

Incremental Relaying for the Gaussian Interference Channel with a Degraded Broadcasting Relay

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Abstract—This paper studies incremental relay strategies for a two-user Gaussian relay-interference channel with an in-band-reception and out-of-band-transmission relay, where the link between the relay and the two receivers is modelled as a degraded broadcast channel. It is shown that generalized hash-and-forward (GHF) can achieve the capacity region of this channel to within a constant number of bits in a certain weak-relay regime, where the transmitter-to-relay link gains are not unboundedly stronger than the interference links between the transmitters and the receivers. The GHF relaying strategy is ideally suited for the broadcasting relay because it can be implemented in an incremental fashion, i.e., the relay message to one receiver is a degraded version of the message to the other receiver. A generalized-degree-of-freedom (GDoF) analysis in the high signal-to-noise ratio (SNR) regime reveals that in the symmetric channel setting, each common relay bit can improve the sum rate roughly by either one bit or two bits asymptotically depending on the operating regime, and the rate gain can be interpreted as coming solely from the improvement of the common messages rate, or alternatively in the very weak interference regime as solely coming from the rate improvement of the private messages. Further, this paper studies an asymmetric case in which the relay has only a single link to one of the destinations. It is shown that with only one relay-destination link, the approximate capacity region can be established for a larger regime of channel parameters. Further, from a GDoF point of view, the sum-capacity gain due to the relay can now be thought as coming from either signal relaying only, or interference forwarding only.

Index Terms—Approximate capacity, generalized hash-and-forward (GHF), generalized degrees of freedom, Han-Kobayashi strategy, interference channel, relay channel.

I. INTRODUCTION

Interference is a key limiting factor in modern communication systems. In a wireless cellular network, the performance of cell-edge users is severely limited by intercell interference. This paper considers the use of relays in cellular networks. The uses of relays to combat channel shadowing and to extend coverage for wireless systems have been widely studied in the literature. The main goal of this paper is to demonstrate the benefit of relaying for interference mitigation in the interference-limited regime.

Consider a two-cell wireless network with two base-stations each serving their respective receivers while interfering with each other, as shown in Fig. 1. The deployment of a *cell-edge* relay, which observes a linear combination of the two transmit

signals from the base-stations and is capable of independently communicating with the receivers over a pair of relay links, can significantly help the receivers mitigate intercell interference. This model is often referred to as an in-band-reception and out-of-band-transmission relay-interference channel, as the relay-to-receiver transmission can be thought of as taking place on a different frequency band.

A particular feature of the channel model considered in this paper is that the relay-to-receivers link is modeled as a Gaussian broadcast channel. This is motivated by the fact that the relay's transmission to the remote receivers often takes place in a wireless medium. Consequently, the same relay message can be heard by both receivers and can potentially help both receivers at the same time. Further, it is convenient (and without loss of generality as shown later for the achievability scheme and the converse proved in this paper) to model the relay-to-receiver links as digital links with capacities C_1 and C_2 respectively, but where one relay message is required to be a degraded version of the other relay message, as in a Gaussian broadcast channel. The goal of this paper is to devise an incremental relaying strategy and to quantify its benefit for this particular relay-interference channel.

A. Related Work

The classic two-user interference channel consists of two transmitter-receiver pairs communicating in the presence of interference from each other. Although the capacity region of the two-user Gaussian interference channel is still not known exactly, it can be approximated to within one bit [1] using a Han-Kobayashi power splitting strategy [2].

The use of cooperative communication for interference mitigation has received much attention recently. For example, [3]–[5] studied the Gaussian Z-interference channel with a unidirectional receiver cooperation link, and [6]–[9] studied the Gaussian interference channel with bi-directional transmitter/receiver cooperation links. In addition, the Gaussian interference channel with an additional relay node has also been studied extensively in the literature. Depending on the types of the links between the relay and the transmitters/receivers, the relay-interference channel can be categorized as having in-band transmission/reception [10]–[17], out-of-band transmission/reception [18]–[20], out-of-band transmission and in-band reception [21]–[24], or in-band transmission and out-of-band reception [25], the last of which is directly related to the channel model in this paper. In the following, we review different transmission schemes and relaying strategies that have emerged for each of these cases.

