# Parameter Tracking via Optimal Distributed Beamforming in an Analog Sensor Network

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Abstract—We consider the problem of optimal distributed beamforming in a sensor network where the sensors observe a dynamic parameter in noise and coherently amplify and forward their observations to a fusion center (FC). The FC uses a Kalman filter to track the parameter using the observations from the sensors, and we show how to find the optimal gain and phase of the sensor transmissions under both global and individual power constraints in order to minimize the mean squared error (MSE) of the parameter estimate. For the case of a global power constraint, a closed-form solution can be obtained. A numerical optimization is required for individual power constraints, but the problem can be relaxed to a semidefinite programming problem (SDP), and we show how the optimal solution can be constructed from the solution to the SDP. Simulation results show that compared with equal power transmission, the use of optimized power control can significantly reduce the MSE.

### I. INTRODUCTION

In an analog-based distributed sensor network, the sensor nodes multiply their noisy observations by a gain and phase and transmit the result to a fusion center (FC). The FC then uses the sum of the received signals to estimate the parameter. The key problem in this setting is to design the optimal gain and phase multiplier for each sensor in order to obtain the most accurate parameter estimate at the FC. Furthermore, these multipliers must be updated in situations where the parameter is time-varying. Some examples of prior work on this type of problem include [1], [2], [3], [4], [5]. In [1], an orthogonal multiple access channel (MAC) was assumed between the sensor nodes and FC. The FC used a best linear unbiased estimator to estimate a static parameter and and the optimal power allocation with both sum and individual power constraints were investigated to minimize the mean square error (MSE). A coherent MAC was considered in [2] and a linear minimum mean square error estimator was adopted at the FC to estimate the Gaussian source. The optimal power allocation problem was solved under a total transmit power constraint. A phase-only optimization problem was formulated in [3] and the phase of the transmitted signal from different sensor nodes was adjusted such that the received signal at the FC can be added coherently to optimize the performance of a maximum likelihood (ML) estimator. In [4] and [5], the parameter of interest was modeled as a dynamic process and the FC employed a Kalman filter to track the parameter. In [4], a power optimization problem was formulated to minimize the MSE under a sum power constraint and an asymptotic expression for the outage probability of the MSE was derived for a large number of sensor nodes. Additionally, the problem of minimizing MSE outage probability was studied in [5].

In this paper, we consider a setup similar to [4] and [5]. We assume that the parameter of interest is a dynamic process and the sensor nodes coherently amplify (gain and phase) and forward their observations of the parameter to the FC. The sensors act like a distributed beamformer, but they are also forwarding their observation of the background noise along with the measured parameter. The FC uses a Kalman filter to track the dynamic process, and we show how the transmit gain and phase of the sensor can be optimized at each time step to minimize the MSE of the parameter estimate. We assume that the optimized gain and phase is fed back to the sensor from the FC at each step, prior to the next measurement. The contributions of this paper are as follows:

- We find a closed-form solution for the optimal transmit gain and phase that minimizes MSE under a sum power constraint. This problem was also solved in [4] using the KKT conditions derived in [2]. However, our approach converts the problem to a Rayleigh quotient maximization problem and results in a simpler and more direct solution.
- 2) The problem of minimizing the MSE under individual sensor power constraints is solved by relaxing it to a semi-definite programming (SDP) problem, and then proving that the optimal solution can be constructed from the SDP solution.
- For a suboptimal case where the sensor nodes use equal power transmission, we derive an exact expression for the MSE outage probability.

#### II. SYSTEM MODEL

We model the complex-valued dynamic parameter  $\theta_n$  as a first-order Gauss-Markov process:

$$\theta_n = \alpha \theta_{n-1} + u_n$$

where the process noise  $u_n$  has distribution  $\mathcal{CN}(0, \sigma_u^2)$ . Assuming the FC and the sensor node are all configured with a single antenna, the received signal at the FC is

$$y_n = \mathbf{a}_n^H \mathbf{h}_n \theta_n + \mathbf{a}_n^H \mathbf{H}_n \mathbf{v}_n + w_n , \qquad (1)$$

where  $\mathbf{h}_n = [h_{1,n}, \dots, h_{N,n}]^T$  and  $h_{i,n} \in \mathbb{C}$  is the channel coefficient between the *i*th sensor and the FC,  $\mathbf{a}_n$  =  $[a_{1,n},\ldots,a_{N,n}]^T$  is the conjugate of the sensor transmit gain and phase,  $\mathbf{H}_n = \text{diag}\{h_{1,n}, \dots, h_{N,n}\}$ ,  $\mathbf{v}_n$  is Gaussian measurement noise at the sensors with covariance  $\mathbf{V}$  =  $\mathbb{E}\{\mathbf{v}_n\mathbf{v}_n^H\} = \operatorname{diag}\left\{\sigma_{v,1}^2, \cdots, \sigma_{v,N}^2\right\}$ , and  $w_n$  is additive white Gaussian noise at the fusion center with variance  $\sigma_w^2$ . The channel parameter is defined as

