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On Iterative Performance of LDPC and Root-LDPC Codes over Block-Fading Channels

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***Abstract*—This paper[1] presents our investigation on iterative decoding performances of some sparse-graph codes on block-fading Rayleigh channels. The considered code ensembles are standard LDPC codes and Root-LDPC codes, first proposed in [1] and shown to be able to attain the full transmission diversity. We study the iterative threshold performance of those codes as a function of fading gains of the transmission channel and propose a numerical approximation of the iterative threshold versus fading gains, both both LDPC and Root-LDPC codes. Also, we show analytically that, in the case of 2 fading blocks, the iterative threshold γ^*_{root} of Root-LDPC codes is proportional to $(\alpha_1\alpha_2)^{-1}$, where α_1 and α_2 are corresponding fading gains. From this result, the full diversity property of Root-LDPC codes immediately follows.**

I. INTRODUCTION

The nonergodic block-fading model is appropriate for a wide variety of wireless communication systems, for example using multiple antennas or retransmission schemes. For such type of channels, it is known that the most probable transmission scenario (i.e. around 70% of cases for the transmission channel with two fading blocks) is the so called unbalanced case, when the fading gains of different transmission blocks differ much from each other. This unbalanced regime happens to be the outage regime for standard sparse-graph codes (namely LDPC codes), and so they show a poor performance. As a consequence, LDPC codes have transmit diversity 1 and their error decoding performance scales as $(SNR)^{-1}$ at high signal-to-noise ratios (SNRs). The last result have been demonstrated in [1] by analyzing the density evolution of LDPC codes.

However, the best decoding performance at high SNRs is SNR^{-d}, if d is the number of fading blocks. Codes that guarantee this performance are said to satisfy the em full-diversity property. The design of full-diversity sparse-graph codes has been first addressed in [1], where the authors proposed a new code family, the *Root-LDPC codes*, which attains the maximal theoretical diversity 2 over a block-fading channel with two independent fading realizations per coded data packet.

In this paper we consider both LDPC and Root-LDPC codes, and study how the iterative-decoding threshold depends on the two fading gains parametrizing the channel. Note that the computation of the iterative-decoding threshold for various fading coefficients is the main part of the computation of the outage probability, attainable for a given code ensemble. This computation is usually performed by heavy numerical simulations, and only the case of ergodic channels (i.e. with one fading realization per coded data packet) is well handled; the iterative decoding threshold is estimated by means of the density evolution under semi-Gaussian approximation [2]. In the case of nonergodic channels the Gaussian approximation is not valid anymore. The approximation, proposed in this paper, enables to estimate the outage probability achievable with a given code ensemble without resorting to extensive, time-consuming simulations. Previous results [2] along this line allow to find only one point of the threshold curve—the one corresponding to equal fading gains—which corresponds to an ergodic block-fading channel [3]. Moreover, we confirm the full-diversity property of Root-LDPC codes by using information-theoretic tools.

Our channel model here is the same as in [1], where one half of the coded data block is affected by a Rayleigh-distributed fading gain α_1, and the other half by a similarly distributed, independent gain α_2. Two different types of code ensembles will be examined: standard random LDPC codes and Root-LDPC codes. We hasten to recall that, while the random LDPC codes diversity is only 1, Root-LDPC codes achieve transmit diversity 2, and hence their error probability after decoding decreases as SNR^{-2}.

This paper is organized as follows. The relevant definitions of code ensembles are given in Section II. Transmission model and threshold curves are introduced in Section III. Numerical and analytical approximations of the threshold curve are derived in Sections IV and V, respectively. Section VI concludes the paper.