For interference channels equipped with an in-band transmission and reception relay, the relay interacts with both trans-

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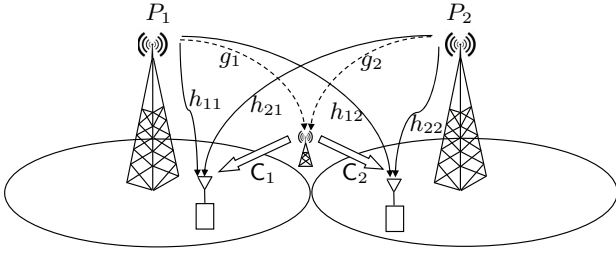


Fig. 1. A two-cell network with an in-band reception and out-of-band-broadcasting relay for interference mitigation

mitters and receivers in the same frequency band. Relaying strategies that have been investigated in the literature include decode-and-forward, compress-and-forward, and amplify-and-forward. For example, [12], [13] show that decoding-and-forwarding either the intended signal or the interfering signal to a receiver can both be beneficial. The former is termed as signal relaying, the latter interference forwarding. Decode-and-forward and half-duplex amplify-and-forward strategies are also studied in [14], [16]. When combining decode-and-forward relaying strategy and the Han-Kobayashi rate splitting input scheme, [17] gives an achievable rate region that has a shape similar to the Chong-Motani-Garg (CMG) region for the interference channel [26]. The exact capacity for this type of relay-interference channel is in general open, but there is a special potent-relay case [11] for which the sum capacity is known in some specific regimes.

The difficulty in establishing the capacity of the interference channel with in-band transmission/reception relay is in part due to the fact that the relay's received and transmit signals intertwine with that of the underlying interference channel. To simplify the matter, the interference channel with an out-of-band transmission/reception relay has been studied in [18]–[20]. In this channel model, the relay essentially operates on a separate set of parallel channels. Based on signal relaying and interference forwarding strategies, [18] identifies the condition under which the capacity region can be achieved with separable or nonseparable coding between the out-of-band relay and the underlying interference channel. Further, [19] studies this channel model in a symmetric setting and characterizes the sum capacity to within 1.15 bits. The transmission scheme of [19] involves further splitting of the common messages in the Han-Kobayashi scheme and a relay strategy that combines nested lattice coding and Gaussian codes. It is shown that in the strong interference regime, the use of structured codes is optimal.

Another variation of the relay-interference channel involves an out-of-band reception and in-band transmission relay. This channel is studied in [21], in which the transmitter further splits the transmit signal according to the Han-Kobayashi scheme; the relay decodes only part of the message depending the capacity of the transmitter-relay links; the rest of the codewords are transmitted directly from the sources to the destinations without the help of the relay. With this partial decode-and-forward relaying scheme, the sum capacity is found under a so-called strong relay-interference condition.

The interference channel with an in-band reception/out-of-band transmission relay has been briefly discussed in [25], and studied in [22], [23] for a case where the relay-destination links are shared between the two receivers. Conventional decode-and-forward and compress-and-forward relay strategies are not well matched for helping both receivers simultaneously with a common relayed message. Thus, [22], [23] consider a generalized hash-and-forward (GHF) strategy, which generalizes the conventional compress-and-forward scheme, and is shown to achieve the capacity region of this channel model to within a constant number of bits in the regime where the shared relay-destination link rate is sufficiently small. The channel model under consideration in this paper further extends the shared relay-destinations link to be a degraded broadcast channel. We focus on a different weak-relay regime. The main objective is similar: to efficiently use the relay bits to simultaneously benefit both users and to achieve capacity to within a constant gap.

Finally, the GHF relay strategy used in this paper is essentially the same as the noisy network coding [27]–[29] and the quantize-map-and-forward relay strategies [30]. The result of this paper can be thought of as an effort in generalizing these relay strategies to a particular case of the multiple unicast setting, for which constant-gap result continues to hold for certain channel-parameter regimes. Related works for the multiple unicast problem include [31]–[33].

B. Main Contributions

This paper considers a relay-interference channel with in-band reception and out-of-band degraded broadcasting links from the relay to the receivers. The key features of the transmission strategy and the main results of the paper are as follows.

1) *Incremental Relaying*: This paper uses a GHF relaying strategy to take advantage of the in-band reception link and the out-of-band broadcasting link from the relay to the receivers. In GHF, the relay quantizes its observation, which is a linear combination of the transmitted signals, using a *fixed* quantizer, then bins and forwards the quantized observation to the receivers. This strategy of fixing the quantization level is near optimal when a certain *weak-relay* condition is satisfied, and is ideally matched to the degraded broadcasting relay-to-receiver links with capacities C_1 and C_2 , because it allows an incremental binning strategy at the relay. Assuming that $C_1 \leq C_2$, the relay may first bin its quantized observation into 2^{nC_1} bins and send the bin index to both receivers, then further divide each bin into $2^{n(C_2-C_1)}$ sub-bins and sends the extra bin index to receiver 2 only. Thus, the relay message to the first receiver is a degraded version of the message to the second receiver.

