

# $K$ -User Fading Interference Channels: The Ergodic Very Strong Case

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**Abstract**—Sufficient conditions required to achieve the interference-free capacity region of ergodic fading  $K$ -user interference channels (IFCs) are obtained. In particular, this capacity region is shown to be achieved when every receiver decodes all  $K$  transmitted messages such that the channel statistics and the waterfilling power policies for all  $K$  (interference-free) links satisfy a set of  $K(K - 1)$  ergodic very strong conditions. The result is also of independent interest in combinatorics.

## I. INTRODUCTION

The  $K$ -user interference channel (IFC) is a network with  $K$  transmitter-receiver pairs (also referred to as users or links) in which each transmitter transmits to its intended receiver while creating interference at one or more of the unintended receivers. In general, the problem of determining the capacity region of a  $K$ -user IFC remains open. Capacity regions are known for the two-user *strong* IFC [1], [2] and the *very strong* IFC [3], and in both cases the capacity region is achieved when both receivers decode both the intended and interfering signals, i.e., the IFC reduces to a compound MAC (C-MAC) [4]. The *very strong* IFC is a sub-class of the class of strong IFCs for which the sum-capacity and the capacity region are determined by the interference-free bottleneck links from the two transmitters to their intended receivers. On the other hand, only the sum-capacity is known for a class of *weak* one-sided non-fading two-user IFCs [5] and is achieved by ignoring interference (i.e., treating it as noise). More recently, for the two-sided model, the sum-capacity of a class of noisy or very weak Gaussian IFCs is determined independently in [6], [7], and [8] and these results have been extended for  $K > 2$  in [8] and [9].

Ergodic fading and parallel Gaussian IFCs model the fading properties of wireless networks. In this paper, we focus on  $K$ -user ergodic fading Gaussian IFCs and seek to determine a set of conditions for which the sum-capacity of  $K$  interference-free links can be achieved. Recently, in [10] which develops sum-capacity and separability results for two-user ergodic fading Gaussian IFCs, an *ergodic very strong* (EVS) sub-class has been identified as a collection of ergodic fading Gaussian IFCs with a weighted mixture of weak and strong sub-channels (fading states) for which the sum of the interference-free capacities of the two user links

can be achieved. Two-user parallel Gaussian IFCs have also been studied in [11], [12] and [13]. For fading IFCs with three or more users, [14] presents an *interference alignment* scheme to show that the sum-capacity of a  $K$ -user IFC scales linearly with  $K$  in the high signal-to-noise ratio (SNR) regime when all links in the network have similar channel statistics.

The sum-capacity and capacity region of two-user EVS IFCs are achieved when each user transmits to its intended receiver as if there were two independent interference-free links. The sum-capacity optimal power policies are thus the classic point-to-point waterfilling solutions developed in [15]. While the sum-rate achieved thus is always an outer bound on the sum-capacity of IFCs, in [10] it is shown that this outer bound can be achieved when both receivers decode both messages, i.e., the IFC converts to a C-MAC, provided the sum of the interference-free capacities of each link is strictly smaller than the multiaccess sum-rates achieved at each receiver. These sufficient conditions does not impose strong or weak conditions on any sub-channel and only involve fading averaged conditions on the waterfilling policies.

For two-user IFCs, the above-mentioned sufficient conditions are obtained simply by enumerating all possible intersections of the two MAC pentagons and identifying the intersection satisfying the EVS definition. They can also be obtained using the fact that the multiaccess rate region at each receiver is a polymatroid [16], and therefore, a single known lemma on the sum-rate of two intersecting polymatroids [17, chap. 46] readily yields a closed-form expression for each possible intersection of the two MAC regions. However, for intersections of three or more polymatroids no such lemma exists that can simplify the sum-capacity analysis of C-MACs with  $K$  transmitters and  $K$  receivers. This in turn makes it difficult to identify a set of sufficient conditions for EVS IFCs when every receiver is allowed to decode the message from every transmitter.

