Infinite-message Distributed Source Coding for Two-terminal Interactive Computing

Nan Ma and Prakash Ishwar

Abstract— **A two-terminal interactive function computation problem with alternating messages is studied within the framework of distributed block source coding theory. For any arbitrary fixed number of messages, a single-letter characterization of the minimum sum-rate function was provided in previous work using traditional information-theoretic techniques. These techniques, however, do not lead to a satisfactory characterization of the infinite-message limit, which is a new, unexplored dimension for asymptotic-analysis in distributed block source coding involving potentially infinitesimal-rate messages. This paper introduces a new convex-geometric approach to provide a blocklength-free single-letter characterization of the infinite-message minimum sum-rate function as a functional of the joint source pmf. This characterization is not obtained by taking a limit as the number of messages goes to infinity. Instead, it is in terms of the least element of a family of partially-ordered marginal-perturbations-concave functionals associated with the functions to be computed. For computing the Boolean AND function of two independent Bernoulli sources at one and both terminals, the respective infinite-message minimum sum-rates are characterized in closed analytic form. These sum-rates are achievable using infinitely many infinitesimal-rate messages. The convex-geometric functional viewpoint also suggests an iterative algorithm for evaluating any finite-message minimum sum-rate function.**

I. Introduction

In this paper we study a two-terminal interactive function computation problem with alternating messages (Fig. 1) within a distributed block source coding framework. Here, n samples of one component of a discrete memoryless multi-source $\mathbf{X} := X^n := (X(1), \ldots, X(n)) \in \mathcal{X}^n$ are available at terminal A and n samples of another component of the multi-source $\mathbf{Y} \in \mathcal{Y}^n$ are available at a different terminal B. The two component sources of the multi-source are statistically dependent. Terminal A is required to compute n samples $\mathbf{f}_A(\mathbf{X}, \mathbf{Y}) := (f_A(X(1), Y(1)), \ldots, f_A(X(n), Y(n)))$ of a samplewise function $f_A : \mathcal{X}\times\mathcal{Y} \to \mathcal{Z}_A$ of the two component sources. Similarly, terminal B is required to compute n samples $\mathbf{f}_B(\mathbf{X}, \mathbf{Y}) := (f_B(X(1), Y(1)), \ldots, f_B(X(n), Y(n)))$ of a samplewise function $f_B : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}_B$ of the two component sources. All alphabets are assumed to be finite. To achieve the desired objective, t coded messages, $M_1, \ldots, M_t$, of respective bit rates (bits per source sample), $R_1, \ldots, R_t$, are sent alternately from the two terminals starting with some terminal. The message sent from a terminal can depend on the source samples at that terminal and on all the previous messages (which are available to both terminals). There is enough memory at both terminals to store all the source samples and messages. After t messages, terminal A produces a sequence $\widehat{\mathbf{Z}}_A \in \mathcal{Z}_A^n$ and terminal B produces a sequence $\widehat{\mathbf{Z}}_B \in \mathcal{Z}_B^n$. The t-message minimum sum-rate $R_{sum,t}$ is the infimum of the sum of all rates $\sum_{i=1}^t R_i$ for which $\mathbb{P}(\mathbf{f}_A(\mathbf{X}, \mathbf{Y}) \neq \widehat{\mathbf{Z}}_A)$ and $\mathbb{P}(\mathbf{f}_B(\mathbf{X}, \mathbf{Y}) \neq \widehat{\mathbf{Z}}_B) \to 0$ as $n \to \infty$.

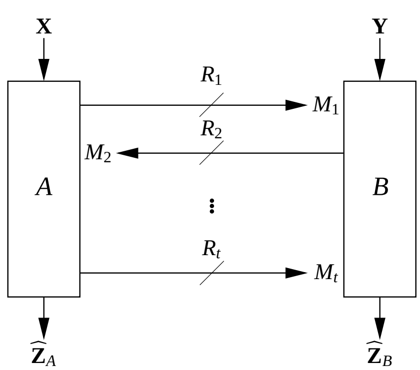


Fig. 1. *Interactive distributed source coding with t alternating messages.*

This material is based upon work supported by the US National Science Foundation (NSF) under awards (CAREER) CCF–0546598 and CCF-0915389. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

The authors are with the Department of Electrical and Computer Engineering, Boston University, Boston, MA 02215, USA {nanma, pi}@bu.edu

For any *fixed number* t, a single-letter characterization of the set of all feasible coding rates (the rate region) and the minimum sum-rate $R_{sum,t}$, for a more general two-terminal interactive rate-distortion problem, was provided in our previous work [1], [2] using traditional information-theoretic techniques. These techniques, however, do not lead to a satisfactory characterization of the *infinite-message limit* $R_{sum,\infty} := \lim_{t\to\infty} R_{sum,t}$. The objective of this paper is to provide a characterization of $R_{sum,\infty}$ which is not obtained by taking a limit as the number of messages goes to infinity and also an iterative algorithm to evaluate it. Understanding the minimum sum-rate in the limit where potentially an infinite number of alternating messages are allowed to be exchanged will shed light on the fundamental benefit of cooperative interaction in two-terminal problems. While asymptotics involving blocklength, rate, quantizer step-size, and network size have been explored in the distributed block source coding literature, asymptotics involving an infinite number of messages, each with potentially infinitesimal rate, has not been studied. The number of messages is a relatively unexplored resource and a new dimension for asymptotic analysis.

This paper introduces a new convex-geometric approach to provide a blocklength-free single-letter characterization of the infinite-message minimum sum-rate as a functional of the multi-source pmf. This characterization is not obtained by taking a limit as the number of messages goes to infinity. Instead, it is in terms of the least element of a family of partially-ordered, marginal-perturbations-concave functionals associated with the functions to be computed. For computing the Boolean AND function of two independent Bernoulli sources at one/both terminals, the respective infinite-message minimum sum-rates are characterized in closed analytic form and shown to be achievable using infinitely many infinitesimal-rate messages. The functional viewpoint also leads to an iterative algorithm for evaluating any finite-message minimum sum-rate.

Related interactive computation problems have been studied extensively in the area of communication complexity [3], [4] where the main focus is on *exact* zero error computation, without regard for the statistical dependencies in samples across terminals, and where computing efficiency is gauged in terms of the *order-of-magnitude* of the *total number of bits* exchanged; not bit-rate (notable exceptions to this main focus are [5], [6]). Two-way distributed block source coding where the goal is to *reproduce* the sources with a non-zero per-sample distortion, as opposed to computing functions, was studied by Kaspi [7] who characterized the t-message sum rate-distortion function in each direction. Orlitsky and Roche [8] studied two-terminal samplewise function computation with a vanishing block-error probability and characterized the feasible rates and the minimum sum-rate for two alternating messages ($t = 1, 2$). A more detailed account of related work appears in [2].

The focus of this paper is on t-message two-terminal samplewise function computation where the probability of computation error for a block of samples vanishes as the blocklength goes to infinity. The results presented here, however, directly extend to the more general two-terminal rate-distortion problem studied in [2] involving coupled single-letter distortion criteria. The generalization is omitted here due to limited space but will be presented in [9].

Notation: Vectors are denoted by boldface letters; the dimension will be clear from the context. The acronym 'iid' stands for independent and identically distributed and 'pmf' stands for probability mass function. With the exception of the symbols R, N, A, and B, random quantities are denoted in upper case and their specific instantiations in lower case. For integers i, j, with $i \le j$, V_i^j denotes the sequence of random variables $V_i, \ldots, V_j$. For $i \ge 1$, V_1^i is abbreviated to V^i. If $j < i$ then "V_i^j" denotes the void expression "". More generally, if $\{Q_i\}_{i \in S}$ is a set of quantities Q indexed by a subset S of integers then for all integers i not in S, "Q_i" = "". For a set $\mathcal{S}$, $\mathcal{S}^n$ denotes the n-fold Cartesian product $\mathcal{S} \times \ldots \times \mathcal{S}$. The support-set of a pmf p is the set over which it is strictly positive and is denoted by $\mathrm{supp}(p)$. If $\mathrm{supp}(q) \subseteq \mathrm{supp}(p)$ then we write $q \ll p$. The set of all pmfs on alphabet $\mathcal{A}$, i.e., the probability simplex in $\mathbb{R}^{|\mathcal{A}|}$, is denoted by $\Delta(\mathcal{A})$. $X \sim \mathrm{Ber}(p)$ means $p_X(1) = 1 - p_X(0) = p$, and $h_2(p)$ denotes its entropy. $X \perp\!\!\!\perp Y$ means X and Y are independent.

