Diversity-Multiplexing Tradeoff of the Half-Duplex Relay Channel

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Abstract—We show that the diversity-multiplexing tradeoff of a half-duplex single-relay channel with identically distributed Rayleigh fading channel gains meets the 2 by 1 MISO bound. We generalize the result to the case when there are N non-interfering relays and show that the diversity-multiplexing tradeoff is equal to the N + 1 by 1 MISO bound.

I. INTRODUCTION

Cooperation between nodes can provide both diversity and degree of freedom gain in wireless fading channels [15], [16], [1]. The diversity-multiplexing tradeoff (DMT) was a metric introduced by Zheng and Tse [3] to evaluate simultaneously the diversity and degrees of freedom gain in general fading channels. Significant effort has been spent in the past few years in computing the DMT of cooperative relay networks. The simplest such network has one relay and a direct link between the source and the destination (Fig. 2, with the channel gains modeled as quasi-static identically distributed Rayleigh faded and known only to the respective receive node. A simple upper bound to performance is the DMT of the 2 by 1 MISO channel obtained when the source and relay can fully cooperate to transmit to the destination:

$$d(r) = 2(1-r)$$
 $0 \le r \le 1$

It is quite easy to see that this upper bound can be achieved if the relay can operate on a full-duplex mode, i.e. transmit and receive at the same time. But most radios can only operate on a half-duplex mode. Somewhat surprisingly, the DMT for the half-duplex single-relay network is still an open problem despite substantial effort.

Figure 1 shows the DMT performance of several schemes and how they compare to the MISO bound. We see that none of the schemes achieves the bound for the



Fig. 1. Diversity multiplexing tradeoff of several schemes for the half-duplex relay channel

entire range of multiplexing gains. The dynamic-decodeand-forward [4] and partial decode-and-forward [13], [17] schemes achieves the MISO DMT for multiplexing gains $r \leq 0.5$ but there is a gap for r > 0.5. Is this gap fundamental or is there a better scheme?

In this paper, we show that indeed there is a scheme that achieves the MISO DMT for *all* multiplexing gains rup to 1. The problem with decode-and-forward schemes is that for r > 0.5, it takes too long for the relay to decode the whole message and there is not enough time for it to forward information. The problem with partialdecode-and-forward scheme is that the source does not know how to split the overall message without knowing the instantaneous channel gains of the various channels. In contrast, the scheme that we propose, which we call *quantize-and-map*, does not decode or partially decode the message. Instead, the relay extracts the significant bits of the received signal above noise level by quantization and re-encodes them to forward to the relay. The destination then combines the received signal from the relay and the direct signal from the source to solve for the information bits. Because there is no need to decode any message, there is also no need for any dynamic adaptation of the listening period for the relay. In fact, it turns out that it suffices for the relay to always listen half of the time and talk half of the time regardless of the channel state.

The quantize-and-map scheme is based on a recent deterministic approach to approximate the capacity of Gaussian relay networks [7], [8], [9], [10]. Inspired by the optimal scheme that was found for the deterministic relay networks [7], [8], the quantize-and-map scheme was shown in [9], [10] to achieve within a constant gap of the capacity of arbitrary Gaussian relay networks, where the constant gap does not depend on the channel parameters. A key observation is that since the scheme does not require any channel information at the nodes, it can also be utilized in a fading scenario in which there is no channel state information available at the transmitter. Now since at high SNR and high rates the approximation gap is negligible, as a corollary one can show that for any listen-transmit schedule, this scheme achieves the diversity-multiplexing tradeoff of the cut-set bound on the capacity. The desired result is obtained when this fact is combined with the observation that the DMT of the cutset bound of the half duplex network matches that of the MISO bound when the relay listens half of the time and talks the other half. This result can also be generalized to more than 1 relay when these relays have no link between themselves.

II. SYSTEM MODEL

Consider a network as shown in Figure 2 with a source S, a destination D, and one relay node R.



Fig. 2. The relay channel.