$$h_{i,n} = \frac{\tilde{h}_{i,n}}{d_i^{\gamma}} \,,$$

where  $h_{i,n}$  is complex Gaussian with zero mean and unit variance,  $d_i$  denotes the distance between sensor i and the FC, and  $\gamma$  is the path-loss exponent.

Based on the above dynamic and observation models, the standard Kalman Filter is defined by the following quantities:

- Prediction Step:  $\hat{\theta}_{n|n-1} = \alpha \hat{\theta}_{n-1|n-1}$  Prediction MSE:  $P_{n|n-1} = \alpha^2 P_{n-1|n-1} + \sigma_u^2$
- Kalman Gain:

$$k_n = \frac{P_{n|n-1}\mathbf{h}_n^H \mathbf{a}_n}{\mathbf{a}_n^H \mathbf{H}_n \mathbf{V} \mathbf{H}_n^H \mathbf{a}_n + P_{n|n-1}\mathbf{a}_n^H \mathbf{h}_n \mathbf{h}_n^H \mathbf{a}_n + \sigma_w^2}$$

• Measurement Update:

$$\hat{\theta}_{n|n} = \hat{\theta}_{n|n-1} + k_n \left( y_n - \mathbf{a}_n^H \mathbf{h}_n \hat{\theta}_{n|n-1} \right)$$

 $P_{n|n} = (1 - k_n \mathbf{a}_n^H \mathbf{h}_n) P_{n|n-1} .$ 

• MSE:

## III. MINIMIZING MSE UNDER GLOBAL POWER CONSTRAINT

In this section, we formulate and solve the problem under the assumption that the sensor nodes have a sum power constraint. The optimization problem is formulated as

$$\min_{\mathbf{a}_{n}} P_{n|n}$$
(3)
$$s.t. \mathbf{a}_{n}^{H} \mathbf{D} \mathbf{a}_{n} \leq P_{\max} ,$$

(2)

where  $\mathbf{D} = \text{diag}\{\sigma_{\theta}^2 + \sigma_{v,1}^2, \cdots, \sigma_{\theta}^2 + \sigma_{v,N}^2\}, \sigma_{\theta}^2$  denotes the variance of  $\theta_n$  and  $P_{\text{max}}$  is the maximum sum transmit power. From (2), to minimize MSE  $P_{n|n}$ , we need to maximize  $k_n \mathbf{a}_n^H \mathbf{h}_n$  which is calculated as

$$k_n \mathbf{a}_n^H \mathbf{h}_n = \frac{P_{n|n-1} \mathbf{a}_n^H \mathbf{h}_n \mathbf{h}_n^H \mathbf{a}_n^H}{\mathbf{a}_n \mathbf{H}_n \mathbf{V} \mathbf{H}_n^H \mathbf{a}_n + P_{n|n-1} \mathbf{a}_n^H \mathbf{h}_n \mathbf{h}_n^H \mathbf{a}_n + \sigma_w^2}$$

Thus, the optimization problem (3) is equivalent to

$$\max_{\mathbf{a}_{n}} \quad \frac{\mathbf{a}_{n}^{H}\mathbf{h}_{n}\mathbf{h}_{n}^{H}\mathbf{a}_{n}}{\mathbf{a}_{n}^{H}\mathbf{H}_{n}\mathbf{V}\mathbf{H}_{n}^{H}\mathbf{a}_{n} + \sigma_{w}^{2}}$$
(4)  
s.t. 
$$\mathbf{a}_{n}^{H}\mathbf{D}\mathbf{a}_{n} \leq P_{\max} .$$

Denote the optimal solution to problem (4) as  $\mathbf{a}_n^*$ . It is easy to verify that the sum transmit power constraint should be met with equality  $\mathbf{a}_n^{*H} \mathbf{D} \mathbf{a}_n^* = P_{\max}$ , so that (4) can be rewritten as

$$\max_{\mathbf{a}_{n}} \quad \frac{\mathbf{a}_{n}^{H}\mathbf{h}_{n}\mathbf{h}_{n}^{H}\mathbf{a}_{n}}{\mathbf{a}_{n}^{H}(\mathbf{H}_{n}\mathbf{V}\mathbf{H}_{n}^{H}+\frac{\sigma_{w}^{2}}{P_{\max}}\mathbf{D})\mathbf{a}_{n}}$$
(5)  
s.t. 
$$\mathbf{a}_{n}^{H}\mathbf{D}\mathbf{a}_{n}=P_{\max}.$$