II. Interactive function computation problem

A. Problem formulation

We consider two statistically dependent discrete memoryless stationary sources taking values in finite alphabets. For $i = 1, \ldots, n$, let $(X(i), Y(i)) \sim$ iid $p_{XY}(x, y), x \in \mathcal{X}, y \in \mathcal{Y}, |\mathcal{X}| < \infty, |\mathcal{Y}| < \infty$. Here, p_{XY} is a joint pmf which describes the statistical dependencies among the samples observed at the two terminals at each time instant i. Let $f_A : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}_A$ and $f_B : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}_B$ be functions of interest at terminals A and B respectively, where $\mathcal{Z}_A$ and $\mathcal{Z}_B$ are finite alphabets. The desired outputs at terminals A and B are $\mathbf{Z}_A$ and $\mathbf{Z}_B$ respectively, where for $i = 1, \ldots, n$, $Z_A(i) := f_A(X(i), Y(i))$ and $Z_B(i) := f_B(X(i), Y(i))$.

Definition 1: A two-terminal interactive distributed source code (for function computation) with initial terminal A and parameters $(t, n, |\mathcal{M}_1|, \ldots, |\mathcal{M}_t|)$ is the tuple $(e_1, \ldots, e_t, g_A, g_B)$ of t block encoding functions $e_1, \ldots, e_t$ and two block decoding functions g_A, g_B, of blocklength n, where for $j = 1, \ldots, t$,

$$\text{(Enc.}j) \quad e_j : \begin{cases} \mathcal{X}^n \times \bigotimes_{i=1}^{j-1} \mathcal{M}_i \to \mathcal{M}_j\ , & \text{if } j \text{ is odd} \\ \mathcal{Y}^n \times \bigotimes_{i=1}^{j-1} \mathcal{M}_i \to \mathcal{M}_j\ , & \text{if } j \text{ is even} \end{cases},$$

$$\text{(Dec.}A) \quad g_A : \quad \mathcal{X}^n \times \bigotimes_{j=1}^{t} \mathcal{M}_j \to \mathcal{Z}_A^n,$$

$$\text{(Dec.}B) \quad g_B : \quad \mathcal{Y}^n \times \bigotimes_{j=1}^{t} \mathcal{M}_j \to \mathcal{Z}_B^n.$$

The output of e_j, denoted by M_j, is called the j-th message, and t is the number of messages. The outputs of g_A and g_B are denoted by $\widehat{\mathbf{Z}}_A$ and $\widehat{\mathbf{Z}}_B$ respectively. For each j, $(1/n) \log_2 |\mathcal{M}_j|$ is called the j-th block-coding rate (in bits per sample). The sum of all the individual rates $(1/n) \sum_{j=1}^t \log_2 |\mathcal{M}_j|$ is called the sum-rate.

Definition 2: A rate tuple $\mathbf{R} = (R_1, \ldots, R_t)$ is admissible for t-message interactive function computation with initial terminal A if, $\forall \epsilon > 0$, $\exists\ N(\epsilon, t)$ such that $\forall n > N(\epsilon, t)$, there exists an interactive distributed source code with initial terminal A and parameters $(t, n, |\mathcal{M}_1|, \ldots, |\mathcal{M}_t|)$ satisfying

$$\frac{1}{n} \log_2 |\mathcal{M}_j| \le R_j + \epsilon, \quad j = 1, \ldots, t,$$
$$\mathbb{P}(\mathbf{Z}_A \ne \widehat{\mathbf{Z}}_A) \le \epsilon, \ \mathbb{P}(\mathbf{Z}_B \ne \widehat{\mathbf{Z}}_B) \le \epsilon.$$

Note that of interest here are the probabilities of block error $\mathbb{P}(\mathbf{Z}_A \ne \widehat{\mathbf{Z}}_A)$ and $\mathbb{P}(\mathbf{Z}_B \ne \widehat{\mathbf{Z}}_B)$ which are multi-letter distortion functions. The set of all admissible rate tuples, denoted by $\mathcal{R}_t^A$, is called the operational rate region for t-message interactive function computation with initial terminal A. The rate region is closed and convex due to the way it has been defined. The minimum sum-rate $R_{sum,t}^A$ is given by $\min\left(\sum_{j=1}^t R_j\right)$ where the minimization is over $\mathbf{R} \in \mathcal{R}_t^A$. For initial terminal B, the rate region and the minimum sum-rate are denoted by $\mathcal{R}_t^B$ and $R_{sum,t}^B$ respectively. The focus of this paper is on the minimum sum-rate rather than the rate region.

We allow the number of messages t to be equal to 0. When $t = 0$, there is no message transfer and the initial terminal is irrelevant. Thus for $t = 0$, in the notation for the minimum sum-rate, we omit the superscript and denote the minimum sum-rate as $R_{sum,0}$.

For a given initial terminal, for $t = 0$ and $t = 1$, function computation may not be feasible for general p_{XY}, f_A, f_B. If the computation is infeasible, $\mathcal{R}_t^A$ is empty and we set $R^A_{sum,t} = +\infty$. If for some specific p_{XY}, f_A, f_B, the computation is feasible, then $R^A_{sum,t}$ will be finite. Note that for $t \geq 2$, the computation is always feasible and $R^A_{sum,t}$ is finite.

For all $j \leq t$, null messages, i.e., messages for which $|\mathcal{M}_j| = 1$, are permitted by Definition 1. Hence, a $(t-1)$-message interactive code is a special case of a t-message interactive code. Thus, $R^A_{sum,(t-1)} \geq R^A_{sum,t}$ and $R^A_{sum,(t-1)} \geq R^B_{sum,t}$ (see [1, Proposition 1] for a detailed discussion). Therefore, $\lim_{t\to\infty} R^A_{sum,t} = \lim_{t\to\infty} R^B_{sum,t} =: R_{sum,\infty}$. The limit $R_{sum,\infty}$ is the infinite-message minimum sum-rate.

Depending on the specific joint pmf p_{XY} and functions f_A and f_B, it may be possible to reach the infinite-message limit $R_{sum,\infty}$ with finite t (see end of Sec. V-B for examples).

For all finite t, a single-letter characterization of the operational rate region $\mathcal{R}_t^A$ and the minimum sum-rate $R^A_{sum,t}$ were respectively provided in Theorem 1 and Corollary 1 of [1]. As discussed in Sec. II-B, this does not, in general, lead to a satisfactory characterization of the infinite-message limit $R_{sum,\infty}$ which is a new, unexplored dimension for asymptotic-analysis in distributed block source coding involving potentially an infinite number of infinitesimal-rate messages. The main goal and contribution of this paper is the development of a general convex-geometric blocklength-free characterization of this infinite-message limit.

B. *Characterization of $R^A_{sum,t}$ for finite t*

Let $\mathcal{U}_1, \ldots, \mathcal{U}_t$ be finite alphabets whose cardinalities are bounded as follows

$$|\mathcal{U}_j| \leq \begin{cases} |\mathcal{X}|\left(\prod_{i=1}^{j-1}|\mathcal{U}_i|\right) + t - j + 3, & j \text{ odd}, \\ |\mathcal{Y}|\left(\prod_{i=1}^{j-1}|\mathcal{U}_i|\right) + t - j + 3, & j \text{ even}. \end{cases} \quad (1)$$

Note that these bounds are independent of blocklength n. For $j = 1, \ldots, t$, j odd, let $p_{U_j|XU^{j-1}}$ denote a conditional pmf where for each $(x, u^{j-1}) \in \mathcal{X} \times \mathcal{U}_1 \times \ldots \times \mathcal{U}_{j-1}$, $p_{U_j|XU^{j-1}}(\cdot|x, u^{j-1}) \in \Delta(\mathcal{U}_j)$. Similarly, for $j = 1, \ldots, t$, j even, let $p_{U_j|YU^{j-1}}$ denote a conditional pmf where for each $(y, u^{j-1}) \in \mathcal{Y} \times \mathcal{U}_1 \times \ldots \times \mathcal{U}_{j-1}$, $p_{U_j|YU^{j-1}}(\cdot|y, u^{j-1}) \in \Delta(\mathcal{U}_j)$. Let $X, Y, U_1, \ldots, U_t$ denote random variables taking values in $\mathcal{X}, \mathcal{Y}, \mathcal{U}_1, \ldots, \mathcal{U}_t$ respectively with joint pmf $p_{XYU^t} = p_{XY} p_{U^t|XY}$ where for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and all $u^t \in \bigotimes_{i=1}^t \mathcal{U}_i$,

$$\begin{aligned} p_{U^t|XY}(u^t|x, y) &= p_{U_1|X}(u_1|x) \cdot p_{U_2|YU_1}(u_2|y, u_1) \cdot \\ &\quad \cdot p_{U_3|XU^2}(u_3|x, u^2) \ldots . \end{aligned} \quad (2)$$

