Cooperative Robust Estimation with Local Performance Guarantees

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***Abstract*— The paper considers the problem of cooperative estimation for a linear uncertain plant observed by a network of communicating sensors. We take a novel approach by treating the filtering problem from the view point of local sensors while the network interconnections are accounted for via an uncertain signals modelling of estimation performance of other nodes. That is, the information communicated between the nodes is treated as the true plant information subject to perturbations, and each node is endowed with certain believes about these perturbations during the filter design. The proposed distributed filter achieves a suboptimal H_∞ consensus performance. Furthermore, local performance of each estimator is also assessed given additional constraints on the performance of the other nodes. These conditions are shown to be useful in tuning the desired estimation performance of the sensor network.**

I. INTRODUCTION

The research on cooperative filtering and estimation of networked systems has gained much momentum during the past decade, aiming at developing efficient estimation algorithms for large assemblies of networked sensors [1]–[5]. The mentioned references reflect the common trend in the literature, where the main objective is to accomplish a globally optimal or suboptimal estimation performance of the network. Usually, the performance of individual sensors is not considered in such problems. This observation motivates the question about a relationship between the estimation performance of the individual filters within a distributed estimation network and the performance of the overall network. This paper considers this problem within the specific framework of distributed H_∞ consensus estimation [4], [5].

Our approach also targets the global convergence problem however not with a brute-force decoupling of the global solution. Rather we define a local objective function in terms of uncertain signals capturing the performance of the other nodes. This leads to decoupling of the distributed filter while implicitly maintaining a meaningful connection to the network. This way, the local objective function abstracts the dependence on other nodes, eliminating the need to consider their exact models or their raw measurements. Nevertheless, the convergence of the local filter is dependent on the rest of the network and we provide conditions that render the H_∞ convergence of the network of the filters. Furthermore, by asserting further conditions on the individual performances of the other nodes a guaranteed H_∞ performance of the individual filters is established. These conditions express that if all the neighbours maintain a certain level of accuracy then the local filter also guarantees a nominated H_∞ performance.

To establish the above relationship, here we analyze the distributed filter network consisting of estimators solving an auxiliary optimal filtering problem at every node. In that sense, our approach bears some resemblance with decentralized control where each controller is constructed to regulate a local subsystem. The mentioned auxiliary filtering problem originates in [6], [7]; it was shown in [8] to yield interconnected consensus-type filters that exchange information between the network nodes although the parameters for each filter could be computed online in the decentralized fashion.

The new element of this paper compared with [8] is how the neighboring information is interpreted by each node. In [8], each node was considered to be agnostic about the amount of energy in the error between the true state of the plant and the neighbours' estimates of that state. In contrast, here we consider a model where each node perceives a relationship between the energy in the neighbours' error and the accuracy of its own filter. We give a detailed discussion of this idea later in the paper; for now we only note that technically our model adds a constraint on the energy in the error inputs arising in the auxiliary minimum energy problems. Such a constraint has the form of an Integral Quadratic Constraint previously used in robust decentralized control problems and filtering problems; e.g., see [9], [10]. However, unlike those problems, the parameters of the constraints used here play the role of tunable parameters which are adjusted according to the desired local and global performance. They also serve as indicators of sensitivity of the individual filters to the neighbours' performance.

The main result of this paper are sufficient conditions on the network parameters that ensure H_∞ performance of the network consisting of the proposed minimum energy filters. As mentioned, not only global disturbance attenuation is guaranteed by these conditions, but also certain local H_∞ properties of the node filters are established. We show that these conditions admit the form of a convex semidefinite program, which enables constructing a filter network yielding a suboptimal disturbance attenuation.

Notation: $\mathbb{R}^n$ is the Euclidean space of vectors, $\|\cdot\|$ is the Euclidean norm, and for any positive semidefinite matrix X, $X = X' \geq 0$, $\|a\|_X \triangleq (a'Xa)^{1/2}$. For $0 < T \leq \infty$, $\mathcal{L}_2[0,T)$ denotes the Lebesgue space of vector-valued signals square-integrable on $[0,T)$. $\mathrm{diag}[X_1,\ldots,X_N]$ denotes the block diagonal matrix with $X_1,\ldots,X_N$ as its diagonal blocks, and $\otimes$ is the Kronecker product of matrices. $\lambda_{\min}(Z)$ is the smallest eigenvalue of a symmetric matrix Z.

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II. Problem Formulation and Preliminaries

A. The plant and the distributed estimator

Consider a linear system

$$\dot{x} = Ax + Bw, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ are, respectively, the state and the unknown modeling disturbance input; the latter is assumed to be $\mathcal{L}_2$ integrable on $[0,\infty)$. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known, however the initial state x_0 is unknown and is considered to be part of the uncertainty about the system (1).