2) *Oblivious Power Splitting*: The transmission scheme used in this paper consists of a Han-Kobayashi power splitting strategy [2] at the transmitter. The common-private power splitting ratio in such a strategy is crucial. In a study of the interference channel with conferencing links [6], Wang and Tse used the power splitting strategy of Etkin, Tse and Wang [1] where the private power is set at the noise level at the

receivers. This is sensible for the conferencing-receiver model considered in [6], but not necessarily so for the interference channel with an independent relay, unless again a certain *weak-relay* condition is satisfied. This strategy of fixing the power splitting at the transmitter to be independent of the relay is termed *oblivious power splitting* in [23]. Oblivious power splitting is used in this paper as well.

3) Constant Gap to Capacity in the weak-relay Regime:

The main result of this paper is that when the relay links are not unboundedly stronger than the interfering links, i.e.,

$$\max \left\{ \frac{|g_1|^2}{|h_{12}|^2}, \frac{|g_2|^2}{|h_{21}|^2} \right\} = \rho < \infty, \quad (1)$$

for some fixed ρ , the capacity of the relay-interference channel with a broadcast link can be achieved to within a constant gap, where the gap is a function of ρ but otherwise independent of channel parameters. This operating regime is called the *weak-relay regime* in this paper.

The main result of this paper is motivated by the results in [22] and [23], which studies a two-user interference channel augmented with a shared digital relay link to the receivers of rate R_0 , and obtains a constant-gap-to-capacity result under a certain small- R_0 condition using GHF and oblivious power splitting. The relay strategy studied in this paper goes one step further in that the relay-to-receivers link is modeled as a degraded broadcast channel. Moreover, the weak-relay regime studied in this paper is a counterpart of the small- R_0 regime studied in [23], as can be visualized in the practical setup of Fig. 1. When the mobiles are close to their respective cell centers, the relay link capacities C_1 and C_2 are small, thereby satisfying the small- R_0 condition of [23]. In the more practically important regime where the mobile terminals are close to the cell edge, the channel falls into the weak-relay regime of this paper. An interesting feature of the result in this paper is that the gap to capacity is a function of ρ , the relative channel strength between the interfering channel and the channel to the relay; the gap becomes smaller as $\rho \rightarrow 1$. In the limiting case with $\rho = 1$, corresponding to the situation where the mobiles are at the cell edge, the capacity region can be achieved to within $\frac{1}{2} \log \frac{5+\sqrt{33}}{2} = 1.2128$ bits.

A technical contribution of this paper is a particular set of capacity region outer bounds which are established by giving different combinations of side information (genies) to the receivers and by applying the known outer-bound results of the Gaussian interference channel [1] and the single-input multiple-output (SIMO) Gaussian interference channel [34]. It is shown that there are two constraints for the individual rates R_1 and R_2 , twelve constraints for the sum rate $R_1 + R_2$, six constraints for $2R_1 + R_2$, and six constraints for $R_1 + 2R_2$. Furthermore, the outer bounds established in this paper hold for all channel parameters. This set of outer bounds is tight to within a constant gap in the weak-relay regime.

To obtain insights from the performance gain brought by the relay, this paper further investigates the improvement in the generalized degrees of freedom (GDoF) per user for the relay-interference channel due to a broadcasting link. In the symmetric setting, it is shown that a common broadcast link can improve the sum capacity by two bits per each relay bit in

the very weak, moderately weak, and very strong interference regimes, but by one bit per each relay bit in other regimes. This asymptotic behavior can be interpreted by noting that the relay link essentially behaves like a deterministic channel in the high signal-to-noise-ratio (SNR) regime. Further, in the symmetric setting, the sum-capacity gain due to the relay can be thought of as solely coming from the rate improvement of the common messages, or alternatively in a very weak interference regime as solely coming from the rate improvement of the private messages.

