

# Coding for Parallel Gaussian Bi-Directional Relay Channels: A Deterministic Approach

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**Abstract**—We study the design of good coding schemes and the achievable exchange rates for the parallel Gaussian bi-directional relay channel. We first consider the corresponding linear deterministic model and propose two different schemes that can achieve the exchange capacity for this linear deterministic model. The insights obtained from this are used to design coding schemes for the original parallel Gaussian bi-directional relay channel. The first coding scheme uses superposition-based coding at both nodes and reorders codewords that cannot be transmitted within their own sub-channels to the sub-channels that can support the transmission at the relay. The second coding scheme employs lattice partition chains proposed by Nam *et al.* in the multiple access phase and then performs coding across sub-channels in the broadcast phase. While both schemes are optimal for the linear deterministic model, the performance of their Gaussian counterparts are different in general and which one performs better depends on the operating SNR and channel coefficients. Numerical results show that both schemes substantially outperform the decode-and-forward scheme and also provide non-trivial gains over the scheme proposed by Huang *et al.* Moreover, it is shown that the performance of both schemes is close to that of the cut-set bound and that the second scheme is asymptotically optimal.

**Index Terms**—Bi-directional relay channels, two way relay channels, inter-symbol interference, and linear deterministic models.

## I. INTRODUCTION

We study the parallel Gaussian bi-directional relay channel shown in Fig. 1 where two nodes  $A$  and  $B$  wish to exchange information through a relay node  $R$  between them via a set of  $L$  parallel bi-directional relay channels. The channel coefficients are assumed to be fixed in each sub-channel but may vary from one sub-channel to another. Many communication channels can be transformed into parallel Gaussian channels and, therefore, the parallel channel considered here represents a canonical model to study such communication channels. For example, inter-symbol interference (ISI) channels and multiple input multiple output (MIMO) channels can be converted into parallel Gaussian channels via multi-carrier systems such as orthogonal frequency division multiplex and via matrix decompositions, respectively.

The problem of communication over the single bi-directional relay channel has been intensively studied and classical information forwarding strategies such as amplify-and-forward and decode-and-forward can be very sub-optimal (see [1] and the reference therein.) Recently, it has been shown that a coding scheme based on the nested lattice codes at both

nodes and compute-and-forward at the relay performs optimally asymptotically for this bi-directional relaying without memory [2]-[4]. The main idea comes from the observation that the relay does not need to decode individual messages; instead, in compute-and-forward the relay is only required to decode to a function of individual messages such that both end nodes are able to figure out the other's message from this function and their own messages as side information. One example of a computation function is the modulo-sum of two lattice points, which is adopted by [2].

Motivated by the success of nested lattice codes and compute-and-forward, in [5], the authors proposed a pre-filtering scheme for the bi-directional relay channel with ISI (a special case of a parallel Gaussian bi-directional relay network) and provided an example showing that the Gaussian bi-directional relay channel is inseparable, i.e., to achieve its capacity, joint processing across sub-channels are required in general. The idea of this coding scheme is to align two ISI channels by performing pre-filtering at both nodes so that one can again use compute-and-forward at the relay. However, the optimal filters design problem seems to be non-convex and very difficult to solve. Coding schemes for the MIMO bi-directional relaying channel have also been considered in [11] and [12].

Different from [5], in this paper, we will start by investigating the linear deterministic model proposed by Avestimehr *et al.* [6] of the considered setup so that we can ignore the background noise and focus on the interaction between signals from different nodes. This approach has been very successful in capturing the features that a good coding scheme should possess for many channels including the bi-directional relay channel without memory [7]. Moreover, it usually leads to a coding scheme that can approach the capacity within constant bits.

For the considered setup, we will study the corresponding linear deterministic model and provide two coding schemes that can achieve the cut-set bound. After that, for each proposed scheme, we will propose a coding scheme for the Gaussian setup based on the insight obtained from studying the linear deterministic model. Interestingly, we will show that although both schemes achieve the exchange capacity of the linear deterministic model, their Gaussian counterparts perform differently. Numerical results show that both proposed schemes can approach the cut-set upper bound well and out-

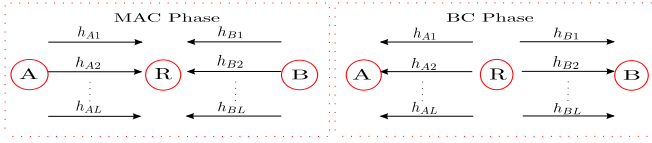


Fig. 1. The parallel Gaussian bi-directional relay channel.

perform the pre-filtering scheme proposed in [5]. This provides a way to circumvent the optimal filters design problem.

## II. CHANNEL MODEL

Consider a set of  $L$  parallel Gaussian bi-directional relay channels where each sub-channel  $l \in \{1, 2, \dots, L\}$  can be modeled as a bi-directional relay channel with sub-nodes  $A_l$ ,  $B_l$ , and  $R_l$  as shown in Fig. 1. In addition, joint processing such as coding across sub-channels is permitted. Also note that each node is assumed to have global channel knowledge and channel reciprocity is assumed, i.e., the channel coefficients in the uplink are equal to that in the downlink and are, respectively, denoted as  $h_{Al} \in \mathbb{C}$  and  $h_{Bl} \in \mathbb{C}$  for all  $l \in \{1, 2, \dots, L\}$ . Nodes  $A$  and  $B$  first split their messages  $w_A, w_B \in \{1, 2, \dots, M\}$  into  $L$  parts, namely  $w_{Al}$  and  $w_{Bl}$  for  $l \in \{1, 2, \dots, L\}$ , respectively. For the sub-channel  $l$ ,  $A_l$  and  $B_l$  map  $w_{Al}$  and  $w_{Bl}$ , independent of each other and of all other sub-channels' messages, to length- $n$  codewords  $\mathbf{x}_{Al}$  and  $\mathbf{x}_{Bl}$ , respectively. Each node is subject to an individual power constraint  $P$ . The transmission protocol we consider is a two phase protocol consisting of a multiple access channel (MAC) phase and a broadcast channel (BC) phase. Each of phases occupies a half of channel uses and is assumed to be orthogonal to each other.

During the MAC phase, both nodes transmit their signals to the relay simultaneously and the relay keeps silent. The received signal at  $R_l$  is then given by

$$\mathbf{y}_{Rl} = \mathbf{x}_{Al}h_{Al} + \mathbf{x}_{Bl}h_{Bl} + \mathbf{z}_{Rl}, \quad (1)$$

$\mathbf{z}_{Rl} \sim \mathcal{CN}(0, \mathbf{I})$  be i.i.d. Gaussian noise. Each  $R_l$  generates the transmitted signal  $\mathbf{x}_{Rl}$  in the BC phase according to all  $\mathbf{y}_{Rl}$ . This mapping depends on the forwarding strategy and will be discussed later in section IV.