Problem (5) maximizes a Rayleigh quotient under a quadratic constraint, which results in a simple closed-form solution. If we define  $\mathbf{B} = \mathbf{H}_n \mathbf{V} \mathbf{H}_n^H + \frac{\sigma_w^2}{P_{\max}} \mathbf{D}$ , the optimal solution is given by

$$\mathbf{a}_n^* = \sqrt{\frac{P_{\max}}{\mathbf{h}_n^H \mathbf{B}^{-1} \mathbf{D}^{-1} \mathbf{B}^{-1} \mathbf{h}_n}} \mathbf{B}^{-1} \mathbf{h}_n$$

and the optimal value of (5) is calculated as

$$\frac{\mathbf{a}_n^{*H}\mathbf{h}_n\mathbf{h}_n^H\mathbf{a}_n^*}{\mathbf{a}_n^{*H}(\mathbf{H}_n\mathbf{V}\mathbf{H}_n^H+\frac{\sigma_w^2}{P_{\max}}\mathbf{D})\mathbf{a}_n^*} \quad = \quad \mathbf{h}_n^H\mathbf{B}^{-1}\mathbf{h}_n \;,$$

which is a random variable that depends on the distribution of  $\mathbf{h}_n$ . An upper bound for  $\mathbf{h}_n^H \mathbf{B}^{-1} \mathbf{h}_n$  is given by

$$\mathbf{h}_{n}^{H} \mathbf{B}^{-1} \mathbf{h}_{n} \stackrel{(a)}{\leq} \mathbf{h}_{n}^{H} (\mathbf{H} \mathbf{V} \mathbf{H}^{H})^{-1} \mathbf{h}_{n}$$
$$= \sum_{i=1}^{N} \frac{1}{\sigma_{v,i}^{2}}, \qquad (6)$$

where (a) follows from  $\mathbf{B}^{-1} \prec (\mathbf{H}\mathbf{V}\mathbf{H}^{H})^{-1}$ . Plugging (6) into (2), we obtain a lower bound on the MSE:

$$P_{n|n} > \left(1 - \frac{1}{1 + \frac{1}{\left(\sum_{i=1}^{N} \frac{1}{\sigma_{v,i}^{2}}\right)P_{n|n-1}}}\right)P_{n|n-1}$$
$$= \frac{P_{n|n-1}}{1 + \left(\sum_{i=1}^{N} \frac{1}{\sigma_{v,i}^{2}}\right)P_{n|n-1}}.$$

This lower bound can be asymptoticly achieved with  $P_{\rm max} \rightarrow$  $\infty$  or  $\sigma_w^2 \ll \sigma_{v,i}^2,$  and the corresponding sensor transmit gain and phase is

$$\mathbf{a}_{n}^{*} = \sqrt{\frac{P_{\max}}{\sum_{i=1}^{N} \frac{1}{\sigma_{v,i}^{2}(\sigma_{\theta}^{2} + \sigma_{v,i}^{2})}} \left[\frac{1}{\bar{h}_{1,n}\sigma_{v,1}^{2}}, \cdots, \frac{1}{\bar{h}_{N,n}\sigma_{v,N}^{2}}\right]}.$$
 (7)

From (7), it can be observed that sensors whose product  $|h_{i,n}|\sigma_{v,i}^2$  is small will be allocated more transmit power.

## IV. MINIMIZING MSE UNDER INDIVIDUAL POWER CONSTRAINTS

When the sensor nodes have individual power constraints, the optimal distributed beamforming problem becomes

$$\max_{\mathbf{a}_{n}} \qquad \frac{\mathbf{a}_{n}^{H}\mathbf{h}_{n}\mathbf{h}_{n}^{H}\mathbf{a}_{n}}{\mathbf{a}_{n}^{H}\mathbf{H}_{n}\mathbf{V}\mathbf{H}_{n}^{H}\mathbf{a}_{n} + \sigma_{w}^{2}} \qquad (8)$$
  
s.t. 
$$|a_{i,n}|^{2}(\sigma_{\theta}^{2} + \sigma_{v,i}^{2}) \leq P_{\max,i}, \quad i = 1, \cdots, N,$$