In asymmetric settings, the improvement in the sum capacity by the relay can be interpreted in different ways. To illustrate this point, this paper investigates a special case of the channel model, where the relay link is available to only one but not both destinations. In this case, the relay may forward information about both the intended signal and the interference, and the capacity can benefit from both signal-relaying and interference-forwarding. This paper shows that a constant-gap-to-capacity result can be derived for this setting under a more relaxed weak-relay condition that requires only $|g_2| \leq \sqrt{\rho}|h_{21}|$ (and not $|g_1| \leq \sqrt{\rho}|h_{12}|$). Moreover, this paper shows that in term of GDoF, when the relay link is above a certain threshold, the sum-capacity gain is equivalent to that of that of a single relay link from user 1. When the relay link is below the threshold, the sum-capacity gain is equivalent to that of a single relay link from user 2.

Finally, the results of this paper show that GHF is sufficient for achieving the approximated capacity region of an in-band reception and out-of-band transmission Gaussian relay-interference channel in the weak-relay regime. Thus, more recently proposed relay techniques based on compute-and-forward [35] or lattice coding [36] is not necessary in this regime as far as constant gap to capacity is concerned. Outside of the weak-relay regime, the optimal relay strategies remain an open problem; lattice coding strategies may be helpful.

C. Organization of This Paper

The rest of the paper is organized as follows. Section II introduces the Gaussian relay-interference channel model, derives capacity region outer bounds that hold for all channel parameters and an achievable rate region, and presents the main constant-gap theorem and the GDoF analysis. Section III deals with the relay-interference channel with a single relay link, derives the corresponding constant-gap result, and gives a quantitative analysis on the relation between signal relaying and interference forwarding. Section IV concludes the paper.

II. GAUSSIAN RELAY-INTERFERENCE CHANNEL: GENERAL CASE

A. Channel Model and Definitions

A Gaussian relay-interference channel consists of two transmitter-receiver pairs and an independent relay. Each transmitter communicates with the intended receiver while causing interference to the other transmitter-receiver pair. The relay receives a linear combination of the two transmit signals and helps the transmitter-receiver pairs by forwarding a message

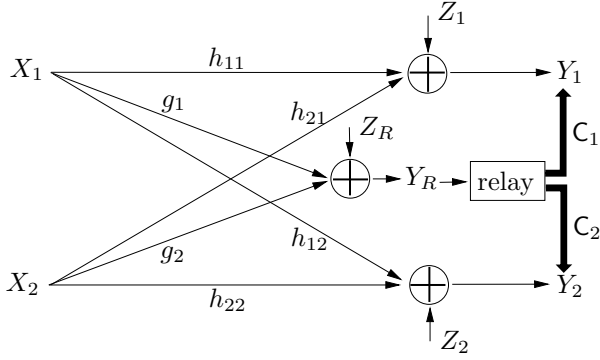


Fig. 2. Gaussian relay-interference channel with two independent digital relay links

to receiver 1 and another message to receiver 2 through rate-limited digital links with capacities C_1 and C_2 respectively. We start by treating a channel model with independent relay links, and later show that requiring one relay message to be a degraded version of the other is without loss of approximate optimality. As shown in Fig. 2, X_1, X_2 and Y_1, Y_2 are real-valued input and output signals, respectively, and Y_R is the observation of the relay. The receiver noises are assumed to be independent and identically distributed (i.i.d.) Gaussian random variables with variance one, i.e., $Z_i \sim \mathcal{N}(0, 1), i = 1, 2$ and R . The input-output relationship can be described by

$$Y_1 = h_{11}X_1 + h_{21}X_2 + Z_1, \quad (2)$$

$$Y_2 = h_{22}X_2 + h_{12}X_1 + Z_2, \quad (3)$$

$$Y_R = g_1X_1 + g_2X_2 + Z_R, \quad (4)$$

where h_{ij} is the channel gain from transmitter i to receiver j , and g_j is the channel gain from transmitter j to the relay, all real valued. The powers of the input signals are normalized to one, i.e., $\mathbb{E}[|X_i|^2] \leq 1, i = 1, 2$.