Here, X and Y are referred to as the source random variables and U^t as the auxiliary random variables. Note that $p_{U^t|XY}$ is a conditional pmf where for each $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $p_{U^t|XY}(\cdot|x, y) \in \Delta(\mathcal{U}_1 \times \ldots \times \mathcal{U}_t)$. The factorization of $p_{U^t|XY}(u^t|x, y)$ in (2) is equivalent to the following Markov chain conditions involving X, Y, U^t: for $i = 1, \ldots, t$, if i is odd, $U_i - (X, U^{i-1}) - Y$ forms a Markov chain, otherwise $U_i - (Y, U^{i-1}) - X$ forms a Markov chain. Let

$$\mathcal{P}^A_{mc,t} := \{\text{all conditional pmfs } p_{U^t|XY} \text{ of the form (2)}\}. \quad (3)$$

Thus, $\mathcal{P}^A_{mc,t}$ is a family of conditional pmfs parameterized (continuously) by the conditional pmfs $p_{U_1|X}$, $p_{U_2|YU_1}$, ... For finite t, $\mathcal{P}^A_{mc,t}$ is a compact subset of a finite-dimensional Euclidean space. Let

$$\begin{aligned} \mathcal{P}_{ent,t}(p_{XY}, f_A, f_B) \quad := \quad \{p_{U^t|XY} : H(f_A(X, Y)|X, U^t) = \\ H(f_B(X, Y)|Y, U^t) = 0\}. \end{aligned} \quad (4)$$

Note that for all $t \geq 2$, the set $\mathcal{P}_{ent,t}$ is not empty because one can choose U_1 and U_2 such that $H(X|U_1) = H(Y|U_2) = 0$: take U_1 (respectively U_2) to be a deterministic one-to-one mapping from $\mathcal{X}$ to $\mathcal{U}_1$ (respectively $\mathcal{Y}$ to $\mathcal{U}_2$) (note that $|\mathcal{X}| \leq |\mathcal{U}_1|$ and $|\mathcal{Y}| \leq |\mathcal{U}_2|$). Also note that $H(f_A(X, Y)|X, U^t)$ and $H(f_B(X, Y)|Y, U^t)$ are continuous functionals of the joint pmf p_{XYU^t}; and for each fixed p_{XY}, they are continuous functionals of $p_{U^t|XY}$. Thus, for finite t, $\mathcal{P}_{ent,t}(p_{XY}, f_A, f_B)$ is a compact subset of a finite-dimensional Euclidean space (since it is the contour of conditional pmfs on which the conditional entropies are equal to zero). Therefore, $\mathcal{P}_t^A(p_{XY}, f_A, f_B) := \mathcal{P}^A_{mc,t} \cap \mathcal{P}_{ent,t}(p_{XY}, f_A, f_B)$ is a compact subset of a finite-dimensional Euclidean space. Generally speaking, $\mathcal{P}_t^A$ is determined by p_{XY}, f_A, and f_B. In the rest of this paper, however, f_A and f_B are fixed (but have general form) and p_{XY} is variable. Therefore, we drop f_A and f_B from the notation and speak of the family of conditional pmfs $\mathcal{P}_t^A(p_{XY})$ associated with p_{XY}. For initial terminal B, the corresponding set is denoted by $\mathcal{P}_t^B(p_{XY})$. We are now ready to state the characterization of $R^A_{sum,t}$ developed in [1].

Fact 1: (Characterization of $R^A_{sum,t}$ [1, Corollary 1])

$$R^A_{sum,t} = \min_{p_{U^t|XY} \in \mathcal{P}_t^A(p_{XY})} [I(X; U^t|Y) + I(Y; U^t|X)]. \quad (5)$$

Note that the conditional mutual information quantities in (5) are continuous functionals of the joint pmf p_{XYU^t}. In the minimization in (5), p_{XY}, f_A, and f_B are fixed. Since we are minimizing a continuous functional over a compact set, a minimizer exists in $\mathcal{P}_t^A(p_{XY})$. Since the arguments live in a finite dimensional Euclidean space, the minimization in (5) is a finite dimensional optimization problem.

The characterization of $R^A_{sum,t}$ in (5) does not directly inform us how quickly $R^A_{sum,t}$ converges to $R_{sum,\infty}$, i.e., bounds on the rate of convergence are unavailable for general p_{XY}, f_A, and f_B. In the absence of such bounds, one pragmatic approach to estimate $R_{sum,\infty}$ is to compute $R^A_{sum,t}$ by numerically solving (with some machine precision) the finite-dimensional optimization problem in (5) for increasing values of t until the difference between $R^A_{sum,t-1}$ and $R^A_{sum,t}$ is smaller than some small number. Although (5) provides a single-letter characterization for $R^A_{sum,t}$ for each finite t, as t increases, an increasing number of auxiliary random variables U^t are involved in the optimization problem. In fact, due to (1), the upper bounds for $|\mathcal{U}_t|$ increase exponentially with respect to t. Therefore, the dimension of the optimization problem in

(5) explodes as t increases. Each iteration is computationally much more demanding than the previous one. To make matters worse, there appears to be no obvious way of re-using the computations done for evaluating $R^A_{sum,t-1}$ when evaluating $R^A_{sum,t}$, i.e., every time t is increased, a new optimization problem needs to be solved all over again. Finally, if we need to estimate $R_{sum,\infty}$ for a different joint pmf p_{XY} (but for the same functions f_A and f_B), we would need to repeat this entire process for the new p_{XY}.

In Sec. III, we take a new fundamentally different approach. We first develop a general convex-geometric blocklength-free characterization of $R_{sum,\infty}$ which does not involve taking a limit as $t \to \infty$. Furthermore, instead of developing the characterization of $R_{sum,\infty}$ for a fixed joint pmf p_{XY} – which is a single nonnegative real number – we characterize the entire infinite-message minimum sum-rate surface $R_{sum,\infty}(p_{XY})$ – which is a functional of the joint pmf p_{XY} – in a single concise description. This leads to a simple test for checking if a given achievable sum-rate functional of p_{XY} coincides with $R_{sum,\infty}(p_{XY})$. It also provides a whole new family of lower bounds for $R_{sum,\infty}$. In Sec. IV, we use the new characterization to develop an iterative algorithm for computing the surfaces $R_{sum,\infty}(p_{XY})$ and $R^A_{sum,t}(p_{XY})$ (for any finite t) in which, crudely speaking, the complexity of computation in each iteration does not grow with iteration number and results from the previous iteration are re-used in the following one. In Sec. V we use the new characterization to evaluate $R_{sum,\infty}$ *exactly*, in closed analytic form, for two specific examples. For one of the examples (Sec. V-A), in an earlier work we had derived an upper bound for $R_{sum,\infty}(p_{XY})$ using an achievable distributed source coding strategy that uses infinitely many infinitesimal-rate messages, but had been unable establish the optimality of that strategy. The new characterization, however, shows this to be optimal.

III. Characterization of $R_{sum,\infty}(p_{XY})$

A. The rate reduction functional $\rho^A_t(p_{XY})$

If the goal is to *losslessly reproduce* the sources ($f_A(x,y) = y$, $f_B(x,y) = x$), the minimum sum-rate is equal to $H(X|Y) + H(Y|X)$ and this can be achieved by Slepian-Wolf coding. The sum-rate needed for computing functions can only be smaller than that needed for reproducing sources losslessly. The reduction in the minimum sum-rate for function computation in comparison to source reproduction is given by

$$\begin{aligned} \rho^A_t &:= H(X|Y) + H(Y|X) - R^A_{sum,t} \\ &= \max_{p_{U^t|XY} \in \mathcal{P}^A_t(p_{XY})} [H(X|Y,U^t) + H(Y|X,U^t)]. \end{aligned} \tag{6}$$

For interactive distributed source codes with initial terminal B, the minimum sum-rate and rate reduction are denoted by $R^B_{sum,t}$ and ρ^B_t respectively. A quantity which plays a key role in the characterization of $R_{sum,\infty}$ is ρ^A_0 corresponding to the "rate reduction" for zero messages (there are no auxiliary random variables in this case). Since the initial terminal has no significance when $t = 0$, $\rho^A_0 = \rho^B_0 =: \rho_0$. Let

$$\mathcal{P}_{f_A f_B} := \{p_{XY} \in \Delta(\mathcal{X}\times\mathcal{Y}) : H(f_A(X,Y)|X) = H(f_B(X,Y)|Y) = 0\}.$$

Error-free computations can be performed without any message transfers if, and only if, $p_{XY} \in \mathcal{P}_{f_A f_B}$. Thus,

$$R_{sum,0} = \begin{cases} 0, & \text{if } p_{XY} \in \mathcal{P}_{f_A f_B}, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\rho_0 = \begin{cases} H(X|Y) + H(Y|X), & \text{if } p_{XY} \in \mathcal{P}_{f_A f_B}, \\ -\infty, & \text{otherwise.} \end{cases} \tag{7}$$

Remark 1: If $f_A(x,y)$ is not a function of x alone and $f_B(x,y)$ is not a function of y alone, then for all $p_{XY} \in \mathcal{P}_{f_A f_B}$, we have $\text{supp}(p_{XY}) \neq \mathcal{X}\times\mathcal{Y}$. Such p_{XY} can only lie on the boundary of the probability simplex $\Delta(\mathcal{X}\times\mathcal{Y})$.

