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Incentive Schemes for Internet Congestion Management: Raffles *versus* Time-of-Day Pricing

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***Abstract*—The Internet is plagued with congestion problems of growing severity which are worst at peak periods. In this paper, we compare two schemes that incentivize users to shift part of their usage from the *peak-time* to the *off-peak time*. The traditional *time-of-day* pricing scheme gives a fixed reward per unit of shifted usage. Conversely, the *raffle-based* scheme provides a random reward distributed in proportion of each user's fraction of the total shifted usage. Using a game-theoretic model, we show that both schemes can achieve an optimal level of decongestion at a unique Nash equilibrium. We provide a comparison of the schemes' sensitivity to uncertainty of the users' utilities.**

***Index Terms*—congestion pricing; raffle-based incentive schemes; public good; probabilistic pricing, demand management**

I. Introduction

A. Motivations

The demand for Internet traffic is constantly growing on both wireless and wired networks. In particular, due to the increasing popularity of bandwidth-intensive mobile devices such as smartphones, mobile data traffic nearly tripled in each of the past 3 years [1]. Numerous independent studies indicate that this growth will continue, as bandwidth intensive applications like streaming video become ever more popular among users [2]. This is forcing providers to address the growing demand for bandwidth via all means, including capacity expansion and pricing schemes aimed at congestion reduction.

Historically, telecommunication providers have found that users prefer flat-rate pricing, and experiments have shown that users will pay a premium to avoid being metered [3], [4]. However, with an increased demand for bandwidth, many service providers are responding by moving away from unlimited data plans to tiered pricing models [2], [5]. The aforementioned user preference for flat-rate pricing makes introducing pricing schemes for congestion management problematic. Thus, the design of decongestion schemes that would be more acceptable for users is of great practical interest.

Network congestion is not uniform throughout the course of a day. It drops at night, after the prime time evening hours. This phenomenon has prompted renewed research interest in using time-of-day pricing to provide users with incentives to shift some of their demand to off-peak hours [6], [7].

However, such schemes are relatively complex and cumbersome for users. Also importantly, for the optimal design of the time-of-day pricing scheme, information about user demand is required, and in particular, the knowledge of how responsive users are in shifting their traffic from peak to off-peak times for a specific scheme [7]. Such information is usually very difficult to obtain with a good accuracy. In light of these considerations, a key property of an incentive scheme is its robustness to incomplete information.

A raffle-based incentive scheme has been recently proposed for decongestion of a shared resource [8]. This scheme is based on economic ideas on how to incentivize contributions to a public good [9]. It works by giving users a reward for reducing their usage of the shared resource, in a lottery-like manner: the expected reward of a user is proportional to his usage reduction and inversely proportional to the total usage reduction.

In this paper, we investigate the benefits of using the raffle-based scheme *in lieu* of time-of-day pricing for demand management in Internet broadband usage. We use a game-theoretic model with a continuum of non-atomic users choosing a fraction of their usage that they are willing to shift from peak to off-peak time. Users are sensitive to the congestion through delays and each user's utility is specified with essentially two terms representing: *(i)* how much he benefits from the peak-time decongestion and *(ii)* his willingness to shift his usage to the off-peak time. This model allows us to compare the performance of the raffle-based scheme and of the time-of-day pricing scheme in terms of their sensitivity to mis-estimation of the users' utilities. Our main conclusions are the following.

- With the raffle-based scheme, the provider knows in advance the total reward that it will give. In contrast, with the time-of-day pricing, it must rely on an estimation of the users' responsiveness to estimate the total reward that it will give.
- The raffle-based scheme possesses a *self-tuning* property which gives it a higher robustness than the time-of-day pricing in some situations that we fully characterize.

B. Related work

Over the past decades, many pricing schemes have been proposed to manage quality of services in networks, mainly based

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on usage and multiclass models [10], [11], [12], [13]. In [14], Paschalidis and Tsitsiklis, propose an optimal congestion-based pricing scheme in the context of phone networks and show that it can be approximated by time-of-day pricing. An alternative to congestion pricing based on a trading scheme has been recently proposed [15].

Two recent papers analyze models of static and dynamic time-of-day pricing over n time slots [6], [7]. In [6], Jiang, Parekh and Walrand consider a model where users have unit demand and choose one time-slot in which to transmit it entirely based on their own utility and on the effect of latency. In [7], Wong, Ha and Chiang consider *sessions* characterized by *waiting functions* representing the willingness of a user to delay its entire session for a given time. However, the effect of congestion on the users' utilities is not considered. In this paper, we consider a model with 2 time slots where users can shift an arbitrary continuous fraction of their demand from a peak to an off-peak time. Our model include the effect of congestion on the users. We show that the problem of decongesting the peak time can be seen as a public good provision problem for which a raffle-based incentive scheme has been recently proposed in [8].

The notion of public good has been used in recent works [16], [17] in the context of optimal power allocation for wireless networks. The idea of lottery-based incentive schemes for congestion management has also been used in prior works. For example in [18], Merugu, Prabhakar and Rama conduct a field study where lotteries are used to decongest transportation networks. Raffle scheduling is also a widely applied technique in resource scheduling in computer operating systems [19].

C. Outline

We introduce the model in Section II. In Section III, we prove existence and uniqueness of the Nash equilibrium and show that social optimum is achievable in Nash equilibrium. In Section IV, we compare the two schemes in terms of robustness. We conclude in Section V.

II. Model

A. Definitions and notations

We consider two usage periods: a peak period and an off-peak period. We denote T_p and T_o their respective durations.