Define the signal-to-noise ratios and interference-to-noise ratios as follows:

$$\begin{aligned} \text{SNR}_i &= |h_{ii}|^2, & \text{SNR}_{ri} &= |g_i|^2, \quad i = 1, 2 \\ \text{INR}_1 &= |h_{12}|^2, & \text{INR}_2 &= |h_{21}|^2. \end{aligned}$$

Define functions $\alpha(\cdot)$ and $\beta(\cdot)$ as

$$\alpha(x) = \frac{1}{2} \log(2x + 2 + \rho), \quad \beta(x) = \frac{1}{2} + \frac{1}{2} \log \left(1 + \frac{1 + \rho}{x} \right), \quad (5)$$

where $\log(\cdot)$ is base 2 and ρ is defined as

$$\rho \triangleq \max \left\{ \frac{|g_1|^2}{|h_{12}|^2}, \frac{|g_2|^2}{|h_{21}|^2} \right\}. \quad (6)$$

This paper considers a weak-relay regime where ρ is a finite constant.

B. Outer Bounds and Achievable Rate Region

We first present outer bounds and achievability results that are applicable to the relay-interference channel model with two independent digital relays as shown in Fig. 2.

Theorem 1 (Capacity Region Outer Bounds). *The capacity region of the Gaussian relay-interference channel as depicted*

in Fig. 2 is contained in the outer bound $\bar{\mathcal{C}}$ given by the set of (R_1, R_2) for which

$$R_1 \leq \frac{1}{2} \log(1 + \text{SNR}_1) + \min \left\{ C_1, \frac{1}{2} \log \left(1 + \frac{\text{SNR}_{r1}}{1 + \text{SNR}_1} \right) \right\} \quad (7)$$

$$R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2) + \min \left\{ C_2, \frac{1}{2} \log \left(1 + \frac{\text{SNR}_{r2}}{1 + \text{SNR}_2} \right) \right\} \quad (8)$$

$$R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) + C_1 + C_2 \quad (9)$$

$$R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + C_1 + C_2 \quad (10)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \text{INR}_2 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) + \frac{1}{2} \log \left(1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + C_1 + C_2 \quad (11)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1 + \text{SNR}_{r1}} \right) + \frac{1}{2} \log(1 + \text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2} + \text{INR}_1 + \text{SNR}_{r1}) + C_1 \quad (12)$$

$$R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2 + \text{SNR}_{r2}}{1 + \text{INR}_2} \right) + C_1 \quad (13)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1 + \text{SNR}_{r1}} + \text{INR}_2 \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} + \text{INR}_1 + \text{SNR}_{r1} \right) + C_1 \quad (14)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2 + \text{SNR}_{r2}} \right) + \frac{1}{2} \log(1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{INR}_2 + \text{SNR}_{r2}) + C_2 \quad (15)$$

$$R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) + C_2 \quad (16)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2 + \text{SNR}_{r2}} + \text{INR}_1 \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1} + \text{INR}_2 + \text{SNR}_{r2} \right) + C_2 \quad (17)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1 + \text{SNR}_{r1}} \right) + \frac{1}{2} \log(1 + \text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2} + \text{INR}_1 + \text{SNR}_{r1}) \quad (18)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2 + \text{SNR}_{r2}}{1 + \text{INR}_2 + \text{SNR}_{r2}} \right) + \frac{1}{2} \log(1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{INR}_2 + \text{SNR}_{r2}) \quad (19)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1 + \text{SNR}_{r1}} \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2 + \text{SNR}_{r2}} \right) + \text{INR}_1 + \text{SNR}_{r1} \quad (20)$$

$$2R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) + 2C_1 + C_2 \quad (21)$$

$$2R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1 + \text{SNR}_{r1}} \right) + \frac{1}{2} \log(1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{INR}_2 + \text{SNR}_{r2}) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2 + \text{SNR}_{r2}} \right) + \text{INR}_1 + \text{SNR}_{r1} \quad (22)$$

$$2R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} \right) + \text{INR}_1 + \text{SNR}_{r1} + C_1 \quad (23)$$

$$2R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2 + \text{SNR}_{r2}} + \text{INR}_1 \right) + \frac{1}{2} \log(1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{INR}_2 + \text{SNR}_{r2}) + C_2 \quad (24)$$

$$2R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1 + \text{SNR}_{r1}} \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} \right) + \text{INR}_1 + \text{SNR}_{r1} + 2C_1 \quad (25)$$

$$2R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) + C_1 + C_2, \quad (26)$$

and $R_1 + 2R_2$ bounded by (21)-(26) with indices 1 and 2 switched, where ϕ_1^2 and ϕ_2^2 are defined as

$$\phi_1^2 = \left| \frac{g_1 h_{21}}{g_2 h_{11}} - 1 \right|^2, \quad \phi_2^2 = \left| \frac{g_2 h_{12}}{g_1 h_{22}} - 1 \right|^2. \quad (27)$$