All the channel links h_{sd} , h_{sr} , h_{rd} are assumed to be flat-fading, i.i.d complex normal $\mathcal{CN}(0,1)$ distribution. It

is assumed that although random, once realized, channel gains remain unchanged for the duration of the codeword and change independently from one codeword to another i.e., quasi-static fading. Noise at all of the receivers is additive i.i.d $\mathcal{CN}(0,1)$ independent of any other form of randomness in the system. All nodes have single antenna and have equal average power constraint specified by average Signal to Noise Ratio (SNR), denoted by ρ . Relay node R is assumed to be in half-duplex operation and for simplicity it is assumed that transmission of source and relay are synchronous at symbol level. Furthermore, channel state information (CSI) is only available at the receivers. So, relay has CSI about h_{sr} , destination has CSI about h_{sd} , h_{rd} and no CSI at all at the source.

III. DIVERSITY-MULTIPLEXING TRADEOFF OF THE HALF-DUPLEX RELAY CHANNEL

In this section we characterize the diversitymultiplexing tradeoff of the half-duplex relay channel, described in section II. First we describe the quantizemap relaying scheme that we proposed earlier in [9] and [10]. As we showed in these references, this relaying scheme achieves a rate within a constant gap to the cutset upper bound of the capacity of the relay channel for all channel gains, where the constant is independent of the channel SNRs. Furthermore, since this relaying scheme does not require any channel information at the source and the relay, it can also be performed in our scenario (i.e. no CSI at the transmitter). Now, since at high SNR and high rates the approximation gap is negligible, as a corollary we will show that this scheme achieves the diversity-multiplexing tradeoff of the cut-set bound on the capacity for any listen-transmit scheduling at the relay. Finally we illustrate that a fixed scheduling that relay listens only half the time and transmits the rest is enough to achieve the diversity-multiplexing tradeoff of the 2×1 MISO channel, hence we find the optimal DMT of the half-duplex relay channel.

A. Description of the relaying scheme

We have a single source S with a sequence of messages $w_k \in \{1, 2, ..., 2^{KTR}\}, k = 1, 2, ...$ to be transmitted. At both the relay and the source we create random Gaussian codebooks. Source randomly maps each message to one of its Gaussian codewords and sends it in KT transmission times (symbols) giving an overall transmission rate of R. Due to half-duplex nature of the relay, it has to do listen-transmit cycles. Relay operates over blocks of time T symbols and since total length of codeword at source is KT we have K blocks in each codeword. Relay listens to the first Tt $(0 \leq t \leq 1)$ time symbols of each block. Let $X_S^{1(k)}$ denote the sequence of these Tt symbols transmitted at the source in block k. Also let $\mathbf{Y}_R^{(k)}$ and $\mathbf{Y}_D^{1(k)}$ be the received signal at relay and destination respectively during this time. Then the relay it quantizes its received signal in the first tT time symbols to $\hat{\mathbf{Y}}_R^{(k)}$ which is then randomly mapped into a Gaussian codeword $\mathbf{X}_R^{(k)}$ using a random mapping function $f_R(\hat{\mathbf{Y}}_R^{(k)})$ and sends it in the next T(1-t) time symbols. Let $\mathbf{Y}_D^{2(k)}$ denote the sequence of symbols received by destination during this time. Given the knowledge of all the encoding functions at the relay and signals received over K blocks, the decoder D, attempts to decode the message sent by the source.

B. DMT of the relaying scheme

For any fixed listen-transmit scheduling strategy (i.e. fixed t), the cut-set upper bound on the capacity of the half-duplex Gaussian relay channel, \overline{C}_{hd} , is given by (1) on the top of next page [11].

Now, as we showed in [10], for any fixed listentransmit scheduling, the quantize-map relaying scheme described in Section III-A, uniformly achieves a rate within a constant gap to the capacity. Therefore by Theorem 4.7 in [10], for all channel gains we have,

$$\overline{C}_{hd}(h_{sr}, h_{sd}, h_{rd}, \rho, t) - \kappa \le R_{\text{quantize-map}}(h_{sr}, h_{sd}, h_{rd}, \rho, t)$$
(3)

where $\kappa \leq 15$ is a constant that does not depend on the channel gains and SNR.