II. LDPC AND ROOT-LDPC CODE ENSEMBLES

LDPC codes are a well-studied family of sparse-graph codes, and Root-LDPC codes are a family specially designed for transmission over block-fading channels with two subblocks. First recall that a classical (λ, ρ) LDPC ensemble is a set of binary codes whose parity-check matrix contains a fraction λ_i of columns with i nonzero entries,

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$i \in \{i_1, i_2, \ldots, i_{max}\}$, and a fraction ρ_j of rows with j nonzero entries, $j \in \{j_1, j_2, \ldots, j_{max}\}$. The LDPC ensemble is completely described by the polynomials

$$\lambda(x) \triangleq \sum_i \lambda_i x^{i-1} \quad \text{and} \quad \rho(x) \triangleq \sum_j \rho_j x^{j-1}.$$

A (λ, ρ) Root-LDPC ensemble is defined with the help of multinomials $\lambda_{\text{root}}(\underline{\mu}, \underline{x})$ and $\rho_{\text{root}}(\underline{\mu}, \underline{x})$ with $\underline{\mu} \triangleq (\mu_1, \mu_2)$ and $\underline{x} \triangleq (x_1, x_2, x_3, x_4, x_5, x_6)$:

$$\lambda_{root}(\underline{\mu}, \underline{x}) = \frac{1}{2}\sum_i \left(\frac{\lambda_i}{i}\mu_1 x_1^i + \frac{(i-1)\lambda_i}{i}\mu_1 x_2^i + \lambda_i \mu_1 x_3^i \right.$$
$$\left. + \lambda_i \mu_2 x_4^i + \frac{(i-1)\lambda_i}{i}\mu_2 x_5^i + \frac{\lambda_i}{i}\mu_2 x_6^i \right), \quad (1)$$

$$\rho_{root}(\underline{\mu}, \underline{x}) = \frac{1}{2}\sum_i \rho_i \left(x_1 \sum_j \binom{i}{j} f_e^j x_4^j g_e^{i-j} x_5^{i-j} \right.$$
$$\left. + x_6 \sum_k \binom{i}{k} f_e^k x_3^k g_e^{i-k} x_2^{i-k} \right). \quad (2)$$

Here the variables μ_1 and μ_2 correspond to the two fading gains, and the variables $x_1, x_2, \ldots, x_6$ to 6 edge types in the bipartite code graph. Note that x_3 and x_4 are related to "parity" edges, and the remaining variables to "information" edges, reflecting the fact that the structure of the root-code graph consists of four types of variable nodes (denoted $1i$, $1p$, $2i$ and $2p$) and two of check nodes (denoted $1c$ and $2c$). Permutations of edges within edge classes are chosen uniformly at random. Variable nodes $1i$ and $2i$ correspond to the information bits in a codeword, while variable nodes $1p$ and $2p$ correspond to the redundant (parity) bits. Also, f_e (resp., g_e) denotes the probability that an edge connected to a given check node is a parity (resp., information) edge. It is easy to se that

$$f_e = \frac{\sum_i (i-1)\frac{\lambda_i}{i}}{\sum_i (i-1)\frac{\lambda_i}{i} + 1} \text{ and } g_e = \frac{1}{\sum_i (i-1)\frac{\lambda_i}{i} + 1} = 1 - f_e.$$

As discussed in detail in [1], this structure generates a code ensemble of rate $1/2$.

III. Transmission over Block-Fading Channels: Performance Parameters

In our model, a codeword is divided into two equal subblocks, transmitted over two Rayleigh fading channels with common signal-to-noise ratio SNR$= \gamma$ and independent fading gains α_1 and α_2. The observation y_i resulting from the transmission of binary symbol $x = \pm 1$ received from the ith channel, $i = 1, 2$, has the form $y_i = \alpha_i x + z_i$, where $\alpha_i \in [0, +\infty)$, and z_i are independent and Gaussian distributed, $z_i \sim \mathcal{N}(0, \sigma^2)$, with $\sigma^2 = 1/\gamma$.

The received data block is decoded using a standard belief propagation algorithm. The asymptotic performance of a given code ensemble is described by the iterative decoding threshold γ^*—the minimum SNR at which reliable transmission is possible, for fixed fading gains α_1 and α_2 and infinite code length. γ^* can be found by means of the density evolution, described below for both LDPC and Root-LDPC ensembles.