In this paper, we determine a set of sufficient conditions for which the sum-capacity is the sum of the capacities of  $K$  interference-free links when all receivers are allowed to decode the messages from all transmitters. This in turn corresponds to determining the conditions for which the intersection of  $K$  rate polymatroids results in a  $K$ -dimensional *box* (hyper-cube) that is uniquely defined by the interference-free point-to-point rates of the  $K$  links. We show that the number of sufficient conditions for EVS IFCs grows

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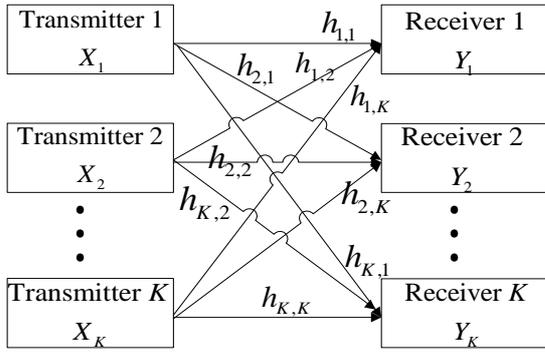


Fig. 1. A  $K$ -user ergodic fading interference channel.

quadratically in  $K$ . As a special case, for two-user IFCs, we show that these conditions are both necessary and sufficient.

Recently, in [18], the authors present sufficient conditions for which a  $K$ -user symmetric non-fading IFC achieves the sum-capacity of  $K$  interference-free links. The conditions are developed using lattice codes which enables complete decoding of the interference but not the message from every interfering transmitter. Using the fact that the non-fading IFC is a special case of an EVS IFC, we compare our results to those in [18] and show that decoding messages from all users is best used in the low power regime or when the symmetric cross-links are relatively closer to unity while lattice codes are advantageous otherwise. Finally, we note that the results developed here are also of independent interest in combinatorics.

The paper is organized as follows. We present channel model and preliminaries in Section II. The main result and the proof are developed in Section III. We discuss the results and present numerical examples in Section IV. We conclude in Section V.

## II. CHANNEL MODEL

A  $K$ -user (or  $K$ -link) ergodic fading Gaussian IFC consists of  $K$  transmitter-receiver pairs, each link pair indexed by  $k$ ,  $k \in \mathcal{K} = \{1, 2, \dots, K\}$ , as shown in Fig. 1. Transmitter  $k$  uses the channel  $n$  times to transmit its message  $W_k$ , which is distributed uniformly in the set  $\{1, 2, \dots, 2^{B_k}\}$  and is independent of the messages from the other transmitters, to its intended receiver  $k$ , at a rate  $R_k = B_k/n$  bits per channel use. In each use of the channel, transmitter  $k$  transmits the signal  $X_k$  while receiver  $k$  receives  $Y_k$ ,  $k \in \mathcal{K}$ . For  $\mathbf{X} = [X_1 \ X_2 \ \dots \ X_K]^T$ , the channel output vector  $\mathbf{Y} = [Y_1 \ Y_2 \ \dots \ Y_K]^T$  in a single channel use is given by

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z} \quad (1)$$

where  $\mathbf{Z} = [Z_1 \ Z_2 \ \dots \ Z_K]^T$  is a noise vector with entries that are zero-mean, unit variance, circularly symmetric complex Gaussian noise variables and  $\mathbf{H}$  is a random matrix of fading gains with entries  $H_{m,k}$ , for all  $m, k \in \mathcal{K}$ , such that  $H_{m,k}$  denotes the fading gain between receiver  $m$  and

transmitter  $k$ . We use  $\mathbf{h}$  to denote a realization of  $\mathbf{H}$ . We assume the fading process  $\{\mathbf{H}\}$  is stationary and ergodic but not necessarily Gaussian. Note that the channel gains  $H_{m,k}$ , for all  $m$  and  $k$ , are not assumed to be independent; however,  $\mathbf{H}$  is assumed to be known instantaneously at all the transmitters and receivers.