During the BC phase, each  $R_l$  broadcasts  $\mathbf{x}_{Rl}$  back to  $A_l$  and  $B_l$  and both nodes keep silent. Then the received signal at end nodes are respectively given by

$$\mathbf{y}_{Al} = \mathbf{x}_{Rl}h_{Al} + \mathbf{z}_{Al}, \quad (2)$$

$$\mathbf{y}_{Bl} = \mathbf{x}_{Rl}h_{Bl} + \mathbf{z}_{Bl}, \quad (3)$$

where again  $\mathbf{z}_{Al}, \mathbf{z}_{Bl} \sim \mathcal{CN}(0, \mathbf{I})$ . After collecting all  $\mathbf{y}_{Al}$ 's and  $\mathbf{y}_{Bl}$ 's, nodes  $A$  and  $B$  form estimates of the collection of other's messages  $w_B$  and  $w_A$ , namely  $\hat{w}_B$  and  $\hat{w}_A$ , respectively. The error probability is given by

$$P_e = \Pr(\{w_A \neq \hat{w}_A\} \cup \{w_B \neq \hat{w}_B\}). \quad (4)$$

Moreover, an exchange rate  $R_{ex}$  is said to be achievable if, for any  $\varepsilon > 0$ , there is an  $(n, M)$  code such that

$$M \geq 2^{nR_{ex}}, \quad P_e \leq \varepsilon. \quad (5)$$

The exchange capacity is then defined as the supremum of exchange rates. Therefore, the exchange capacity can be interpreted as the maximum possible rate for both nodes to exchange their information reliably with the same rate.

The cut-set upper bound for the exchange capacity is given by [8]

$$\overline{C}_{ex} = \min(C_{AB}, C_{BA}), \quad (6)$$

where  $C_{ij} = \min(C_{iR}, C_{Rj})$ ,  $i, j \in \{A, B\}$ ,  $i \neq j$ , and

$$C_{iR} = \sum_{l=1}^L \frac{1}{2} \log(1 + |h_{il}|^2 P), \quad (7)$$

$$C_{Rj} = \sum_{l=1}^L \frac{1}{2} \log(1 + |h_{jl}|^2 P), \quad (8)$$

where the  $\frac{1}{2}$  is due to the fact that each phase occupies a half of channel uses. Our goal in this paper is to study the parallel Gaussian bi-directional relay channel from a deterministic view and try to build coding and forwarding schemes to approach the cut-set upper bound for the Gaussian channel based on the insight obtained from the deterministic model.

## III. DETERMINISTIC PARALLEL BI-DIRECTIONAL RELAY CHANNEL

### A. Linear Deterministic Model

The linear deterministic model of this setup is described as a collection of  $L$  parallel deterministic bi-directional relay channels where at the  $l^{th}$  sub-channel, there are  $n_l$  links connected from  $A_l$  to  $R_l$  and  $m_l$  links from  $B_l$  to  $R_l$  during the MAC phase. Also, since we consider the case when channels are reciprocal; therefore in the BC phase, the number of links connected remains the same as those in the MAC phase. Further, joint processing across sub-channels is permitted.

In the MAC phase, at the  $l^{th}$  sub-channel,  $A_l$  and  $B_l$  send  $\mathbf{x}_{Al} \in \mathbb{F}_2^q$  and  $\mathbf{x}_{Bl} \in \mathbb{F}_2^q$ , respectively. Here  $q = \max(n_1, n_2, \dots, n_L, m_1, m_2, \dots, m_L)$ . The received signal at  $R_l$  is given by

$$\mathbf{y}_{Rl} = \mathbf{S}^{q-n_l} \mathbf{x}_{Al} \oplus \mathbf{S}^{q-m_l} \mathbf{x}_{Bl}. \quad (9)$$

where  $\oplus$  is the XOR operation and

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \quad (10)$$

is a  $q \times q$  downshift matrix. Let  $\mathbf{y}_R = [\mathbf{y}_{R1}^t, \mathbf{y}_{R2}^t, \dots, \mathbf{y}_{RL}^t]^t$  be the collection of received signals.

### B. Proposed Processing at the Relay

The relay first maps the received signal  $\mathbf{y}_R$  to the transmitted signal in the BC phase at the  $R_l$  as

$$\mathbf{x}_{Rl} = \mathbf{G}_l \mathbf{y}_R, \quad (11)$$

where  $\mathbf{G}_l$  represents the proposed linear information forwarding strategy for generating transmitted signals at  $R_l$ . Note

that since we allow joint processing across sub-channels,  $\mathbf{x}_{Rl}$  depends on the whole  $\mathbf{y}_R$  instead of just  $\mathbf{y}_{Rl}$ .

In the BC phase, at the  $l^{th}$  sub-channel,  $R_l$  broadcasts  $\mathbf{x}_{Rl}$  to both nodes and the received signals are given by

$$\mathbf{y}_{Al} = \mathbf{S}^{q-n_l} \mathbf{x}_{Rl}, \quad (12)$$

$$\mathbf{y}_{Bl} = \mathbf{S}^{q-m_l} \mathbf{x}_{Rl}. \quad (13)$$

### C. Decoding at the Nodes

Each node first cancels their own messages and then tries to decode the other's messages. The equivalent signal model after canceling their own messages is given by

$$\begin{aligned} \tilde{\mathbf{y}}_{Al} &= \mathbf{S}^{q-n_l} \mathbf{G}_l \underbrace{\begin{pmatrix} \mathbf{S}^{q-m_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{q-m_2} & \cdots & \mathbf{0} \\ \vdots & \cdots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{S}^{q-m_L} \end{pmatrix}}_{\mathbf{S}_B} \underbrace{\begin{pmatrix} \mathbf{x}_{B1} \\ \mathbf{x}_{B2} \\ \vdots \\ \mathbf{x}_{BL} \end{pmatrix}}_{\mathbf{x}_B} \\ &= \underbrace{\mathbf{S}^{q-n_l} \mathbf{G}_l \mathbf{S}_B}_{\mathbf{G}_{Al}} \mathbf{x}_B = \mathbf{G}_{Al} \mathbf{x}_B, \end{aligned} \quad (14)$$

and

$$\begin{aligned} \tilde{\mathbf{y}}_{Bl} &= \mathbf{S}^{q-m_l} \mathbf{G}_l \underbrace{\begin{pmatrix} \mathbf{S}^{q-n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{q-n_2} & \cdots & \mathbf{0} \\ \vdots & \cdots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{S}^{q-n_L} \end{pmatrix}}_{\mathbf{S}_A} \underbrace{\begin{pmatrix} \mathbf{x}_{A1} \\ \mathbf{x}_{A2} \\ \vdots \\ \mathbf{x}_{AL} \end{pmatrix}}_{\mathbf{x}_A} \\ &= \underbrace{\mathbf{S}^{q-m_l} \mathbf{G}_l \mathbf{S}_A}_{\mathbf{G}_{Bl}} \mathbf{x}_A = \mathbf{G}_{Bl} \mathbf{x}_A. \end{aligned} \quad (15)$$