Now since this relaying scheme does not require any channel information at the source and the relay, it can also be performed in our scenario in which there is no channel state information available at the transmitter. Furthermore, as at high SNR and high data rates the approximation gap is negligible, as a corollary we will now show that for any fixed listen-transmit scheduling, this scheme achieves the diversity-multiplexing tradeoff of the cut-set bound.

Theorem 3.1: For any fixed scheduling t, the quantize-map relaying scheme achieves the diversitymultiplexing tradeoff of \overline{C}_{hd} , where \overline{C}_{hd} is defined by (1).

Proof: Assume a targeted communication rate R. By (3), we know that the destination will be able to decode the information sent by the source as long as

$$\overline{C}_{hd}(h_{sr}, h_{sd}, h_{rd}, \rho, t) - \kappa > R \tag{4}$$

Therefore for any scheduling t, we have

$$\mathcal{P}_{\text{outage}}(\rho) \le \mathbb{P}\left\{C_{hd} - \kappa < R\right\}$$
(5)

where the probability is calculated over the randomness of channel gain realizations. Now by definition, for any scheduling t, the achievable diversity of quantize-map scheme is

$$d_{QM}(r) = -\lim_{\rho \to \infty} \frac{\log \left(\mathcal{P}_{\text{outage}}(\rho)\right)}{\log \rho}$$
(6)

$$\stackrel{(5)}{\geq} -\lim_{\rho \to \infty} \frac{\log \left(\mathbb{P} \left\{ \overline{C}_{hd} - \kappa < r \log \rho \right\} \right)}{\log \rho} (7)$$

$$\stackrel{*}{\approx} -\lim_{\rho \to \infty} \frac{\log \left(\mathbb{P} \left\{ \overline{C}_{hd} < r \log \rho \right\} \right)}{\log \rho} (8)$$

where * is true since κ is a constant and does not scale with ρ . Therefore for any t, the quantize-map relaying strategy achieves the diversity-multiplexing tradeoff of $\overline{C}_{hd}(t)$.

Next, we will show that with t = 0.5, the diversitymultiplexing tradeoff of \overline{C}_{hd} matches the diversitymultiplexing tradeoff of the 2×1 MISO channel, and hence we complete the proof of our main Theorem.

First we give some intuition on why this is true.



Fig. 3. Two scheduling modes of the system: relay listens t fraction of the time and relay transmits (1 - t) fraction of the time

First note that in equation (2), the first term corresponds to the information flowing through cut $\{S\}, \{R, D\}$ (see Figure 3) and the second term corresponds to the information flowing through the cut $\{S, R\}, \{D\}$. Now, the value of the first cut $\{S\}, \{R, D\}$ corresponds to the capacity of a SIMO system with 1 transmit antenna and 2 receive antennas where one

$$\overline{C}_{hd}(h_{sr}, h_{sd}, h_{rd}, \rho, t) = \max_{p(x_{S}^{1}, x_{S}^{2}, x_{R})} \min\{tI(X_{S}^{1}; Y_{R}, Y_{D}^{1} | X_{R}) + (1-t)I(X_{S}^{2}; Y_{D}^{2} | X_{R}), tI(X_{S}^{1}; Y_{D}^{1}) + (1-t)I(X_{S}^{2}, X_{R}; Y_{D}^{2})\}(1) \\
\leq \min\{t\left(\log(1+\rho(|h_{sr}|^{2}+|h_{sd}|^{2}))\right) + (1-t)\left(\log(1+\rho|h_{sd}|^{2}), (1-t)\left(\log(1+\rho(|h_{rd}|+|h_{sd}|)^{2})\right) \\
+ t\left(\log(1+\rho|h_{sd}|^{2})\right\} \tag{2}$$

receive antenna (corresponding to relay) is listening only t amount of time. Similarly, the value of the second cut i.e., $\{S, R\}, \{D\}$ corresponds to the capacity of a MISO system with 2 transmit antennas and 1 receive antenna, where one transmit antenna (corresponding to relay) is transmitting only 1 - t amount of time. Since we are limited by the minimum of these two values optimal strategy is to try to make them equal. Also since DMT of 1×2 SIMO is same as that of 2×1 MISO, a natural choice is to set t = 0.5.