Over  $n$  uses of the channel, the transmit sequences  $\{X_{k,i}\}$  are constrained in power according to

$$\sum_{i=1}^n |X_{k,i}|^2 \leq n\bar{P}_k, \text{ for } k \in \mathcal{K}. \quad (2)$$

Since the transmitters know the fading states of the links on which they transmit, they can allocate their transmitted signal power according to the channel state information. A power policy  $\underline{P}(\mathbf{h})$  with entries  $P_k(\mathbf{h})$  for  $k \in \mathcal{K}$  is a mapping from the fading state space consisting of the set of all fading states (instantiations)  $\mathbf{h}$  to the set of non-negative real values in  $\mathcal{R}_+^K$ . We write  $\underline{P}(\mathbf{H})$  to describe explicitly the policy for the entire set of random fading states. For an ergodic fading channel, (2) then simplifies to

$$\mathbb{E}[P_k(\mathbf{H})] \leq \bar{P}_k \text{ for } k \in \mathcal{K}, \quad (3)$$

where the expectation in (3) is taken over the distribution of  $\mathbf{H}$ .

For the special case in which all receivers decode the messages from all transmitters, we obtain a compound MAC. We write  $\mathcal{C}_{\text{IFC}}(\underline{P})$  and  $\mathcal{C}_{\text{C-MAC}}(\underline{P})$  to denote the capacity regions of an ergodic fading IFC and C-MAC, respectively, where  $\underline{P}$  is a vector whose entries are the average power constraints  $\bar{P}_k$ , for  $k \in \mathcal{K}$ . Our definitions of average error probabilities, capacity regions, and achievable rate pairs  $(R_1, R_2, \dots, R_K)$  for both the IFC and C-MAC mirror the standard information-theoretic definitions [19, Chap. 14].

Throughout the sequel, we use the terms fading states and sub-channels interchangeably.  $C(x)$  denotes  $\log(1+x)$  where the logarithm is to the base 2 and  $R_S$  denotes  $\sum_{k \in S} R_k$  for any  $S \subseteq \mathcal{K}$ . We assume that the reader is familiar with sub-modular functions and polymatroids (see, for example, [17]).

## III. ACHIEVING THE INTERFERENCE-FREE CAPACITY REGION

The following theorem summarizes the main result of this paper.

*Theorem 1:* A  $K$ -user ergodic fading IFC achieves the interference-free capacity region of  $K$  independent links if the waterfilling solutions  $P_k^{(wf)}(H_{kk})$  for the (interference-free) ergodic fading point-to-point links between transmitters  $k$  and receivers  $k$ ,  $k = 1, 2, \dots, K$ , satisfy

$$\mathbb{E} \left[ C \left( |H_{k,k}|^2 P_k^{(wf)}(H_{kk}) \right) \right] < C_{sum}^{(j)}(\mathcal{K}) - C_{sum}^{(j)}(\mathcal{K} \setminus \{k\}), \text{ for all } j, k \in \mathcal{K}, j \neq k, \quad (4)$$

where for any  $\mathcal{A} \subseteq \mathcal{K}$

$$C_{sum}^{(j)}(\mathcal{A}) = \mathbb{E} \left[ C \left( \sum_{m \in \mathcal{A}} |H_{j,m}|^2 P_m^{(wf)}(H_{kk}) \right) \right]. \quad (5)$$

The capacity region of the resulting ergodic very strong IFC is

$$\mathcal{C}_{IFC}^{EVS} = \{(R_1, R_2, \dots, R_K) : R_k \leq \mathbb{E} \left[ C \left( |H_{k,k}|^2 P_k^{wf}(H_{k,k}) \right) \right], k \in \mathcal{K}\} \quad (6)$$

and the sum-capacity is

$$\sum_{k=1}^K \mathbb{E} \left[ C \left( |H_{k,k}|^2 P_k^{wf}(H_{k,k}) \right) \right]. \quad (7)$$

*Remark 2:* The conditions in (4) involve averaging over all channel states and do not require every sub-channel to be strong. As with the two-user ergodic fading IFCs, the capacity achieving scheme for  $K$ -user EVS IFCs requires coding jointly across all sub-channels. For  $K = 2$ , (4) simplifies to the EVS conditions in [10, Theorem 2]. The conditions for a  $K$ -user non-fading very strong IFC are simply a special case of (4) obtained for a constant  $\mathbf{H}$ .