The main objective of the paper is to determine conditions under which the plant state $x(t)$ can be estimated by a network of filters each using its plant measurement

$$y_i = C_i x + D_i v_i, \tag{2}$$

where $i = 1, 2, \ldots, N$ indicates the measurement taken at node i of the network. Each measurement $y_i \in \mathbb{R}^{p_i}$ is imperfect, it is subject to a measurement disturbance v_i taking values in $\mathbb{R}^{m_i}$ that also belongs to the space $\mathcal{L}_2[0,\infty)$ by assumption. The coefficients of each measured output are matrices of the matching dimensions, $C_i \in \mathbb{R}^{p_i\times n}$, $D_i \in \mathbb{R}^{p_i\times m_i}$, with $E_i \triangleq D_i D_i' > 0$.

In addition to its direct measurements of the plant, each node receives information from other nodes of the network, of the form

$$c_{ij} = W_{ij}\hat{x}_j + F_{ij}\epsilon_{ij}, \tag{3}$$

where $\hat{x}_j$ is the estimate of state x at the neighbouring node j. The signal ϵ_{ij} with values in $\mathbb{R}^{m_{ij}}$ represents the communication errors or uncertainty in the communication channel, $\epsilon_{ij} \in \mathcal{L}_2[0,\infty)$. We assume that $G_{ij} \triangleq F_{ij}F_{ij}' > 0$.

The network graph describing communications between the filtering nodes is assumed to be directed, its node and edge sets are denoted $\mathbf{V} = \{1,\ldots,N\}$ and $\mathbf{E} \subseteq \mathbf{V}\times\mathbf{V}$, respectively. The neighborhood of node i, i.e., the set of nodes which send information to node i, is denoted by $\mathbf{N}_i = \{j : (i,j) \in \mathbf{E}\}$ and its cardinality is denoted l_i. The Laplace matrix of the network graph is denoted $\mathbf{L}$ [11].

Following [1], [2], [4] and many other papers on distributed estimation, we consider a class of consensus-based interconnected filters each processing the direct measurements y_i and neighbours' information c_{ij} by means of a Luenberger-type observer of the form

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(y_i - C_i\hat{x}_i) + \sum_{j\in\mathbf{N}_i} K_{ij}(c_{ij} - W_{ij}\hat{x}_i), \tag{4}$$
$$\hat{x}_i(0) = \xi_i.$$

The estimation problem in this paper is to determine coefficients L_i, K_{ij} (which can be time-varying) that ensure convergence of the network to trajectories of the plant, and also guarantee an acceptable attenuation of the detrimental effects of disturbances on the estimation error. Formally, these properties are formulated as follows. Given a positive semidefinite matrix $P \in \mathbb{R}^{nN\times nN}$ and a collection of positive semidefinite matrices $\mathcal{X}_i \in \mathbb{R}^{n\times n}$, and constants γ^2 and $\bar{\gamma}_i^2$, $i \in \mathbf{V}$, we wish to determine a collection of filters of the form (4) that guarantee the following properties:

P1. In the absence of disturbances w, v_i and ϵ_{ij}, $j \in \mathbf{N}_i$, $i = 1,\ldots,N$ the estimation error of the filter $e_i(t) = \hat{x}_i(t) - x(t)$ converges to zero asymptotically.

P2. In the presence of disturbances, the network of filters (4) attains the type of H_∞ disturbance attenuation property

$$\int_0^\infty \|e\|_P^2 dt \le \gamma^2\Bigg(\sum_{i=1}^N \|x(0)-\xi_i\|_{\mathcal{X}_i}^2 + N\|w\|_2^2 + \sum_{i=1}^N\Big(\|v_i\|_2^2 + \sum_{j\in\mathbf{N}_i}\|\epsilon_{ij}\|_2^2\Big)\Bigg), \tag{5}$$

where $e = [(\hat{x}_1 - x)', \ldots, (\hat{x}_N - x)']'$ and $\|.\|_2^2$ is the $\mathcal{L}_2$ norm.

P3. Provided the neighbours of node i contribute a sufficient effort (this will be quantitatively defined later) to assist i, it is also guaranteed that at that node

$$\int_0^\infty \|e_i\|^2 ds \le \bar{\gamma}_i^2\Big[\beta_i + \|x_0 - \xi_i\|_{\mathcal{X}_i}^2 + \int_0^\infty\big[\|w\|^2 + \|v_i\|^2 + \sum_{j\in\mathbf{N}_i}\|\epsilon_{ij}\|^2\big]ds\Big]; \tag{6}$$

$\beta_i > 0$ is a constant which will be determined later.

These properties formalize the desired attributes of a distributed filter that we want to achieve. In particular, property **P2** specifies the desired global disturbance attenuation performance across the sensor network using a network of decoupled filter equations (4). Note that decoupled equations governing the gains in filters (4) will be provided later. Furthermore, property **P3** articulates the desired local disturbance attenuation provided that there is sufficient contribution from the neighbours. The sufficient contribution condition is quantitatively defined later in the paper.