We consider a set of users, sharing a common access point (wired or wireless) to the Internet. We denote by C the capacity of this access point. In a real situation, there would be a finite number of users, each having his own time preferences. However, to account for a large number of users, we consider a continuum of non-atomic users (*i.e.,* each user contributes a negligible fraction of the total usage). Users are indexed by their type denoted θ which characterizes their time preference for Internet usage. Let $(\Theta, \mathcal{F}, \mu)$ be a measured space; where Θ is the set of users' types, $\mathcal{F}$ is a $\sigma-$algebra and μ is a finite measure accounting for the distribution of the users' types. Note that we work here directly with the distribution of types, instead of working with the distribution of users as in [20]. Therefore, measure μ can have atoms without violating the atomless assumption on the distribution of users. This would simply mean that a fraction of users of positive measure would have the same type corresponding to the atom of measure μ.

We assume that all users have identical peak-time demand and off-peak time demand[1]. For simplicity, we assume that the peak-time demand of each user is 1, and his off-peak time demand is $d_o < 1$. Part of the peak-time demand is *shiftable* to the off-peak time. We assume that all users have identical shiftable demand, denoted $d_s < 1$. We denote by

$$D = \int_\Theta \mu(\mathrm{d}\theta)$$

the aggregate peak-time demand and by

$$D_o = d_o \cdot D \text{ and } D_s = d_s \cdot D$$

the aggregate off-peak time and shiftable demand respectively.

Each user of type $\theta \in \Theta$ chooses a fraction $x_\theta \in [0, d_s]$ of his demand that he shifts to off-peak time. This defines a measurable function $x : \Theta \to [0, d_s]$ on $(\Theta, \mathcal{F}, \mu)$. We introduce the *public good*:

$$G = \int_\Theta x_\theta \mu(\mathrm{d}\theta) \in [0, D_s] \tag{1}$$

which corresponds to the total shifted demand. It is a public good in the sense that when a user shifts part of his demand, it reduces the congestion at peak-time and therefore it benefits to all the users. We assume that the total demand is uniformly spread within each period, so that the effective loads are

$$\rho_p = \frac{D - G}{CT_p} \text{ and } \rho_o = \frac{D_o + G}{CT_o}$$

in the peak and off-peak time periods respectively. For ease of exposition, we assume that $\rho_p < 1$ and $\rho_o < 1$ for all $G \geq 0$[2].

B. Utilities

We assume that the utility of a user of type $\theta \in \Theta$ who shifts a fraction x_θ of his demand is

$$\begin{aligned} U_\theta \quad = \quad & P_\theta\left(1 - x_\theta\right) + O_\theta\left(d_o + x_\theta\right) \\ & - \left(1 - x_\theta\right) \cdot L\left(\rho_p\right) - \left(d_o + x_\theta\right) \cdot L\left(\rho_o\right) - p \end{aligned} \tag{2}$$

where $P_\theta(\cdot)$ and $O_\theta(\cdot)$ are the utilities that the user gets for his usage in the peak and off-peak periods respectively, $L(\cdot)$ is a disutility function due to congestion and p is a fixed subscription price for the Internet access.

In our context, we assume that the utility at peak time is higher than the utility at off-peak time, so that a user looses utility when shifting part his demand. We define for all $\theta \in \Theta$

[1] If the users differ by their demand in volume, each user could be viewed as an appropriate number of users with identical demand and the proposed model still applies with the measure μ defined for all subset $\Theta_1 \in \mathcal{F}$ by $\mu(\Theta_1) = \int_\Delta d \cdot \nu(\Theta_1, \mathrm{d}d)$ where Δ is the set of demands and the measure ν on $\Theta \times \Delta$ represents the joint distribution of types and demand.

[2] It is straightforward to generalize our results to the case where $\rho_p < 1$ and $\rho_o < 1$ only beyond a minimal value $G_{\min} \in (0, D_s)$. Clearly, in this case, there would not be any equilibrium possible in the range $[0, G_{\min})$.

the *cost of shifting* as the loss of utility that a user of type θ incurs when shifting a fraction x_θ of his demand:

$$c_\theta(x_\theta) = \bar{U}_\theta - \left(P_\theta\left(1 - x_\theta\right) + O_\theta\left(d_o + x_\theta\right)\right), \quad (3)$$

where

$$\bar{U}_\theta = P_\theta(1) + O_\theta(d_o) \quad (4)$$

is the maximal utility that a user could get without shifting any usage if there was no congestion. The cost of shifting characterizes the time preferences of a user of type θ. We assume that it is increasing and convex.

Similarly, we assume that despite the effect of users shifting part of their usage, the off-peak period remains much less congested that the peak-time, *i.e.,* $\rho_o \ll \rho_p$. Therefore,

$$L\left(\rho_o\right) \simeq 0. \quad (5)$$

Following [10], we focus on the delay as a measure of the network quality, and we assume that the disutility function $L(\cdot)$ is an increasing convex function of the average delay δ. We will use the most classical model for 3G and 4G networks, the processor sharing queue [21], for which the average delay is itself an increasing convex function of the load:

$$\delta\left(\rho_p\right) = \frac{\delta_0}{1 - \rho_p}, \quad (6)$$

where δ_0 is a constant. Therefore, the disutility function $L(\cdot)$ is also an increasing convex function of the load ρ_p (it is also what is assumed in the model of [6]). In [10], Honig and Steiglitz use the M/D/1 model, more common for wired networks, for which the average delay is also an increasing convex function of the load: $\delta\left(\rho_p\right) = \frac{\delta_0 \rho_p}{2(1-\rho_p)}$.

In the rest of the paper, we use the following example:

Example 1. The distribution of types is uniform on $\Theta = [0, 1]$, *i.e.,* $\mu(\mathrm{d}\theta) = D \cdot \mathrm{d}\theta$, with $D = 99$, $d_s = 0.2$, $d_o = 0.5$, $p = 0$, $L(\rho_p) = L_0 \cdot \delta(\rho_p)^n$, where $L_0 = 0.5 \cdot 10^{-5}$, $n = 2$ and $\delta(\rho_p)$ is given by (6) with $\delta_0 = 1$, $C = 50$, $T_p = 2$, $T_o = 22$, $O_\theta(\cdot) = 0$, and $P_\theta(y) = \theta\left(1 - (1-y)^2\right), \forall y \in [0, 1]$, which gives $\bar{U}_\theta = \theta$ and $c_\theta(x_\theta) = \theta \cdot x_\theta^2$.