where  $P_{\max,i}$  is the maximum transmit power of the *i*th sensor node. Problem (8) is a quadratically constrained ratio of two quadratic functions (QCRQ). Using the approach proposed in [6], the QCRQ problem can be relaxed to an SDP problem. Introduce a real auxiliary variable t and define  $\tilde{\mathbf{a}}_n = t\mathbf{a}_n$ , so that problem (8) is equivalent to

$$\max_{\mathbf{a}_{n},t} \frac{\tilde{\mathbf{a}}_{n}^{H}\mathbf{h}_{n}\mathbf{h}_{n}\mathbf{h}_{n}^{H}\tilde{\mathbf{a}}_{n}}{\tilde{\mathbf{a}}_{n}^{H}\mathbf{H}_{n}\mathbf{V}\mathbf{H}_{n}^{H}\tilde{\mathbf{a}}_{n} + \sigma_{w}^{2}t^{2}} \qquad (9)$$
s.t.  $\tilde{\mathbf{a}}_{n}^{H}\mathbf{D}_{i}\tilde{\mathbf{a}}_{n} \leq t^{2}P_{\max,i}, \quad i = 1, \cdots, N,$ 
 $t \neq 0,$ 

where  $\mathbf{D}_i = \text{diag}\{0, \dots, 0, \sigma_{\theta}^2 + \sigma_{v,i}^2, 0, \dots, 0\}$ . Then, we can further rewrite problem (9) as

$$\max_{\mathbf{a}_{n},t} \quad \tilde{\mathbf{a}}_{n}^{H}\mathbf{h}_{n}\mathbf{h}_{n}^{H}\tilde{\mathbf{a}}_{n} \tag{10}$$

$$s.t. \quad \tilde{\mathbf{a}}_{n}^{H}\mathbf{H}_{n}\mathbf{V}\mathbf{H}_{n}^{H}\tilde{\mathbf{a}}_{n} + \sigma_{w}^{2}t^{2} = 1,$$

$$\tilde{\mathbf{a}}_{n}^{H}\mathbf{D}_{i}\tilde{\mathbf{a}}_{n} \leq t^{2}P_{\max,i}, \quad i = 1, \cdots, N.$$

Note that the constraints in problem (10) already guarantee that  $t \neq 0$ , so the constraint  $t \neq 0$  is removed.

Define  $\bar{\mathbf{a}}_n = [\tilde{\mathbf{a}}_n^H, t]^H$ ,  $\bar{\mathbf{H}}_n = \begin{bmatrix} \mathbf{h}_n \mathbf{h}_n^H & 0\\ \mathbf{0}^T & 0 \end{bmatrix}$ ,  $\bar{\mathbf{C}}_n = \begin{bmatrix} \mathbf{H}_n \mathbf{V} \mathbf{H}_n^H & 0\\ \mathbf{0}^T & \sigma_w^2 \end{bmatrix}$ , and  $\bar{\mathbf{D}}_i = \begin{bmatrix} \mathbf{D}_i & 0\\ \mathbf{0}^T & -P_{\max,i} \end{bmatrix}$ , so that problem (10) can be written in the compact form

$$\begin{array}{ll}
\max_{\bar{\mathbf{a}}_n} & \bar{\mathbf{a}}_n^H \bar{\mathbf{H}}_n \bar{\mathbf{a}}_n & (11) \\
s.t. & \bar{\mathbf{a}}_n^H \bar{\mathbf{C}}_n \bar{\mathbf{a}}_n = 1 , \\
& \bar{\mathbf{a}}_n^H \bar{\mathbf{D}}_i \bar{\mathbf{a}}_n \leq 0 , \quad i = 1, \cdots, N .
\end{array}$$

Problem (11) is equivalent to

$$\begin{array}{ll}
\max_{\bar{\mathbf{A}}} & \operatorname{tr}(\bar{\mathbf{A}}\bar{\mathbf{H}}_{n}) & (12) \\
s.t. & \operatorname{tr}(\bar{\mathbf{A}}\bar{\mathbf{C}}_{n}) = 1 , \\
& \operatorname{tr}(\bar{\mathbf{A}}\bar{\mathbf{D}}_{i}) \leq 0 , \quad i = 1, \cdots, N , \\
& \operatorname{rank}(\bar{\mathbf{A}}) = 1 .
\end{array}$$

