Random Access on Graphs: A Survey and New Results

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***Abstract*—This paper overviews the recently proposed coded slotted ALOHA (CSA) random access scheme and illustrates some new results in this area. In CSA, a randomly picked linear block code is employed by each user to encode segments of its bursts prior to transmission, where the choice of the code is performed with no coordination with the other users. On the receiver side iterative interference cancellation combined with decoding of the local codes is performed to recover from collisions. This process may be represented as an iterative decoding algorithm over a sparse bipartite graph.**

I. Introduction

Recently, a renewed interest in high-throughput random access (RA) techniques [1], [2] has been originated by observing how iterative signal processing can largely improve the transmission efficiency up to levels that are typical of coordinated medium access control (MAC) schemes [3]–[13]. The principal practical applications are i) satellite networks [5], [12]; ii) terrestrial wireless networks [6], [10] including cellular, ad-hoc, and sensor networks; iii) machine-to-machine communications [14]; iv) shared memories. In all cases, the main problem is the contention among the sources and the need to share the channel resource [1], [2], [15], [16].

Random multiple access is one of the indispensable medium access schemes, and it is used for example in the control channel of the cellular system and ad hoc networks. The earliest and still most important example is the ALOHA protocol [1]. In general, random multiple access has the main advantage of allowing a common channel to be dynamically shared by a group of users, assuming minimal or no coordination. Such an assumption is justified in practical scenarios either due to the lack of global information, or due to the intolerable delay associated with coordination establishment. Due to the lack of coordination and opportunistic transmission, packet reception can experience severe interference from overlapping transmissions (collisions), traditionally requiring the retransmission of the involved packets [1], [2], [15], [16].

Since the introduction of the ALOHA protocol [1], several RA protocols have been introduced. Among them, the specific sub-class of feedback-free RA protocols [17], [18] re-gained attention in the recent past (see e.g. [19]–[21]). In a framed slotted ALOHA (SA) protocol M users attempt the transmission of a *burst* (i.e., packet) of time duration T_{slot} over a MAC frame of time duration T_{frame}. Neglecting guard times, the MAC frame is composed of $N = T_{\text{frame}}/T_{\text{slot}}$ slots and the offered (channel) traffic is

$$G = \frac{M}{N}\,.$$

Within a MAC frame, each of the M active users sends its packet on a randomly chosen slot. We assume that, when more users pick the same slot, their packets are lost in the collision. The users whose packets have been collided must be informed by a feedback channel and retransmitted in a subsequent frame. The probability of success transmission in one slot $S = \Pr\{\text{success in one slot}\}$ is therefore

$$S = M\frac{1}{N}\left(1-\frac{1}{N}\right)^{M-1} \to \frac{M}{N}e^{-\frac{M}{N}} = Ge^{-G} \qquad (1)$$

where the limit is for large M, N and fixed $G = M/N$. The maximum throughput S of this scheme is $\max\{S\} = 1/e \simeq 0.37$ obtained with offered traffic $G = 1$. Despite their relatively low throughput, the simplicity of ALOHA-like random multiple access schemes has made them extremely popular in communication networks for over thirty years.

In this paper, we review a recent modification of the SA protocol dubbed coded slotted Aloha (CSA) [7], [8] which allows attaining a throughput S arbitrarily close to 1 [packet/slot] [11], [22]. We further provide an alternative proof of the capacity bound for CSA of [23], and we discuss a cooperative version of CSA, particularly appealing in wireless sensor network applications.

II. Coded Slotted Aloha

CSA works as follows. When a user wishes to transmit its burst of time duration T_{slot} over the MAC frame, the burst is splitted into k information *slices*, each of time duration $T_{\text{slice}} = T_{\text{slot}}/k$. The k information slices are then encoded by the user via a packet-oriented linear block code which generates n_h encoded slices, each of time duration T_{slice}. For each transmission, the code to be employed is drawn randomly by the user from a set $\mathcal{C}$ of n_c possible codes. For $h \in \{1, \ldots, n_c\}$ the h-th code, denoted by $\mathscr{C}_h \in \mathcal{C}$, is an (n_h, k, d_h) code, that is it has length n_h, dimension k, and minimum distance d_h. We further impose that, for all $h \in \{1, \ldots, n_c\}$ $\mathscr{C}_h$ has no idle symbols and fulfills $d_h \geq 2$.