Evaluating $R^A_{sum,t}$ is equivalent to evaluating the rate reduction ρ^A_t. Notice, however, that in (6), all the auxiliary random variables appear only as conditioned random variables whereas this is not the case in (5). As discussed in Sec. III-C, this difference is critical as it enables us to characterize $\rho_\infty := \lim_{t\to\infty} \rho^A_t = \lim_{t\to\infty} \rho^B_t$ which then gives us a characterization of $R_{sum,\infty}$ as $R_{sum,\infty} = H(X|Y) + H(Y|X) - \rho_\infty$. The rate reduction functional is the key to the characterization.

B. Marginal-perturbations-closed family of joint pmfs $\mathcal{P}_{XY}$

Generally speaking, $R^A_{sum,t}$, ρ^A_t, $R_{sum,0}$ and ρ_∞ are functionals of p_{XY}, f_A, and f_B. We will view $R^A_{sum,t}(p_{XY})$, $\rho^A_t(p_{XY})$, $R_{sum,\infty}(p_{XY})$ and $\rho_\infty(p_{XY})$ as functionals of p_{XY} with f_A and f_B fixed to emphasize the dependence of p_{XY}. Instead of evaluating $\rho_\infty(p_{XY})$ for one particular p_{XY} as it is done in the numerical evaluation of single-terminal and Wyner-Ziv rate-distortion functions, our approach is to evaluate $\rho_\infty(p_{XY})$ for all p_{XY} belonging to $\mathcal{P}_{XY}$ – a collection of joint pmfs of interest which is closed in the sense of Definition 4. We will develop a characterization of $\rho_\infty(p_{XY})$ for the entire pmf-collection $\mathcal{P}_{XY}$; not just for one particular p_{XY}. Central to the definition of $\mathcal{P}_{XY}$ is the idea of a marginal perturbation set which is discussed next.

Definition 3: (X-marginal and Y-marginal perturbation sets $\mathcal{P}_{Y|X}(p_{XY})$ and $\mathcal{P}_{X|Y}(p_{XY})$) The set of X-marginal perturbations of a pmf $p_{XY} \in \Delta(\mathcal{X}\times\mathcal{Y})$ is defined as

$$\mathcal{P}_{Y|X}(p_{XY}) := \{p'_{XY} \in \Delta(\mathcal{X}\times\mathcal{Y}) : p'_{XY} \ll p_{XY}, p'_{XY} p_X = p_{XY} p'_X\}$$

where p_X and p'_X denote the X-marginals of p_{XY} and p'_{XY} respectively. Similarly, let

$$\mathcal{P}_{X|Y}(p_{XY}) := \{p'_{XY} \in \Delta(\mathcal{X}\times\mathcal{Y}) : p'_{XY} \ll p_{XY}, p'_{XY} p_Y = p_{XY} p'_Y\}$$

denote the set of Y-marginal perturbations of p_{XY} where p_Y and p'_Y denote the Y-marginals of p_{XY} and p'_{XY} respectively.

The sets $\mathcal{P}_{Y|X}(p_{XY})$ and $\mathcal{P}_{X|Y}(p_{XY})$ are nonempty as they contain p_{XY}. Notice that a pmf $p'_{XY} \in \mathcal{P}_{Y|X}(p_{XY})$ iff $p'_X \ll p_X$ and $\forall (x,y) \in \text{supp}(p'_X)\times\mathcal{Y}$, $p'_{Y|X}(y|x) = p_{Y|X}(y|x)$, where $p'_X, p'_{Y|X}(y|x)$ and $p_X, p_{Y|X}(y|x)$ are X-marginal and conditional pmfs of p'_{XY} and p_{XY} respectively. Essentially, $\mathcal{P}_{Y|X}(p_{XY})$ is the collection of all joint pmfs p'_{XY} which have the same conditional pmf $p_{Y|X}$ or $p'_{XY} = p_{Y|X}\cdot p'_X$ on $\text{supp}(p'_{XY})$. The subtlety is that the conditional pmf $p'_{Y|X}$ of the joint pmf p'_{XY} is well-defined only on $\text{supp}(p'_X)\times\mathcal{Y}$. Corresponding statements can be made for $\mathcal{P}_{X|Y}(p_{XY})$. Marginal perturbation sets can be viewed as neighborhoods of p_{XY}.

Remark 2: For all p_{XY}: (i) $\mathcal{P}_{Y|X}(p_{XY})$ and $\mathcal{P}_{X|Y}(p_{XY})$ are convex sets of joint pmfs; (ii) if $p'_{XY} \in \mathcal{P}_{Y|X}(p_{XY})$ then $\mathcal{P}_{Y|X}(p'_{XY}) \subseteq \mathcal{P}_{Y|X}(p_{XY})$; and (iii) if $p'_{XY} \in \mathcal{P}_{X|Y}(p_{XY})$ then $\mathcal{P}_{X|Y}(p'_{XY}) \subseteq \mathcal{P}_{X|Y}(p_{XY})$.

We will develop a characterization of $\rho_\infty(p_{XY})$ for all p_{XY} belonging to any family of joint pmfs $\mathcal{P}_{XY}$ which is closed with respect to X-marginal and Y-marginal perturbations.

Definition 4: (Marginal-perturbations-closed family of joint pmfs $\mathcal{P}_{XY}$) A family of joint pmfs $\mathcal{P}_{XY} \subseteq \Delta(\mathcal{X}\times\mathcal{Y})$ will be called marginal-perturbations-closed if for all $p_{XY} \in \mathcal{P}_{XY}$, $\mathcal{P}_{Y|X}(p_{XY}) \cup \mathcal{P}_{X|Y}(p_{XY}) \subseteq \mathcal{P}_{XY}$.

Examples of such marginal-perturbations-closed families of joint pmfs include (i) the set of all joint pmfs with supports contained in a specified subset of $\mathcal{X} \times \mathcal{Y}$, i.e., $\mathcal{P}_{XY} = \Delta(S)$ where $S \subseteq \mathcal{X} \times \mathcal{Y}$ and (ii) the set of all joint pmfs of all independent sources: $\mathcal{P}_{XY} = \{p_X p_Y | p_X \in \Delta(\mathcal{X}), p_Y \in \Delta(\mathcal{Y})\}$ (see Sec. V). In fact, if $q_X q_Y$ belongs to any marginal-perturbations-closed family with $\mathrm{supp}(q_X) = \mathcal{X}$ and $\mathrm{supp}(q_Y) = \mathcal{Y}$, then the family contains $\Delta(\mathcal{X}) \times \Delta(\mathcal{Y})$, that is, all product pmfs on $\mathcal{X}\times\mathcal{Y}$.

C. Main result

To describe the characterization of the functional $R_{sum,\infty}(p_{XY})$, it is convenient to define the following family of functionals associated with computing f_A and f_B.

Definition 5: (Marginal-perturbations-concave, ρ_0-majorizing family of functionals $\mathcal{F}(\mathcal{P}_{XY})$) Let $\mathcal{P}_{XY}$ be any marginal-perturbations-closed family of joint pmfs on $\Delta(\mathcal{X} \times \mathcal{Y})$. The set of marginal-perturbations-concave, ρ_0-majorizing family of functionals $\mathcal{F}(\mathcal{P}_{XY})$ is the set of all the functionals $\rho : \mathcal{P}_{XY} \to \mathbb{R}$ satisfying the following three conditions:

1) ρ_0-majorization: $\forall p_{XY} \in \mathcal{P}_{XY}$, $\rho(p_{XY}) \geq \rho_0(p_{XY})$.
2) Concavity with respect to X-marginal perturbations: $\forall p_{XY} \in \mathcal{P}_{XY}$, ρ is concave on $\mathcal{P}_{Y|X}(p_{XY})$.
3) Concavity with respect to Y-marginal perturbations: $\forall p_{XY} \in \mathcal{P}_{XY}$, ρ is concave on $\mathcal{P}_{X|Y}(p_{XY})$.