By relaxing the rank-one constraint on  $\overline{A}$ , we convert problem (12) to a standard SDP problem:

$$\begin{array}{ll}
\max_{\bar{\mathbf{A}}} & \operatorname{tr}(\bar{\mathbf{A}}\bar{\mathbf{H}}_{n}) & (13) \\
s.t. & \operatorname{tr}(\bar{\mathbf{A}}\bar{\mathbf{C}}_{n}) = 1 , \\
& \operatorname{tr}(\bar{\mathbf{A}}\bar{\mathbf{D}}_{i}) \leq 0 , \quad i = 1, \cdots, N , \\
& \bar{\mathbf{A}} \succeq 0 .
\end{array}$$

The above problem can be solved in polynomial time using the interior point method. Due to the relaxation of the rankconstraint on  $\bar{\mathbf{A}}$ , the optimal value of problem (13) provides an upper bound for problem (8). After obtaining the optimal solution  $\bar{\mathbf{A}}^*$ , a rank-one solution  $\mathbf{a}_n^*$  can be recovered for the original problem (8). In the following, we show that based on  $\bar{\mathbf{A}}^*$  a rank-one solution  $\mathbf{a}_n^*$  can be constructed such that  $\mathbf{a}_n^*$  is the optimal solution to problem (8).

Defining  $\bar{\mathbf{A}}_{l,m}^*$  as the (l,m)th element of  $\bar{\mathbf{A}}^*$  and  $\bar{\mathbf{A}}_N^*$  as the *N*th order leading principal submatrix of  $\bar{\mathbf{A}}^*$  formed by deleting the (N+1)st row and column of  $\bar{\mathbf{A}}^*$ , we propose the following theorem:

**Theorem 1.** Define the optimal solution to problem (13) as  $\bar{\mathbf{A}}^*$ , then  $\bar{\mathbf{A}}^*_N = \mathbf{a}\mathbf{a}^H$  and the optimal solution to problem (8) is given by  $\mathbf{a}^*_n = \frac{1}{\sqrt{\mathbf{A}^*_{N+1,N+1}}}\mathbf{a}$ .

*Proof:* We first utilize the strong duality between problem (13) and its dual problem to show the property of the optimal solution  $\bar{\mathbf{A}}^*$ . The dual problem of problem (13) is given by [7]:

$$\min_{y_i,z} \qquad z \qquad (14)$$

$$s.t. \qquad \sum_{i=1}^N y_i \bar{\mathbf{D}}_i + z \bar{\mathbf{C}}_n - \bar{\mathbf{H}}_n \succeq 0 ,$$

$$y_1, \dots, y_N, z \ge 0 .$$

It is easy to verify that there exists strictly feasible points for problem (13) and problem (14). For problem (13), we can construct

$$\bar{\mathbf{A}}^f = \operatorname{diag}\{ab, \cdots, ab, b\}$$
,

where  $0 < a < \min_i \frac{P_{\max,i}}{\sigma_{\theta}^2 + \sigma_{v,i}^2}$  and  $b = \frac{1}{\sum_{i=1}^N a |h_{n,i}|^2 \sigma_{v,i}^2 + \sigma_w^2}$ . For problem (14), we can randomly select  $y_i^f > 0$ , and set  $z^f$  large enough such that

$$z^{f} > \max\left\{\frac{\mathbf{h}_{n}^{H}\mathbf{h}_{n} + \sum_{i=1}^{N} y_{i}^{f} P_{\max,i}}{\sigma_{w}^{2}}, \frac{\mathbf{h}_{n}^{H}\mathbf{h}_{n} - y_{i}^{f}(\sigma_{\theta}^{2} + \sigma_{v,i}^{2})}{|h_{n,i}|^{2}\sigma_{v,i}^{2}}\right\}$$

Then, according to Slater's theorem, strong duality holds between the primal problem (13) and the dual problem (14) and we have the following complementary condition:

$$\operatorname{tr}(\bar{\mathbf{A}}^*\mathbf{G}^*) = 0 , \qquad (15)$$

where  $\mathbf{G}^* = \sum_{i=1}^N y_i^* \bar{\mathbf{D}}_i + z^* \bar{\mathbf{C}}_n - \bar{\mathbf{H}}_n$  and  $y_i^*$  and  $z^*$  denote the optimal solution to problem (14). Due to the special structure of  $\bar{\mathbf{D}}_i$ ,  $\bar{\mathbf{C}}_n$  and  $\bar{\mathbf{H}}_n$ ,  $\mathbf{G}^*$  can be expressed as

$$\mathbf{G}^* = \left[ \begin{array}{cc} \mathbf{G}_N^* & \mathbf{0} \\ \mathbf{0}^T & \mathbf{G}_{N+1,N+1}^* \end{array} \right] \; ,$$