Once we set t = 0.5, DMT of cut-set bound is just DMT of a 2×1 MISO system with 1 transmit antenna being used only half the time, but this system is *strictly better* than a system with 2 transmit 1 receive antennas and where each of the two transmit antennas are used only half the time in an alternate fashion i.e., parallel channel with rate r on each channel. It is well known and easy to compute that DMT of this parallel channel is 2(1-r). Also we have obvious upper bound of DMT of MISO system which is again 2(1-r). Thus the cutset bound achieves the optimal DMT for t = 0.5. The formal proof of this is given in Appendix A.

IV. EXTENSION TO MULTIPLE-RELAY NETWORK

In this section we extend our result to general multiple-relay networks. The listen-transmit scheduling model that we use to study this problem is the same as [11]. In this model the network has finite modes of operation. Each mode of operation (or state of the network), denoted by $m \in \{1, 2, \dots, M\}$, is defined as a valid partitioning of the nodes of the network into two sets of "sender" nodes and "receiver" nodes such that there is no active link that arrives at a sender node¹. For each node *i*, the transmit and the receive signal at mode m are respectively shown by x_i^m and y_i^m . Also t_m defines the portion of the time that network will operate in state m, as the network use goes to infinity. As shown in [11], the cut-set upper bound on the capacity of the Gaussian relay network with half-duplex constraint, C_{hd} , is given by (9) on the top of next page.

Now we describe the quantize-map relaying scheme that we proposed in [9], [10] for general half-duplex relay networks.

A. Description of the relaying scheme

We have a single source S with a sequence of messages $w_k \in \{1, 2, ..., 2^{KTR}\}, k = 1, 2, ...$ to be transmitted. At all nodes we create a random Gaussian codebook. Source randomly maps each message to one of its Gaussian codewords and sends it in KT transmission times (symbols) giving an overall transmission rate of R. Relays operate over blocks of length T symbols. Starting from the beginning of the block each relay i spends a total of $t_m T$ symbols in state $m, m = 1, \ldots, M$. In each state, if it is assigned to listen, it receives a sequence $\mathbf{Y}_{i}^{(k,m)}$. Otherwise, if it is assigned to transmit, it quantizes all received signals in the previous block (i.e. $\mathbf{Y}_i^{(k-1,m)}$, $m = 1, \dots, M$) to $\hat{\mathbf{Y}}_i^{(k,m)}$ which is then randomly mapped into a Gaussian codeword $\mathbf{X}_{i}^{(k,m)}$ using a random mapping function $f_i(\hat{\mathbf{Y}}_R^{(k,m)})$ and sends it in that $t_m T$ time symbols. Given the knowledge of all the encoding functions at the relay and signals received over K + |V| - 2 blocks, the decoder D, attempts to decode the message W sent by the source.

B. DMT of the relaying scheme

As we showed in [10], for any fixed listen-transmit scheduling, the quantize-map relaying scheme described above, achieves within a constant gap to the capacity. Therefore by Theorem 4.7 in [10], for all channel gains we have,

$$\overline{C}_{hd} - \kappa \le R_{\text{quantize-map}} \tag{10}$$

where $\kappa \leq 5|\mathcal{V}|$ is a constant that does not depend on the channel gains and SNR.

Therefore similar to Theorem 3.1 we can show the following theorem:

Theorem 4.1: For any fixed scheduling, the quantizemap relaying scheme achieves the diversity-multiplexing tradeoff of \overline{C}_{hd} , where \overline{C}_{hd} is defined by (9).