*Remark 3:* The conditions in (4) are equivalent to the requirements that the rate achieved by each transmitter in the presence of interference from all other users at each of the unintended receivers is at least as large as the interference-free rate achieved at its intended receiver.

*Corollary 4:* For a class of symmetric non-fading IFCs with  $H_{k,k} = 1$ ,  $H_{j,k} = a$  for all  $j, k \in \mathcal{K}$ ,  $j \neq k$ , and  $\bar{P}_k = \bar{P}$ , (4) reduces to the condition

$$\bar{P} < \frac{a^2 - 1}{1 + (K-2)a^2} \quad (8)$$

or equivalently

$$a^2 > \frac{1 + \bar{P}}{1 - (K-2)\bar{P}}. \quad (9)$$

Thus, for any  $a > 0$  and  $K > 2$ ,  $\bar{P} < 1$ ; furthermore, for large  $K$ ,  $\bar{P}$  scales inversely with  $K$ . Conversely, for large  $K$ ,  $a^2$  scales linearly with  $K$ .

*Remark 5:* A very strong condition for  $K$ -user symmetric IFCs is presented in [18, eqn. (5)] which requires that  $a^2$  grow exponentially with  $K$  when each receiver decodes all the unintended messages before decoding its intended message. It is unclear whether the condition in [18, eqn. (5)] ensures that the intersection of the MAC polymatroids, one at each receiver, is a box. In contrast, the condition in (9) only grows linearly in  $K$  and ensures a box intersection.

*Remark 6:* [18] also presents a sufficient condition using lattice codes for interference-free communications in symmetric  $K$ -user IFCs as  $a^2 > (\bar{P} + 1)^2 / \bar{P}$  which is independent of the number of users.

*Proof: Outer Bound:* An outer bound on the sum-capacity of an IFC results from eliminating interference at all the receivers thereby reducing it to  $K$  interference-free point-to-point links. From [15, Appendix], the capacity achieving policy for each link requires each transmitter to waterfill over

its fading link to its receiver, and thus, we have that any achievable rate tuple  $(R_1, R_2, \dots, R_K)$  must satisfy

$$\sum_{k=1}^K R_k \leq \sum_{k=1}^K \mathbb{E} \left[ C \left( |H_{k,k}|^2 P_k^{wf}(H_{k,k}) \right) \right]. \quad (10)$$

*Inner Bound:* Consider the achievable scheme in which every receiver decodes all the interfering signals, i.e., the IFC is converted to a C-MAC. Assuming every transmitter encodes its message across all sub-channels and every receiver jointly decodes all messages across all sub-channels, the Gaussian MAC rate region achieved at receiver  $k$  when the power policy at transmitter  $m$  is  $P_m^{(wf)}$ , for all  $k, m \in \mathcal{K}$  is given by [10, Theorem 1]

$$\mathcal{R}_k(\underline{P}^{(wf)}) = \{(R_1, R_2, \dots, R_K) : R_S \leq f_k^*(S), \text{ for all } S \subseteq \mathcal{K}\} \quad (11)$$

where

$$f_k^*(S) = \mathbb{E} \left[ C \left( \sum_{m \in S} |H_{k,m}|^2 P_m^{wf}(H_{m,m}) \right) \right]. \quad (12)$$