Remark 3: Since $\rho_0(p_{XY}) = -\infty$ for all $p_{XY} \notin \mathcal{P}_{f_A f_B}$, condition 1) of Definition 5 is trivially satisfied for all $p_{XY} \in \mathcal{P}_{XY} \setminus P_{f_A f_B}$ (we use the convention that $\forall a \in \mathbb{R}$, $a > -\infty$). Thus the statement that ρ majorizes ρ_0 on the set $\mathcal{P}_{XY}$ is equivalent to the statement that ρ majorizes $H(X|Y)+H(Y|X)$ on the set $\mathcal{P}_{f_A f_B} \bigcap \mathcal{P}_{XY}$.

Remark 4: Conditions 2) and 3) do not imply that ρ is concave on $\mathcal{P}_{XY}$. In fact, $\mathcal{P}_{XY}$ itself may not be convex. For example, the set $\mathcal{P}_{XY} = \{p_X p_Y | p_X \in \Delta(\mathcal{X}), p_Y \in \Delta(\mathcal{Y})\}$ is not convex.

We now state and prove the main result of this paper.

Theorem 1: (i) $\rho_\infty \in \mathcal{F}(\mathcal{P}_{XY})$. (ii) For all $\rho \in \mathcal{F}(\mathcal{P}_{XY})$, and all $p_{XY} \in \mathcal{P}_{XY}$, we have $\rho_\infty(p_{XY}) \leq \rho(p_{XY})$.

The set $\mathcal{F}(\mathcal{P}_{XY})$ is partially ordered with respect to majorization. The theorem says that $\mathcal{F}(\mathcal{P}_{XY})$ has a least element and that ρ_∞ is the least element. Note that there is no parameter t which needs to be sent to infinity in this characterization of ρ_∞.

To prove Theorem 1 we will establish a connection between the t-message interactive coding problem and a $(t-1)$-message interactive coding subproblem. Intuitively, to construct a t-message interactive code with initial terminal A, we need to begin by choosing the first message. This corresponds to choosing the auxiliary random variable U_1. Then for each realization $U_1 = u_1$, constructing the remaining part of the code becomes a $(t-1)$-message subproblem with initial terminal B with the same desired functions, but with a different source pmf $p_{XY|U_1}(\cdot,\cdot|u_1) \in \mathcal{P}_{Y|X}(p_{XY})$. We can repeat this procedure recursively to construct a $(t-1)$-message interactive code with initial terminal B. After t steps of recursion, we will be left with the trivial 0-message problem.

Proof: (i) We need to verify that ρ_∞ satisfies all three conditions in Definition 5:

1) Since $\forall p_{XY} \in \mathcal{P}_{XY}$, $R_{sum,\infty}(p_{XY}) \leq R_{sum,0}(p_{XY})$, we have $\rho_\infty(p_{XY}) \geq \rho_0(p_{XY})$. Thus ρ_∞ is ρ_0-majorizing.

2) For an arbitrary $q_{XY} \in \mathcal{P}_{XY}$, consider two arbitrary joint pmfs $p_{XY,1}, p_{XY,0} \in \mathcal{P}_{Y|X}(q_{XY})$. For every $\lambda \in (0,1)$, let $p_{XY,\lambda} := \lambda p_{XY,1} + (1-\lambda) p_{XY,0}$. Let $p_{X,0}(x), p_{Y|X,0}(y|x)$ and $p_{X,1}(x), p_{Y|X,1}(y|x)$ and $p_{X,\lambda}, p_{Y|X,\lambda}(y|x)$ denote the X-marginal and conditional pmfs of $p_{XY,0}$ and $p_{XY,1}$ and $p_{XY,\lambda}$ respectively. Due to Remark 2(i), $p_{XY,\lambda} \in \mathcal{P}_{Y|X}(q_{XY})$. We need to show that $\rho_\infty(p_{XY,\lambda}) \geq \lambda\rho_\infty(p_{XY,1}) + (1-\lambda)\rho_\infty(p_{XY,0})$.

Let (X,Y) be a pair of source random variables with joint pmf $p_{XY,\lambda}$. Consider an auxiliary random variable U_1^* taking values in $\mathcal{U}_1^* := \{0,1\}$ such that $(X,Y,U_1^*) \sim p_{XY,\lambda} p_{U_1^*|X}$ where $\forall x \in \mathrm{supp}(p_{X,\lambda})$, $p_{U_1^*|X}(1|x) := \lambda p_{X,1}(x)/p_{X,\lambda}(x)$ and $p_{U_1^*|X}(0|x) := (1-\lambda) p_{X,0}(x)/p_{X,\lambda}(x)$.

It follows that the marginal pmf of U_1^* is $\mathrm{Ber}(\lambda)$ and $Y - X - U_1^*$ is a Markov chain. Consequently, $\forall (x,u_1) \in \mathrm{supp}(p_{X,\lambda}) \times \mathcal{U}_1^*$, $p_{X|U_1^*}(x|u_1) = p_{X,u_1}(x)$ and $\forall (x,y,u_1) \in \mathrm{supp}(p_{XY,\lambda}) \times \mathcal{U}_1^*$, $p_{Y|X,U_1^*}(y|x,u_1^*) = p_{Y|X,\lambda}(y|x)$.

The key implication is that $\forall (x,y,u_1) \in \mathrm{supp}(p_{XY,\lambda}) \times \mathcal{U}_1^*$, $p_{XY|U_1^*}(x,y|u_1) = p_{XY,u_1}(x,y)$. This is because $p_{XY|U_1^*}(x,y|u_1) = p_{X,u_1}(x) \cdot p_{Y|X,\lambda}(y|x) = p_{X,u_1}(x) \cdot p_{Y|X,u_1}(y|x) = p_{XY,u_1}(x,y)$ where in the last but one equality we used the crucial property that all joint pmfs in $\mathcal{P}_{Y|X}(q_{XY})$ have the same conditional pmf.

Now, for all $t \in \mathbb{Z}^+$ we have,

$$
\begin{aligned}
\rho_t^A(p_{XY,\lambda}) &= \max_{p_{U^t|XY} \in \mathcal{P}_t^A(p_{XY,\lambda})} \left\{ H(X|Y,U^t) + H(Y|X,U^t) \right\} \\
&= \max_{p_{U_1|X}} \left\{ \max_{\substack{p_{U_2^t|XYU_1}: \\ p_{U_1|X} p_{U_2^t|XYU_1} \in \mathcal{P}_t^A(p_{XY,\lambda})}} \left\{ H(X|Y,U^t) + H(Y|X,U^t) \right\} \right\} \\
&\overset{(a)}{\geq} \max_{\substack{p_{U_2^t|XYU_1^*}: \\ p_{U_1^*|X} p_{U_2^t|XYU_1^*} \in \mathcal{P}_t^A(p_{XY,\lambda})}} \left\{ H(X|Y,U_2^t,U_1^*) + H(Y|X,U_2^t,U_1^*) \right\} \\
&\overset{(b)}{=} \lambda \cdot \max_{\substack{p_{U_2^t|XYU_1^*}(\cdot|\cdot,\cdot,1): \\ p_{U_1^*|X} p_{U_2^t|XYU_1^*} \in \mathcal{P}_t^A(p_{XY,1})}} \left\{ H(X|Y,U_2^t,U_1^*=1) + H(Y|X,U_2^t,U_1^*=1) \right\} \\
&\quad + \quad (1-\lambda) \cdot \\
&\quad \cdot \max_{\substack{p_{U_2^t|XYU_1^*}(\cdot|\cdot,\cdot,0): \\ p_{U_1^*|X} p_{U_2^t|XYU_1^*} \in \mathcal{P}_t^A(p_{XY,0})}} \left\{ H(X|Y,U_2^t,U_1^*=0) + H(Y|X,U_2^t,U_1^*=0) \right\}
\end{aligned}
$$

$$\overset{(c)}{=} \quad \lambda\, \rho^B_{t-1}(p_{XY,1}) + (1-\lambda)\, \rho^B_{t-1}(p_{XY,0}). \tag{8}$$

In step (a) we replaced $p_{U_1|X}$ with the particular $p_{U_1^*|X}$ defined above. Step (b) follows from the "law of total conditional entropy" with the additional observations that conditioned on $U_1^* = u_1$, $p_{XY|U_1^*}(x, y|u_1) = p_{XY,u_1}(x, y)$ and $(H(X|Y, U_2^t, U_1^* = u_1) + H(Y|X, U_2^t, U_1^* = u_1))$ only depends on $p_{U_2^t|XYU_1^*}(\cdot|\cdot,\cdot,u_1)$. Step (c) is due to the observation that for a fixed $p_{U_1^*|X}$, conditioned on $U_1^* = u_1$, (i) $p_{U_1^*|X} p_{U_2^t|XYU_1^*} \in \mathcal{P}^A_{mc,t}$ iff $p_{U_2^t|XYU_1^*} \in \mathcal{P}^B_{mc,t-1}$ and (ii) $p_{U_1^*|X} p_{U_2^t|XYU_1^*} \in \mathcal{P}_{ent,t}(p_{XY,u_1}, f_A, f_B)$ iff $p_{U_2^t|XYU_1^*} \in \mathcal{P}_{ent,t-1}(p_{XY,u_1}, f_A, f_B)$. Therefore, $p_{U_1^*|X} p_{U_2^t|XYU_1^*} \in \mathcal{P}^A_t(p_{XY,u_1})$ iff $p_{U_2^t|XYU_1^*} \in \mathcal{P}^B_{t-1}(p_{XY,u_1})$. Now send t to infinity in both the left and right sides of (8). Since $\lim_{t\to\infty} \rho^A_t = \lim_{t\to\infty} \rho^B_t = \rho_\infty$, we have $\rho_\infty(p_{XY,\lambda}) \geq \lambda\, \rho_\infty(p_{XY,1}) + (1-\lambda)\, \rho_\infty(p_{XY,0})$. Therefore, ρ_∞ satisfies condition 2) in Definition 5.