A. Density Evolution

From now on, let the function $\gamma^*(\alpha_1, \alpha_2)$ denote the iterative threshold as a function of the fading gains realizations. Also, let denote the probability density functions (pdfs) of channel LLR outputs from the two transmission subchannels by $\mu_1(x)$ and $\mu_2(x)$, respectively. These are normal pdfs with means $2\alpha_1^2/\gamma$ and $2\alpha_2^2/\gamma$ and variances $4\alpha_1^2/\gamma$ and $4\alpha_2^2/\gamma$, respectively.

The density evolution of LDPC and Root-LDPC codes is defined with the help of two following operations: convolution and R-convolution.

Definition 1: The convolution of two infinite-support pdfs $\alpha(x)$ and $\beta(x)$ i defined as

$$\alpha \otimes \beta(x) = \int_R \alpha(t)\beta(x-t)dt.$$

Definition 2: The R-convolution of two pdfs $\alpha(x)$ and $\beta(x)$ is

$$\alpha \odot \beta(x) = f(\hat{\alpha}(x) \otimes \hat{\beta}(x)),$$

where

$$\hat{\alpha}(x) \triangleq \frac{2\alpha(2\text{th}^{-1}(x))}{1-x^2}, \quad \hat{\beta}(x) = \frac{2\beta(2\text{th}^{-1}(x))}{1-x^2}$$

and

$$f(x) = \cosh^2\left(\frac{\hat{\alpha} \otimes \hat{\beta}(x)}{2}\right) \text{th}^{-1}(\hat{\alpha} \otimes \hat{\beta}(x)).$$

Note that the convolution of pdfs corresponds to the operation, performed over LLRs at variable nodes; the R-convolution of pdfs corresponds to the operation $2\text{th}^{-1}\big(\text{th}(A/2) + \text{th}(B/2)\big)$ over the corresponding random variables A and B, which is exactly the operation performed at the check nodes.

For LDPC codes, let $q^m(x)$ denote the pdf of LLR messages from variable nodes to check nodes at decoding iteration m. With some abuse of notation, let

$$\lambda(x) = \sum_i \lambda_i x^{\otimes(i-1)} \text{ and } \rho(x) = \sum_i \rho_i x^{\otimes(i-1)}.$$

Then the density evolution operation is written as

$$q^{m+1}(x) = \left(\frac{\mu_1(x)}{2} + \frac{\mu_2(x)}{2}\right) \otimes \lambda(\rho(q^m(x))).$$

The iterative threshold γ^* for an (λ, ρ) LDPC ensemble is then the maximum SNR, for which, under all-zero codeword assumption, $q^\infty(x) = \delta_\infty$. Here δ_∞ denotes the Dirac function at the infinity.

Now consider Root-LDPC codes. Let us denote the average pdfs for 6 edge sets by $q_1(x)$, $f_1(x)$, $g_1(x)$, $g_2(x), f_2(x)$, and $q_2(x)$, which correspond to variables $x_1, \ldots, x_6$ in the description (1). It can be shown that the evolution of the pdfs at the iteration $m+1$ can be described by the following recursions:

$$\begin{aligned}
q_1^{m+1}(x) &= \mu_1(x) \otimes \mathring{\lambda}(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \\
f_1^{m+1}(x) &= \mu_1(x) \otimes \tilde{\lambda}(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \\
&\quad \otimes \mathring{\rho}(f_e f_2^m(x) + g_e g_2^m(x)) \\
g_1^{m+1}(x) &= \mu_1(x) \otimes \lambda(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \\
g_2^{m+1}(x) &= \mu_2(x) \otimes \lambda(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x))) \\
f_2^{m+1}(x) &= \mu_2(x) \otimes \tilde{\lambda}(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x))) \\
&\quad \otimes \mathring{\rho}(f_e f_1^m(x) + g_e g_1^m(x)) \\
q_2^{m+1}(x) &= \mu_2(x) \otimes \mathring{\lambda}(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x)))
\end{aligned}$$

where we have borrowed from [3] the following notation:

$$\tilde{\lambda}(x) \triangleq \frac{\bar{d}_b}{\bar{d}_b - 1} \sum_i \frac{\lambda_i (i-1)}{i} x^{\otimes(i-2)}; \qquad \bar{d}_b \triangleq 1/\sum_i \lambda_i / i;$$

$$\tilde{\rho}(x) \triangleq \frac{\bar{d}_c}{\bar{d}_c - 1} \sum_i \frac{\rho_i (i-1)}{i} x^{\odot(i-2)}; \qquad \bar{d}_c \triangleq 1/\sum_i \rho_i / i;$$

$$\mathring{\lambda}(x) \triangleq \bar{d}_b \sum_i \frac{\lambda_i}{i} x^{\otimes(i-1)}; \qquad \mathring{\rho}(x) \triangleq \bar{d}_c \sum_i \frac{\rho_i}{i} x^{\odot(i-1)}.$$

Also, we define

$$\tilde{\rho}(q, x) \triangleq \frac{\bar{d}_c}{\bar{d}_c - 1} \sum_i \frac{\rho_i (i-1)}{i} q \odot x^{\odot(i-3)}.$$

We have the following result on the degradation of pdfs q's, f's and g's:

Lemma 1: For a pdf $\alpha(x)$ for all $x \in R$, define $P_{e,\alpha}$ as $P_{e,\alpha} = \int_{R^-} \alpha(x) dx$. Then it can be easily shown that, for any iteration m,

$$\begin{aligned}
P_{e,q_1} \leq P_{e,f_1} \leq P_{e,g_1}, \\
P_{e,q_2} \leq P_{e,f_2} \leq P_{e,g_2}.
\end{aligned}$$

The proof is straightforward and follows from density evolution equations.

The lemma above shows that the pdfs $q(x)$ are "better" when compared to corresponding pdfs $f(x)$ and $q(x)$. Hence, we define the iterative threshold γ^*_{root} as the maximum SNR for which both $q_1^\infty(x)$ and $q_2^\infty(x)$ are δ_∞.

B. Useful Lemma on γ^*

We have the following lemma on $\gamma(\alpha_1, \alpha_2)$, simplifying a lot our further calculation:

Lemma 2:

$$\gamma^*(\alpha_1, \alpha_2) = \frac{\gamma^*(\alpha_1/\alpha_2, 1)}{\alpha_2^2} = \frac{\gamma^*(1, \alpha_2/\alpha_1)}{\alpha_1^2}.$$

Proof: By direct calculation, the pdf of the LLR channel estimate corresponding to the threshold is

$$\begin{aligned}
p_{\Lambda_0^*(\alpha_1,\alpha_2)} &= \frac{1}{2}\mathcal{N}(2\alpha_1^2\gamma^*, 4\alpha_1^2\gamma^*) + \frac{1}{2}\mathcal{N}(2\alpha_2^2\gamma^*, 4\alpha_2^2\gamma^*) \\
&= \frac{1}{2\alpha_1^2}\mathcal{N}(2\gamma^*, 4\gamma^*) + \frac{1}{2\alpha_1^2}\mathcal{N}(2\frac{\alpha_2^2}{\alpha_1^2}\gamma^*, 4\frac{\alpha_2^2}{\alpha_1^2}\gamma^*) \\
&= \frac{p_{\Lambda_0^*(1,\alpha_2/\alpha_1)}}{\alpha_1^2}.
\end{aligned}$$

Similarly, $p_{\Lambda_0^*(\alpha_1,\alpha_2)} = p_{\Lambda_0^*(\alpha_1/\alpha_2)}/\alpha_2^2$ by symmetry. □

This lemma shows that, to obtain the iterative decoding threshold for any pair of fading gains (α_1, α_2), it is sufficient to compute the single-argument function $\gamma^*(1, \alpha)$, where $\alpha \triangleq \alpha_2/\alpha_1$, $\alpha \in R^+$. For notational simplicity, we write $\gamma^*(\alpha)$ in lieu of $\gamma^*(1, \alpha)$.