To use the public goods framework, we introduce function

$$h(G) = -L\left(\frac{D - G}{C T_p}\right). \quad (7)$$

It is an increasing concave function of G, which translates the notion of how much users benefit from the network decongestion at peak-time. Then, the term $-(1 - x_\theta)L(\rho_p) = (1 - x_\theta)h(G)$ in (2) has the interpretation that the benefit a user gets from the peak period is the product of the fraction of demand $(1 - x_\theta)$ he places in that period times the benefit per unit demand $h(G)$ in the period. Notice that $h(G)$ is typically negative, but its important characteristic is that it is increasing and concave with G.

In summary, in view of (2), (3), (4), (5), (7), our peak-time decongestion model reduces to a public good problem similar to [8]: the utility of a user of type $\theta \in \Theta$ is

$$U_\theta = \bar{U}_\theta + (1 - x_\theta)h(G) - c_\theta(x_\theta) - p, \quad (8)$$

where $h(\cdot)$ and $c_\theta(\cdot)$ are twice continuously differentiable functions such that

(A1) $h'(\cdot) > 0$ and $h''(\cdot) < 0$,
(A2) $c'_\theta(\cdot) > 0$ and $c''_\theta(\cdot) > 0$.

We shall also use the following technical assumption:

(A3) $\sup_{\theta \in \Theta} c'_\theta(d_s) < \infty$.

C. Incentive schemes

We define the aggregate user welfare as

$$\begin{aligned} W &= \int_\Theta U_\theta \mu(\mathrm{d}\theta) \quad (9) \\ &= (D - G)h(G) - \int_\Theta c_\theta(x_\theta)\mu(\mathrm{d}\theta) + \int_\Theta \bar{U}_\theta \mu(\mathrm{d}\theta) - pD. \end{aligned}$$

Individual users maximize their own utility (8), which differs from maximizing (9). Thus, in general, the level of public good and the aggregate user welfare achieved in the individual maximization and in the social optimum differ.

In this paper, we compare two different incentive schemes to align both objectives: a raffle-based scheme and a time-of-day pricing scheme. Each scheme introduces a reward for the shifted demand and an increase in the flat subscription price for the service provider to finance the respective reward. The reward may be given to users with some randomization. The expected utility with respect to this randomization becomes for a user of type $\theta \in \Theta$:

$$U_\theta = \bar{U}_\theta + (1 - x_\theta)h(G) - c_\theta(x_\theta) - p + S_i(x_\theta, G), \quad (10)$$

where the subscript i refers to the scheme considered. The raffle-based scheme ($i = L$) consists in giving each user of type $\theta \in \Theta$ an expected reward of the functional form:

$$S_L(x_\theta, G) = R \cdot \frac{x_\theta}{G} - \Delta p_L. \quad \text{[raffle]} \quad (11)$$

With a finite number of users, it is the simplest type of lottery that could correspond to organizing a raffle where each user wins the prize R with a probability equal to his percentage contribution to the total amount of shifted demand. An alternative implementation of the scheme for the case of an infinite number of users would be to give each user a reward R/G with probability x_θ and a reward 0 with probability $(1 - x_\theta)$. The time-of-day pricing scheme ($i = T$) corresponds to a fixed reward r per unit of shifted demand:

$$S_T(x_\theta, G) = r \cdot x_\theta - \Delta p_T. \quad \text{[time-of-day pricing]} \quad (12)$$

Clearly, this scheme is a variation of a conventional time-of-day pricing scheme, with an off-peak price subsidy.

In (11) and (12), Δp_i denotes the increase in the subscription price that the service provider imposes to finance the reward mechanism. In the next section, we demonstrate that both schemes have a unique Nash equilibrium for any parameters r, R. Let $G^{(\mathrm{eq})}$ denote the level of public good (1) achieved at the respective equilibrium. We assume that the price Δp_i is fixed in advance by the service provider to compensate the reward, *i.e.,* $\int_\Theta S_i(x_\theta, G^{(\mathrm{eq})}) = 0$. Then,

$$\begin{cases} \Delta p_L = \frac{R}{D}, \\ \Delta p_T = r \cdot \frac{G^{(eq)}}{D}. \end{cases} \quad (13)$$

Notice that due to this choice of Δp_i, the expression of the aggregate welfare (9) is not directly modified by the schemes, but only through the chosen contributions x_θ.

From (13), we immediately see that the service provider has to know the equilibrium to determine the price Δp_T for the time-of-day pricing. An error in the estimation of $G^{(\text{eq})}$ may have a dramatic effect on the service provider budget. In contrast, such knowledge is not necessary for the raffle-based scheme where Δp_L only depends on the parameter R chosen by the service provider.

The derivatives of the utilities in both schemes are

$$\frac{\partial U_\theta}{\partial x_\theta} = -h(G) - c'_\theta(x_\theta) + S'_i(G), \tag{14}$$

where

$$\begin{cases} S'_L(G) = \frac{R}{G}, \\ S'_T(G) = r. \end{cases} \tag{15}$$

For each scheme $i \in \{L, T\}$, the marginal expected reward S'_i does not depend on x_θ, due to the non-atomic assumption. Due to the term $-h(G)$ in (14), the marginal utility decreases when G increases. Intuitively, if the congestion is lower in the peak period, a user would want to use it more. Hence he would want to shift less of his demand. This decrease of the marginal utility is accentuated by the term $S'_L(G) = R/G$ in the case of the raffle-based scheme.