It can be easily verified that the functions  $f_k^*(S)$ , for all  $k$ , are sub-modular functions and the MAC rate regions  $\mathcal{R}_k(\underline{P}^{(wf)})$  are polymatroids (see for e.g., [20]). For any  $S, \mathcal{A} \subseteq \mathcal{K}$  such that  $S \cap \mathcal{A} = \emptyset$ , we write  $f_k^*(S|\mathcal{A})$  as

$$f_k^*(S|\mathcal{A}) \equiv \mathbb{E} \left[ C \left( \frac{\sum_{m \in S} |H_{k,m}|^2 P_m^{wf}(H_{m,m})}{1 + \sum_{m \in \mathcal{A}} |H_{k,m}|^2 P_m^{wf}(H_{m,m})} \right) \right], \quad (13)$$

i.e.,  $f_k^*(S|\mathcal{A})$  is the rate achieved by the users in  $S$  at receiver  $k$  in the presence of interference from the users in a disjoint set  $\mathcal{A}$ .

We now show that when (4) is satisfied the intersection of  $\mathcal{R}_k(\underline{P}^{(wf)})$  for all  $k$  is a  $K$ -dimensional hyper-cube. To this end, for ease of analysis, we first write (4) in terms of  $f_k^*(\mathcal{K})$  as

$$f_k^*({k}) < f_j^*(\mathcal{K}) - f_j^*(\mathcal{K} \setminus {k}), \text{ for all } j, k \in \mathcal{K}, j \neq k. \quad (14)$$

Thus, given (14), we now prove that

$$\sum_{k \in S} f_k^*({k}) \leq f_j^*(S), \text{ for all } j \text{ and } S \subseteq \mathcal{K}. \quad (15)$$

Without loss of generality, let  $j = 1$  and  $S = \mathcal{K}$ . Thus, we have

$$\sum_{k \in \mathcal{K}} f_k^*({k}) = f_1^*({1}) + f_2^*({2}) + \dots + f_K^*({K}) \quad (16a)$$

$$\leq f_1^*({1}) + f_1^*({1,2}) - f_1^*({1}) \quad (16b)$$

$$+ f_1^*({1,2,3}) - f_1^*({1,2}) + \dots$$

$$+ f_1^*(\mathcal{K}) - f_1^*(\mathcal{K} \setminus {k}) \quad (16c)$$

$$= f_1^*(\mathcal{K}) \quad (16d)$$

where (16b) follows from (14) and the fact that for any  $S \subset \mathcal{K}$  such that  $k \notin S$ , using chain rule for mutual

information, we have

$$\begin{aligned}
& f_j^*(\mathcal{K}) - f_j^*(\mathcal{K} \setminus \{k\}) \\
&= f_j^*(\mathcal{S} \cup \{k\}) + f_j^*(\mathcal{K} \setminus (\mathcal{S} \cup \{k\}) | \mathcal{S} \cup \{k\}) \\
&- f_j^*(\mathcal{S}) - f_j^*(\mathcal{K} \setminus (\mathcal{S} \cup \{k\}) | \mathcal{S}) \quad (17) \\
&\leq f_j^*(\mathcal{S} \cup \{k\}) - f_j^*(\mathcal{S}) \quad (18)
\end{aligned}$$

where (18) follows from the fact that due to additional interference from user  $k$ , the second term to the right of the equality in (17) is smaller than the fourth term where all terms in (17) can be expanded using (13).

Following steps similar to (16), one can show that  $\sum_{k \in \mathcal{S}} f_k^*(\{k\}) \leq f_1^*(\mathcal{S})$  for all  $\mathcal{S} \subset \mathcal{K}$ . Furthermore, the same steps can also be used to show that (15) holds for all  $j \in \mathcal{K}$ . Let  $R_k^* = f_k^*(\{k\})$  for all  $k$ . Thus, from (15) and (11), we have that

$$(R_1^*, R_2^*, \dots, R_K^*) \in \mathcal{R}_j \left( \underline{P}^{(wf)} \right), \quad \text{for all } j \in \mathcal{K}, \quad (19)$$

$$\implies (R_1^*, R_2^*, \dots, R_K^*) \in \bigcap_{j=1}^K \mathcal{R}_j \left( \underline{P}^{(wf)} \right). \quad (20)$$