C. Outage Probability versus Iterative Threshold γ^*

If one knew $\gamma^*(\alpha_1, \alpha_2)$ for a given code ensemble, the outage probability P_{out} could be obtained by observing that, with α_1 and α_2 independent unit-mean Rayleigh random variables,

$$P_{\text{out}} = \int_{(\alpha_1,\alpha_2)\in R^*} e^{-\alpha_1^2 - \alpha_2^2}\, d\alpha_1^2\, d\alpha_2^2,$$

where R^* denotes the region below the threshold curve $\gamma^*(\alpha_1, \alpha_2)$ (see [3] for illustration). In general, $\gamma^*(\alpha_1, \alpha_2)$ can be obtained by performing the density evolution for each point of the curve.

IV. Numerical Approximation of $\gamma^*(\alpha)$

Extensive numerical simulations have consistently shown a behavior of $\gamma^*(\alpha)$ for random codes which differs considerably from that associated with root codes with the same $\lambda(x)$ and $\rho(x)$. The approximations $\widehat{\gamma}^*_{\text{rand}}(\alpha)$ and $\widehat{\gamma}^*_{\text{root}}(\alpha)$ that follow were seen to match well the numerical results.

For the (λ, ρ) random LDPC ensemble,

$$\widehat{\gamma}^*_{\text{rand}}(\alpha) = \frac{a(\alpha)}{\alpha^2} + \frac{b(\alpha)}{\alpha} + a(\alpha), \tag{3}$$

where

$$\begin{aligned}
a(\alpha) &\triangleq K_a e^{-\tau_a \alpha}, \\
b(\alpha) &\triangleq K_b (1 - e^{-\tau_b \alpha})
\end{aligned}$$

for some constants K_a, K_b, τ_a, and τ_b depending on the code ensemble. Also, for the (λ, ρ) root-LDPC ensemble,

$$\widehat{\gamma}^*_{\text{root}}(\alpha) = \frac{c(\alpha)}{\alpha}, \tag{4}$$

with

$$c(\alpha) \triangleq K_a (1 - e^{-\tau_a \alpha}) + K_b e^{-\tau_b \alpha}.$$

The above approximations are remarkably close to the numerical results obtained for all the distributions $\lambda(x)$ and $\rho(x)$ we used in simulations. Moreover, they capture the difference in diversity for random and root ensembles, and in addition the root-code boundary is seen to approach outage capacity better than its random-code counterpart.

As an example, Figs. 1 and 2 compare simulation and approximation results for random and Root (x^2, x^5) LDPC codes, respectively. In both cases, a very good match is observed between numerical results and approximations. The constants estimated are $K_a = 10^{0.11}$, $K_b = 10^{0.65}$, $\tau_a = 18$, and $\tau_b = 18$.

We claim that (3) and (4) give a very accurate approximation of $\gamma(\alpha)$ for random and root LDPC codes respectively. Note

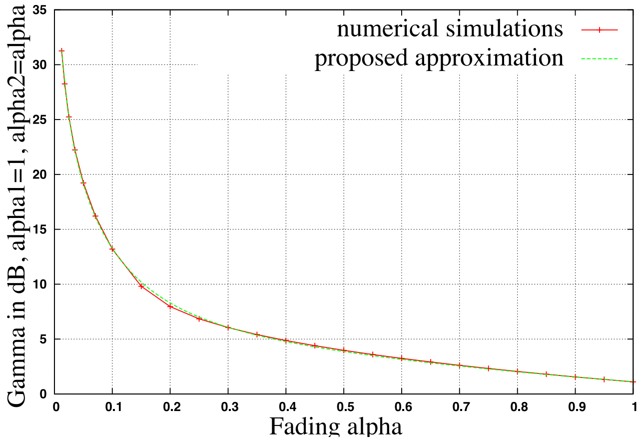

Fig. 1. Approximation $\widehat{\gamma}^*_{\rm rand}(\alpha)$ (dashed line) compared with $\gamma^*_{\rm rand}(\alpha)$ (continuous line) obtained by simulations for the (x^2, x^5) random LDPC ensemble.