To find the optimal performance of this scheme, one should optimize over all possible scheduling strategies. In general we don't know the optimizing strategy, however as we show in Section IV-C, in a special case of

¹Active link is defined as a link which is departing from the set of sender nodes

$$C_{hd} \leq \overline{C}_{hd} = \max_{\substack{p(\{x_j^m\}_{j \in \mathcal{V}, m \in \{1, \dots, M\}})\\t_m: \ 0 \leq t_m \leq 1, \ \sum_{m=1}^M t_m = 1}} \min_{\Omega \in \Lambda_D} \sum_{m=1}^M t_m I(Y_{\Omega^c}^m; X_{\Omega}^m | X_{\Omega^c}^m)$$
(9)

two hop network with N non interfering relays, a fixed uniform scheduling (i.e. $t_m = 2^{-N}$, $m = 1, \ldots, 2^N$) achieves the optimal DMT.

C. Optimal DMT of two hop network with N noninterfering half duplex relays

Consider a two hop network with single source S, destination D and N half-duplex relays R_i , $1 \le i \le N$, as shown in Figure 4.

All the assumptions in section II are carried over with additional assumption that there is no link among any two relays. Let h_{sr_i} be the link from source S to relay i and h_{r_id} be the link from i^{th} relay to destination D.

Here is our main result for this relay network.

Theorem 4.2: The optimal diversity-multiplexing tradeoff of a two-hop relay network, with N non-interfering half-duplex relays is equal to the diversity-multiplexing tradeoff of the $(N+1) \times 1$ MISO channel. Furthermore, it is achieved by the quantize-map relaying strategy define in Section IV-A with fixed and uniform scheduling, $t_m = 2^{-N}$, $m = 1, \ldots, 2^N$.

Proof: See Appendix B



Fig. 4. Two-hop network with N half-duplex relays

APPENDIX A Achieving the MISO bound in the relay channel with t = 0.5

From theorem 3.1, it is sufficient to show that DMT of \overline{C}_{hd} is equal to that of MISO.

For ease of computation define $n_{sd} := \log(1+|h_{sd}|^2\rho)$ and α_{sd} as its exponential order i.e.,

$$\alpha_{sd} := \lim_{\rho \to \infty} \frac{\log(1 + |h_{sd}|^2 \rho)}{\log \rho}$$

then the probability density function (pdf) of α_{sd} can be shown to be

$$f_{\alpha_{sd}}(\alpha_{sd}) = \lim_{\rho \to \infty} \exp(-\rho^{-(1-\alpha_{sd})})\rho^{-(1-\alpha_{sd})}\log \rho$$
$$= \rho^{-(1-\alpha_{sd})} \quad 0 \le \alpha_{sd} \le 1$$

Consider first term in inequality (2), using t = 0.5

$$0.5 \left(\log(1 + \rho(|h_{sr}|^2 + |h_{sd}|^2)) \right) + 0.5 \left(\log(1 + \rho|h_{sd}|^2) \right) \doteq 0.5 \left(\max\{ \log(1 + \rho|h_{rd}|^2), \log(1 + \rho|h_{sd}|^2) \} \right) + 0.5 \left(\log(1 + \rho|h_{sd}|^2) \right) = n_{sd} + 0.5(n_{rd} - n_{sd})^+$$

similarly second term can be simplified resulting in

$$\overline{C}_{hd} \doteq n_{sd} + 0.5 \min\{(n_{sr} - n_{sd})^+, (n_{rd} - n_{sd})^+\}$$