We remark that properties **P1**, **P2** jointly generalize the property of H_∞ consensus introduced in [4]; also see [5], [12]. For example, let $P = (\mathbf{L} + \mathbf{L}_\top)\otimes P_0$ where $P_0 = P_0' \ge 0$, and $\mathbf{L}_\top$ is the Laplacian matrix of the graph obtained from the network graph by reversing its edges. This choice of P results in the left hand side of (5) being equal to the weighted H_∞ disagreement cost between the nodes, $\int_0^\infty \sum_i\sum_{j\in\mathbf{N}_i}\|\hat{x}_i - \hat{x}_j\|_{P_0}^2 ds$ [4], [5], [13]. More generally, letting $P = (\mathbf{L}+\mathbf{L}_\top)\otimes P_0 + \mathrm{diag}[P_1 \ \ldots \ P_N]$, $P_i = P_i' > 0$, reduces **P1**, **P2** to the property of strong robust synchronization introduced in [12]. In addition, property **P3** describes H_∞ attenuation properties of individual node filters. Including such property into analysis constitutes the main difference between the problem posed above and the previous work in the area of distributed estimation.

B. Representation of the neighboring information

To make performance analysis of individual filters possible, let us introduce the mismatch between the disturbance-free information contained in the signal c_{ij} and the corresponding true version of this information,

$$\eta_{ij} = W_{ij}(\hat{x}_j - x) = -W_{ij}e_j \in \mathbb{R}^{p_{ij}}, \quad j \in \mathbf{N}_i. \tag{7}$$

With these signals the information received by sensor i can be represented as

$$c_{ij} = W_{ij}x + \eta_{ij} + F_{ij}\epsilon_{ij}, \quad j \in \mathbf{N}_i. \tag{8}$$

Equation (8) can be regarded as an additional measurement of the plant affected by disturbances ϵ_{ij} and η_{ij}.

Treating the signals η_{ij}, *for the purpose of filter derivation*, as the disturbances additional to w, v_i and ϵ_{ij} has an effect of decoupling node i from its neighbours. Indeed, consider the error dynamics of the filter (4) at node i,

$$\begin{aligned} \dot{e}_i &= \left(A - L_iC_i - \sum_{j\in\mathbf{N}_i} K_{ij}W_{ij}\right)e_i - Bw \\ &\quad + L_iD_iv_i + \sum_{j\in\mathbf{N}_i} K_{ij}(\eta_{ij} + F_{ij}\epsilon_{ij}), \qquad (9) \\ \eta_{ki} &= W_{ki}e_i, \quad i \in \mathbf{N}_k. \end{aligned}$$

In this system, the signals η_{ij}, $j \in \mathbf{N}_i$, play the role of exogenous disturbances and each signal $\eta_{ki} = W_{ki}e_i$ represents the output used by agent k for whom i is the neighbour, i.e., $i \in \mathbf{N}_k$. This interpretation allowed us to construct in [8] minimum energy filters of the form (4) with the property that for any initial condition x_0, arbitrary $\mathcal{L}_2$-integrable disturbances w, v_i, ϵ_{ij}, η_{ij} and an arbitrary $T > 0$

$$\begin{aligned} \int_0^T \|e_i\|^2_{R_i}dt &\le \gamma^2\Bigg(\|x_0 - \xi_i\|^2_{\mathcal{X}_i} + \int_0^T \Big[\|w\|^2 + \|v_i\|^2 \\ &\quad + \sum_{j\in\mathbf{N}_i}(\|\epsilon_{ij}\|^2 + \|\eta_{ij}\|^2_{Z_{ij}^{-1}})\Big]dt\Bigg). \end{aligned} \tag{10}$$

Here, γ^2 and $R_i = R_i' > 0$ are a positive constant and matrices whose existence is determined by certain LMI conditions in [8]. Also, $Z_{ij} = Z_{ij}' > 0$ are *given* matrices; in [8] they were associated with the confidence of node i about performance of node j.

Condition (10) provides an H_∞ type bound on the energy in the filter estimation errors at node i expressed in terms of the energy of the disturbances affecting that node, and is similar to (6) in property **P3**. The important difference between (10) and (6) is that the former condition includes the energy in the signals η_{ij} that depend on the neighbours' accuracy. Also, according to (10), the *same level of disturbance attenuation* γ^2 is stated for all nodes. Our goal is to revisit the design of the filters (4) to obtain a possibly sharper H_∞ property for at least some of the local filters, and for other filters, to provide a means for assessing their local performance and sensitivity to the neighbours' errors.

Owing to the relation $\eta_{ij} = -W_{ij}e_j$, from the viewpoint of node i, the error dynamics of the network can be seen as an interconnection of two systems, representing, respectively, i's own error dynamics and the errors dynamics of the rest of the system; see Figure 1. Motivated by (10), we propose the following condition to formally capture the sensitivity of each node to the accuracy of its neighbours' filters:

For every i, there exist positive definite symmetric matrices $\bar{Z}_{ij}$ and constants $d_{ij} \ge 0$, $j \in \mathbf{N}_i$, such that for all $t \in [0,\infty)$ and $w, v_i, \epsilon_{ij} \in \mathcal{L}_2[0,t)$,

$$\int_0^t \|\eta_{ij}\|^2_{\bar{Z}_{ij}^{-1}}dt \le \int_0^t (\|e_i\|^2 + \|w\|^2)dt + d_{ij}, \qquad \forall j \in \mathbf{N}_i, i = 1,\dots,N. \tag{11}$$

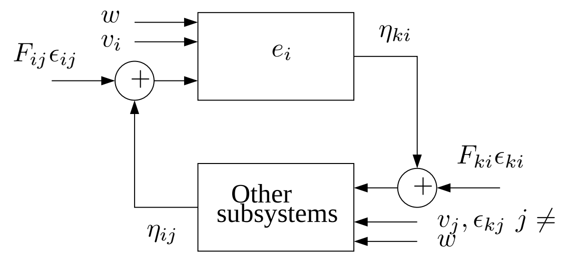

Fig. 1. A two-block representation of the error dynamics system.