where  $\mathbf{G}_{N}^{*} = \sum_{i=1}^{N} y_{i}^{*} \mathbf{D}_{i} + z^{*} \mathbf{H}_{n} \mathbf{V} \mathbf{H}_{n}^{H} - \mathbf{h}_{n} \mathbf{h}_{n}^{H}$  and  $\mathbf{G}_{N+1,N+1}^{*} = z^{*} \sigma_{w}^{2} - \sum_{i=1}^{N} y_{i}^{*} P_{\max,i}$ . Since both  $\bar{\mathbf{A}}^{*}$  and  $\mathbf{G}^{*}$  are positive semidefinite, Eq. (15) is equivalent to

$$\bar{\mathbf{A}}^*\mathbf{G}^*=0.$$

Additionally, with consideration of the structure of  $\mathbf{G}^*$ , we have

$$\mathbf{A}_N^*\mathbf{G}_N^* = 0$$

Define  $\mathbf{V}_G$  as a set of vectors orthogonal to the row space of  $\mathbf{G}_N^*$ . Then the column vectors of  $\bar{\mathbf{A}}_N^*$  must belong to  $\operatorname{span}(\mathbf{V}_G)$  and  $\operatorname{rank}(\bar{\mathbf{A}}_N^*) \leq \operatorname{rank}(\mathbf{V}_G)$ . Given two matrices  $\mathbf{M}$  and  $\mathbf{N}$ , we have  $\operatorname{rank}(\mathbf{M} + \mathbf{N}) \geq |\operatorname{rank}(\mathbf{M}) - \operatorname{rank}(\mathbf{N})|$ [8], thus, a lower bound for  $\operatorname{rank}(\mathbf{G}_N^*)$  is calculated as

$$\operatorname{rank}(\mathbf{G}_{N}^{*}) \geq \operatorname{rank}\left(\sum_{i=1}^{N} y_{i}^{*}\mathbf{D}_{i} + z^{*}\mathbf{H}_{n}\mathbf{V}\mathbf{H}_{n}^{H}\right) - \operatorname{rank}(\mathbf{h}_{n}\mathbf{h}_{n}^{H})$$
$$= N - 1.$$

An upper bound for  $rank(\mathbf{V}_G)$  is then given by

$$\operatorname{rank}(\mathbf{V}_G) = N - \operatorname{rank}(\mathbf{G}_N^*) \quad (16)$$
  
$$\leq 1.$$

Since  $\operatorname{tr}(\bar{\mathbf{A}}^*\bar{\mathbf{H}}) = \mathbf{h}_n^H \bar{\mathbf{A}}_N^* \mathbf{h}_n$  and  $\operatorname{tr}(\bar{\mathbf{A}}^*\bar{\mathbf{H}}) > \operatorname{tr}(\bar{\mathbf{A}}^f \bar{\mathbf{H}}) > 0$ , we have

$$\bar{\mathbf{A}}_N^* \neq 0 \qquad \operatorname{rank}(\bar{\mathbf{A}}_N^*) \ge 1$$
 (17)

Combining Eqs. (16) and (17), we have

$$\operatorname{rank}(\mathbf{A}_N^*) = 1$$

Define the rank-one decomposition of  $\bar{\mathbf{A}}_N^*$  as  $\bar{\mathbf{A}}_N^* = \mathbf{a}\mathbf{a}^H$ , so that the optimal rank-one solution to problem (13) is

$$\bar{\mathbf{a}}^* = [\mathbf{a}^H, \sqrt{\bar{\mathbf{A}}^*_{N+1,N+1}}]^H \ .$$

If the optimal solution of problems (8) is  $a_n^*$ , then

$$\frac{\mathbf{a}_{n}^{*H}\mathbf{h}_{n}\mathbf{h}_{n}^{H}\mathbf{a}_{n}^{*}}{\mathbf{a}_{n}^{*H}\mathbf{H}\mathbf{V}\mathbf{H}^{H}\mathbf{a}_{n}^{*}+\sigma_{w}^{2}} \leq \operatorname{tr}(\bar{\mathbf{A}}^{*}\bar{\mathbf{H}})$$

where equality can be achieved provided that an optimal rankone solution exists for problem (13). Since  $\operatorname{tr}(\bar{\mathbf{A}}^*\bar{\mathbf{D}}_i) \leq 0$ ,  $\bar{\mathbf{A}}^* \neq 0$  and  $\mathbf{D}_i \succ 0$ , then we have  $\bar{\mathbf{A}}^*_{N+1,N+1} > 0$ , otherwise  $\operatorname{tr}(\bar{\mathbf{A}}^*\bar{\mathbf{D}}_i) > 0$ , which contradicts the constraints in problem (13). Based on  $\bar{\mathbf{a}}^*$ , the optimal solution to problem (8) is given by