The specifics of the environment might require modifications of the proposed scheme. For instance, a provider facing competition may not be able to increase the monthly price. Instead the reward could be financed by the provider being able to accommodate more users with the same infrastructure due to the reduction of congestion. In addition, the reduction of congestion could reduce the maintenance cost.

Our setup does not include an explicit model of providers competition. Still, for any competition structure, the provider profit maximizing objective forces them to produce efficiently. Since reducing congestion improves efficiency by means of reducing the gap between individually and socially optimal user incentives, our scheme works with any competition structure.

For clarity, we will use the following additional notation: $\Gamma_L\left(\Theta, \mu, h, \{c_\theta\}_{\theta\in\Theta}, R\right)$ and $\Gamma_T\left(\Theta, \mu, h, \{c_\theta\}_{\theta\in\Theta}, r\right)$ are the games where users selfishly optimize their own utility (10) in the raffle scheme ($i = L$) and in the time-of-day pricing scheme ($i = T$) respectively. We denote with the superscript $^{(\text{eq})}$ the quantities at equilibrium in both games and we explicitly write their dependence on r and R or on other parameters whenever necessary to avoid ambiguity. Similarly, we denote with the superscript * the social optimum quantities corresponding to the maximization of (9), and denote explicitly their dependence on parameters whenever necessary.

III. Nash equilibrium and social optimum

In this section, we show that both schemes have a unique Nash equilibrium. Then, we show that for appropriate values of the schemes parameters, they achieve social optimum. The results of this section are similar to those given in [8] for the raffle-based scheme. We extend them here to the time-of-day pricing scheme and compare the results of both schemes.

In our non-atomic model, a Nash equilibrium is a function $x : \Theta \to [0, d_s]$ such that for almost all $\theta \in \Theta$, $U_\theta(a, G) \leq U_\theta(x_\theta, G), \forall a \in [0, d_s]$ where G is defined by (1) (see [20]). Due to the strict concavity of the utility U_θ, it is equivalent to finding a function x that satisfies almost-surely the first-order conditions (FOCs)

$$-h(G) - c'_\theta(x_\theta) + S'_i(G) \quad \begin{cases} \leq 0, & \forall \theta : x_\theta = 0, \\ = 0, & \forall \theta : x_\theta \in (0, d_s), \\ \geq 0, & \forall \theta : x_\theta = d_s, \end{cases} \tag{16}$$

and such that (1) is satisfied.

Remark 1. Due to assumption **(A2)**, it cannot exist a Nash equilibrium of our model where two users of the same type choose different contributions. This justifies that we can work directly at the level of types instead of users.

A. Nash equilibrium: existence and uniqueness

The first proposition establishes existence and uniqueness of the Nash equilibrium in the games corresponding to both incentive schemes.

Theorem 1. *For any $R \geq 0$, there exists a Nash equilibrium $x^{(eq)}(R)$ of the raffle-based scheme, uniquely determined almost-everywhere.*
The same result holds for the time-of-day pricing scheme, for any $r \geq 0$.

For the intuition behind Theorem 1, consider scheme $i \in \{L, T\}$. For a given G, each type chooses its best response contribution $x^{(\text{resp})}(G) \in [0, d_s]$ to maximize his utility. Then, integrating the contribution of each type gives the amount of public good $G^{(\text{resp})}(G)$ that users want to provide in response to a given G. An equilibrium occurs when both quantities are equal, which corresponds to solving the fixed-point equation

$$G^{(\text{resp})}(G) = G. \tag{17}$$

Fig. 1 illustrate the two terms of the fixed-point equation for both schemes. As we mentioned, a key feature of our model is that the higher the public good G is, the fewer users are willing to shift their traffic (the marginal utilities of (14) are decreasing with G). This is accentuated for the raffle scheme for which the marginal utilities are decreasing faster. Therefore, when the total amount of public good provided by all users G increases, the amount of public good $G^{(\text{resp})}(G)$ that users are willing to shift decreases. Moreover, $G^{(\text{resp})}(G)$ is continuous, which leads to a unique fixed point. The continuity of $G^{(\text{resp})}(G)$ is due to the strict convexity of the functions c_θ (assumption **(A2)**), which prevents situations where a slight modification of G would make some users to go from wanting to shift none of their traffic to wanting to shift all of their traffic.

B. Social optimum

We now show that the social optimum is unique and coincides with the Nash equilibrium of both schemes for parameters R^* and r^* given in the next theorem.

Theorem 2. *The following characterize the social optimum:*

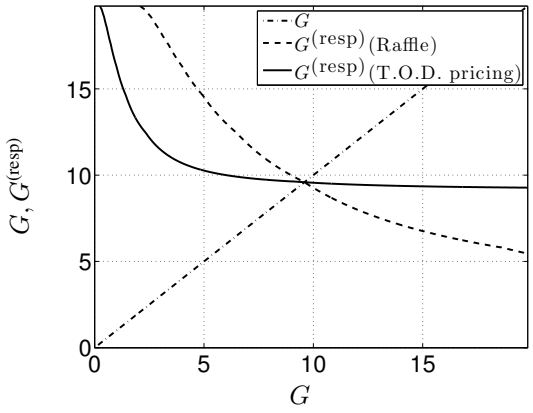

Fig. 1. Illustration of the fixed-point equation (17) for Example 1. The dashdotted line corresponds to the amount G that users shifts (r.h.s. of (17)). The dashed and solid lines correspond to the aggregate amount $G^{(\mathrm{resp})}$ that users would want to shift (l.h.s. of (17)) given G has been shifted for the raffle-based and time-of-day pricing schemes respectively. To obtain $G^{(\mathrm{eq})} = G^* = 9.59$, parameters were set to $R = 0.72$ and $r = 0.075$.