3) In a similar manner, by reversing the roles of terminals A and B in the above proof, it can be shown that ρ_∞ also satisfies condition 3) in Definition 5. Thus, $\rho_\infty \in \mathcal{F}(\mathcal{P}_{XY})$.

(ii) It is sufficient to show that: $\forall \rho \in \mathcal{F}(\mathcal{P}_{XY})$, $\forall p_{XY} \in \mathcal{P}_{XY}$, $\forall t \in \mathbb{Z}^+ \bigcup\{0\}$, $\rho^A_t(p_{XY}) \leq \rho(p_{XY})$ and $\rho^B_t(p_{XY}) \leq \rho(p_{XY})$. We prove this by induction on t. For $t = 0$, the result is true by condition 1) in Definition 5: $\rho^A_0(p_{XY}) = \rho^B_0(p_{XY}) = \rho_0(p_{XY}) \leq \rho(p_{XY})$. Now assume that for an arbitrary $t \in \mathbb{Z}^+$, $\rho^A_{t-1}(p_{XY}) \leq \rho(p_{XY})$ and $\rho^B_{t-1}(p_{XY}) \leq \rho(p_{XY})$ hold. We will show that $\rho^A_t(p_{XY}) \leq \rho(p_{XY})$ and $\rho^B_t(p_{XY}) \leq \rho(p_{XY})$ hold.

$$\begin{aligned}
\rho^A_t(p_{XY}) &= \max_{p_{U^t|XY} \in \mathcal{P}^A_t(p_{XY})} \left\{ H(X|Y, U^t) + H(Y|X, U^t) \right\} \\
&= \max_{p_{U_1|X}} \left\{ \max_{\substack{p_{U_2^t|XYU_1}: \\ p_{U_1|X} p_{U_2^t|XYU_1} \in \mathcal{P}^A_t(p_{XY})}} \left\{ H(X|Y, U^t) + H(Y|X, U^t) \right\} \right\} \\
&\overset{(d)}{=} \max_{p_{U_1|X}} \left\{ \sum_{u_1 \in \mathrm{supp}(p_{U_1})} p_{U_1}(u_1) \left\{ \max_{\substack{p_{U_2^t|XYU_1}(\cdot|\cdot,\cdot,u_1): \\ p_{U_1|X} p_{U_2^t|XYU_1} \in \mathcal{P}^A_t(p_{XY|U_1}(\cdot,\cdot|u_1))}} \left\{ H(X|Y, U_2^t, U_1 = u_1) \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \; H(Y|X, U_2^t, U_1 = u_1) \right\} \right\} \right\} \\
&\overset{(e)}{=} \max_{p_{U_1|X}} \left\{ \sum_{u_1 \in \mathrm{supp}(p_{U_1})} p_{U_1}(u_1)\, \rho^B_{t-1}(p_{XY|U_1}(\cdot,\cdot|u_1)) \right\} \qquad (9) \\
&\overset{(f)}{\leq} \max_{p_{U_1|X}} \left\{ \sum_{u_1 \in \mathrm{supp}(p_{U_1})} p_{U_1}(u_1)\, \rho(p_{XY|U_1}(\cdot,\cdot|u_1)) \right\} \\
&\overset{(g)}{\leq} \max_{p_{U_1|X}} \left\{ \rho\left(\sum_{u_1 \in \mathrm{supp}(p_{U_1})} p_{U_1}(u_1) p_{XY|U_1}(\cdot,\cdot|u_1) \right) \right\} \\
&= \rho(p_{XY}).
\end{aligned}$$

The reasoning for steps (d) and (e) are similar to those for steps (b) and (c) respectively in the proof of part (i) (see equation array (8)) but for step (e) we need to also confirm that $p_{XY|U_1}(\cdot,\cdot|u_1) \in \mathcal{P}_{Y|X}(p_{XY})$ for all $u_1 \in \mathrm{supp}(p_{U_1})$. This is confirmed by noting that since $Y - X - U_1$ is a Markov chain, $\forall u_1 \in \mathrm{supp}(p_{U_1})$ and $\forall x \in \mathrm{supp}(p_X)$, we have $p_{Y|XU_1}(y|x, u_1) = p_{Y|X}(y|x)$ (see para after Definition 3). Step (f) is due to the inductive hypothesis $\rho^B_{t-1}(p_{XY}) \leq \rho(p_{XY})$. Step (g) is Jensen's inequality applied to $\rho(p_{XY})$ which is concave on $\mathcal{P}_{Y|X}(p_{XY})$. Using similar steps as above, we can also show that $\rho^B_t(p_{XY}) \leq \rho(p_{XY})$. ■

Remark 5: In the proof of Theorem 1, there are only two places where the marginal-perturbations-closed property of $\mathcal{P}_{XY}$ is used. It is first used in part (i) to show that $p_{XY|U_1^*}(x, y|u_1) = p_{XY,u_1}(x, y)$. It is used in part (ii) to show that $p_{XY|U_1}(\cdot,\cdot|u_1) \in \mathcal{P}_{Y|X}(p_{XY})$.

Remark 6: It can be verified that the functional $(H(X|Y) + H(Y|X))$ belongs to $\mathcal{F}(\Delta(\mathcal{X}\times\mathcal{Y}))$. Whereas both $(H(X|Y) + H(Y|X))$ and $\rho_\infty(p_{XY})$ are concave on X-marginal and Y-marginal perturbation sets of p_{XY}, it cannot be claimed that $R_{sum,\infty}(p_{XY}) = (H(X|Y) + H(Y|X)) - \rho_\infty(p_{XY})$ will be *convex* on the marginal perturbation sets of p_{XY}. For each t, ρ^A_t is the maximum of $(H(X|Y, U^t) + H(Y|X, U^t))$, where U^t appear only as conditioned random variables. This enables us to use the "law of total conditional entropy" (which corresponds to convexification) and arrive at (8) and (9). Notice, however, that $R_{sum,\infty}$ is the minimum value of $(I(X; U^t|Y) + I(Y; U^t|X))$ over all U^t where U^t are not conditioned. Therefore, $R^A_{sum,t}$ cannot be expressed as a convex combination of $R^B_{sum,t-1}$. Due to these reasons, although evaluating ρ_∞ is equivalent to evaluating $R_{sum,\infty}$, the rate reduction functional is the key to the characterization as remarked in Sec. III-A.

Since every $\rho \in \mathcal{F}(\mathcal{P}_{XY})$ gives an upper bound for ρ_∞, $(H(X|Y) + H(Y|X) - \rho)$ gives a lower bound for $R_{sum,\infty}$. This fact provides a way testing if an achievable sum-rate functional is optimal. If R^* is a sum-rate functional which is achievable then $\forall p_{XY} \in \mathcal{P}_{XY}$, $R^*(p_{XY}) \geq R_{sum,\infty}(p_{XY})$. If it can be verified that $\rho^* := (H(X|Y) + H(Y|X) - R^*)$ belongs to $\mathcal{F}(\mathcal{P}_{XY})$, then by Theorem 1, $R^* = R_{sum,\infty}$. The nontrivial part of the test is to verify if R^* is concave on X-marginal and Y-marginal perturbation sets. We will demonstrate this test on two examples in Sec. V.

IV. Iterative algorithm for computing $R^A_{sum,t}(\cdot)$ and $R_{sum,\infty}(\cdot)$

Although Theorem 1 provides a characterization of ρ_∞ and $R_{sum,\infty}$ that is not obtained by taking a limit, it does not directly provide an algorithm to evaluate $R_{sum,\infty}$. To efficiently represent and search for the least element of $\mathcal{F}(\mathcal{P}_{XY})$ is nontrivial because each element is a functional; not a scalar. The proof of Theorem 1, however, inspires an iterative algorithm for evaluating $R^A_{sum,t}$ and $R_{sum,\infty}$.