As a generalized form of the property (10), condition (11) reflects how the neighbours' accuracy influences the local disturbance attenuation property (6) at every node. Therefore in what follows, we will use condition (11) to establish (6), i.e., (11) is the quantitative characteristic of the neighbours' effort mentioned in **P3**.

To demonstrate the role of (11) more vividly, take for example $\bar{Z}_{ij} = \bar{z}_{ij}I$, with a scalar $\bar{z}_{ij} > 0$ and suppose (6) holds provided (11) is satisfied with a very small $\bar{z}_{ij}$. Since according to (6), the energy in e_i is bounded, (11) suggests that node i can only tolerate relatively 'small' mismatch inputs η_{ij} to be able guarantee (6). However, a small energy in η_{ij} can only be accomplished by the corresponding neighbour j. This suggests that the eigenvalues of $\bar{Z}_{ij}$ may be indicative of sensitivity of the local filters to fidelity of its neighbours' estimates. In Section IV we will show that the matrices $\bar{Z}_{ij}$ can be computed jointly with the attenuation levels γ^2, $\bar{\gamma}_i^2$. This provides the means for performance tuning of the local filters.

Similarly, the constants d_{ij} in (11) describe the bound on the estimation error energy that node i is prepared to tolerate from its neighbour j, in response to (hypothetically) estimating the perfectly known plant ($w = 0$) with the utmost precision ($e_i = 0$). Indeed, in this hypothetical case, condition (11) reduces to a bound on the energy in the mismatch disturbance signal η_{ij} of the neighbour j.

C. Distributed estimation problem

We are now in a position to present a formal definition of the distributed estimation problem described in Section II-A.

Problem 1: Determine a collection of filters of the form (4) and matrices $\bar{Z}_{ij} \in \mathbb{R}^{p_{ij}\times p_{ij}}$, $j \in \mathbf{N}_i$, $i \in \mathbf{V}$, and constants γ^2 and $\bar{\gamma}_i^2$ such that the following conditions hold:

(i) Given a positive semidefinite matrix $P \in \mathbb{R}^{nN\times nN}$, the network of filters (4) achieves properties **P1** and **P2** with this P and the found γ^2.

(ii) The following implication holds with the found $\bar{Z}_{ij}$ and $\bar{\gamma}_i^2$: If signals $\eta_{ij}(t)$, $j \in \mathbf{N}_i$, satisfy (11), then the filter (4) guarantees the satisfaction of condition (6), i.e., **P3** is satisfied.

We stress that the global performance properties **P1** and **P2** of the proposed distributed filter will be proved without using condition (11). The IQC (11) will only be used to guarantee certain local performance of each node i subject to acceptable performance of its neighbours. The latter development will be analogous to how IQCs were used in the derivation of decentralized robust controllers to quantify the uncertainty arising from system interconnections; e.g, see [9], [14]. However, different from decentralized controllers in those references, our aim is to maintain coupling between the filters, to ensure cooperation between them.

III. Distributed minimum energy filtering with local performance guarantees

In this section, our main results are presented. As was explained in Section II-B, our goal is to obtain a converging (in the H_∞ sense) distributed filter which provides global estimation performance described in item (i) of Problem 1, and also characterize quantitatively the connection between local H_∞ properties of the filters and their sensitivity to estimation accuracy of their neighbours.

To solve Problem 1, we first introduce an auxiliary robust minimum energy filtering problem involving a modified version of the standard minimum energy cost [6]. This cost functional, depending on the signals w and $\eta_i \triangleq [\eta'_{ij_1} \dots \eta'_{ij_{l_i}}]'$, affecting the measurements $y_i|_{[0,t]}$ and $\{c_{ij}|_{[0,t]}\ j \in \mathbf{N}_i\}$ available at node i is as follows:

$$\begin{aligned}
&\bar{J}_{i,t}(x,w,\eta_i) \\
&= \frac{1}{2}\|x^{t,x}(0)-\xi_i\|^2_{\mathcal{X}_i} + \frac{1}{2}\int_0^t \Bigg(\|w\|^2 + \|y_i - C_i x^{t,x}\|^2_{E_i^{-1}} \\
&\quad + \sum_{j\in\mathbf{N}_i} \|c_{ij} - W_{ij}x^{t,x} - \eta_{ij}\|^2_{G_{ij}^{-1}} \\
&\quad -\gamma^{-2}\|x^{t,x}-\hat{x}_i\|^2_{R_i}\Bigg)ds; \qquad (12)
\end{aligned}$$