We assume that, at any transmission, each user picks individually his local code according to a probability mass

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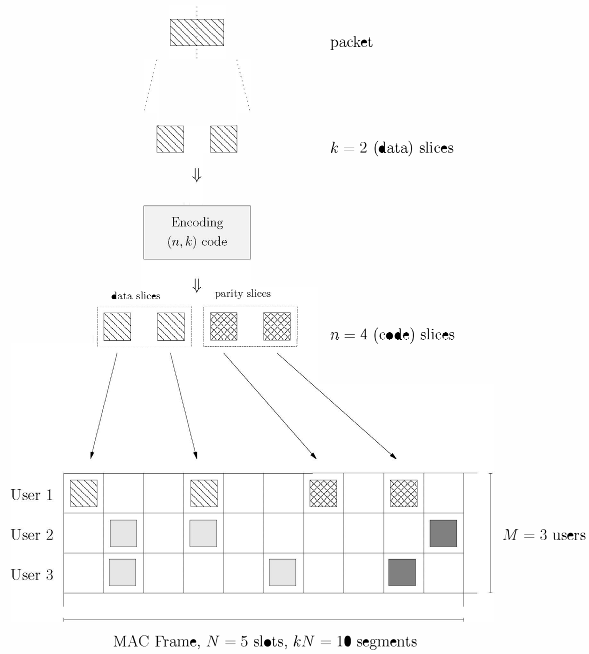

Fig. 1. Model of the access scheme. Each user splits its packets into $k = 2$ fragments. User 1 encodes them by a $(4, 2)$ linear block code. Users 2 and 3 adopt a $(3, 2)$ SPC code.

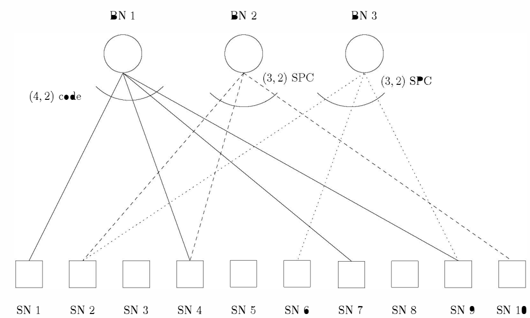

Fig. 2. Bipartite graph representation of the access scheme of Fig. 1. Segments are represented by SNs, packets (bursts) are represented by BNs. Each slice is associated with an edge.

function (p.m.f.) $\mathbf{\Lambda}= [\Lambda_h]_{h=1}^{n_c}$ which is the same for all users. Denoting again by T_{frame} the MAC frame duration, the MAC frame is composed of kN *segments*, each of time duration T_{slice}. The n_h coded slices are then transmitted by the user over n_h segments picked uniformly at random. An example for $k = 2$ is provided in Fig. 1. For the special case of $k = 1$, $T_{\text{slice}} = T_{\text{slot}}$ and each $\mathscr{C}_h$ is a repetition code of length n_h. The overall *rate* of the proposed scheme is defined as $R = k/\bar{n}$, where $\bar{n} := \sum_{h=1}^{n_c} \Lambda_h n_h$ is the expected length of the code employed by the generic user.

It is now convenient to introduce a graph representation of the RA scheme, depicted in Fig. 2 (with reference to the case provided in Fig. 1). The MAC frame status can be represented by a bipartite graph, $\mathcal{G} = (\mathcal{B}, \mathcal{S}, \mathcal{E})$, consisting of a set $\mathcal{B}$ of M *burst nodes* (one for each burst transmitted in the MAC frame), a set $\mathcal{S}$ of kN *sum nodes* (one for each segment), and a set $\mathcal{E}$ of edges. An edge connects a burst node (BN) $b_i \in \mathcal{B}$ to a sum node (SN) $s_j \in \mathcal{S}$ if and only if an encoded slice associated with the i-th burst is transmitted in the j-th segment. In other words, BNs correspond to bursts, SNs to segments, and edges to encoded slices. Therefore, a burst split into k information slices and encoded via the code $\mathscr{C}_h$ is represented as a BN with n_h neighbors. Correspondingly, a segment where d slices collide is represented as a SN with d connections. The number of edges emanating from a node is the node degree. Moreover, a BN where $\mathscr{C}_h$ is employed during the current transmission is referred to as a BN of type h.