Equation (9) states that $\rho^A_t(p_{XY})$ is the maximum value of $\rho \in \mathbb{R}$ such that (p_{XY}, ρ) is a finite convex combination of $\{(p_{XY|U_1}(\cdot,\cdot|u_1), \rho^B_{t-1}(p_{XY|U_1}(\cdot,\cdot|u_1))\}_{u_1 \in \mathrm{supp}(p_{U_1})}$, where $p_{XY|U_1}(\cdot,\cdot|u_1)$ belongs to $\mathcal{P}_{Y|X}(p_{XY})$ for all u_1 in $\mathrm{supp}(p_{U_1}) \subseteq \mathcal{U}_1$. Consider the hypograph of $\rho^B_{t-1}(\cdot)$ on $\mathcal{P}_{Y|X}(p_{XY})$: $\mathrm{hyp}_{\mathcal{P}_{Y|X}(p_{XY})}\rho^B_{t-1} := \{(p_{XY}, \rho) : p_{XY} \in \mathcal{P}_{Y|X}(p_{XY}),\ \rho \leq \rho^B_{t-1}(p_{XY})\}$. Due to (9), the convex hull of $\mathrm{hyp}_{\mathcal{P}_{Y|X}(p_{XY})}\rho^B_{t-1}$ is $\mathrm{hyp}_{\mathcal{P}_{Y|X}(p_{XY})}\rho^A_t$. This enables us to evaluate ρ^A_t from ρ^B_{t-1} on the set $\mathcal{P}_{Y|X}(p_{XY})$: ρ^A_t is the least concave functional on

$\mathcal{P}_{Y|X}(p_{XY})$ that majorizes ρ^B_{t-1}. In the convex optimization literature, $(-\rho^A_1)$ is called the double Legendre-Fenchel transform or convex biconjugate of $(-\rho^B_{t-1})$ [10]. Thus ρ^A_t can be determined through a convex biconjugation operation (taking a convex hull of a hypograph) on any given X-marginal perturbation set. To determine $\rho^A_t(p_{XY})$ for all $p_{XY} \in \mathcal{P}_{XY}$, we can, in principle, first choose a cover for $\mathcal{P}_{XY}$ made up of X-marginal perturbation sets, say $\{\mathcal{P}_{Y|X}(p_{XY})\}_{p_{XY}\in\mathcal{A}}$, where $\mathcal{A} \subseteq \mathcal{P}_{XY}$, and then perform the convex biconjugation operation in every X-perturbation set in the cover. This relationship between ρ^A_t and ρ^B_{t-1} leads to the following iterative algorithm.

Algorithm to evaluate $R^A_{sum,t}$ and $R^B_{sum,t}$

- **Initialization:** Choose a marginal-perturbations-closed family $\mathcal{P}_{XY}$ containing all source joint pmfs of interest. Define $\rho^A_0(p_{XY}) = \rho^B_0(p_{XY}) = \rho_0(p_{XY})$ by equation (7) in the domain $\mathcal{P}_{XY}$. Choose a cover for $\mathcal{P}_{XY}$ made up of X-marginal perturbation sets, denoted by $\{\mathcal{P}_{Y|X}(p_{XY})\}_{p_{XY}\in\mathcal{A}}$, where $\mathcal{A} \subseteq \mathcal{P}_{XY}$. Also choose a cover for $\mathcal{P}_{XY}$ made up of Y-marginal perturbation sets, denoted by $\{\mathcal{P}_{X|Y}(p_{XY})\}_{p_{XY}\in\mathcal{B}}$, where $\mathcal{B} \subseteq \mathcal{P}_{XY}$.
- **Loop:** For $\tau = 1$ through t do the following.
  For every $p_{XY} \in \mathcal{A}$, do the following in the set $\mathcal{P}_{Y|X}(p_{XY})$.
  - Construct $\mathrm{hyp}_{\mathcal{P}_{Y|X}(p_{XY})}\rho^B_{\tau-1}$.
  - Let ρ^A_τ be the upper boundary of the convex hull of $\mathrm{hyp}_{\mathcal{P}_{Y|X}(p_{XY})}\rho^B_{\tau-1}$.

  For every $p_{XY} \in \mathcal{B}$, do the following in the set $\mathcal{P}_{X|Y}(p_{XY})$.
  - Construct $\mathrm{hyp}_{\mathcal{P}_{X|Y}(p_{XY})}\rho^A_{\tau-1}$.
  - Let ρ^B_τ be the upper boundary of the convex hull of $\mathrm{hyp}_{\mathcal{P}_{X|Y}(p_{XY})}\rho^A_{\tau-1}$.
- **Output** $R^A_{sum,t}(p_{XY}) = H(X|Y) + H(Y|X) - \rho^A_t(p_{XY})$ and $R^B_{sum,t}(p_{XY}) = H(X|Y) + H(Y|X) - \rho^B_t(p_{XY})$.

To make numerical computation feasible, $\mathcal{P}_{XY}$ has to be discretized. Once discretized, however, in each iteration, the amount of computation is the same and is fixed by the discretization step-size. Also note that results from each iteration are re-used in the following one. Therefore, for large t, the complexity to compute $R^A_{sum,t}$ grows linearly with respect to t.

$R_{sum,\infty}$ can also be evaluated to any precision, in principle, by running this iterative algorithm for $t = 1, 2, \ldots$, until some stopping criterion is met, e.g., the maximum difference between ρ^A_{t-1} and ρ^A_t on $\mathcal{P}_{XY}$ falls below some threshold. Developing stopping criteria with precision guarantees requires some knowledge of the rate of convergence which is not established in this paper; the rate may, however, be empirically estimated. When the objective is to evaluate $R_{sum,\infty}(p_{XY})$ for all pmfs in $\mathcal{P}_{XY}$, this iterative algorithm is much more efficient than using (5) to solve for $R^A_{sum,t}$ for each p_{XY} for $t = 1, 2, \ldots$, an approach which follows the definition of $R_{sum,\infty}$ literally as the limit of $R^A_{sum,t}$ as $t \to \infty$. Our iterative algorithm is based on Theorem 1 which is a characterization of $R_{sum,\infty}$ *without taking a limit involving* t.

Since $-\rho^A_t$ is the convex biconjugate of $-\rho^B_{t-1}$ on all X-marginal perturbation sets and $-\rho^B_t$ is the convex biconjugate of $-\rho^A_{t-1}$ on all Y-marginal perturbation sets, it follows that for all $t > 0$, ρ^A_t satisfies conditions 1) and 2) in Definition 5 (ρ_0-majorization and concavity with respect to X-marginal perturbations), and ρ^B_t satisfies satisfies conditions 1) and 3) (ρ_0-majorization and concavity with respect to Y-marginal perturbations). By Theorem 1, ρ_∞ satisfies all three conditions of Definition 5 and is not larger than any ρ which satisfies all three conditions. Also for all t, by definition, $\rho^A_t \le \rho_\infty$ and $\rho^A_t \le \rho_\infty$. Hence, if for some t, ρ^A_t satisfies 3) then $\rho^A_t = \rho_\infty$. Similarly, if for some t, ρ^B_t satisfies 2) then $\rho^B_t = \rho_\infty$. Thus, ρ^A_t and ρ^B_t equal ρ_∞ iff they satisfy all three conditions. If all three conditions are not satisfied (two are always satisfied), it is beneficial to increase the number of messages. Specifically, if ρ^A_t is not concave on a Y-marginal perturbation set, then for some p_{XY}, $\rho^A_t(p_{XY}) < \rho^B_{t+1}(p_{XY}) \le \rho^A_{t+2}(p_{XY})$.