We can further write the collection of received signals at all sub-channels as

$$\tilde{\mathbf{y}}_A = \underbrace{\mathbf{S}_A \mathbf{G} \mathbf{S}_B}_{\mathbf{G}_A} \mathbf{x}_B = \mathbf{G}_A \mathbf{x}_B, \quad (16)$$

$$\tilde{\mathbf{y}}_B = \underbrace{\mathbf{S}_B \mathbf{G} \mathbf{S}_A}_{\mathbf{G}_B} \mathbf{x}_A = \mathbf{G}_B \mathbf{x}_A, \quad (17)$$

where  $\mathbf{G} = [\mathbf{G}_1^t \mathbf{G}_2^t \dots \mathbf{G}_L^t]^t$ .

**Main Result:** The exchange capacity of the linear deterministic parallel bi-directional relay channel with reciprocity assumption is

$$C_{ex}^d = \min \left( \sum_{l=1}^L n_l, \sum_{l=1}^L m_l \right) \text{ per two channel uses} \quad (18)$$

(one for each phase). Moreover, only a very simple joint processing across sub-channels at the relay is required to achieve the exchange capacity.

From the cut-set upper bound, we have  $C_{ex}^d \leq \min \left( \sum_{l=1}^L n_l, \sum_{l=1}^L m_l \right)$ . Thus, it remains to show that there is at least one scheme can achieve this upper bound. In Section III-D and Section III-E, we propose two such schemes. Further, notice that we only have to consider the case when  $\sum_{l=1}^L n_l = \sum_{l=1}^L m_l$  since otherwise, we can always discard some links such that these two values are equal and this will not change the cut-set bound.

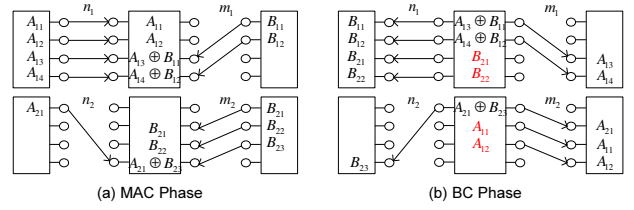


Fig. 2. An example of a linear deterministic model of the considered setup.

### D. Proposed Scheme 1

We first consider a small example to demonstrate the main idea behind the first scheme.

**Example 1:** Let us consider the example with  $L = 2$  given in Fig. 2 whose cut-set bound is 5 bits per two channel uses (one for each phase). We provide a coding and forwarding scheme that can achieve the cut-set bound. In the MAC phase,  $A_l$  ( $B_l$ ) first sends  $n_l$  ( $m_l$ ) bits to  $R_l$ . The received signals at the relay are as shown in the figure. At each sub-channel, the relay first shifts the aligned (XOR) part of received signals to its top. After this, the relay still needs to broadcast  $A_{11}, A_{12}$  to the node  $B$  and  $B_{21}, B_{22}$  to the node  $A$ . It then *reorders* those remaining bits across sub-channels such that it will broadcast  $A_{11}, A_{12}$  at the second sub-channel and broadcast  $B_{21}, B_{22}$  at the first sub-channel. Each node then cancels out their own messages for the aligned (XOR) part and recovers 5 bits from the other node.

The above example suggests that a simple routing scheme may be able to achieve the cut-set upper bound if one routes the bits carefully. In what follows, we summarize the first proposed scheme and then show that this scheme indeed can achieve the cut-set upper bound:

**step 1:** In the MAC phase,  $A_l$  ( $B_l$ ) sends  $n_l$  ( $m_l$ ) data stream to the relay.

**step 2:** Each  $R_l$  shifts the aligned part to the top as in [7] because it contains information intended for both directions.

**step 3:** For the non-aligned part, the relay *reorders* the signals to other sub-channels that can support the transmission to the desired destination.

**Theorem 1:** The scheme described above achieves the cut-set upper bound.

**Proof: case 1:**(no interference) We first consider channel realizations where no interference has occurred at the relay for all sub-channels. i.e., for all  $l$ , the number of vertices connected to  $R_l$  from  $(A_l, B_l)$  is either  $(n_l, 0)$  or  $(0, m_l)$ . For this case, we can divide the transmission into two disjoint unicast sessions, one for each direction. It is well-known that for unicast sessions, routing achieves the cut-set bound. Moreover, for no interference cases, our proposed forwarding scheme is nothing but routing. Therefore, this scheme achieves the cut-set bound for this case.

**case 2:**(general case) Now, we consider general cases that we may have interference. In other words, at  $R_l$ , we will have  $\min\{n_l, m_l\}$  signals belonging to the aligned part. Also, we will have  $(n_l - m_l)^+$  extra bits only intended for the node  $B$  and  $(m_l - n_l)^+$  extra bits only intended for the node  $A$ , where

$(\cdot)^+$  stands for  $\max\{\cdot, 0\}$ . Our proposed forwarding strategy will first shift this aligned part to the top. Due to channel reciprocity, this aligned part can always be broadcasted to both node  $A$  and  $B$ . Both nodes can then cancel their own messages and obtain  $\min\{n_l, m_l\}$  bits.