(i) There exists a function x^*, uniquely determined almost-everywhere, for which the aggregate welfare (9) *is maximized.*
(ii) We have $x^{(eq)}(R) = x^*$ almost-everywhere and hence $G^{(eq)}(R) = G^*$ with the raffle-based scheme for $R = R^*$, where

$$R^* = G^* h'(G^*)(D - G^*). \tag{18a}$$

The same result holds for the time-of-day pricing scheme for $r = r^*$, where

$$r^* = h'(G^*)(D - G^*). \tag{18b}$$

Intuitively, this result holds because the externality faced by a user $(-h(G) + S_i')$ in the game corresponding to any scheme is independent of its type θ. Therefore, by fixing a reward that is also independent of the type θ, it is possible to achieve social optimum by making users effectively pay a Pigovian tax [22].

C. Nash equilibrium: variation with the scheme parameters

Proposition 1. *With the raffle scheme, $G^{(eq)}(R) > 0$ for any $R > 0$.*

The intuition behind Proposition 1 is that if $R > 0$, then the reward per unit of shifted demand is infinite if $G = 0$. Then, all users want to contribute. This does not mean that in equilibrium, all the users will contribute. As G grows, some users may stop contributing. A similar result does not hold for the time-of-day pricing scheme where the reward per unit of shifted demand is constant. There, if for some $r \geq 0$, the marginal utility of almost-all types is negative at zero, the equilibrium level of public good will be $G^{(\mathrm{eq})}(r) = 0$. This can happen even it would be socially optimal that some users contribute positively, *i.e.,* $G^* > 0$.

Note that Proposition 1 is consistent with the result (18a) of Theorem 2 showing that if $G^* = 0$, then $R^* = 0$ and it is the only possible value of R for which the social optimum is achieved at Nash equilibrium with the raffle-based scheme. In contrast, if $G^* = 0$, the social optimum is achieved at Nash equilibrium in the time-of-day pricing scheme for any r below a threshold.

The last proposition of this section shows that the level of public good at equilibrium increases with the reward.

Proposition 2. *For the raffle scheme, we have:*

(i) For any $R' > R$, $G^{(eq)}(R') \geq G^{(eq)}(R)$; and the inequality is strict if $0 < G^{(eq)}(R) < D_s$.
(ii) There exists a threshold $\bar{R}$ such that $G^{(eq)}(R) = D_s$ for all $R \geq \bar{R}$.

The same results hold for the time-of-day pricing scheme by changing R to r everywhere.

The existence of the thresholds $\bar{R}$ and $\bar{r}$ is due to assumption **(A3)** which means that shifting even the last shiftable bit implies a finite marginal cost which can be compensated by a large-enough reward. Due to space limitations, we do not explicitly model the participation constraint, *i.e.,* the constraint that each user type must have positive utility (if not, such a user would not buy the service subscription). Instead, we implicitly assume that the utility $\bar{u}_\theta$ is large enough and the maximum shiftable demand d_s is small enough, so that each user type still has positive utility when shifting all of his shiftable demand. Hence all the users will participate for any parameter R or r. This strong assumption could be relaxed in many situations. For instance, if users are homogeneous on the timescale of a month, but heterogeneous in their willingness to shift on any given day, then all users will have the same utility on a monthly timescale. If this common utility value were positive before the introduction of the scheme, implementing the scheme with reward close to the optimal reward R^* or r^* can only increase social welfare and hence this common utility value, thus users will continue to participate.

One implication of Proposition 2 is that both schemes can "overshoot": if R or r is too large (larger that R^* or r^* defined in (18)), $G^{(\mathrm{eq})}$ can be larger than G^* and the aggregate user welfare is suboptimal. In an competitive environment, a provider would not intentionally choose an overshooting parameter because it would be a competitive disadvantage. In our model, the reward is financed by increasing the subscription price. Therefore, this "overshooting" would be also be limited in a real situation with competition because the service provider would loose its subscribers.

IV. Comparison of the two incentive schemes

In the previous section, we have shown that both schemes have a unique Nash equilibrium which achieves social optimum for parameters R^* and r^* (see (18)). Next, we consider the robustness of each scheme when the service provider picks R^* and r^* based on erroneous data. In particular, let the games $\Gamma_L\left(\Theta, \mu, h, \{c_\theta\}_{\theta\in\Theta}, R^*\right)$ and $\Gamma_T\left(\Theta, \mu, h, \{c_\theta\}_{\theta\in\Theta}, r^*\right)$ correspond to the baseline case of perfect information considered in Section III and suppose that R^* and r^* have been chosen according to (18) to induce an equilibrium amount G^*, the socially optimal amount of public good. We assume that $G^* \in (0, D_s)$. We study the change in equilibrium and

in social optimum when R^* and r^* are maintained for the respective schemes and utilities are perturbed.

We consider here the simple case where functions $c_\theta(\cdot)$ of the utilities (10) are scaled by a factor γ, that is where the cost of shifting varies. We denote by $G_L^{(\text{eq})}(\gamma)$ and $G_T^{(\text{eq})}(\gamma)$, and by $W_L^{(\text{eq})}(\gamma)$ and $W_T^{(\text{eq})}(\gamma)$, the equilibrium amount of public good and popular welfare in the new games $\Gamma_L\left(\Theta, \mu, h, \{\gamma c_\theta\}_{\theta\in\Theta}, R^*\right)$ and $\Gamma_T\left(\Theta, \mu, h, \gamma\{c_\theta\}_{\theta\in\Theta}, r^*\right)$ respectively. Let $G^*(\gamma)$ and $W^*(\gamma)$ denote the socially optimal level of public good with perturbed utility, and the corresponding population welfare, resulting from the maximization of the aggregate utility (9) when $c_\theta(\cdot)$ is replaced by $\gamma c_\theta(\cdot)$.