Since the intersection of  $K$  orthogonal rate planes  $R_k^*$  yields a box (a hyper-cube), the C-MAC sum-capacity when (4) holds is given by (7). Combining this achievable sum-rate with the outer bounds in (10), we have that (7) is also the sum-capacity of an EVS IFC for which the channel statistics and optimal power policy satisfy (4). Finally, since the sum-capacity also achieves the interference-free capacity of each user, the capacity region of EVS IFCs is given by (6). ■

#### IV. DISCUSSION

Theorem 1 summarizes a set of sufficient conditions for which the interference-free capacity of  $K$  transmit-receive pairs can be achieved when every receiver decodes the messages from all transmitters, an achievable scheme we henceforth refer to as the C-MAC scheme. In this section, we present numerical examples of fading and non-fading IFCs and the feasible power and channel gains regime for which the EVS IFC conditions in (4) are satisfied.

We first consider a three-user ergodic fading IFC with non-fading unit-gain direct links and cross-links that are independent and identically distributed Rayleigh faded links, i.e.,  $H_{j,k} \sim \mathcal{CN}(0, \sigma^2)$  for all  $j \neq k, j, k \in \{1, 2, 3\}$ , and  $\overline{P}_k = \overline{P}$  for all  $k$ . The resulting channel is a mix of weak and strong sub-channels for which each user transmits at  $\overline{P}$  in every sub-channel if the EVS conditions in (4) are satisfied. The feasible  $\overline{P}$  vs.  $\sigma^2$  region and the maximum  $\overline{P}(\sigma^2)$  for which (4) holds is plotted in Fig. 2(a). In Fig. 2(b), the EVS sum-capacity when each user transmits at the maximum  $\overline{P}(\sigma^2)$  is plotted as a function of the fading variance  $\sigma^2$ . For this  $\overline{P}(\sigma^2)$ , also plotted in Fig. 2(b) is the sum-rate achieved by ergodic interference alignment in which knowledge of the channel states is used by the transmitters to enable the cancellation of interference at all receivers simultaneously [21]. As shown in both subplots, as the variance of the cross-links increases, thereby increasing the probability of strong

fading states, the largest  $\overline{P}$ , and hence the sum-capacity, for which the EVS sum-capacity is achievable also increases.

Next we consider a non-fading three-user symmetric IFC with unit gains on the intended links, a real positive channel gain  $a$  on the cross-links, and  $\overline{P}_k = \overline{P}$ , for  $k = 1, 2, 3$ . In Fig. 3(a), as a function of  $a^2$ , we plot the maximum feasible  $\overline{P}$  in (8) for which a very strong IFC results using a C-MAC achievable scheme. As observed in Corollary 4, we require  $\overline{P} < 1$ . Also included are plots of the upper  $\overline{P}_u$  and lower  $\overline{P}_l$  bounds on the feasible power with lattice codes for which a very strong IFC results. Thus, as  $a^2$  increases, decoding the interference using lattice codes allows a larger class of three-user symmetric IFCs to be considered very strong relative to decoding the message from every user. On the other hand, only the C-MAC scheme achieves the VS condition for  $a^2 < 4$ , i.e., the C-MAC achievable scheme is more appropriate in the low-power regime in achieving the sum-capacity of  $K$  interference free point-to-point links.

A set of sufficient EVS conditions given by (4) in Theorem 1 prompt the question of whether these conditions are also necessary, i.e., whether the intersection of  $K$  polymatroids would cease to be a box if one or more conditions were violated. We now present a three-user example that shows that when all six conditions in (4) for  $K = 3$  are not satisfied, the intersection of the  $K$  MAC polymatroids is a box, i.e., the  $K$ -user interference-free sum capacity can still be achieved.