For $R = r \log \rho$ the cut-set bound is in outage if
 $n_{sd} + 0.5 \min\{(n_{sr} - n_{sd})^+, (n_{rd} - n_{sd})^+\} \le r \log \rho$
 $i.e., \alpha_{sd} + 0.5 \min\{(\alpha_{sr} - \alpha_{sd})^+, (\alpha_{rd} - \alpha_{sd})^+\} \le r$

$$\mathcal{O}(r) = \{ \alpha_{sd}, \alpha_{sr}, \alpha_{rd} \mid \alpha_{sd} + 0.5 \min\{(\alpha_{sr} - \alpha_{sd})^+, (\alpha_{rd} - \alpha_{sd})^+\} \le r \}$$

$$P_{\mathcal{O}}(r) = \int_{\overline{\alpha} \in \mathcal{O}(r)} f_{\overline{\alpha}}(\overline{\alpha}) d\overline{\alpha}$$

$$= \int_{\overline{\alpha} \in \mathcal{O}(r)} f_{\alpha_{sd}}(\alpha_{sd}) f_{\alpha_{sr}}(\alpha_{sr}) f_{\alpha_{rd}}(\alpha_{rd}) d\overline{\alpha}$$

$$= \int_{\overline{\alpha} \in \mathcal{O}(r)} \rho^{-3 + (\alpha_{sd} + \alpha_{sr} + \alpha_{rd})} d\overline{\alpha}$$

$$= \rho^{-d(r)}$$

where

$$d(r) = \inf_{\substack{(\alpha_{sd}, \alpha_{sr}, \alpha_{rd}) \in \mathcal{O}(r) \\ 0 \le \alpha_{sd}, \alpha_{sr}, \alpha_{rd} \le 1}} 3 - (\alpha_{sd} + \alpha_{sr} + \alpha_{rd})$$

- 1) If $\alpha_{sd} \geq \min\{\alpha_{sr}, \alpha_{rd}\}$: Then outage implies $\min\{\alpha_{sr}, \alpha_{rd}\} \leq \alpha_{sd} \leq r$. And since $\max\{\alpha_{sr}, \alpha_{rd}\} \leq 1$ we have $(\alpha_{sd} + \alpha_{sr} + \alpha_{rd}) \leq 1 + 2r$.
- 2) If $\alpha_{sd} \leq \min\{\alpha_{sr}, \alpha_{rd}\}$: Then Outage implies

$$\begin{aligned} \alpha_{sd} + 0.5(\min\{\alpha_{sr}, \alpha_{rd}\} - \alpha_{sd}) &\leq r\\ \alpha_{sd} + \min\{\alpha_{sr}, \alpha_{rd}\} &\leq 2r\\ \alpha_{sd} + \min\{\alpha_{sr}, \alpha_{rd}\} + \max\{\alpha_{sr}, \alpha_{rd}\} &\leq 1 + 2r \end{aligned}$$

Therefore

$$d(r) = 3 - (1 + 2r) = 2(1 - r)$$

Thus quantize-map relaying scheme achieves the optimal DMT of 2×1 MISO system.

APPENDIX B DMT FOR TWO-HOP NETWORK WITH NNON-INTERFERING HALF-DUPLEX RELAYS

We prove theorem 4.2 in two steps, we first show that DMT of cut-set bound with fixed uniform scheduling achieves DMT of $(N + 1) \times 1$ MISO system and then apply theorem 3.1.

A. DMT of cut-set bound for fixed uniform scheduling

In subsection IV-B theorem 3.1 showed that for any fixed scheduling quantize-map relaying scheme achieves the DMT of the cut-set for that scheduling. We make use of this fact to show the achievability of MISO performance in N relay case. First we note that since there are N half-duplex relays, each relay has a choice to be either in receiving mode or in transmitting mode, accordingly we have $M = 2^N$ states. Then following the lead from our single relay case and using the fact the everything in network is nicely symmetrical we operate network in each of these states for equal amount of time i.e., $t_m = 2^{-N}$ for $m = 1, \dots, 2^N$. Now if we show that DMT of cut-set for this scheduling is equal to (N+1)(1-r) we are done.

We first derive a lower bound on the cut-set. Any cut in a network partitions all nodes into two groups Ω with

 $S \in \Omega$ and its compliment Ω^c with $D \in \Omega^c$, each relay has a choice of being in a either Ω or Ω_c , thus we have 2^N total possible cuts and the cut-set bound of network is equal to the minimum of mutual information flowing through each of these 2^N possible cuts.