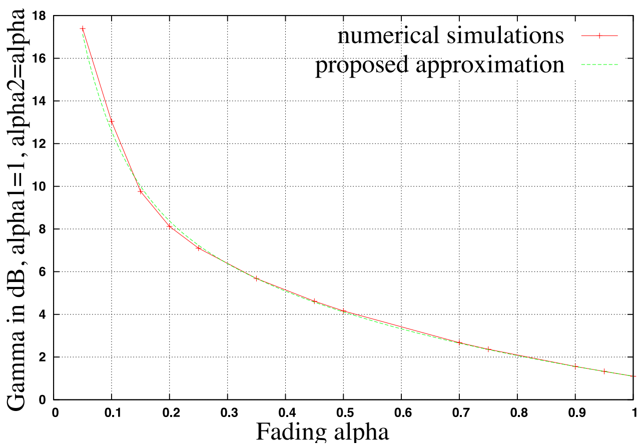

Fig. 2. Approximation $\widehat{\gamma}^*_{\rm root}(\alpha)$ (dashed line) compared with $\gamma^*_{\rm root}(\alpha)$ (continuous) obtained by simulations for the (x^2, x^5) root-LDPC ensemble.

that to obtain the approximation $\gamma(\alpha)$ for all α's, one needs to simulate at most four points for random ensembles, as there are four unknowns to be estimated. For root-LDPC ensembles the maximum numimber of points is 2 (two unknown parameters in (4)).

V. Analytical Approximation of $\gamma^*_{\rm root}(\alpha_1, \alpha_2)$

In this section we develop an analytic approximation to the iterative-decoding threshold curve. This is based on the assumption that the behavior of $\gamma^*_{\rm root}(\alpha_1, \alpha_2)$ in the high SNR regime is similar to the behavior of the outage boundary, as derived from the outage capacity of the corresponding block-fading channel. An approximation to the outage boundary is determined below.

A. Approximation of the Outage Boundary

We have an outage whenever α_1^2 and α_2^2 are such that the mutual information between channel input and output is lower than the code rate. Under our assumption of rate $1/2$, an outage occurs when α_1 and α_2 are such that

$$\mathbb{E}_X \log_2\left(1 + e^{-2\alpha_1^2 X}\right) + \mathbb{E}_X \log_2\left(1 + e^{-2\alpha_2^2 X}\right) > 1 \quad (5)$$

where X is the random variable related to the instantaneous channel SNR, $X \sim \mathcal{N}(\gamma, \gamma)$ [3]. Defining

$$g(\alpha) \triangleq \mathbb{E}_X \log_2\left(1 + e^{-2\alpha^2 X}\right) \quad (6)$$

the condition for an outage becomes

$$g(\alpha_1) + g(\alpha_2) > 1.$$

Explicitly, the function $g(\cdot)$ has the general form

$$g(\alpha) = \frac{1}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} \log_2(1 + e^{-2\alpha^2 x}) e^{-(x-\gamma)^2/\gamma}\, dx \quad (7)$$

A good approximation to (7) comes from the observation that the integrand function has a maximum around $x = 0.5$, a value which does not change much for different values of γ. Thus,

$$g(\alpha) \approx \log_2(1 + e^{-\alpha^2\gamma}) \quad (8)$$

Under the approximation above, the outage boundary equation is specified by

$$\log_2(1 + e^{-\alpha_1^2\gamma}) + \log_2(1 + e^{-\alpha_2^2\gamma}) = 1$$

and, hence,

$$\alpha_2^2 = \frac{1}{\gamma} \ln \frac{1 + e^{-\alpha_1^2\gamma}}{1 - e^{-\alpha_1^2\gamma}}$$