For the analysis, we will restrict to small perturbations, that is γ close to 1. A key element to compare the variation of each scheme's equilibrium is the unit reward. We introduce the following notation:

$$r_L(G) = \frac{R}{G}, \tag{19a}$$

$$r_T(G) = r, \tag{19b}$$

$$r_{SO}(G) = h'(G)(D - G), \tag{19c}$$

and we denote by r'_L, r'_T, r'_{SO} the respective derivatives w.r.t. to G. By definition (see (15)), $r_L(G)$ and $r_T(G)$ correspond to the unit reward of the raffle-based and time-of-day pricing scheme respectively. From Theorem 2, the social optimum can also be seen as a Nash equilibrium in a scheme with unit reward given by (19c).

To evaluate the variations of $G^{(\text{eq})}$ which is defined as the fixed-point of $G^{(\text{resp})}$ (see (17)), one has to evaluate the variations of the aggregate best response $G^{(\text{resp})}$. For this purpose, we introduce for each scheme $i \in \{L, T, SO\}$ the quantity α_i corresponding to the opposite of the derivative of $G^{(\text{resp})}$ at the common equilibrium point G^* (when $\gamma = 1$). We define the conditions:

(C1) $\left|\frac{1}{1+\alpha_L} - \frac{1}{1+\alpha_{SO}}\right| < \left|\frac{1}{1+\alpha_T} - \frac{1}{1+\alpha_{SO}}\right|$,

(C2) $\left|\frac{1}{1+\alpha_L} - \frac{1}{1+\alpha_{SO}}\right| > \left|\frac{1}{1+\alpha_T} - \frac{1}{1+\alpha_{SO}}\right|$.

If the slopes α_i's for the different schemes i's are close enough, these conditions reduces to the following more intuitive conditions:

(C1′) $|r'_L(G) - r'_{SO}(G)| < |r'_T(G) - r'_{SO}(G)|$, at $G = G^*(1)$,
(C2′) $|r'_L(G) - r'_{SO}(G)| > |r'_T(G) - r'_{SO}(G)|$, at $G = G^*(1)$.

Then we have the following results.

Proposition 3. *For any γ close enough to 1, we have:*

*(i) If condition **(C1)** is realized, then*
(a) $G_T^{(eq)}(\gamma) = G_L^{(eq)}(\gamma) = G^*(\gamma)$ if $\gamma = 1$ (baseline case);
(b) $G_T^{(eq)}(\gamma) < G_L^{(eq)}(\gamma) < G^*(\gamma)$ if $\gamma > 1$;
(c) $G_T^{(eq)}(\gamma) > G_L^{(eq)}(\gamma) > G^*(\gamma)$ if $\gamma < 1$.

*(ii) If condition **(C2)** is realized, then*
(a) $G_L^{(eq)}(\gamma) = G_T^{(eq)}(\gamma) = G^*(\gamma)$ if $\gamma = 1$ (baseline case);
(b) $G_L^{(eq)}(\gamma) < G_T^{(eq)}(\gamma) < G^*(\gamma)$ if $\gamma > 1$;
(c) $G_L^{(eq)}(\gamma) > G_T^{(eq)}(\gamma) > G^*(\gamma)$ if $\gamma < 1$.

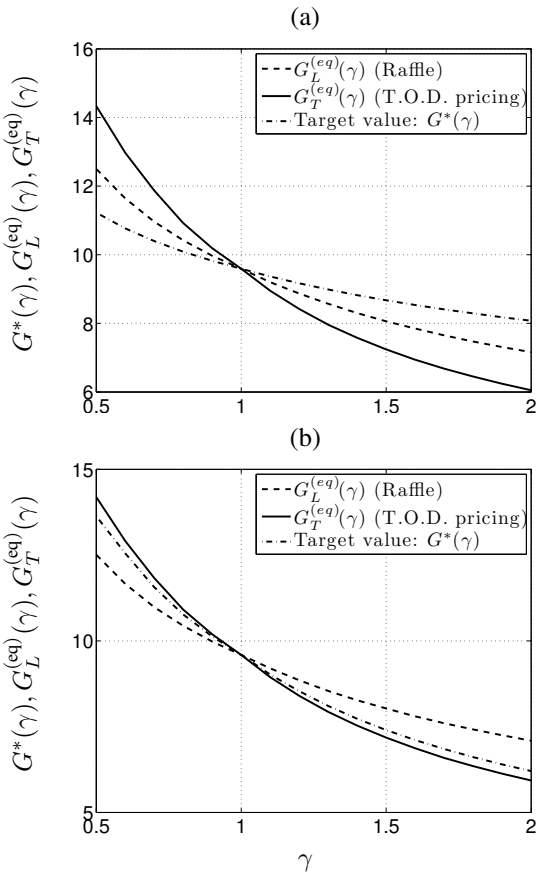

Fig. 2. Variation of the equilibrium amount of public good $G^{(\text{eq})}$ when functions $c_\theta(\cdot)$ are scaled by a factor γ starting from the baseline case: (a) for Example 1 – (b) for Example 2.

Proof: See Appendix A. ∎

The intuition behind Proposition 3 is as follows. The scheme with the unit reward closer to the optimal unit reward $r_{SO}(G)$ will have an equilibrium closer to the social optimum equilibrium $G^*(\gamma)$. Since $r_L(G)$ and $r_{SO}(G)$ are both decreasing functions, one might expect $r_L(G)$ to be closer to $r_{SO}(G)$ than $r_T(G)$. The fact that $r_L(G)$ decreases when G increases is what we call the *closed-loop* effect: the more users shift, the lower the incentive to shift is. However, if $r_L(G)$ decreases much more quickly that $r_{SO}(G)$, it can be that $r_T(G)$ is actually closer to $r_{SO}(G)$. This possibility is covered by case *(ii)* of Proposition 3 above.