Fig. 5. m^{th} state of Network



Fig. 6. Z- channel

Consider a cut Ω in the network which is operating

in state m it looks as shown in fig. 5. Let $V_m \subseteq \Omega - S$ be the set of relays $R_j \in \Omega$ which are transmitting and $W_m \subseteq \Omega^c - D$ be the set of relays $R_j \in \Omega^c$ which are receiving in state m. Let R_{V_m} be a relay with strongest channel say $h_{rd}^* := \max_j \{h_{rjd}\} \ j \in V_m$ to the destination and analogously let R_{W_m} be a relay with strongest channel say $h_{sr}^* := \max_j \{h_{srj}\} \ j \in W_m$ from source. We can lower bound the total mutual information flowing across this cut in fig 5 by the the mutual information flowing across the same cut $\{S, R_{V_m}\}\{R_{W_m}, D\}$ in the Z-channel formed by these nodes, see fig 6. This Zchannel can be viewed as MIMO system with upper triangular channel matrix $H = \begin{pmatrix} h_{rd}^* \ h_{sd} \\ 0 \ h_{sr}^* \end{pmatrix}$ So mutual information flow across this cut in Z-channel is given by

$$\log \det \left(I_{2\times 2} + \rho H H^{+} \right) \qquad \text{wher}$$

$$= \log \left(1 + \rho (|h_{rd}^{*}|^{2} + |h_{sd}|^{2} + |h_{sr}^{*}|^{2} + \rho |h_{rd}^{*}|^{2} |h_{sr}^{*}|^{2}) \right)$$

$$\geq \max \{ \log(1 + \rho |h_{sd}|^{2}), \log(1 + \rho |h_{rd}^{*}|^{2}) \} \qquad d(r)$$

$$= \max \{ \log(1 + \rho |h_{sd}|^{2}), \log(1 + \rho \max_{j \in V_{m}} (|h_{rjd}|^{2})) + \log(1 + \rho \max_{j \in W_{m}} (|h_{srj}|^{2})) \}$$

$$= \max \{ n_{sd}, \max_{j \in V_{m}} (n_{rjd}) + \max_{j \in W_{m}} (n_{srj}) \}$$

Thus for each cut Ω the cut value is,

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$$\overline{C}_{\Omega} \ge \frac{1}{2^{N}} \sum_{i=1}^{2^{N}} \max\{n_{sd}, \max_{j \in V_{m}}(n_{r_{j}d}) + \max_{j \in W_{m}}(n_{sr_{j}})\}$$
(11)

Now the cut-set bound is simply,

$$\overline{C}_{hd} \ge \min_{\Omega} \overline{C}_{\Omega} \tag{12}$$

Lemma B.1: For any cut Ω , there are N + 1 distinct links flowing across the cut and the mutual information flowing through it given by (11) can be further lower bounded by their average

$$\overline{C}_{\Omega} \ge \frac{n_{sd} + \sum_{j \in \Omega - \{S\}} n_{r_jd} + \sum_{j \in \Omega^c - \{D\}} n_{sr_j}}{N+1} \quad (13)$$

B. Optimality of DMT of each cut \overline{C}_{Ω}

Following Appendix A for each j, we define Then $\alpha_{sd}, \alpha_{r_jd}, \alpha_{sr_j}$ as exponential order's of $n_{sd}, n_{r_jd}, n_{sr_j}$ respectively.

From lemma B.1 outage is equal to set

$$\mathcal{O}(r) = \{\overline{\alpha} \mid \alpha_{sd} + \sum_{j \in \Omega - \{S\}} \alpha_{r_jd} + \sum_{j \in \Omega^c - \{D\}} \alpha_{sr_j} \le (N+1)r\}$$
(14)

$$P_{\mathcal{O}}(r) = \int_{\overline{\alpha} \in \mathcal{O}(r)} f_{\overline{\alpha}}(\overline{\alpha}) d\overline{\alpha}$$