We then remove those bidirectional links that has already been used for the aligned part. It should be noted that

$$\begin{aligned} \sum_{l=1}^L (n_l - m_l)^+ + \min\{m_l, n_l\} &= \sum_{l=1}^L n_l \\ &\stackrel{(a)}{=} \sum_{l=1}^L m_l = \sum_{l=1}^L (m_l - n_l)^+ + \min\{m_l, n_l\}, \end{aligned} \quad (19)$$

where (a) is due from the assumption. This implies that the total number of bits belonging to the non-aligned part intended for the node  $A$  is equal to that intended for the node  $B$ , i.e.,  $\sum_{l=1}^L (n_l - m_l)^+ = \sum_{l=1}^L (m_l - n_l)^+$ . Therefore, after removing those bidirectional links, the resulting network lies in the case 1 in which we have shown that our scheme achieves the cut-set bound. Thus, we have shown that the rate achieved by this scheme for node  $A$  and  $B$  are

$$R_A^d = \sum_{l=1}^L (n_l - m_l)^+ + \min\{m_l, n_l\} = \sum_{l=1}^L n_l, \quad (20)$$

$$R_B^d = \sum_{l=1}^L (m_l - n_l)^+ + \min\{m_l, n_l\} = \sum_{l=1}^L m_l, \quad (21)$$

respectively. Thus, this coding scheme indeed achieves the exchange capacity given in (18). ■

#### E. Proposed Scheme 2

Although we have already provided a scheme that can achieve the cut-set bound, it is not trivial to extend this scheme to the case when channels may not be reciprocal. In fact, in Section VI, we provide an example showing that, without channel reciprocity, the first proposed scheme is not optimal in general. Further, despite the fact that the first coding scheme is very simple and is optimal for the liner deterministic model, there is no guarantee that its Gaussian interpretation is also optimal. These motivate us to look for another scheme which will also work for non-reciprocal channels and whose Gaussian interpretation will perform better than the first one for some cases.

We first notice that in the previous scheme, we only use the channel once. However, in Gaussian channels, we are usually allowed to use the channel  $N$  times and  $N \rightarrow \infty$  is required for approaching the capacity even for very simple networks. Therefore, we consider using the linear deterministic channel  $N$  times and propose a coding scheme that can approach the exchange capacity almost surely as  $N \rightarrow \infty$ .

The corresponding signal model can still be described by (16) and (17) except that now  $\mathbf{x}_{Al}, \mathbf{x}_{Bl} \in \mathbb{F}_{2^N}^q$  and all operations are over the extension field  $\mathbb{F}_{2^N}$ . The scheme we propose here simply choose elements in  $\mathbf{G}$  (where  $\mathbf{x}_R = \mathbf{G}\mathbf{y}_R$ ) randomly from  $\mathbb{F}_{2^N}$  with all elements equiprobable.

*Theorem 2:* The second proposed scheme achieves the cut-set upper bound almost surely as  $N \rightarrow \infty$ .

*Proof:* Taking a closer look at the  $\mathbf{G}_A$  in (16), we observe that

$$\mathbf{G}_A = \begin{pmatrix} \mathbf{G}_{11}^A & \cdots & \mathbf{G}_{1L}^A \\ \vdots & \ddots & \vdots \\ \mathbf{G}_{L1}^A & \cdots & \mathbf{G}_{LL}^A \end{pmatrix}, \quad (22)$$

where

$$\mathbf{G}_{ij}^A = \begin{pmatrix} \mathbf{0}_{(q-n_i) \times q} \\ \mathbf{G}_{ij}^{A'} \mathbf{0}_{n_i \times (q-m_j)} \end{pmatrix}, \quad (23)$$

and  $\mathbf{G}_{ij}^{A'}$  is a  $n_i \times m_j$  matrix. Similarly, we have

$$\mathbf{G}_B = \begin{pmatrix} \mathbf{G}_{11}^B & \cdots & \mathbf{G}_{1L}^B \\ \vdots & \ddots & \vdots \\ \mathbf{G}_{L1}^B & \cdots & \mathbf{G}_{LL}^B \end{pmatrix}, \quad (24)$$

where

$$\mathbf{G}_{ij}^B = \begin{pmatrix} \mathbf{0}_{(q-m_i) \times q} \\ \mathbf{G}_{ij}^{B'} \mathbf{0}_{m_i \times (q-n_j)} \end{pmatrix}, \quad (25)$$

and  $\mathbf{G}_{ij}^{B'}$  is a  $m_i \times n_j$  matrix. Let us exclude all-zero rows and all-zero columns in  $\mathbf{G}_A$  and  $\mathbf{G}_B$  to form  $\sum n_l \times \sum n_l$  (since  $\sum n_l = \sum m_l$ ) sub-matrices  $\mathbf{D}_A$  and  $\mathbf{D}_B$ , respectively. Clearly, the rank of  $\mathbf{G}_A$  and  $\mathbf{G}_B$  are equal to the rank of their corresponding sub-matrices, respectively. Also, since  $\mathbf{G}$  is chosen randomly, these sub-matrices are also random. Define  $E_A$  and  $E_B$  as the events that  $\mathbf{D}_A$  and  $\mathbf{D}_B$  are full rank, respectively. We have [9]

$$Pr(E_A) = Pr(E_B) = \prod_{i=0}^{\sum n_l - 1} \left(1 - 2^{-N(\sum n_l - i)}\right), \quad (26)$$

which converges to 1 as  $N \rightarrow \infty$ . Moreover, the probability that both  $\mathbf{D}_A$  and  $\mathbf{D}_B$  are full rank simultaneously is

$$\begin{aligned} Pr((E_A \cap E_B)) &= 1 - Pr((E_A \cap E_B)^c) \\ &= 1 - Pr(E_A^c \cup E_B^c) \\ &\stackrel{(a)}{\geq} 1 - (Pr(E_A^c) + Pr(E_B^c)) \\ &\stackrel{(b)}{\rightarrow} 1, \end{aligned} \quad (27)$$

where (a) follows from the union bound and (b) is due from (26). This result tells us that if we randomly choose  $\mathbf{G}$  at the relay, both the resulting  $\mathbf{G}_A$  and  $\mathbf{G}_B$  will simultaneously have rank  $\sum n_l$  as the number of channel uses tends to infinity. Therefore, this scheme achieves the cut-set bound almost surely as  $N \rightarrow \infty$ . ■

*Remark 1:* Note that in the above proof, we do not use the assumption that channels are reciprocal. Therefore, this coding scheme also works for channels without reciprocity.

#### IV. PROPOSED SCHEMES FOR THE GAUSSIAN SETUP

In this section, we go back to study the parallel Gaussian bi-directional relay channel and propose two coding schemes based on the insight obtained from the linear deterministic model discussed previously.

### A. Proposed Scheme 1 - Reordering

The first coding scheme in Section III suggests the following. For the case that one link is stronger than the other, the node having a stronger link should send two codewords and one of these should be perfectly aligned with the signal sent from the node with a weaker link. On the other hand, if both links have the same channel gain, both nodes should only send one codeword since the signals will be automatically aligned. The relay then *reorders* the non-aligned codeword to another sub-channel that can support transmitting this codeword to the desired end node.