Compared to the standard minimum energy cost functional, it includes the additional weighted penalty on the tracking error at node i; see the last term in (12). It was shown in [8] that the inclusion of this term enforces a guaranteed H_∞-type performance of the filter while a minimum energy estimate is sought; cf. [15]. The weight matrix of this term, $R_i = R_i' > 0$, $R_i \in \mathbb{R}^{n\times n}$ was regarded as parameters of the filter, and a process of selecting those matrices to optimize γ^2 was proposed in [8]. However, different from [8], the cost (12) does not include a direct quadratic penalty on η_{ij}. Instead, our derivation of the local filters will impose the constraint (11) on the mismatch signals η_{ij}, $j \in \mathbf{N}_i$.

With these modifications, the auxiliary robust minimum-energy filtering problem consists of determining a set of the unknowns x, w, η_i compatible with the measurements y_i and the communications c_{ij} and minimizing the energy cost (12) subject to the constraint (11):

$$\inf_x \left(\inf_{w\in\mathcal{L}_2[0,t]} \inf_{\eta_i\in\Xi_{i,t}} \bar{J}_{i,t}(x,w,\eta_i)\right). \qquad (13)$$

Here $\Xi_{i,t}$ denotes the class of vector signals η_i obtained by stacking up all η_{ij}, $j \in \mathbf{N}_i$, satisfying (11). Originated from the minimum energy filtering [6] and least square fitting, this problem will lead to the 'most likely' minimum-energy trajectory $x^*_{i,t}(\cdot)$ compatible with the data at node i, $y|_{[0,t]}$, $c_{ij}|_{[0,t]}$ [6]. The subscripts i,t at $x^*_{i,t}(\cdot)$ are to highlight that the trajectory $x^*_{i,t}(\cdot)$ is consistent with the data collected on the interval $[0,t]$ at node i. By definition, the end point of this trajectory is the minimum-energy estimate of the state $x(t)$, given the measurement data $y|_{[0,t]}$, $c_{ij}|_{[0,t]}$: $\hat{x}_i(t) \triangleq x^*_{i,t}(t)$.

To solve the constrained optimization problem (13), we apply the method of S-procedure [14]. In fact, since the cost $\bar{J}_{i,t}(x,w,\eta_i)$ itself depends on $\hat{x}_i(\cdot)$, this requires us to solve a family of minimum energy filtering problems, in which $\hat{x}_i$ is replaced with an arbitrary signal $\bar{x}_i$. Then we take the fixed point of the mapping $\bar{x}_i(t) \to x^*_{i,t}(t)$ generated by this family of minimum energy filtering problems, as $\hat{x}_i(t)$. Due to lack of space, we omit the details and proceed assuming that $\hat{x}_i(t)$ is such a fixed point.

Let $\tau_i \in \mathbb{R}^n$ be a vector $\tau_i = [\tau_{i1}\ \dots \tau_{iN}]'$ such that $\tau_{ij} > 0$ if $j \in \mathbf{N}_i$, and $\tau_{ij} = 0$ otherwise. Then define

$$\begin{aligned}
\bar{J}^{\tau_i}_{i,t}(x,w,\eta_i) &= \bar{J}_{i,t}(x,w,\eta_i) + \sum_{j\in\mathbf{N}_i}\frac{\tau_{ij}}{2}\int_0^t \Big(\|\eta_{ij}\|^2_{\bar{Z}_{ij}^{-1}} \\
&\quad - \|x^{t,x}-\hat{x}_i\|^2 - \|w\|^2 - \|y_i - C_i x^{t,x}\|^2 \\
&\quad - \sum_{r\in\mathbf{N}_i}\|c_{ir} - W_{ir}x^{t,x} - \eta_{ir}\|^2\Big)ds \qquad (14)
\end{aligned}$$

and for fixed t and $\hat{x}_i(\cdot)$, x, consider the unconstrained optimization problem

$$\bar{V}^{\tau_i}_i(x,t) = \inf_{w,\eta_i\in\mathcal{L}_2[0,t]} \bar{J}^{\tau_i}_{i,t}(x,w,\eta_i). \qquad (15)$$

For each t, x, the optimization problem (15) is a standard optimal tracking problem with a fixed terminal condition $x(t) = x$, which has a unique solution under the condition

$$\sum_{j\in\mathbf{N}_i}\tau_{ij} < 1. \qquad (16)$$

We now establish a relationship between this problem and the constrained inner optimization problem in (13).

Let $\mathcal{T}_i(t,x) \triangleq \{\tau_i\colon (16) \text{ holds and } \bar{V}^{\tau_i}_i(x,t) > -\infty\}$. Also for convenience, define a vector $d_i \in \mathbb{R}^n$ whose jth component is d_{ij} if $j \in \mathbf{N}_i$ and is 0 otherwise.