=
$$\int_{\overline{\alpha} \in \mathcal{O}(r)} \rho^{-(N+1)}$$
$$0 \le \overline{\alpha} \le 1$$
$$\cdot \rho^{\alpha_{sd} + \sum_{j \in \Omega^{-}\{S\}} \alpha_{r_{j}d} + \sum_{j \in \Omega^{c} - \{D\}} \alpha_{sr_{j}}} d\overline{\alpha}$$
$$\doteq \rho^{-d(r)}$$

where

$$= \inf_{\substack{\overline{\alpha} \in \mathcal{O}(r) \\ 0 \le \overline{\alpha} \le 1}} (N+1) - \left(\alpha_{sd} + \sum_{j \in \Omega^{-}\{S\}} \alpha_{r_j d} + \sum_{j \in \Omega^{c} - \{D\}} \alpha_{sr_j} \right)$$
$$= (N+1)(1-r)$$

last equality follows from the equation 14. Now since each cut has optimal DMT, from inequality (12) it is clear that cut-set bound also achieves optimal DMT. And then we use theorem 3.1.

C. Proof of Lemma B.1

To prove this, first we show the following lemma,

Lemma B.2: Consider a set of numbers a, s_1, \ldots, s_n . Assume function f is such that for any set $V \subseteq \{1, \ldots, n\}$ we have,

$$f(V) \ge \max(a, s_V) \tag{15}$$

where

$$s_V = \{s_i | i \in V\} \tag{16}$$

$$\frac{1}{2^n} \sum_{V \subseteq \{1,\dots,n\}} f(V) \ge \frac{a + \sum_{i=1}^n s_i}{n+1}$$
(17)

Proof: Without loss of generality assume that s_i 's are ordered (i.e. $s_1 \leq s_2 \leq \ldots \leq s_n$). Then we have

$$\frac{\frac{1}{2^{n}} \sum_{V \subseteq \{1,...,n\}} f(V)}{\sum_{V \subseteq \{1,...,n\}} f(V)} \\
\geq \frac{a + \max(a, s_{1}) + \ldots + 2^{n-1} \max(a, s_{n})}{2^{n}} \\
\stackrel{*}{\geq} \frac{a + \max(a, s_{1}) + \ldots + \max(a, s_{n})}{n+1} \\
\geq \frac{a + s_{1} + s_{2} + \ldots + s_{n}}{n+1}$$

where * is true by applying two sequences (a, s_1, \ldots, s_m) and $(2^{-n}, 2^{-n}, 2^{-n+1}, \ldots, 2^{-1})$ to the Tchebychef's inequality,

Tchebychef's inequality: Assume two sequences (a_1, \ldots, a_n) and (b_1, \ldots, b_n) are similarly ordered (i.e. $(a_u - a_v)(b_u - b_v) \ge 0$, for all u and v). Then

$$\frac{1}{n}\sum_{i=1}^{n}a_{i}b_{i} \ge \left(\frac{1}{n}\sum_{i=1}^{n}a_{i}\right)\left(\frac{1}{n}\sum_{i=1}^{n}b_{i}\right)$$
(18)

Now we prove Lemma B.1.

Proof: (proof of Lemma B.1)

First note that for any $V_m \subseteq \Omega - \{S\}$ and $W_m \subseteq \Omega^c - \{D\}$ we have

$$f(V_m, W_m) = \max\left(n_{sd}, \max_{i \in V_m} (n_{r_id}) + \max_{i \in W_m} (n_{sr_i})\right)$$

$$\geq \max(n_{sd}, n_{3V_m}, n_{2W_m})$$

where

$$n_{3V_m} = \{n_{r_i d} | i \in V_m\}$$

 $n_{2W_m} = \{n_{sr_j} | j \in W_m\}$

Now by Lemma B.2 we know that

$$\geq \frac{\frac{1}{2^{N}} \sum_{V_{m} \subseteq \Omega - \{S\}} \sum_{W_{m} \subseteq \Omega^{c} - \{D\}} f(V_{m}, W_{m})}{N + 1}$$

hence the proof is complete.

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