Based on the above insight, a superposition-based coding scheme is proposed where at each sub-channel, the node having a stronger link transmits the superposition of a lattice codeword and an extra random codeword and the other node sends only a lattice codeword. Moreover, two lattice codewords are chosen from an identical nested lattice code and to be perfectly aligned at the relay. In the BC phase, this scheme then reorders the extra codeword to the sub-channel that can support the transmission to the desired destination.

In general, the transmitted signals of the first proposed scheme at the  $m^{th}$  sub-channel from node  $i \in \{A, B, R\}$  is given by

$$\mathbf{x}_{im} = \sqrt{\alpha_{im}}\mathbf{x}_{im}^{(1)} + \sqrt{1 - \alpha_{im}}\mathbf{x}_{im}^{(2)}, \quad (28)$$

where  $\mathbf{x}_{im}^{(1)}$  is codeword chosen from nested lattice codes with codebook size  $2^{nR_{im}^{(1)}}$  and  $\mathbf{x}_{im}^{(2)}$  is codeword chosen from a random codebook with size  $2^{nR_{im}^{(2)}}$ .

In the MAC phase, at the  $l^{th}$  sub-channel, three possible cases are listed as follows,

case 1 ( $|h_{Al}| = |h_{Bl}|$ ): Node  $A$  and  $B$  only send  $\mathbf{x}_{Al}^{(1)}$  and  $\mathbf{x}_{Bl}^{(1)}$ , respectively, chosen from an identical nested lattice code and set  $\alpha_{Al} = 1$ ,  $\alpha_{Bl} = 1$ ,  $R_{Al}^{(2)} = 0$  and  $R_{Bl}^{(2)} = 0$ . The relay tries to decode to the modulo-sum of two lattice points  $\mathbf{f}_l$  (see [2] or [3] for the details). From the result in [2], any rate satisfies

$$R_{Al}^{(1)} = R_{Bl}^{(1)} \leq \frac{1}{2} \log \left( \frac{1}{2} + P|h_{Al}|^2 \right), \quad (29)$$

is achievable. Note that, in all rate expressions in this section, there is a half in front of log as each phase occupies a half of channel uses.

case 2 ( $|h_{Al}| > |h_{Bl}|$ ): For this case, node  $A$  can have one extra codeword sent to the relay in the MAC phase. We therefore set  $\alpha_{Bl} = 1$  and  $R_{Bl}^{(2)} = 0$ . The  $\alpha_{Al}$  is then chosen to make sure that two lattices are perfectly aligned at the relay as

$$\alpha_{Al} = \frac{|h_{Bl}|^2}{|h_{Al}|^2}. \quad (30)$$

The relay first decodes the extra codeword  $\mathbf{x}_{Al}^{(2)}$  by treating the aligned part as noise. This gives us

$$R_{Al}^{(2)} \leq \frac{1}{2} \log \left( 1 + \frac{P|h_{Al}|^2(1 - \alpha_{Al})}{1 + 2P|h_{Bl}|^2} \right). \quad (31)$$

It then subtracts the decoded codeword and tries to decode the lattice function  $\mathbf{f}_l$ . This results in

$$R_{Al}^{(1)} = R_{Bl}^{(1)} \leq \frac{1}{2} \log \left( \frac{1}{2} + P|h_{Bl}|^2 \right). \quad (32)$$

case 3 ( $|h_{Al}| < |h_{Bl}|$ ): Switch the role of the node  $A$  and the node  $B$  in the previous case. We have  $\alpha_{Al} = 1$ ,  $R_{Al}^{(2)} = 0$ ,

$$\alpha_{Bl} = \frac{|h_{Al}|^2}{|h_{Bl}|^2}, \quad (33)$$

$$R_{Bl}^{(2)} \leq \frac{1}{2} \log \left( 1 + \frac{P|h_{Bl}|^2(1 - \alpha_{Bl})}{1 + 2P|h_{Al}|^2} \right), \quad (34)$$

and

$$R_{Al}^{(1)} = R_{Bl}^{(1)} \leq \frac{1}{2} \log \left( \frac{1}{2} + P|h_{Al}|^2 \right). \quad (35)$$

We remark that the order of decoding matters and since the lattice function contains information for both direction, we always decode it last. After correctly decoding signals at all sub-channels  $l \in \{1, 2, \dots, L\}$  in the MAC phase, the relay reorders the extra codewords as mentioned. Suppose now the relay reorders the extra codeword at the  $k^{th}$  sub-channel (if there is one) to the  $l^{th}$  sub-channel, the achievable rate can be computed as follows.

case 1 ( $R_{Ak}^{(2)} > 0$ ,  $|h_{Al}| < |h_{Bl}|$ ): Since  $R_{Ak}^{(2)} > 0$ , this extra codeword is intended for the node  $B$ . Fortunately, now the link to the node  $B$  is stronger than that to the node  $A$ ; therefore, we may be able to send this extra codeword to the desired destination. The relay maps its own  $\mathbf{f}_l$  to the codeword  $\mathbf{x}_{Rl}^{(1)}$  and the reordered  $\mathbf{x}_{Ak}^{(2)}$  to  $\mathbf{x}_{Rl}^{(2)}$  and then broadcasts the superposition. At the node  $A$ , since it knows its own message  $\mathbf{x}_{Ak}^{(2)}$ , it can first cancel out  $\mathbf{x}_{Rl}^{(2)}$  and then decodes  $\mathbf{f}_l$ . The achievable rate is given by

$$R_{Rl}^{(1)} \leq \frac{1}{2} \log (1 + P|h_{Al}|^2 \alpha_{Rl}). \quad (36)$$

At the node  $B$ , it first decodes the  $\mathbf{f}_l$ , the rate constraint is given by

$$R_{Rl}^{(1)} \leq \frac{1}{2} \log \left( 1 + \frac{P|h_{Bl}|^2 \alpha_{Rl}}{1 + P|h_{Bl}|^2(1 - \alpha_{Rl})} \right). \quad (37)$$