Each coded slice associated with a BN of type h is equipped with information about the relevant user and with a pointer to the other $n_h - 1$ slices. On the receiver side, slices that collided in some segments with those sent by another user are marked as lost, so that a BN is connected to "known" edges and to "unknown" ones. Hence, some of its information slices are known, and the others unknown. At the generic BN (say of type h), erasure decoding of the code $\mathscr{C}_h$ may allow to recover some of the unknown encoded and information slices. It is now possible to subtract the interference contribution of the newly recovered encoded slices from the signal received in the corresponding segment. If $d-1$ slices that collided in a SN of degree d have been recovered by the corresponding BNs, the remaining slice becomes known. The *successive interference subtraction* process combined with local decoding at the BNs proceeds iteratively, i.e., cleaned slices may allow solving other collisions. This procedure is equivalent to iterative decoding of a doubly-generalized low-density parity-check (D-GLDPC) code over the erasure channel [24], [25], where variable nodes are generic linear block codes and check nodes are SPC codes.

In our analysis, we rely on three assumptions, namely: i) *Sufficiently high signal-to-noise ratio (SNR)*. This allows to claim that, when a slice is received in a clean segment, it is known at the receiver. ii) *Ideal channel estimation*. Under this assumption (and the previous one), ideal interference subtraction is possible, allowing the recovery of collided slices with a probability that is essentially one. iii) *Destructive collisions*. Slices that collide in a segment are treated as erasures. These assumptions simplify the analysis without substantially affecting the obtained results, as shown for example in [5] [7].

Under the previous assumptions we can model the channel as the general "XOR" multiple access channel depicted in Fig. 3. This channel could also apply to those situations where different processes share in a uncoordinated manner a common memory (*shared memory*). In the XOR multiple access channel, collisions result in a bit-wise sum (over the binary field) of the interfering slices. We shall see in the following that this simplified model allows to derive a simple capacity bound in an alternative manner with respect to the extrinsic information transfer (EXIT)-chart based proof proposed in [23].

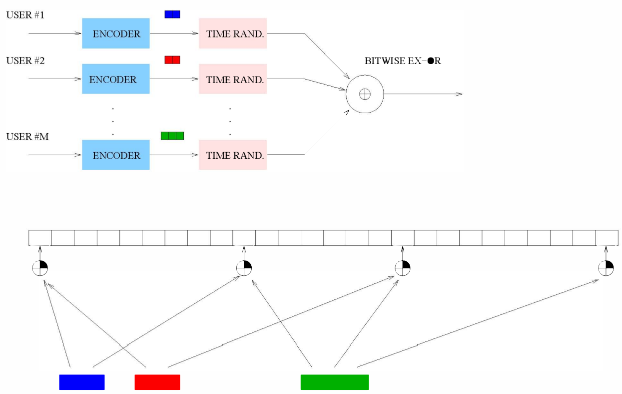

Fig. 3. The XOR multiple access channel

III. Asymptotic Analysis

In this section, the evolution of the decoding process for given k and G, in the asymptotic case where $M \to \infty$ (and $N \to \infty$), is reviewed [8].

We start by briefly recalling the definition of information function of a linear block code. Consider an (n, k) linear block code $\mathscr{C}$, where n is the codeword length and k the code dimension, and let $\mathbf{G}$ be any generator matrix of $\mathscr{C}$. Then, the g-th un-normalized information function of $\mathscr{C}$, denoted by $\tilde{e}_g$, is defined as the summation of the ranks over all the possible submatrices obtained selecting g columns (with $0 \le g \le n$) out of $\mathbf{G}$.