Using the series expansion [4, 1.513.1], we obtain the outage boundary for large α_1:

$$\alpha_2^2 = \frac{2}{\gamma} e^{-\alpha_1^2\gamma}. \quad (9)$$

Similarly, for large α_2^2,

$$\alpha_1^2 = \frac{2}{\gamma} e^{-\alpha_2^2\gamma}. \quad (10)$$

To verify that the approximation found captures well the behavior of the outage boundary, we compute the outage probability using approximation (8):

$$\begin{aligned} P_{\rm out} &\approx \mathbb{P}\left(\log_2(1 + e^{-\alpha_1^2\gamma})(1 + e^{-\alpha_2^2\gamma}) > 1\right) \\ &= \mathbb{P}\left((1 + e^{-\alpha_1^2\gamma})(1 + e^{-\alpha_2^2\gamma}) > 2\right) \\ &= \mathbb{P}\left(1 + e^{-\alpha_1^2\gamma} > \frac{2}{1 + e^{-\alpha_2^2\gamma}}\right) \\ &= \mathbb{P}\left(\alpha_1^2 < \frac{1}{\gamma} \ln \frac{1 + e^{-\alpha_2^2\gamma}}{1 - e^{-\alpha_2^2\gamma}}\right) \end{aligned}$$

Under the assumption of Rayleigh fading, α_1^2 has an exponential density, and hence we may use the approximation, valid for small x,

$$\mathbb{P}(\alpha_1^2 < x) \approx x.$$

which yields

$$P_{\rm out} \approx \frac{1}{\gamma} \mathbb{E}\left[\ln \frac{1 + e^{-\alpha_2^2\gamma}}{1 - e^{-\alpha_2^2\gamma}}\right]. \quad (11)$$

Using the fact that $\ln \frac{1+e^{-\alpha_2^2\gamma}}{1-e^{-\alpha_2^2\gamma}} \approx 2e^{-\alpha_2^2\gamma}$, we obtain

$$\begin{aligned} P_{\text{out}} &\approx \frac{2}{\gamma}\mathbb{E}\left[e^{-\alpha_2^2\gamma}\right] \\ &= \frac{2}{\gamma}\int_0^\infty e^{-x(\gamma+1)}\,dx \\ &= \frac{2}{\gamma(\gamma+1)}. \end{aligned}$$

Finally, for high SNR (large γ)

$$P_{\text{out}} \approx \frac{2}{\gamma^2}, \tag{12}$$

which is consistent with the result given in [3].

B. $\gamma^*(\alpha_1, \alpha_2)$ *for Root-LDPC Codes at High SNR*

Here we derive an approximation of the threshold curve for Root-LDPC ensembles, assuming that its behavior at high SNRs is the same as the one of the outage boundary. We have the following theorem:

Theorem 1: Consider a (λ, ρ) Root-LDPC ensemble. Then

$$\gamma^*_{\text{root}}(\alpha_1, \alpha_2) \approx \frac{1}{\alpha_1^2} W(2\alpha_1^2/\alpha_2^2)$$

or, equivalently,

$$\gamma^*_{\text{root}}(\alpha_1, \alpha_2) \approx \frac{1}{\alpha_2^2} W(2\alpha_2^2/\alpha_1^2),$$

where $W(\cdot)$ is the Lambert function [5].

Proof : We are interested in studying the behavior of the function

$$\gamma = \gamma^*_{\text{root}}(\alpha_1, \alpha_2)$$

where γ, α_1^2, and α_2^2 are related by the outage boundary condition

$$\mathbb{E}_X \log_2\left(1 + e^{-2\alpha_1^2 X}\right) + \mathbb{E}_X \log_2\left(1 + e^{-2\alpha_2^2 X}\right) = 1 \tag{13}$$

with $X \sim \mathcal{N}(\gamma, \gamma)$. As before, we may approximate (13) with

$$\log_2\left(1 + e^{-\alpha_1^2\gamma}\right) + \log_2\left(1 + e^{-\alpha_2^2\gamma}\right) = 1 \tag{14}$$