$$\mathbf{a}_n^* = rac{1}{\sqrt{ar{\mathbf{A}}_{N+1,N+1}^*}} \mathbf{a} \ ,$$

and we have

$$\frac{\mathbf{a}_n^{*H}\mathbf{h}_n\mathbf{h}_n^H\mathbf{a}_n^*}{\mathbf{a}_n^{*H}\mathbf{H}\mathbf{V}\mathbf{H}^H\mathbf{a}_n^*+\sigma_w^2}=\mathrm{tr}(\bar{\mathbf{A}}^*\bar{\mathbf{H}})$$

which verifies the optimality of  $\mathbf{a}_n^*$ .

### V. EQUAL POWER ALLOCATION

Here we calculate the MSE outage probability of a suboptimal solution in which each sensor transmits with the same power. The outage probability derived here can serve as an upper bound for the outage performance of the optimal algorithm with individual power constraints. The transmit gain is given by

$$\mathbf{a}_e = \sqrt{\frac{P_{\max}}{N}} \left[ \frac{1}{\sqrt{\sigma_{\theta}^2 + \sigma_{v,1}^2}}, \cdots, \frac{1}{\sqrt{(\sigma_{\theta}^2 + \sigma_{v,N}^2)}} \right]$$

For this suboptimal approach, the MSE is calculated as

$$P_{n|n} = \left(1 - \frac{P_{n|n-1}\mathbf{a}_e^H \mathbf{h}_n \mathbf{h}_n^H \mathbf{a}_e}{\mathbf{a}_e^H \mathbf{H}_n \mathbf{V} \mathbf{H}_n \mathbf{a}_e + P_{n|n-1} \mathbf{a}_e^H \mathbf{h}_n \mathbf{h}_n^H \mathbf{a}_e + \sigma_w^2}\right) P_{n|n-1} ,$$

which is a random variable depending on the distribution of the channel parameter  $\mathbf{h}_n$ . Define the outage probability as  $P_{out} = \Pr \{ P_{n|n} > \epsilon \}$ , so that

$$P_{out} = \Pr\left\{\frac{\mathbf{a}_{e}^{H}\mathbf{h}_{n}\mathbf{h}_{n}^{H}\mathbf{a}_{e}}{\mathbf{a}_{e}^{H}\mathbf{H}_{n}\mathbf{V}\mathbf{H}_{n}^{H}\mathbf{a}_{e} + \sigma_{w}^{2}} < \beta\right\}$$
  
$$= \Pr\left\{\mathbf{a}_{e}^{H}\mathbf{h}_{n}\mathbf{h}_{n}^{H}\mathbf{a}_{e} - \beta\mathbf{a}_{e}^{H}\mathbf{H}_{n}\mathbf{V}\mathbf{H}_{n}^{H}\mathbf{a}_{e} < \beta\sigma_{w}^{2}\right\}$$
  
$$= \Pr\left\{\tilde{\mathbf{h}}_{n}^{H}\left(\bar{\mathbf{D}}\mathbf{a}_{e}\mathbf{a}_{e}^{H}\bar{\mathbf{D}} - \beta\mathbf{E}\right)\tilde{\mathbf{h}}_{n} \leq \beta\sigma_{w}^{2}\right\},$$

where 
$$\beta = \frac{P_{n|n-1}-\epsilon}{\epsilon P_{n|n-1}}, \quad \bar{\mathbf{D}} = \operatorname{diag}\left\{\frac{1}{d_{1}^{\gamma}}, \cdots, \frac{1}{d_{N}^{\gamma}}\right\},$$
  
 $\mathbf{E} = \operatorname{diag}\left\{\frac{P_{\max}\sigma_{v,i}^{2}}{N(\sigma_{\theta}^{2}+\sigma_{v,i}^{2})d_{i}^{2\gamma}}, \cdots, \frac{P_{\max}\sigma_{v,N}^{2}}{N(\sigma_{\theta}^{2}+\sigma_{v,N}^{2})d_{i}^{2\gamma}}\right\}, \quad \tilde{\mathbf{h}}_{n} = [\tilde{h}_{1,n}, \cdots, \tilde{h}_{N,n}].$ 