Thus,  $R_{Rl}^{(1)} \leq \min\{(36), (37)\}$ . It then cancels the decoded signals and tries to decode  $\mathbf{x}_{Ak}^{(2)}$ . Therefore, we have

$$R_{Rl}^{(2)} \leq \frac{1}{2} \log (1 + P|h_{Bl}|^2(1 - \alpha_{Rl})). \quad (38)$$

where  $\alpha_{Rl}$  is obtained by setting  $R_{Ak}^{(2)} = R_{Rl}^{(2)}$  at the relay and is given by

$$\alpha_{Rl} = 1 - \frac{|h_{Ak}|^2(1 - \alpha_{Ak})}{|h_{Bl}|^2(1 + 2P|h_{Bk}|^2)}. \quad (39)$$

case 2 ( $R_{Bk}^{(2)} > 0$ ,  $|h_{Al}| > |h_{Bl}|$ ): Switch the role of the node  $A$  and  $B$  in the previous case. We obtain

$$R_{Rl}^{(1)} \leq \frac{1}{2} \log (1 + P|h_{Bl}|^2 \alpha_{Rl}), \quad (40)$$

$$R_{Rl}^{(1)} \leq \frac{1}{2} \log \left( 1 + \frac{P|h_{Al}|^2 \alpha_{Rl}}{1 + P|h_{Al}|^2(1 - \alpha_{Rl})} \right). \quad (41)$$

Thus,  $R_{Rl}^{(1)} \leq \min\{(40), (41)\}$ . Also,

$$R_{Rl}^{(2)} \leq \frac{1}{2} \log (1 + P|h_{Al}|^2(1 - \alpha_{Rl})), \quad (42)$$

where

$$\alpha_{Rl} = 1 - \frac{|h_{Bk}|^2(1 - \alpha_{Bk})}{|h_{Al}|^2(1 + 2P|h_{Ak}|^2)}. \quad (43)$$

*case 3* (otherwise): The relay is not able to transmit the extra codeword to the desired destination; therefore, the relay simply sets  $\alpha_{Rl} = 1$  and  $R_{Rl}^{(2)} = 0$  and only broadcasts its own lattice function. This leads to

$$R_{Rl}^{(1)} \leq \frac{1}{2} \log (1 + P \cdot \min(|h_{Al}|^2, |h_{Bl}|^2)). \quad (44)$$

Now the problem becomes how to assign  $L$  extra codewords (if no extra codeword at this sub-channel, simply set it to be 0) to  $L$  sub-channels efficiently. Since this scheme does not allow multiple codewords to be assigned to the same sub-channel, this problem reduces to the weighted bipartite matching problem and can be solved in polynomial time by the *Hungarian method* [10]. We now define the weight (cost)  $c_{lk}$  of assigning the extra codeword at the  $k^{th}$  sub-channel to the  $l^{th}$  sub-channel as

$$c_{lk} = \min\{R_{Al}^{(1)}, R_{Rl}^{(1)}\} + \min\{R_{Ak}^{(2)}, R_{Rl}^{(2)}\} + \min\{R_{Bk}^{(2)}, R_{Rl}^{(2)}\}. \quad (45)$$

Note that for each channel realization, at most one of  $R_{Ak}^{(2)} > 0$  and  $R_{Bk}^{(2)} > 0$  can be true for this scheme. Therefore, at least one of the second term and the third term in (45) is 0. The optimal assignment strategy for the reordering is the one such that the weight is maximized. Let us denote the optimal reordering strategy obtained by the Hungarian method as  $H$  where the extra codeword in the  $H(l)^{th}$  sub-channel is assigned to the  $l^{th}$  sub-channel.

The achievable rate from the node  $A$  to the node  $B$  and that from the node  $B$  to the node  $A$  are given by, respectively,

$$R_{AB} = \sum_{l=1}^L \min\{R_{Al}^{(1)}, R_{Rl}^{(1)}\} + \min\{R_{AH(l)}^{(2)}, R_{Rl}^{(2)}\}, \quad (46)$$

and

$$R_{BA} = \sum_{l=1}^L \min\{R_{Bl}^{(1)}, R_{Rl}^{(1)}\} + \min\{R_{BH(l)}^{(2)}, R_{Rl}^{(2)}\}. \quad (47)$$

The achievable exchange rate is then given by

$$R_{ex,1} = \min\{R_{AB}, R_{BA}\}. \quad (48)$$

### B. Proposed Scheme 2 - Coding Across Sub-channels

We now proposed another coding scheme based on the insight obtained from the second proposed scheme in Section III. Intuitively, one can think of randomly choosing  $\mathbf{G}$  as performing random coding across sub-channels since it completely messes up the structure inherited in the deterministic model.

The second proposed scheme uses a sequence of lattice partition chains proposed by Nam *et al.* [4] in the MAC phase

since it achieve the largest computation rate currently known. Specifically, at  $l^{th}$  sub-channel, let  $\Lambda_{1l}^n \subseteq \Lambda_{2l}^n \subseteq \Lambda_{Cl}^n$ , and  $\Lambda_{1l}^n$  and  $\Lambda_{2l}^n$  have second moments  $P \cdot \max\{|h_{Al}|^2, |h_{Bl}|^2\}$  and  $P \cdot \min\{|h_{Al}|^2, |h_{Bl}|^2\}$ . Let the node with a stronger link uses the nested lattice code  $\Lambda_{Cl}^n/\Lambda_{1l}^n$  and the one with a weaker link uses  $\Lambda_{Cl}^n/\Lambda_{2l}^n$  as channel coding. Note that, for  $|h_{Al}| = |h_{Bl}|$ , it makes no difference which nested lattice code is used by whom since in this case,  $\Lambda_{1l}^n = \Lambda_{2l}^n$ . The transmitted signals are further scaled for satisfying the power constraint.

At  $R_l$ , it tries to decode to the modulo-sum ( $\bmod \Lambda_{1l}$ ) of a quantization index of the signal from the weaker node and two messages. From the result in [4], for the node  $i \in \{A, B\}$  at the  $l^{th}$  sub-channel, any rate satisfies the following is achievable

$$R_{il} \leq \frac{1}{2} \log \left( \frac{|h_{il}|^2}{|h_{il}|^2 + |h_{jl}|^2} + P|h_{il}|^2 \right), \quad (49)$$

where  $j \in \{A, B\}$  and  $i \neq j$ . Denote the total rate achieved in the MAC phase from the node  $i$  to the relay as  $R_{iR} = \sum_{l=1}^L R_{il}$ .