Fig. 2 illustrates the result of Proposition 3 and shows that the result seem to hold for large perturbations also (γ not close to 1). As it turns out, Example 1 falls in case *(i)*, where the raffle-based scheme remains closer to social optimum than the time-of-day pricing scheme. For the sole purpose of illustrating numerically case *(ii)*, where the time-of-day pricing scheme is closer to social optimum, we construct the following example:

Example 2. Everything is defined as in Example 1, but the disutility function is artificially contrived to have $h(G) = 1.1 \cdot 10^{-3} \cdot (G^{0.9} - D_s^{0.9})$. (The factor 1.1 has been chosen to yield the same social optimum level of public good G^* than in Example 1 when $\gamma = 1$.)

Clearly, Example 2 is a contrived example where $h(G)$ is almost linear so that the optimal unit reward r_{SO} is almost constant, as in the time-of-day pricing scheme.

From Proposition 3, we deduce the following robustness result.

Theorem 3. *For any $\gamma \neq 1$ close enough to 1, we have:*

*(i) If condition **(C1)** is realized, then the raffle-based scheme is more robust than the time-of-day pricing scheme in the sense that*

$$W_T^{(eq)}(\gamma) < W_L^{(eq)}(\gamma) < W^*(\gamma);$$

*(ii) If condition **(C2)** is realized, then the time-of-day pricing scheme is more robust than the raffle-based scheme in the sense that*

$$W_L^{(eq)}(\gamma) < W_T^{(eq)}(\gamma) < W^*(\gamma).$$

Proof: See Appendix B. ∎

Theorem 3, illustrated on Fig. 3, is our main result. It establishes the ranking of the two incentive schemes in terms of their robustness with respect to a multiplicative perturbation of functions $c_\theta(\cdot)$ of the utilities (10). If the rewards parameters R and r are set using an erroneous estimation of parameter γ (*i.e.,* a erroneous estimation of the willingness of the users to shift their demand to off-peak time), the aggregate welfare is closer to optimal with the raffle-based scheme in case *(i)* and with the time-of-day pricing scheme in case *(ii)*. If the uncertainty on parameter γ were modeled probabilistically and parameters R and r were chosen based on its expectation, a similar result would hold for the average aggregate user welfare. Finally, note that Fig. 3 shows that the result holds for large perturbations also (γ not close to 1).

Proposition 3 and Theorem 3 constitute a first robustness result, when function $c_\theta(\cdot)$, representing the willingness of users to shift their demand, is multiplicatively perturbed. In future work, we will consider more general perturbations of the utility function.

V. Concluding discussion

Our raffle-based scheme can be viewed as a time-of-day pricing scheme with probabilistic prices, since the price depends on the demand realization. This scheme has two main advantages over standard time-of-day pricing with fixed prices. Firstly, it is easily implementable via lottery-like mechanisms with a total given reward known in advance by the provider. Secondly, it has built-in self-tuning, which we believe is attractive in environments with imperfectly known demand.

Practical implementation of our scheme requires discerning how much demand each user shifted from the peak to the off-peak period. Simply measuring a user's off-peak traffic may be insufficient, since a user may try to "game the system" by generating extra off-peak traffic to increase his expected reward. However, one could prevent the gaming by keeping usage statistics and punishing users whose off-peak usage rises sharply without a commensurate fall in their peak usage. Also, the presence of a usage cap or of a pricing scheme based on total demand can also limit gaming behavior.

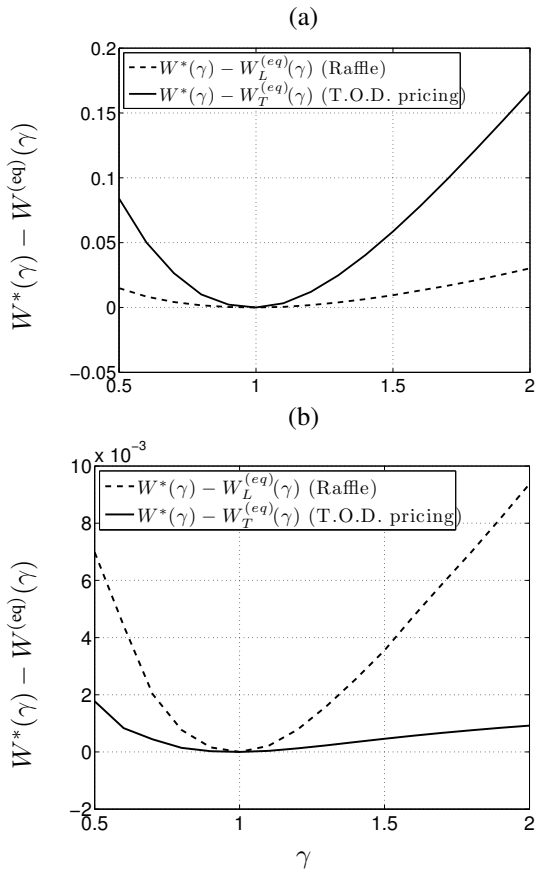

Fig. 3. Variation of the aggregate welfare $W^{(\mathrm{eq})}$ when functions $c_\theta(\cdot)$ are scaled by a factor γ starting from the baseline case: (a) for Example 1 – (b) for Example 2. For readability and robustness comparison, the difference $W^*(\gamma) - W^{(\mathrm{eq})}(\gamma)$ is plotted.

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APPENDIX A
PROOF OF PROPOSITION 3

We prove the results in the case $\gamma < 1$. The case $\gamma > 1$ is similar and the case $\gamma = 1$ is trivial.

For ease of notation, we use the notation $f_\theta(\cdot) = (c'_\theta)^{-1}(\cdot)$. Recall that due to assumption **(A2)**, it is a strictly increasing continuously differentiable function.

