, the response has large oscillations with a very slow decay rate; when $\zeta = 100$, there are no oscillations but a slow decay rate. Hence, a range of values of F can be used to stability but a smaller range provides acceptable performance. As expected, a slightly over damped AW compensator provides the most appealing time-response.

B. Quadrotor example

Consider the quadrotor system taken from [26], [13] and depicted in Figure 4. This is a multivariable system, but one

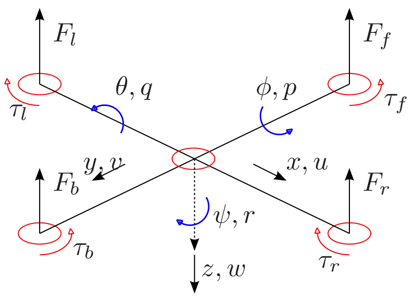


Fig. 4. Force, Torque and States definition of a Quadrotor

which has much structure. A linear model of the quadrotor at hover is given by

$$G(s) = G_D(s)X$$

where

$$G_D(s) = \text{diag}(\frac{1}{J_x s^2}, \frac{1}{J_y s^2}, \frac{1}{J_z s^2}, \frac{1}{ms^2})$$

and J_x, J_y, J_z are moments of inertia in the x, y, z axes and m is the quadrotor's mass. The relationship between the body forces (F) and torques ($\tau_\phi, \tau_\theta, \tau_\psi$) generated by the motors and the motor speed squared is given by the matrix X:

$$\begin{bmatrix} F \\ \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \underbrace{\begin{bmatrix} k_1 & k_1 & k_1 & k_1 \\ 0 & -\alpha k_1 & 0 & \alpha k_1 \\ \alpha k_1 & 0 & \alpha k_1 & 0 \\ -k_2 & k_2 & -k_2 & k_2 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \delta_f \\ \delta_r \\ \delta_b \\ \delta_l \end{bmatrix}}_{u} \tag{38}$$

where $k_1 > 0$ and $k_2 > 0$ are constants that need to be determined experimentally; α is the distance between the motor and centre of mass; and δ_* is the motor angular velocity. Because X is invertible , a nominal controller can be designed on a loop-by-loop basis and has the structure

$$K(s) = X^{-1}K_D(s) \tag{39}$$

where $K_D(s)$ is a block diagonal transfer function matrix, with each element consisting of a PD controller, which has been tuned for good nominal performance - see [13].

Saturation is present on each of the motor velocities, resulting in the scenario depicted in Figure 5. Notice that because the saturation element destroys the decoupling of the system, hence the system may exhibit traditional windup effects as well as directionality issues [27]. The nonlinearity $\chi(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ in Figure 5 is not a pure saturation function as in equations (19)-(20) but instead has the form

$$\chi(u) := X\text{sat}(X^{-1}u) \tag{40}$$

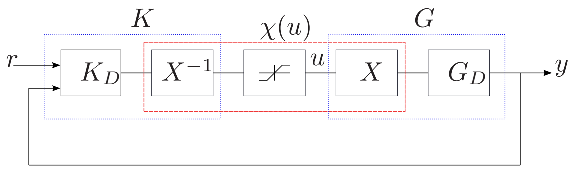


Fig. 5. Plant structure

However, it transpires, via analysis similar to that given in [13], that stability of the above system can be guaranteed by implementing an AW compensator of the form

$$\Theta(s) = \begin{bmatrix} M_D(s) - I \\ N_D(s) \end{bmatrix} X \tag{41}$$

where $G_D(s) = N_D(s)X(M_D(s)X)^{-1}$ is a right coprime factorisation with

$$N_D(s) = \text{blockdiag}\,(N_1(s), \dots, N_m(s)) \tag{42}$$
$$M_D(s) = \text{blockdiag}\,(M_1(s), \dots, M_m(s)) \tag{43}$$

This means that the dynamics of the i'th nonlinear loop are given by the equations ([13])

$$\begin{aligned} \dot{x}_i &= (A_i + B_i F_i)x_i - B_i\tilde{\chi}_i(u) \\ u_i &= F_i x_i \end{aligned} \tag{44}$$

where $\tilde{\chi}(u) = u - \chi(u)$. Equation (44) is in the form of the system in Corollary 1 and it transpires (full analysis omitted) that selecting each element of F_i to be negative for all $i \in \{1, 2, \dots, m\}$ guarantees global asymptotic stability of the closed-loop.

To illustrate AW design we examine the roll channel of the quadrotor; the state space dynamics are described by

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_i = \begin{bmatrix} 0 \\ \frac{1}{J_i} \end{bmatrix} \quad C_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the analysis in Section III-C, setting $\sigma_0 = 1$, F was chosen to have the various damping and undamped natural frequency characteristics shown in Table II.