Consider a three-user non-fading IFC with  $H_{k,k} = 1$  and  $\overline{P}_k = 1$  for all  $k$ . Thus, if the intersection of the MAC rate regions at the receivers results in a box, each user transmits at  $\overline{P}_k$  in every use of the channel. The cross-link gains  $H_{j,k}$  for all  $j, k, j \neq k$ , are such that

$$f_1^*(\mathcal{S}) = \max_{i \in \mathcal{S}} (i) + 0.5 |\mathcal{S}| \quad (21)$$

where  $|\mathcal{S}|$  denotes the cardinality of the set  $\mathcal{S}$  and  $f_1^*(\mathcal{S})$  is obtained by evaluating the rate bounds at  $P_k^{(wf)}(H_{k,k}) = \overline{P}_k = 1$  for all  $k$ . Thus, from (21),  $f_1(\{1\}) = 1.5$ ,  $f_1(\{2\}) = 2.5$ ,  $f_1(\{3\}) = 3.5$ ,  $f_1(\{1, 2\}) = 3$ ,  $f_1(\{1, 3\}) = f_1(\{2, 3\}) = 4$ , and  $f_1(\{1, 2, 3\}) = 4.5$ .

The bounds at the other two receivers are given by  $f_2^*(\mathcal{S}) = f_1^*(\pi(\mathcal{S}))$  and  $f_3^*(\mathcal{S}) = f_1^*(\pi^2(\mathcal{S}))$  where  $\pi$  is a cyclic permutation of the indexes such that each index is decreased by 1 in a cyclic manner such that  $(1, 2, 3)$  map to  $(3, 1, 2)$  and  $\pi^2(\mathcal{S})$  is obtained by applying  $\pi(\mathcal{S})$  twice. Thus,  $f_2(\{2\}) = f_3(\{3\}) = f_1(\{1\}) = 1$ .

For the three rate regions defined thus, one can verify that none of the six conditions in (14) is satisfied. Furthermore, from (21), we have that the bounds for every rate region satisfy  $f_k^*(\mathcal{S}) \geq |\mathcal{S}|$ . Consider the rate tuple  $(R_1, R_2, R_3) = (1, 1, 1)$ . This tuple satisfies  $\sum_{k \in \mathcal{S}} R_k = |\mathcal{S}| \leq f_k^*(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{K}$  and  $j = 1, 2, 3$ , i.e.,  $(1, 1, 1)$  satisfies the rate constraints for each of the three MAC rate regions and therefore lies in their intersection, i.e., the intersection of the three rate regions is a box.