From Theorem 1, we know that for scheme $i \in \{L, T\}$, $G_i^{(\mathrm{eq})}$ is the fixed-point solution of (17). The individual best responses for scheme i are defined for all $\theta \in \Theta$ by

$$x_{i,\theta}^{(\mathrm{resp})}(G) = \begin{cases} 0, \text{ if } r_i(G) - h(G) - c'_\theta(0) \le 0, \\ d_s, \text{ if } r_i(G) - h(G) - c'_\theta(d_s) \ge 0, \\ (c'_\theta)^{-1}(r_i(G) - h(G)), \text{ otherwise,} \end{cases} \tag{20}$$

and the aggregate best response for scheme i is defined by

$$G_i^{(\mathrm{resp})}(G) = \int_\Theta x_{i,\theta}^{(\mathrm{resp})}(G)\mu(\mathrm{d}\theta). \tag{21}$$

From of Theorem 2, we know that G^* is the fixed-point solution of the same equation (17) with $r_i(G) = r_{SO}(G)$ defined by (19c). We use the notation $G^* = G_{SO}^{(\mathrm{eq})}$ which emphasizes this similarity and helps shorten the proof's notation.

Before evaluating the equilibrium variations with γ, note that when $\gamma = 1$, we have the same equilibrium points:

$$G_L^{(\mathrm{eq})}(1) = G_T^{(\mathrm{eq})}(1) = G_{SO}^{(\mathrm{eq})}(1), \tag{22}$$

and the same unit rewards at equilibrium:

$$r_L(G) = r_T(G) = r_{SO}(G) \quad \text{if} \quad G = G_L^{(\mathrm{eq})}(1). \tag{23}$$

When $\gamma \neq 1$, functions c_θ are multiplied by γ and functions f_θ are multiplied by $\frac{1}{\gamma}$. For a given $G \in [0, D_s]$, the aggregate best response is modified accordingly. We denote by $G_{i,\gamma}^{(\mathrm{resp})}(G)$ the new aggregate best response. Recall that we also denote by $G_i^{(\mathrm{eq})}(\gamma)$ the new equilibrium point which is the fixed point of $G_{i,\gamma}^{(\mathrm{resp})}(\cdot)$.

The next lemma readily implies the result of Proposition 3.

Lemma 1. *For any $i \in \{L, T, SO\}$, we have*

$$G_i^{(eq)}(\gamma) = G_i^{(eq)}(1) + \frac{J_\gamma}{1+\alpha_i} + o(\gamma - 1), \tag{24}$$

where

$$J_\gamma = G_{i,\gamma}^{(resp)}\left(G_L^{(eq)}(1)\right) - G_{i,1}^{(resp)}\left(G_L^{(eq)}(1)\right)$$

is a first-order quantity in $(\gamma - 1)$ independent of the scheme i, and

$$\alpha_i = -\frac{dG_i^{(resp)}}{dG}\left(G_L^{(eq)}(1)\right). \tag{25}$$

Proof: To evaluate the variation of $G_i^{(\mathrm{eq})}$ defined as a fixed-point of $G_{i,\gamma}^{(\mathrm{resp})}(\cdot)$, we must evaluate the variations of the function $G_{i,\gamma}^{(\mathrm{resp})}(\cdot)$ when γ moves. When γ goes from 1 to a value $\gamma < 1$, the aggregate best response for a given G goes from $G_{i,1}^{(\mathrm{resp})}(G)$ to $G_{i,\gamma}^{(\mathrm{resp})}(G)$. At G corresponding to the common equilibrium point (22) (for $\gamma = 1$), the aggregate best response increases from the same amount for all the schemes due to (23). This "jump" is

$$\begin{aligned} J_\gamma &= G_{i,\gamma}^{(\mathrm{resp})}\left(G_L^{(\mathrm{eq})}(1)\right) - G_{i,1}^{(\mathrm{resp})}\left(G_L^{(\mathrm{eq})}(1)\right) \\ &= -(\gamma - 1) \cdot G_L^{(\mathrm{eq})}(1) + o(\gamma - 1). \end{aligned}$$

Starting from this point, at the first order, $G_{i,\gamma}^{(\mathrm{resp})}(G)$ decreases linearly when G increases. Therefore it can be seen geometrically that it will cross again the first bisector at the new equilibrium point

$$G_i^{(\mathrm{eq})}(\gamma) = G_i^{(\mathrm{eq})}(1) + \frac{J_\gamma}{1+\alpha_{i,\gamma}} + o(\gamma - 1), \tag{26}$$

where $-\alpha_{i,\gamma}$ is the slope of the curve $G_{i,\gamma}^{(\mathrm{resp})}(G)$ at $G = G_L^{(\mathrm{eq})}(1)$, *i.e.,*

$$\alpha_{i,\gamma} = -\frac{\mathrm{d}G_{i,\gamma}^{(\mathrm{resp})}}{\mathrm{d}G}\left(G_L^{(\mathrm{eq})}(1)\right). \tag{27}$$

From (26), it is easy to see that since J_γ is first-order in $(\gamma - 1)$, the first-order term in the Taylor expansion of $\alpha_{i,\gamma}$ will give a second-order term in the Taylor expansion of $G_i^{(\mathrm{eq})}(\gamma)$. Therefore, we can restrict the expansion of $\alpha_{i,\gamma}$ (27) at the order zero: $\alpha_{i,\gamma} = \alpha_i + o(1)$, which directly gives (24). ■

APPENDIX B
PROOF OF THEOREM 3

We consider the aggregate welfare (9) as a function of G: $W(G) = W(x^{(\mathrm{resp})}(G))$. We have $\frac{\mathrm{d}W}{\mathrm{d}G}(G^*(\gamma)) = 0$. The result of Theorem 3 is then deduced from Proposition 3 using a taylor expansion around $G^*(\gamma)$: for $i \in \{L, T\}$,

$$W(G_i^{(\mathrm{eq})}(\gamma)) = W(G^*(\gamma)) + O\left((G_i^{(\mathrm{eq})}(\gamma) - G^*(\gamma))^2\right).$$