Define  $\mathbf{B} = \mathbf{D}\mathbf{a}_e \mathbf{a}_e^H \mathbf{D} - \beta \mathbf{E}$ , and denote the eigenvalues of **B** as  $\lambda_1, \dots, \lambda_N$ , then the random variable  $\tilde{\mathbf{h}}_n^H \mathbf{B} \mathbf{h}_n$  can be viewed as the weighted sum of independent chi-square random variables  $\sum_i^N \lambda_i \chi_i(2)$ . Based on the results in [9], we have

$$P_{out} = 1 - \sum_{i=1}^{N} \frac{\lambda_i^N}{\prod_{l \neq i} (\lambda_i - \lambda_l)} \frac{1}{|\lambda_i|} e^{-\frac{(P_n|_n - 1^{-\epsilon})\sigma_w^2}{\epsilon P_n|_{n-1}\lambda_i}} u(\lambda_i) ,$$
(18)

where  $u(\cdot)$  is the unit step function. If we let  $e_1 \ge \cdots \ge e_N$  denote the eigenvalues of **E**, then from Weyl's inequality [10] we have the following bounds:

$$\mathbf{a}_{e}^{H}\bar{\mathbf{D}}^{2}\mathbf{a}_{e} - \beta e_{1} \leq \lambda_{1} \leq \mathbf{a}_{e}^{H}\bar{\mathbf{D}}^{2}\mathbf{a}_{e} - \beta e_{N} -\beta e_{N-i+1} \leq \lambda_{i} \leq -\beta e_{N-i+2} , \quad 2 \leq i \leq N ,$$

where  $\mathbf{a}_{e}^{H} \bar{\mathbf{D}}^{2} \mathbf{a}_{e} = \sum_{i=1}^{N} \frac{P_{\max}}{N(\sigma_{\theta}^{2} + \sigma_{v,i}^{2}) d_{i}^{2\gamma}}$ . Since only  $\lambda_{1}$  can be positive, equation (18) can be simplified as

$$P_{out} = \begin{cases} 1 - \frac{\lambda_1^{N-1}}{\prod_{l \neq 1} (\lambda_1 - \lambda_l)} e^{-\frac{(P_n|_{n-1} - \epsilon)\sigma_w^2}{\epsilon P_n|_{n-1}\lambda_1}}, & \lambda_1 > 0, \\ 1, & \lambda_1 \le 0. \end{cases}$$

Since it is not possible to evaluate the  $\lambda_i$  in closed-form, the above outage probability expression must be calculated numerically.



Fig. 1: MSE vs. number of sensors for  $P_{\text{max}} = 300$  or 3000.

## VI. SIMULATION RESULTS

To verify the performance of the proposed optimization approaches, the results of several simulation examples are described here. In the simulation, the distance to the sensors  $d_i$ is uniformly distributed over [2, 8] and the path loss exponent  $\gamma$  is set to 1. The variance  $\sigma_{\theta}^2$  is set to 1, and the  $P_{n|n-1}$ is initialized as 1. The MSE is obtained by averaging over 300 realizations of  $\mathbf{h}_n$ . The observation noise power  $\sigma_{v_i}^2$  is uniformly distributed over [0, 0.5] and the power of the additive noise at the FC is set to  $\sigma_w^2 = 0.5$ . Two different sum power constraints are considered  $P_{\text{max}} = 300$  or 3000. To fairly compare the results under the sum power constraint and the individual power constraint, we set  $P_{\max,i} = \frac{P_{\max}}{N}$ . In Fig. 1, the results show that compared with equal power allocation, the optimized power allocation significantly reduces the MSE. In fact, adding sensors with equal power allocation actually increases the MSE, while the MSE always decreases for the optimal methods. The extra flexibility of the global power constraint leads to better performance compared with individual power constraints, but the difference is not large. The lower bound shows the performance that could be achieved with  $P_{\rm max} \rightarrow \infty$ . The theoretical and simulated outage probabilities of the equal power allocation is presented in Fig 2. The results show that the theoretical analysis matches well with the simulations.

## VII. CONCLUSION

In this paper, we considered optimal distributed beamforming for an analog sensor network attempting to track a dynamic parameter under both global and individual power constraints. For the sum power constraint case, we derived a closedform solution for the optimal sensor transmit gain and phase. For individual power constraints, we developed a numerical



Fig. 2: MSE outage probability for equal power allocation vs. sum transmit power for N = 10 sensors.

optimization procedure that is guaranteed to find the optimal sensor gains and phases. We also derived an exact expression for the MSE outage probability of a suboptimal scheme in which each sensor transmits with equal power. Simulations were presented to verify the performance of the optimal algorithms and the accuracy of the MSE outage probability expression.

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