According to the insight obtained from the deterministic model, the relay collects all the computed functions and broadcasts them by a capacity-achieving random code performing coding across sub-channels. This leads to an achievable rate from the relay to the node  $i \in \{A, B\}$

$$R_{Ri} \leq \sum_{l=1}^L \frac{1}{2} \log (1 + P|h_{il}|^2). \quad (50)$$

Both nodes then identify the other's message from the decoded functions and their own message as side information (The interested reader is referred to [4] for details.) The achievable exchange rate of this coding scheme is then given by

$$R_{ex,2} = \min\{R_{AR}, R_{BR}, R_{RA}, R_{RB}\}. \quad (51)$$

*Remark 2:* Different from the joint matrix decomposition scheme proposed by Khina *et al.* [11] which only works for the case when all channel links are non-zero (i.e., two MIMO channels are full-rank), both two schemes proposed in this paper work for all channel gains. Moreover, when assuming full-rank, one can easily see from (49) that if we make the same assumption  $\det H_i H_i^* = 1$  as that was made in [11], where  $H_i = \text{diag}\{h_{i1}, h_{i2}, \dots, h_{iL}\}$  and  $i \in \{A, B\}$ , we have

$$\begin{aligned} R_{iR} &= \sum_{l=1}^L \frac{1}{2} \log \left( \frac{|h_{il}|^2}{|h_{il}|^2 + |h_{jl}|^2} + P|h_{il}|^2 \right) \\ &\geq \frac{1}{2} \log \left( \prod_{l=1}^L P|h_{il}|^2 \right) = \frac{L}{2} \log \left( \frac{LP}{L} \right). \end{aligned} \quad (52)$$

The RHS of (52) is the rate achieved in the MAC phase by the scheme in [11] (with  $L$  receive antenna and total power  $LP$ ) and has been shown asymptotically optimal. This result implies that the second proposed scheme is asymptotically optimal and is at least as good as the one in [11] when the corresponding  $H_A$  and  $H_B$  are full-rank.

*Remark 3:* The second proposed scheme turns out to be identical with the one proposed independently in [12]. The motivation and focus are quite different in that we are motivated by and largely focus on the linear deterministic model; however, the authors in [12] mainly focus on how to create parallel bi-directional relay channels from MIMO bi-directional relay channels.

*Remark 4:* Compared to the scheme proposed in this subsection, the first proposed scheme is conceptually simpler and more structured in the sense that no coding across sub-channels is required. However, the performance in the BC phase of the first coding scheme can be improved by replacing the superposition-based transmission at the relay by the joint encoding scheme adopted in the BC phase of the second proposed scheme.

## V. NUMERICAL RESULT

Two numerical examples are presented in this section. In these examples, we compare the average (over all sub-channels) achievable exchange rate as the power increases for two proposed schemes and the pre-filtering scheme proposed in [5]. Also, we plot the cut-set bound as an upper bound and the decode-and-forward scheme as a baseline scheme. In both Fig. 3 and Fig. 4, the number of sub-channels is  $L = 512$  and the channel coefficients are described in the inverse discrete Fourier transform (DFT) domain  $\tilde{\mathbf{h}}_A$  and  $\tilde{\mathbf{h}}_B$  for the sake of convenience. i.e.,  $[h_{A1}, h_{A2}, \dots, h_{AL}] = \text{DFT}\{\tilde{\mathbf{h}}_A\}$  and  $[h_{B1}, h_{B2}, \dots, h_{BL}] = \text{DFT}\{\tilde{\mathbf{h}}_B\}$ . The channel parameters are set to be  $\tilde{\mathbf{h}}_A = [1 \ 1]$  and  $\tilde{\mathbf{h}}_B = [1 \ -1]$  in Fig. 3 and  $\tilde{\mathbf{h}}_A = [1 \ 0.5]$  and  $\tilde{\mathbf{h}}_B = [1 \ 0.4 \ 0.3]$  in Fig. 4.

In both figures, we observe that both proposed schemes outperform the decode-and-forward scheme and approach the cut-set bound very well. Also, both figures show that the second proposed scheme employing coding across sub-channels provides a better asymptotic result than that provided by the first proposed scheme. This may be explained by the fact that in a superposition-based coding scheme, there always is a fraction of power which is regarded as an extra noise when the strong receiver tries to decode the cloud center. On the other hand, the lattice-based scheme has been shown providing asymptotically optimal performance for the bi-directional relay channel [2], [4]; therefore, when extended to the setup having a set of parallel bi-directional relay channel, one can expect a fairly good performance in high SNR regime. However, despite this, the reordering scheme still provides a performance whose gap from the cut-set bound is quite small. This implies that a very simple joint processing across sub-channel such as reordering can perform reasonably good.

In the low SNR regime, we observe in Fig. 3 that the first proposed scheme performs better than the second one. This may be due from the inefficiency in the low SNR regime inherited in lattice-based compute-and-forward schemes [2]-[4]. However, in Fig. 4, this rate degradation in the low SNR regime is relatively small compared to that in the Fig. 3. This is because the channel coefficients in Fig. 4 is less mismatched than that in Fig. 3. One can see this from the

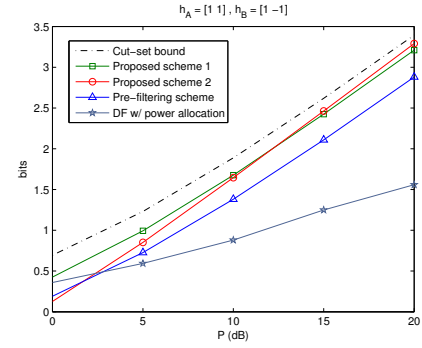


Fig. 3. Performance comparison when  $\tilde{\mathbf{h}}_A = [1 \ 1]$ ,  $\tilde{\mathbf{h}}_B = [1 \ -1]$ .

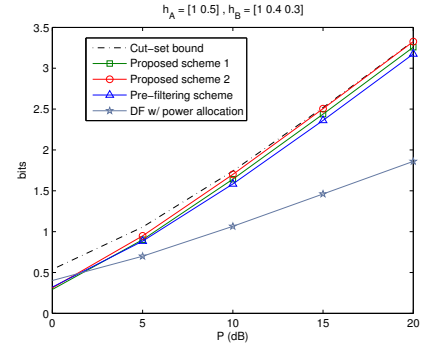


Fig. 4. Performance comparison when  $\tilde{\mathbf{h}}_A = [1 \ 0.5]$ ,  $\tilde{\mathbf{h}}_B = [1 \ 0.4 \ 0.3]$ .

example when channel coefficients are highly mismatched, say  $|h_{Al}|^2 \gg |h_{Bl}|^2$  for some  $l$ . Inside the log in (49), we have  $|h_{Al}|^2 / (|h_{Al}|^2 + |h_{Bl}|^2) \rightarrow 1$  and  $|h_{Bl}|^2 / (|h_{Al}|^2 + |h_{Bl}|^2) \rightarrow 0$ ; therefore, the exchange rate (minimum achievable rate between two nodes) is reduced.