Lemma 1: For every $\hat{x}_i(\cdot)$, $x \in \mathbb{R}^n$, if the corresponding set $\mathcal{T}_i(t,x)$ is nonempty, then the value of the inner optimization problem in (13) is finite,

$$\inf_{\substack{w\in\mathcal{L}_2[0,t],\\ \eta_i\in\Xi_{i,t}}} \bar{J}_{i,t}(x,w,\eta_i) \geq \sup_{\tau_i\in\mathcal{T}_i(t,x)}\left(\bar{V}^{\tau_i}_i(x,t) - \frac{\tau_i' d_i}{2}\right). \qquad (17)$$

From Lemma 1, a lower bound on the value of the problem (13) follows:

$$\begin{aligned}
&\inf_x \inf_{\substack{w\in\mathcal{L}_2[0,t],\\ \eta_i\in\Xi_{i,t}}} \bar{J}_{i,t}(x,w,\eta_i) \\
&\quad \geq \inf_x \sup_{\tau_i\in\mathcal{T}_i(t,x)}\left(\bar{V}^{\tau_i}_i(x,t) - \frac{\tau_i' d_i}{2}\right). \qquad (18)
\end{aligned}$$

We now consider the following optimization problem

$$\inf_x \bar{V}_i^{\tau_i}(x,t) = \inf_x \inf_{w,\eta_i\in\mathcal{L}_2} \bar{J}_{i,t}^{\tau_i}(x,w,\eta_i). \qquad (19)$$

A solution to this problem involves the differential Riccati equation

$$\begin{aligned}\dot{Q}_i^{\tau_i} &= Q_i^{\tau_i}A' + AQ_i^{\tau_i} - Q_i^{\tau_i}\Big(C_i'E_i^{-1}C_i \\ &\quad + \sum_{j\in\mathbf{N}_i} W_{ij}'\bar{U}_{ij}^{-1}W_{ij} - \gamma^{-2}R_i - \bar{W}_i\Big)Q_i^{\tau_i} + S_i, \quad (20)\\ Q_i^{\tau_i}(0) &= \mathcal{X}_i^{-1},\end{aligned}$$

where $\bar{W}_i = (\sum_{j\in\mathbf{N}_i}\tau_{ij})I_n$, $\bar{U}_{ij} \triangleq G_{ij} + \tau_{ij}^{-1}\bar{Z}_{ij}$, $S_i = \Big(1 - \sum_{j\in\mathbf{N}_i}\tau_{ij}\Big)^{-1}BB'$.

Lemma 2: Given fixed $\tau_i \in \mathcal{T}_i(t,x)$ and $T>0$. Suppose the differential Riccati equation (20) has a symmetric nonsingular solution $Q_i^{\tau_i} = Q_i^{\tau_i}(t)$ on the interval $[0,T]$. Then the following filter computes recursively the minimizer $\hat{x}_i^{\tau_i}(t)$ of the optimization problem (19) on the interval $[0,T]$,

$$\begin{aligned}\dot{\hat{x}}_i^{\tau_i} \;&=\; A\hat{x}_i^{\tau_i} + Q_i^{\tau_i}\Bigg(C_i'E_i^{-1}(y_i - C_i\hat{x}_i^{\tau_i}) \\ &\quad + \sum_{j\in\mathbf{N}_i} W_{ij}'\bar{U}_{ij}^{-1}(c_{ij} - W_{ij}\hat{x}_i^{\tau_i})\Bigg), \qquad (21)\\ \hat{x}_i^{\tau_i}(0) \;&=\; \xi_i.\end{aligned}$$

The value of the optimization problem (19) is finite and for $\hat{x}_i = \hat{x}_i^{\tau_i}$ is given by

$$\bar{\rho}_{i,t}^{\tau_i} \triangleq \frac{1}{2}\int_0^t \Bigg[\|y_i - C_i\hat{x}_i^{\tau_i}\|^2_{E_i^{-1}} + \sum_{j\in\mathbf{N}_i}\|c_{ij} - W_{ij}\hat{x}_i^{\tau_i}\|^2_{\bar{U}_{ij}^{-1}}\Bigg]ds.$$

Let

$$\bar{\mathcal{T}}_i(T) \triangleq \left\{\begin{array}{ll}\tau_i: & \text{(16) holds and the DRE (20)}\\ & \text{has a bounded positive definite}\\ & \text{solution on } [0,T].\end{array}\right\}.$$

Lemma 3: For all $T>0$, $\bar{\mathcal{T}}_i(T) \subseteq \bigcap_{t\in[0,T],\ x\in\mathbb{R}^n} \mathcal{T}_i(t,x)$.

The following theorem summarizes the above discussion.

Theorem 1: Given constants γ^2 and γ_i^2 and matrices $\bar{Z}_{ij}$, $j\in\mathbf{N}_i$, suppose the set $\bar{\mathcal{T}}_i(+\infty) = \bigcap_{T>0}\bar{\mathcal{T}}_i(T)$ is not empty. Then for any η_{ij} for which condition (11) holds, the filter (21) computes recursively the process $\hat{x}_i^{\tau_i}(t)$ which satisfies condition (6) with $\beta_i = \tau_i' d_i$.