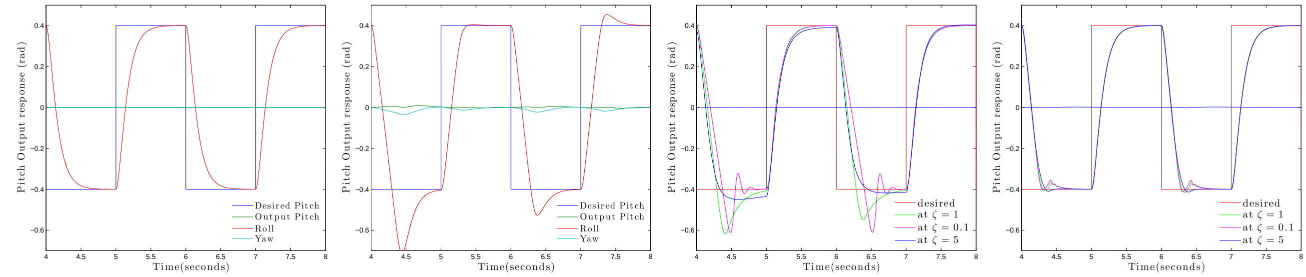
Damping Ratio	Natural Frequency	F_a	F_b	Remark
$\zeta = 0.1$	$\omega_n = 115.47$	-100	-0.1732	$\zeta < 1$
$\zeta = 1$	$\omega_n = 115.47$	-100	-1.7321	$\zeta = 1$
$\zeta = 5$	$\omega_n = 115.47$	-100	-8.6603	$\zeta > 1$
$\zeta = 0.1$	$\omega_n = 36.51$	-10	-0.0548	$\zeta < 1$
$\zeta = 1$	$\omega_n = 36.51$	-10	-0.5477	$\zeta = 1$
$\zeta = 5$	$\omega_n = 36.51$	-10	-2.7386	$\zeta > 1$

TABLE II
ANTI-WINDUP GAINS AND DAMPING/SPEED PROPERTIES

Figure 6 shows the roll attitude response for a pulse demand of $0.4rad$; Figure 7 shows the corresponding control signal response. Figure 6a shows the nominal (un-saturated) response and Figure 6b shows the response with saturation: performance degradation can be observed. Figure 6c shows the AW response using F corresponding to $\omega_n = 36.51$ and various damping ratios. Notice that the response improves as ζ increases from 0.1 to 5 with the best response at $\zeta = 5$. Figure 6d shows the AW response using F corresponding to $\omega_n = 115.47$ and various damping ratios. For this higher undamped natural frequency, improved responses are obtained for all damping ratios, compared to the response for $\omega_n = 36.51$. Again, the best response is for the slightly overdamped case, $\zeta = 5$.

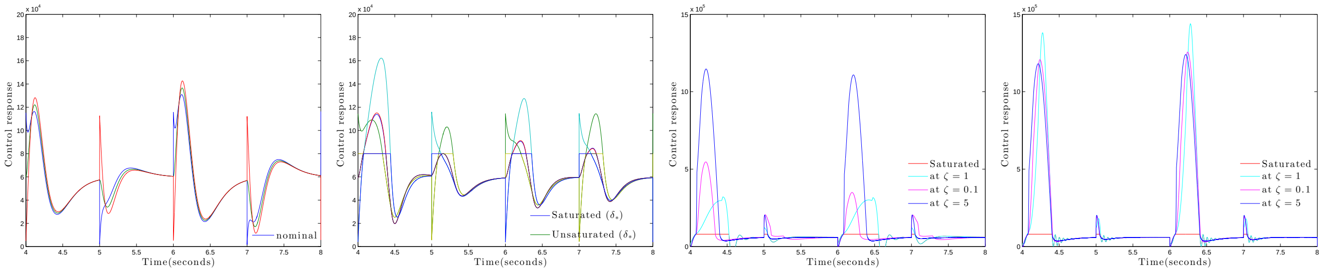
V. CONCLUSION

This paper has proposed a simple method for synthesizing AW compensators for systems containing double integrators, based on a Popov-like sufficient condition presented in [1]



(a) (b) (c) (d)

Fig. 6. Output response:(a) Nominal; (b) Saturation, no AW; (c) Saturation, $\omega_n = 36.51 rad/s$ with AW at different ζ; (d) Saturation, $\omega_n = 115.47 rad/s$ with AW at different ζ



(a) (b) (c) (d)

Fig. 7. Control response:(a) Nominal; (b) Saturation, no AW; $[(c), (d)]$ Saturation, $\omega_n = 36.51 rad/s$ and $\omega_n = 115.47 rad/s$ with AW at different ζ

and an approximate linear analysis of the AW compensator. The main appeal of the approach is that global stability is guaranteed for a large range of F and then F is selected based on the AW compensator's desired speed and damping characteristics. Due to the approach's direct nature, F can be chosen based on the designer's need, and in real time, without repeating the stability analysis.

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