Next, assume that MAP decoding is used locally at each BN. At the i-th iteration of the interference subtraction process, let p_{i-1} be the average probability that an edge is connected to a SN associated with a segment where a collision still persists, before MAP decoding is performed by the BN set. Moreover, let q_i be the average probability that an edge is connected to a BN whose contribution of interference on the corresponding SN cannot be yet cancelled, after maximum a-posteriori (MAP) decoding has been performed at the BN set. Then:

$$\begin{aligned} q_i = \frac{1}{\bar{n}} \sum_{h=1}^{n_c} \Lambda_h \sum_{t=0}^{n_h-1} p_{i-1}^t (1-p_{i-1})^{n_h-1-t} [(n_h - t)\tilde{e}^{(h)}_{n_h-t} \\ - (t+1)\tilde{e}^{(h)}_{n_h-1-t}] \qquad (2) \\ =: f_{\mathsf{b}}(p_{i-1}) \,. \end{aligned}$$

The function $f_b(p)$ is referred to as the average EXIT function of the BN set. It is easy to verify that

$$f_{\mathsf{b}}(p) = \sum_{h=1}^{n_c} \lambda_h f_{\mathsf{b}}^{(h)}(p) \,, \qquad (3)$$

where

$$\begin{aligned} f_{\mathsf{b}}^{(h)}(p) := \frac{1}{n_h} \sum_{t=0}^{n_h-1} p^t (1-p)^{n_h-1-t} [(n_h-t)\tilde{e}^{(h)}_{n_h-t} \\ - (t+1)\tilde{e}^{(h)}_{n_h-1-t}] \qquad (4) \end{aligned}$$

is the average EXIT function of a type-h BN, under MAP decoding.

Eq. (2) allows to update q_i given p_{i-1}. The dependence of p_i on q_i is instead expressed by the relationship

$$\begin{aligned} p_i &= 1 - \rho(1 - q_i) \\ &= 1 - \exp\left(-\frac{G}{R} q_i\right) =: f_{\mathsf{s}}(q_i) \,, \qquad (5) \end{aligned}$$

where the function $f_{\mathsf{s}}(q)$ is called the average EXIT function of the SN set. The *density evolution* equations (2) and (5) define a discrete dynamical system $q_i = q_i(q_{i-1})$ with starting point $q_1 = f_{\mathsf{b}}(0)$, whose stability was analyzed in [8]. Note that the normalized offered traffic G is involved in the recursion through the SN distribution $\rho(\cdot)$ in (5). The *asymptotic threshold* of the interference subtraction process, denoted by $G^*(\mathcal{C}, \mathbf{\Lambda})$, is defined as

$$G^*(\mathcal{C}, \mathbf{\Lambda}) := \sup\{G \ge 0 \text{ s.t. } q_i \to 0 \text{ as } i \to \infty\} \,.$$

In the asymptotic setting $M \to \infty$, for all $G < G^*(\mathcal{C}, \mathbf{\Lambda})$ we have $S = G$, i.e., all collisions are resolved without retransmissions.

In [23] the following fundamental limit was proved.

Theorem 1: For rational R and $0 < R \le 1$, let $\mathbb{G}(R)$ be the unique positive solution to the equation

$$G = 1 - e^{-G/R}$$

in $[0, 1)$. Then, the threshold $G^*(\mathcal{C}, \mathbf{\Lambda})$ fulfills

$$G^*(\mathcal{C}, \mathbf{\Lambda}) < \mathbb{G}(R)$$

for *any* choice of $\mathcal{C} = \{\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_{n_c}\}$ and $\mathbf{\Lambda}$ associated with a rate R.

The previous theorem provides an upper bound to the capacity achievable by the proposed RA scheme when operated without retransmissions. The proof proposed in [23] relies on the EXIT chart analysis and on the Area Theorem for the erasure channel [26]. An alternative proof can be obtained by the XOR multiple access channel model as follows. When transmission occurs over a MAC frame, the average number of idle segments (i.e., segments where no transmission is performed) is $k\,N\,e^{-G/R}$. Thus, the average number of observable linear combinations of information slices is $k\,N\,(1 - e^{-G/R})$. Since the objective is to recover the $k\,M$ information slices, we need that number of equations to be equal to or larger than the number of unknowns, i.e., $k\,N\,(1 - e^{-G/R}) \ge k\,M$. Dividing both sides by $k\,N$ yields $1 - e^{-G/R} \ge G$.