Compared with the distributed minimum energy filter in [8], we have now obtained a family of suboptimal minimum energy filters for each node parametrized by $\tau_i \in \bar{\mathcal{T}}_i(+\infty)$. To be able to apply Theorem 1, it is necessary to have a method for computing at least one such vector τ_i for every node i. In the next section, we will present an algorithm that accomplishes this task. In addition, this algorithm obtains the matrices $\bar{Z}_{ij}$ and constants γ_i^2 consistent with the found γ^2, thus providing a complete solution to Problem 1.

IV. Design of a robust distributed estimator

The algorithm to compute a solution to Problem 1 utilizes a collection of linear matrix inequalities (LMIs) including the condition (16) and the following matrix inequalities:

$$\begin{bmatrix} A'\bar{Y}_i + \bar{Y}_iA + (\bar{\gamma}_i^{-2} + \sum_{j\in\mathbf{N}_i}\tau_{ij})I & \\ -\Big(C_i'E_i^{-1}C_i + \sum_{j\in\mathbf{N}_i}W_{ij}'\Upsilon_{ij}W_{ij}\Big) & \bar{Y}_iB \\ B'\bar{Y}_i & \big(\sum_{j\in\mathbf{N}_i}\tau_{ij} - 1\big)I\end{bmatrix} < 0,$$

$$\begin{aligned}&\bar{Y}_i = \bar{Y}_i' > 0, \quad \Upsilon_{ij} = \Upsilon_{ij}' > 0, \quad \Upsilon_{ij} < G_{ij}^{-1}, \quad \tau_{ij} > 0,\\ &\qquad\qquad j\in\mathbf{N}_i, \quad i = 1,\ldots,N,\\ &\bar{\Theta} > 0. \qquad (22)\end{aligned}$$

Here, the symmetric matrix $\bar{\Theta}$ is composed as follows. Its diagonal blocks $\bar{\Theta}_{ii}$ are defined as

$$\begin{aligned}\bar{\Theta}_{ii} &= \begin{bmatrix}\bar{\theta}_0^{ii} & \bar{\theta}_1^{ii} & \ldots & \bar{\theta}_{l_i}^{ii}\\ (\theta_1^{ii})' & G_{ij_1}^{-1} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ (\theta_{l_i}^{ii})' & 0 & \ldots & G_{ij_{l_i}}^{-1}\end{bmatrix},\\ \bar{\theta}_0^{ii} &= \sum_{j\in\mathbf{N}_i}W_{ij}'\Upsilon_{ij}W_{ij} + (\bar{\gamma}_i^{-2} + \sum_{j\in\mathbf{N}_i}\tau_{ij})I - \gamma^{-2}P_{ii},\\ \bar{\theta}_k^{ii} &= W_{ij_k}'\Upsilon_{ij_k}, \quad k = 1,\ldots,l_i.\end{aligned}$$

Also, its off-diagonal blocks $\bar{\Theta}_{ij}$, $i,j = 1,\ldots,N$, $j\neq i$, are

$$\bar{\Theta}_{ij} = \begin{cases}\begin{bmatrix}\Psi_{ij} - \gamma^{-2}P_{ij} & \mathbf{0}_{n\times M_j}\\ \mathbf{0}_{n\times M_i} & \mathbf{0}_{M_i\times M_j}\end{bmatrix}, & i<j,\\ \bar{\Theta}_{ji}', & i>j,\end{cases}$$

where

$$\Psi_{ij} = \begin{cases}-W_{ij}'\Upsilon_{ij}W_{ij} - W_{ji}'\Upsilon_{ji}W_{ji}, & j\in\mathbf{N}_i, i\in\mathbf{N}_j;\\ -W_{ij}'\Upsilon_{ij}W_{ij}, & j\in\mathbf{N}_i, i\notin\mathbf{N}_j;\\ -W_{ji}'\Upsilon_{ji}W_{ji}, & j\notin\mathbf{N}_i, i\in\mathbf{N}_j;\\ 0 & j\notin\mathbf{N}_i, i\notin\mathbf{N}_j.\end{cases}$$

The LMIs (22), (16) represent a linear constraint on the variables $\bar{Y}_i = \bar{Y}_i' > 0$, $\bar{\gamma}_i^{-2}$, Υ_{ij}, $\tau_{ij} > 0$ ($j\in\mathbf{N}_i$, $i = 1,\ldots,N$), and γ^{-2}. Since γ^2 represents the disturbance attenuation level in the distributed filter, a suitable set of filter parameters is of interest which minimizes this variable. This can be numerically achieved by solving the convex optimization problem

$$\sup\gamma^{-2} \quad \text{subject to (22), (16).} \qquad (23)$$

Let γ^{*2} be the value of the supremum in (23).