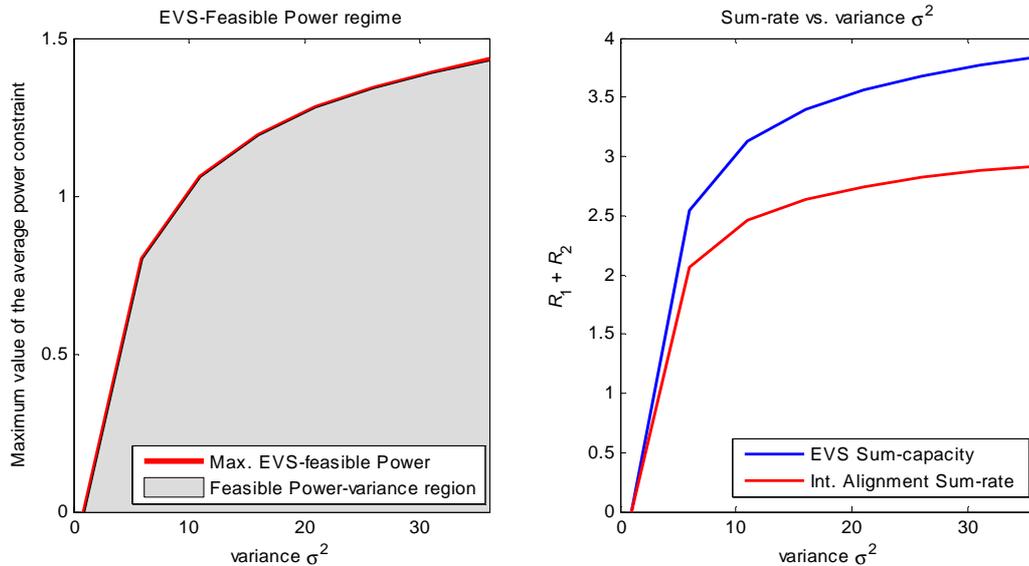


Fig. 2. EVS feasible power-variance region and sum-capacity.

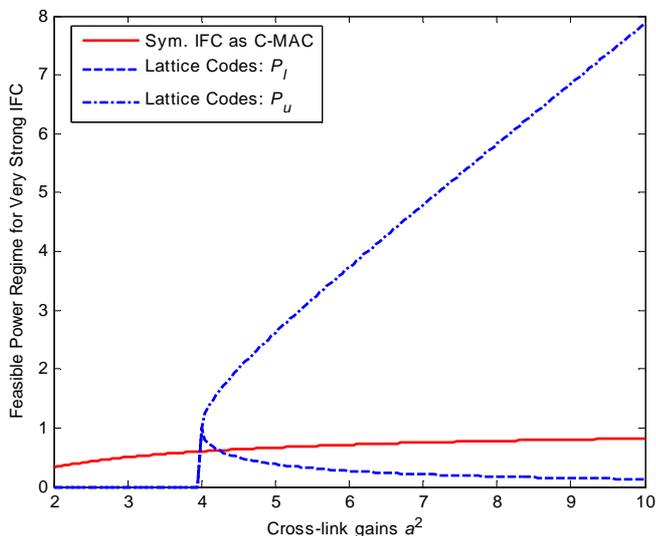


Fig. 3. Bounds on feasible power vs. cross-link gains for C-MAC- and lattice codes-based achievable schemes.

This is so because, for the considered example, while (4) is not satisfied for all  $j$  and  $k$ ,  $j \neq k$ , the conditions  $f_k(\{k\}) \leq f_j(\mathcal{S}) - f_j(\mathcal{S} \setminus \{k\})$  for all  $\mathcal{S} \subset \mathcal{K}$ ,  $j \neq k$ , are satisfied and suffice to achieve the interference-free sum-capacity. Thus, one can conclude that for the C-MAC achievable scheme, an EVS IFC will not result if and only if  $f_k(\{k\})$  does not satisfy all  $(K-1)(2^K-1)$  conditions, i.e., when  $f_k(\{k\}) > f_j(\mathcal{S}) - f_j(\mathcal{S} \setminus \{k\})$  for all  $j, k \in \mathcal{K}$ ,  $j \neq k$  and  $\mathcal{S} \subseteq \mathcal{K}$ . The proof follows in a straightforward manner from showing that  $(f_1^*(1), f_2^*(2), \dots, f_K^*(K)) \notin \mathcal{R}_k(P_k^{(wf)}(H_{kk}))$ , for all  $k$ , using steps analogous to (16) in Theorem 1. Note that only for  $K=2$  are the sufficient conditions in (4) also necessary.

## V. CONCLUDING REMARKS

We have obtained sufficient conditions for achieving the interference-free sum-capacity and capacity region of a  $K$ -link ergodic fading IFC when all receivers are allowed to decode the messages from all transmitters. In particular, we have shown that an EVS IFC results if the channel statistics and the interference-free capacity optimal water-filling policies for all links satisfy  $K(K-1)$  conditions. For  $K=2$ , we have shown that these conditions are both necessary and sufficient. Our result that a quadratic number of conditions suffice for the intersection of  $K$  polymatroids to form a box is also of independent interest in combinatorics where few results are known on intersection of three or more polymatroids. Finally, our results suggest that decoding interference using schemes such as lattice codes may impose less stringent conditions on the average power and channel statistics.

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