V. Examples

A. *$R_{sum,\infty}$ for independent binary sources and Boolean AND function computed at both terminals*

In [1, Sec. IV.F], we studied the samplewise computation of the Boolean AND function at both terminals for independent Bernoulli sources, i.e., $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, $X \perp\!\!\!\perp Y$, $X \sim \mathrm{Ber}(p)$, $Y \sim \mathrm{Ber}(q)$, and $f_A(x, y) = f_B(x, y) = x \wedge y$. An interesting interactive coding scheme was described in [1] where the individual rate for each message vanished as the number of messages went to infinity. The (achievable) infinite-message sum-rate of this scheme, denoted by R^*, was evaluated in closed form as

$$R^*(p, q) = \begin{cases} h_2(p) + ph_2(q) + p\log_2 q + p(1-q)\log_2 e & \\ & \text{if } 0 \le p \le q \le 1, \\ R^*(q, p) & \text{if } 0 \le q \le p \le 1. \end{cases} \tag{10}$$

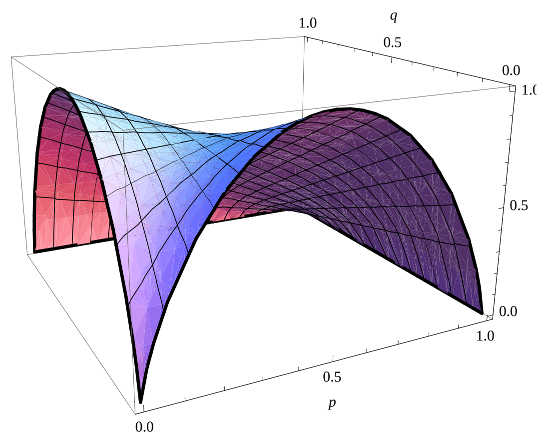
This expression was derived in [1, Sec. IV.F] for the situation $0 < p \le q < 1$. The situation $0 < q \le p < 1$ follows by symmetry. The remaining situations $pq = 0$ and $(1-p)(1-q) = 0$ easily follow using zero or one message. Since $R^*(p, q)$ is an achievable sum-rate, $R^* \ge R_{sum,\infty}$. Using Theorem 1, we shall now prove that R^* is, in fact, equal to $R_{sum,\infty}$. We will verify that $\rho^* := H(X|Y) + H(Y|X) - R^*$ belongs to $\mathcal{F}(\mathcal{P}_{XY})$ for the product pmf family $\mathcal{P}_{XY}$, which will imply, by Theorem 1(ii), that $\rho^* \ge \rho_\infty$, i.e., $R^* \le R_{sum,\infty}$. Note that $R_{sum,\infty}$ is not evaluated using Theorem 1. Only part (ii) of Theorem 1 is used as a converse proof to show that the achievable sum-rate R^* is $R_{sum,\infty}$.

Since the sources are independent, we take the marginal-perturbations-closed family to be $\mathcal{P}_{XY} = \{p_X p_Y | p_X \in \Delta(\mathcal{X}), p_Y \in \Delta(\mathcal{Y})\}$. For each product pmf $p_X p_Y$, the X-marginal and Y-marginal perturbation sets are $\mathcal{P}_{Y|X}(p_X p_Y) = \{p'_X p_Y : p'_X \ll p_X\}$ and $\mathcal{P}_{X|Y}(p_X p_Y) = \{p_X p'_Y : p'_Y \ll p_Y\}$ respectively. Since p_X and p_Y are parameterized by p and q respectively, each product pmf $p_X p_Y$ can be represented by a point $(p, q) \in [0, 1]^2$. For all pmfs $(p, q) \in (0, 1)^2$, the X-marginal and Y-marginal perturbation sets are line segments $[0, 1] \times \{q\}$ and $\{p\} \times [0, 1]$ respectively. For all pmfs $(0, q)$, where $q \in (0,1)$, the X-marginal and Y-marginal perturbation sets are $(0,q)$ and $\{0\}\times[0,1]$ respectively. For the pmfs $(0,0)$, both the X-marginal and Y-marginal perturbation sets are $(0,0)$. The marginal perturbation sets of remaining pmfs on the boundary of $[0,1]^2$ can be derived using symmetry (swap p and q; then swap symbols 0 and 1).

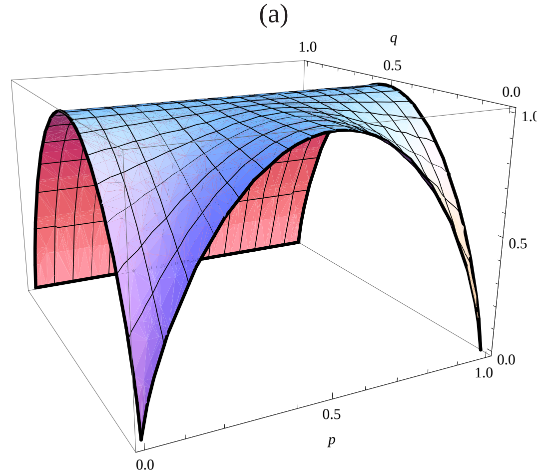
It is easy to see that

$$R_{sum,0}(p,q) = \begin{cases} 0, & \text{if } (p,q) \in \mathcal{P}_{f_A f_B}, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\mathcal{P}_{f_A f_B} = \{(p,q) : p = 0 \text{ or } q = 0 \text{ or } p = q = 1\}$. It is also easy to verify that for all (p,q), $R^*(p,q) \le R_{sum,0}(p,q) = 0$, or equivalently, $\rho^*(p,q) \ge \rho_0(p,q)$. By taking the first and second-order partial derivatives of $\rho^*(p,q) = h_2(p) + h_2(q) - R^*(p,q)$ with respect to p and q, we can verify that for any fixed q, $\rho^*(p,q)$ is concave with respect to p, and for any fixed p, $\rho^*(p,q)$ is concave with respect to q. Therefore, $\rho^*(p,q)$ is concave in every X-marginal and Y-marginal perturbation set. Therefore, $\rho^*(p,q) \in \mathcal{F}(\mathcal{P}_{XY})$, which implies that $R_{sum,\infty}(p,q) \ge R^*(p,q)$ due to Theorem 1(ii). Since $R^*(p,q)$ is both an upper bound and a lower bound of $R_{sum,\infty}(p,q)$, we have $R_{sum,\infty}(p,q) = R^*(p,q)$. Fig. 2(a) shows a plot of $\rho_\infty(p,q) = \rho^*(p,q)$. Note that $\rho_\infty(p,q)$ is concave in p and q separately but not jointly concave in the pair (p,q).



(a)



(b)

Fig. 2. *(a) $\rho_\infty(p,q)$ for $x \wedge y$ computed at both terminals (Sec. V-A). (b) $\rho_\infty(p,q)$ for $x \wedge y$ computed only at terminal B (Sec. V-B).*

B. $R_{sum,\infty}$ for independent binary sources and Boolean AND function computed at only terminal B

We change the problem in Sec. V-A to the problem of computing the Boolean AND function at only terminal B, i.e., $f_A(x,y) = 0$ and $f_B(x,y) = x \wedge y$. The source statistics are unchanged: $X \perp\!\!\!\perp Y$, $X \sim \text{Ber}(p)$, $Y \sim \text{Ber}(q)$. An achievable sum-rate R^* can be derived using the same technique presented in [1, Sec. IV.F]. The derivation is omitted in this paper due to limited space but will be presented in [9]. The expression for R^* is

$$R^*(p,q) = \begin{cases} h_2(p) + p h_2(q) + p \log_2 q + p(1-2q)\log_2 e & \\ \qquad\qquad\qquad\qquad \text{if } 0 \le p \le q \le 1/2, \\ R^*(q,p) & \text{if } 0 \le q \le p \le 1/2, \\ R^*(1-p,q) & \text{if } 0 \le q \le 1/2 \le p \le 1, \\ h_2(p) & \text{if } 1/2 \le q \le 1. \end{cases}$$

Following the method in Sec. V-A, it can verified that $\mathcal{P}_{f_A f_B} = \{(p,q) : p = 0 \text{ or } q = 0 \text{ or } p = 1\}$ and $\rho^*(p,q) = (h_2(p)+h_2(q)-R^*(p,q))$ belongs to $\mathcal{F}(\mathcal{P}_{XY})$, where $\mathcal{P}_{XY} = \{p_X p_Y | p_X \in \Delta(\mathcal{X}), p_Y \in \Delta(\mathcal{Y})\}$ is the same marginal-perturbations-closed family used in Sec. V-A. Therefore, $R^* = R_{sum,\infty}$. Fig. 2(b) shows a plot of $\rho_\infty(p,q) = \rho^*(p,q)$.

Note that for all $(p,q) \in \mathcal{P}_{f_A f_B}$, $R_{sum,0} = 0 = R_{sum,\infty}$ and no message needs to be sent. For all $(p,q) \in (0,1)\times[1/2,1]$, $R_{sum,\infty} = h_2(p)$ and this sum-rate can be achieved with $t = 1$ message from A to B, thus $R^A_{sum,1} = R_{sum,\infty}$. Note that $R_{sum,0} = \infty$ because $(p,q) \notin \mathcal{P}_{f_A f_B}$ and $R^B_{sum,1} = \infty$. For $(p,q) \in \{1/2\} \times (0,1/2)$, $R_{sum,\infty} = h_2(q)$. In [8, Sec. V.C] it was shown that this sum-rate can be achieved with $t = 2$ messages, the first from B to A and the second from A to B. Thus $R^B_{sum,2} = R_{sum,\infty}$. Note that $R^B_{sum,1} = \infty$ and in [1, Sec. IV.C] we showed that $R^A_{sum,1} = \log_2 2 = 1$. These examples show that depending on the specific joint pmf and functions, it may be possible to reach the infinite-message limit $R_{sum,\infty}$ with finite t.

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