Theorem 2: Let the pair (A,B) be stabilizable. Given a positive semidefinite weighting matrix $P = P' \in \mathbb{R}^{nN\times nN}$, suppose $\gamma^2 > \gamma^{*2}$, τ_{ij}, Υ_{ij}, $\bar{\gamma}_i^{-2}I$ and $\bar{Y}_i$, $j\in\mathbf{N}_i$, $i = 1,\ldots,N$, are a feasible collection of matrices and scalars that satisfy the constraints of the convex optimization problem (23). Then each Riccati equation (20) with $R_i = (\gamma/\bar{\gamma}_i)^2$ has a positive definite bounded solution on $[0,\infty)$. Furthermore, the corresponding filtering algorithm (21), (20) verifies claims (i) and (ii) of Problem 1.

TABLE I

SOLUTIONS TO THE PROBLEM (23)

	Simulation 1: $\bar{Z}_{ij} > 0$		Simulation 2: $\bar{Z}_{ij} > 0.1I$	
	$\gamma^2 = 0.2500$		$\gamma^2 = 0.3116$	
Node	$\bar{\gamma}_i^2$	$\min_j \lambda_{\min}(\bar{Z}_{ij})$	$\bar{\gamma}_i^2$	$\min_j \lambda_{\min}(\bar{Z}_{ij})$
1	0.2643	2.6219×10^{-4}	0.6288	0.1074
2	0.0185	0.0250	0.0260	0.3416
3	0.0181	0.0158	0.0395	0.1788
4	0.1313	2.7548×10^{-4}	0.2904	0.1000
5	0.0176	0.0263	0.0265	0.2682

As Theorem 2 shows, solving the SDP problem (23) allows us to determine the suboptimal γ^2 as well as the local disturbance attenuation levels $\bar{\gamma}_i^2$ that characterize local performance of the node filters (see (6)) as well as the matrices $\bar{Z}_{ij}$ in condition (11) consistent with that performance. Then sensitivity of performance of the obtained local filters to the neighbours' accuracy can be assessed using, e.g., the eigenvalues of $\bar{Z}_{ij}$, as explained in Section II-B. This process is illustrated in the example presented next.

V. ILLUSTRATIVE EXAMPLE

In this section, a simulated network of five sensor nodes is considered that are to estimate a three-dimensional plant. The plant's state matrix and the input matrix are

$$A = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.4 \end{bmatrix}. \tag{24}$$

The matrix A corresponds to one of the regimes of the controlled Chua electronic circuit considered in [13].

The network consists of five nodes, its connectivity is described by the set of directed edges $\mathbf{E} = \{(1,3),(2,3),(3,1),(3,2),(3,4),(4,3),(4,5),(5,4)\}$. The matrices C_i were taken from [13] to be $C_1 = C_4 = 0.001 \times [3.1923 \ -4.6597 \ 1]$ and $C_2 = C_3 = C_5 = [-0.8986 \ 0.1312 \ -1.9703]$. Note that none of the pairs (A, C_i) are observable, with (A, C_1) and $A, C_4)$ being not detectable. Also following [13], all communication matrices are taken to be $W_{ij} = I_{3\times 3}$ if $(i,j) \in \mathbf{E}$. Also, we let $D_i = 0.025 I_{1\times 3}$ and $F_{ij} = 0.5 I_{3\times 3}$.

For the above system two distributed filter designs were compared. Both filters were designed to achieve a suboptimal H_∞ consensus performance, that is, in (5) we selected $P = (\mathbf{L}+\mathbf{L}_\top)\otimes I$, cf. [4], [5]. First, the optimization problem (23) was solved with the above parameters. Next, an additional constraint $\bar{Z}_{ij} > 0.1I$ was imposed. The computed levels of local H_∞ attenuation $\bar{\gamma}_i^2$ and the minimum eigenvalues of the computed matrices $\bar{Z}_{ij}$ with which the Property **P3** is guaranteed by Theorem 2 are shown in Table I. One can see that in the first case, the filters at nodes 1 and 4 have much larger constants $\bar{\gamma}_i^2$ and substantially smaller values of eigenvalues of matrices $\bar{Z}_{ij}$. Together these features indicate that these filters are significantly more sensitive to accuracy of their neighbours. This is not unexpected given that the pairs (A, C_1), (A, C_4) are not detectable. The second simulation indicates that robustness of the estimators with respect to accuracy of their neighbours can be improved by moderately increasing γ^2 and $\bar{\gamma}_i^2$.

VI. CONCLUSIONS

In this paper we proposed a distributed filtering algorithm by utilizing an H_∞ minimum-energy filtering approach to the design of constituent filters. The algorithm employs a decoupled computation of the individual filter coefficients. This is achieved by considering the estimation error of neighbouring agents as additional exogenous disturbances weighted according to the nodes' confidence in their neighbours' estimates. The conditions are obtained under which the proposed filter to provides guaranteed internal stability and desired disturbance attenuation of the network error dynamics. In addition each local filter guarantees certain disturbance attenuation when assisted by the neighbours. We have also provided a simulation example that confirms convergence of the proposed filter in the case a system has undetectable pairs (A, C_i) at some of the nodes. Tuning of the filter is discussed to reduce the dependence of the local filters from neighbours accurate estimates.

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