From (14), we can see that the function $g(\,\cdot\,)$ satisfies

$$\gamma^*_{\text{root}}(\alpha_1, \alpha_2) = \frac{1}{\alpha_1^2}\gamma^*_{\text{root}}(1, \alpha_2/\alpha_1) \triangleq \frac{1}{\alpha_1^2}\gamma^*_{\text{root}}(\alpha_2/\alpha_1)$$

which is consistent with Lemma 2. So, our goal is tantamount to studying the single-argument function $\beta(\gamma)$, where

$$(1 + e^{-\gamma})(1 + e^{-\beta\gamma}) = 2$$

and have defined $\beta \triangleq \alpha_2^2/\alpha_1^2$. Given the symmetry of the problem with respect to α_1 and α_2, we may constrain ourselves to the consideration of either $\alpha_2 \le \alpha_1$ or $\alpha_1 \le \alpha_2$. Thus, $\beta \in [0, 1]$ corresponds to $\alpha_2 \le \alpha_1$, while $\beta \in [1, \infty)$ corresponds to $\alpha_1 \le \alpha_2$.

We have the pair of values $\beta(\infty) = 0$ and $\beta(0) = \infty$, and the expression

$$\beta = \frac{1}{\gamma}\ln\frac{1 + e^{-\gamma}}{1 - e^{-\gamma}} \tag{15}$$

Using again the series expansion [4, 1.513.1]

$$\ln\frac{1+x}{1-x} = 2\sum_{k=1}^{\infty}\frac{1}{2k-1}x^{2k-1}, \qquad |x| < 1$$

we have

$$\beta = \frac{2}{\gamma}\sum_{k=1}^{\infty}\frac{1}{2k-1}e^{-(2k-1)\gamma} = \frac{2}{\gamma}\left(e^{-\gamma} + \frac{1}{3}e^{-3\gamma} + \cdots\right) \tag{16}$$

and we can finally obtain the sought function $\gamma(\beta)$ by inverting (16).

This can be done by observing that

$$\beta = \frac{2}{\gamma}e^{-\gamma}$$

is equivalent to

$$\frac{2}{\beta} = \gamma e^{\gamma}$$

Inversion of this function yields

$$\gamma = W(2/\beta)$$

so that, finally,

$$\gamma^*_{\text{root}}(\alpha_1, \alpha_2) = \frac{1}{\alpha_1^2}\,\gamma^*_{\text{root}}(\alpha_2/\alpha_1) \approx \frac{1}{\alpha_1^2}W(2\alpha_1^2/\alpha_2^2).$$

□

Comparing the approximation of Theorem 1 against numerical results, we can see that indeed it gives a good match in the high-SNR region. However, the approximation of the threshold function, derived through the outage boundary approach, is not satisfactory at intermediate-to-low SNRs, as it does not depend on the ensemble parameters.

VI. Discussion

In this paper we have presented estimates of the iterative threshold behavior of a sparse-graph code ensemble for the transmission over a nonergodic block-fading Rayleigh channel with two blocks affected by two independent Rayleigh-distributed fading gains. The first approximation comes from the interpolation of numerical data, and allows one to obtain a close estimate of the whole threshold behavior by simulating only several points of the threshold curve, for both random LDPC codes and rate-$1/2$ Root-LDPC codes, based on degree distributions $\lambda(x)$ and $\rho(x)$. The second approximation concerns rate-$1/2$ $(\lambda(x), \rho(x))$-Root-LDPC codes, it is analytic, and works well in the high-SNR regime.

References

[1] J. Boutros, A. Guillén i Fàbregas, E. Biglieri, and G. Zémor, "Low-density parity-check codes for nonergodic block-fading channels," 2007, *submitted to the IEEE Trans. Inform. Theory*.

[2] X. Jin, A. Eckford, and T. Fuja, "Design of good low-density parity-check codes for block fading channels," in *MILCOM'04*, October 2004.

[3] J. Boutros, "Diversity and coding gain evolution in graph codes," in *ITA'09*, San Diego, CA, February 2009.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. Amsterdam, The Netherlands: Academic Press, 2007.

[5] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function," *Advances in Computational Mathematics*, vol. 5, pp. 329–359, 1996.