## VI. DETERMINISTIC MODEL FOR NON-RECIPROCAL CHANNELS

Although the main focus of this paper is on reciprocal channels, we study a bit of nonreciprocal channels. We assume the uplink channels from node  $A$  and  $B$  to be  $n_l$  and  $m_l$ , respectively, and the downlink channels from node  $A$  and  $B$  to be  $r_l$  and  $k_l$ , respectively.

*Theorem 3:* The exchange capacity of the linear deterministic parallel bi-directional relay without reciprocity assumption is

$$C_{ex,nr}^d = \min \left( \sum_{l=1}^L n_l, \sum_{l=1}^L m_l, \sum_{l=1}^L r_l, \sum_{l=1}^L k_l \right). \quad (53)$$

Moreover, this exchange capacity can be achieved by the linear network coding.

Before we prove this theorem, we again consider an example to convey the main feature behind the proposed scheme.

*Example 2:* Let us consider the example with  $L = 2$  given in Fig. 5. The cut-set bound of this setup is 5 bits per two channel uses. We now provide a coding scheme that can achieve this bound.



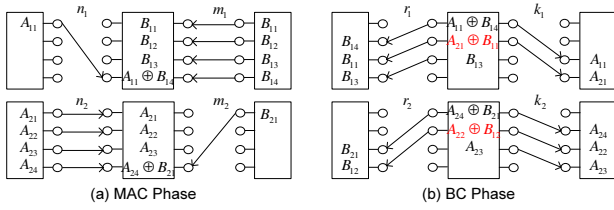


Fig. 5. Example 2.

As shown in the part (a) of Fig. 5, in the MAC phase,  $A_l$  and  $B_l$  send  $n_l$  and  $m_l$  data streams, respectively. Each sub-channel first shifts bits belonging to the aligned part to the top. After this, the relay still wants to broadcast  $B_{11}, B_{12}, B_{13}$  to the node  $A$  and  $A_{21}, A_{22}, A_{23}$  to the node  $B$ . Moreover, the relay still has 2 bi-directional links, 1 unidirectional link to the node  $A$ , and 1 unidirectional link to the node  $B$ . Since this is the case, the relay shifts  $B_{13}$  to the unidirectional link to the node  $A$  and also  $A_{23}$  to that to the node  $B$ . Further, the relay performs linear network coding and send  $A_{21} \oplus B_{11}$  and  $A_{22} \oplus B_{12}$  through two bi-directional links, respectively. This coding and forwarding scheme is illustrated in Fig. 5 (b) where one can verify that indeed we can achieve 5 bits.

*Remark 5:* This example suggest that if the number of aligned bits in the MAC phase is less than the number of bi-directional links in the BC phase as in Example 2, we should perform linear network coding to make use of the fact that some remaining links are bi-directional. Also, one can easily verify that for this example, the first proposed scheme in Section III is unable to achieve the cut-set bound.

*Sketch of the proof:* Only the sketch is provided due to the space limitation. The converse is simply the cut-set upper bound; therefore, it remains to show the achievability. Again, without loss of generality, we can assume that  $\sum_{l=1}^L n_l = \sum_{l=1}^L m_l = \sum_{l=1}^L r_l = \sum_{l=1}^L k_l$ . In the MAC phase, at the  $l^{th}$  sub-channel, let  $A_l$  and  $B_l$  transmit  $n_l$  and  $m_l$  data streams, respectively. Thus, there are  $f_{MAC,l} = \min(n_l, m_l)$  bits received by  $R_l$  belonging to the aligned part. In addition to that, there are  $(n_l - m_l)^+$  bits belonging to the non-aligned part sent by  $A_l$  and  $(m_l - n_l)^+$  non-aligned bits from  $B_l$ .

In the BC phase, at  $R_l$ , there are  $f_{BC,l} = \min(r_l, k_l)$  bi-directional links to both  $A_l$  and  $B_l$ . Moreover, there are  $(r_l - k_l)^+$  unidirectional links to  $A_l$  and  $(k_l - r_l)^+$  unidirectional links to  $B_l$  at the  $l^{th}$  sub-channel. The relay first shifts  $\min(f_{MAC,l}, f_{BC,l})$  bits to the top and also tries to use unidirectional links to send bits belonging to the non-aligned part. After this, we have the number of bits sent by  $A_l$  and  $B_l$  that have not been forwarded by  $R_l$ , respectively,

$$q_{Al} = (f_{MAC,l} - f_{BC,l})^+ + (n_l - m_l)^+ - (k_l - r_l)^+, \quad (54)$$

and

$$q_{Bl} = (f_{MAC,l} - f_{BC,l})^+ + (m_l - n_l)^+ - (r_l - k_l)^+. \quad (55)$$

Note that both  $q_{Al}$  and  $q_{Bl}$  can be either positive or negative integers where being positive means that we have more bits than links and being negative represents the reverse. We further

notice that if  $f_{MAC,l} > f_{BC,l}$ , the term  $(f_{MAC,l} - f_{BC,l})$  appears in both  $q_{Al}$  and  $q_{Bl}$  since those bits are intended for both directions. The relay then routes those remaining bits to other sub-channels that have more links than bits to the desired end node. Denote the collection of  $q_{Al}$  and  $q_{Bl}$  as  $q_A$  and  $q_B$ , respectively. One can verify that  $q_A = q_B = \sum_{l=1}^L (f_{BC,l} - f_{MAC,l})^+$ .

Until now, only  $\min(f_{MAC,l}, f_{BC,l})$  bi-directional links are used; therefore, for sub-channels that  $f_{MAC,l} < f_{BC,l}$ , the relay uses those unused  $\sum_{l=1}^L (f_{BC,l} - f_{MAC,l})^+$  bi-directional links to broadcast the XOR of the remaining bits in  $q_A$  and those in  $q_B$ .

## VII. CONCLUSIONS AND FUTURE WORK

In this paper, we studied the exchange rate for the parallel Gaussian bi-directional relay channel. We first investigated the corresponding linear deterministic model and proposed two coding schemes that can achieve the exchange capacity for the linear deterministic model. Two coding schemes for the original Gaussian setup are then proposed according to the insight obtained from the coding schemes, respectively, for the linear deterministic model. This provides a detour to avoid the optimal linear filter designs for the scheme in [5]. Numerical results showed that both two schemes substantially outperform the decode-and-forward scheme and provide nontrivial gains over the scheme in [5]. One potential future work is to characterize the exchange capacity for the considered setup via showing that either one or both of two schemes can approach the cut-set bound within a constant bit after averaging over sub-channels.

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