In [8], [23] we designed several distributions by means of differential evolution (DE) [27], able to approach closely in some cases the fundamental limit. More specifically, for a given k, a given target rate R, and a given set $\mathcal{C}$ of component codes, we generated by DE optimization the distribution $\mathbf{\Lambda}$ which maximizes the threshold $G^*(\mathcal{C}, \mathbf{\Lambda})$. The thresholds for some of these distributions are reported in Fig. 4. Although not practical due to the vanishing rate, schemes achieving the upper bound on the capacity $G = 1$ for $R \to 0$ were

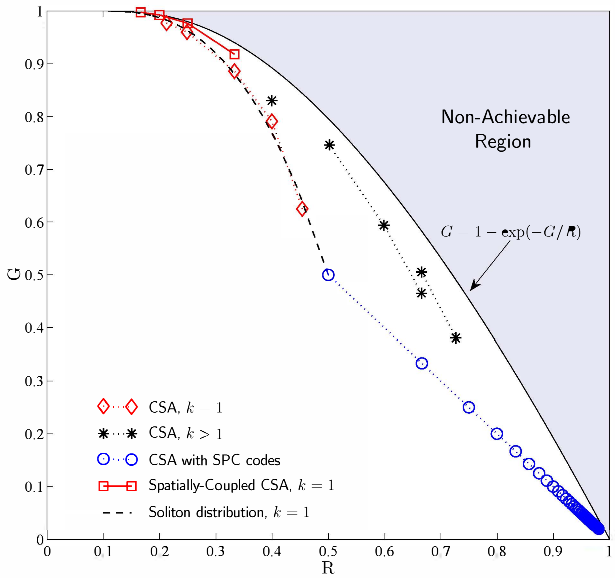

Fig. 4. Upper bound to the capacity vs. overall rate R. Asymptotic thresholds are reported for CSA relying on repetition codes ($k = 1$), on MDS codes with $k > 1$ and SPC codes. Moreover, thresholds for spatially-coupled CSA and for the (truncated) soliton distribution of [11] are provided.

proposed in [11], [22]. While in [22] the limit is achieved with a spatially-coupled version of CSA, [11] relies on a truncated ideal soliton distribution. The thresholds for the two schemes are provided in Figure 4, too. Interestingly, the spatially-coupled scheme converges to the bound faster than the one based on the soliton distribution.

The iterative procedure described in Section III can also be used to estimate the probability that a user's burst is lost, i.e., the packet loss rate (PLR). In fact, if the erasure probability of encoded slices after a given number of iterations is p_i, the probability that the burst for type-h users is lost (i.e., that it is not possible to reconstruct its entire original information burst) is

$$\mathrm{PLR}_h(p_i) = \sum_{m=1}^{n_h} w_m p_i^m (1-p_i)^{n_h - m} \tag{6}$$

where w_m is the number of configurations with m erasures which are not correctable by the h-th code. It is easy to show that w_m is the number of the submatrices with rank strictly less than k obtained by taking in all possible ways $n_h - m$ columns of any generator matrix of code $\mathscr{C}_h$. Note that $w_m = 0 \; \forall m < d_h$ and $w_{n_h} = 1$.

A. Energy and Delay

It could be observed that, depending on the rate R, with CSA the same information is sent together with redundancy slices, so requiring higher energy. For example, assuming all users use codes with the same rate $R = 1/2$, there is a factor 2 in energy spent per information packet. However, a proper comparison with, e.g., SA, should take into account that this latter protocol needs retransmissions, and that the average number of transmission per successful delivering of an information packet can be quite large. The same observation is valid for the delay, since a proper comparison should consider the delay due to retransmissions in SA, compared with the delay associated to the frame length for CSA.

IV. A Cooperative Random Access Scheme

The gains promised by CSA may be obtained in a cooperative setting. We consider next a network of nodes that need to communicate with a centralized gateway through a random access protocol. The nodes are grouped in clusters, such that nodes within each cluster can listen to each other, whereas a node cannot detect any transmission performed in another cluster.

Without loss of generality, we assume that in each cluster there are k nodes, the extension of the protocol to various number of nodes per cluster being trivial. In each cluster a node (cluster head, in the following) is selected, which has the task to coordinate the transmissions of the other nodes belonging to its cluster. The access is organized in slots, where slots are grouped in frames. Each frame is divided in two sub-frame, i.e., an information sub-frame and a parity sub-frame. The first slot in the information sub-frame is reserved for a transmission from the cluster head, that provides:

i. The schedule of the transmission of the k nodes within its cluster;
ii. The specification of a (n, k) systematic binary linear block code to be used to perform a distributed encoding process at the k nodes.

The schedule of the transmissions is obtained by selecting k slots with uniform probability within the information sub-frame, and by associating each of them to a different node. Additionally, $n - k$ slots are selected with uniform probability within the parity sub-frame, and then associated to $n - k$ nodes. During the first phase of the protocol, referred to as *gossip phase*, the k nodes transmit their packets in the allocated slots within the information sub-frame. At the same time, they detect and decode the transmissions of their cluster mates.

Given the knowledge of k information packets, $n - k$ parity packets are computed in a distributed manner by the nodes involved in the second phase. More specifically, a node which has a slot allocated within the parity sub-frame computes a redundant packet as a linear combination of the k information packets, according to the rules specified by the (n, k) binary linear block code. The packet is then transmitted in the allocated slot. Note that the nodes have sufficient information to encode in the packet headers the signalling information required by the interference cancelation process to determine the positions, within the MAC frame, of the overall n transmitted packets.

Fig. 5 reports the throughput achievable when the clusters are composed by $k = 4$ members, and the code adopted at each cluster in a $(7, 4)$ Hamming code. The frame is composed by 400 slots, with 229 allocated to the information sub-frame, 171 slots to the parity sub-frame, and the 1-st slot in the information sub-frame is reserved for the scheduling distribution.

Albeit simple, the scheme achieves a peak throughput close to 0.55 [packets/slot], whereas a conventional SA approach may support a peak throughput of ~ 0.37 [packets/slot].

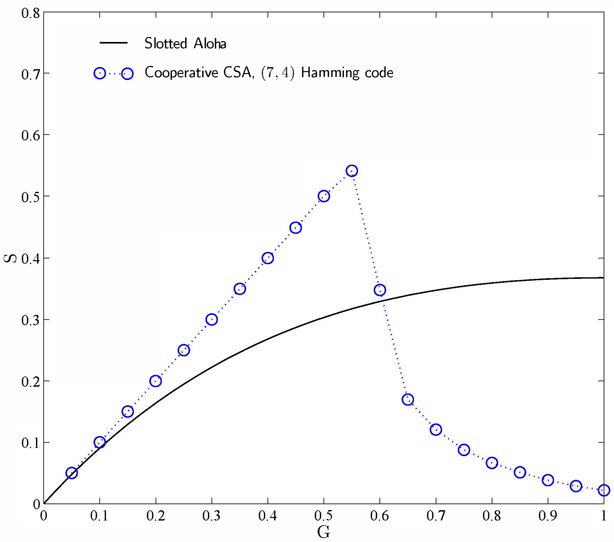

Fig. 5. Throughput versus channel load for a cooperative CSA scheme employing a $(7,4)$ Hamming code at each cluster, 400 slots per frame.

V. Conclusions

In this paper, recent results on the coded slotted ALOHA random access protocol, exploiting codes on graphs to achieve reliability without retransmissions, have been overviewed. The scheme combines erasure correcting codes and interference subtraction to resolve collisions. Asymptotic analysis for throughput and packet loss rate calculation have been presented. The proposed protocol has been also casted in network where nodes are allowed to cooperate, showing remarkable gains with respect to slotted ALOHA. The scheme is very promising in contexts where a little amount of coordination among the users is required, or in delay constrained applications (as the need for retransmissions is largely reduced).

Acknowledgement

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