Proof: The above outer bounds can be proved in a genie-aided approach. See Appendix A for details. ■

Theorem 2 (Achievable Rate Region). *Let \mathcal{P} denote the set of probability distributions $P(\cdot)$ that factor as*

$$P(q, w_1, w_2, x_1, x_2, y_1, y_2, y_R, \hat{y}_{R1}, \hat{y}_{R2}) = p(q)p(x_1, w_1|q)p(x_2, w_2|q)p(y_1, y_2, y_R|x_1, x_2, q)p(\hat{y}_{R1}, \hat{y}_{R2}|y_R, q). \quad (28)$$

For a fixed distribution $P \in \mathcal{P}$, let $\mathcal{R}(P)$ be the set of all rate pairs (R_1, R_2) satisfying

$$0 \leq R_1 \leq d_1 + \min \{ (C_1 - \xi_1)^+, \Delta d_1 \}, \quad (29)$$

$$0 \leq R_2 \leq d_2 + \min \{ (C_2 - \xi_2)^+, \Delta d_2 \}, \quad (30)$$

$$R_1 + R_2 \leq a_1 + g_2 + \min \{ (C_1 - \xi_1)^+, \Delta a_1 \} + \min \{ (C_2 - \xi_2)^+, \Delta g_2 \}, \quad (31)$$

$$R_1 + R_2 \leq a_2 + g_1 + \min \{ (C_1 - \xi_1)^+, \Delta g_1 \} + \min \{ (C_2 - \xi_2)^+, \Delta a_2 \}, \quad (32)$$

$$R_1 + R_2 \leq e_1 + e_2 + \min \{ (C_1 - \xi_1)^+, \Delta e_1 \} + \min \{ (C_2 - \xi_2)^+, \Delta e_2 \}, \quad (33)$$

$$2R_1 + R_2 \leq a_1 + g_1 + e_2 + \min \{ (C_1 - \xi_1)^+, \Delta a_1 \} + \min \{ (C_1 - \xi_1)^+, \Delta g_1 \} + \min \{ (C_2 - \xi_2)^+, \Delta e_2 \}, \quad (34)$$

$$R_1 + 2R_2 \leq a_2 + g_2 + e_1 + \min \{ (C_2 - \xi_2)^+, \Delta a_2 \} + \min \{ (C_2 - \xi_2)^+, \Delta g_2 \} + \min \{ (C_1 - \xi_1)^+, \Delta e_1 \}, \quad (35)$$

where

$$a_1 = I(X_1; Y_1|W_1, W_2, Q), \quad (36)$$

$$d_1 = I(X_1; Y_1|W_2, Q), \quad (37)$$

$$e_1 = I(X_1, W_2; Y_1|W_1, Q), \quad (38)$$

$$g_1 = I(X_1, W_2; Y_1|Q), \quad (39)$$

$$\Delta a_1 = I(X_1; \hat{Y}_{R1}|Y_1, W_1, W_2, Q), \quad (40)$$

$$\Delta d_1 = I(X_1; \hat{Y}_{R1}|Y_1, W_2, Q), \quad (41)$$

$$\Delta e_1 = I(X_1, W_2; \hat{Y}_{R1}|Y_1, W_1, Q), \quad (42)$$

$$\Delta g_1 = I(X_1, W_2; \hat{Y}_{R1}|Y_1, Q), \quad (43)$$

$$\xi_1 = I(Y_R; \hat{Y}_{R1}|Y_1, X_1, W_2, Q), \quad (44)$$

and $a_2, \Delta a_2, d_2, \Delta d_2, e_2, \Delta e_2, g_2, \Delta g_2$, and ξ_2 are defined by (36)-(44) with indices 1 and 2 switched. Then

$$\mathcal{R} = \bigcup_{P \in \mathcal{P}} \mathcal{R}(P) \quad (45)$$

is an achievable rate region for the Gaussian relay-interference channel as shown in Fig. 2.

Proof: The achievable scheme consists of a Han-Kobayashi strategy at the transmitters and a generalized hash-and-forward strategy at the relay. They are the same strategies as adopted in [23] except that unlike the GHF relaying scheme in [23, Theorem 2], where the relay quantizes the received signal and broadcasts its bin index to both receivers through a shared digital link, the relay here quantizes the received signal with two different quantization resolutions, then sends the bin indices of the quantized signals to the receivers through separated digital links of rates C_1 and C_2 . The following is a sketch of the encoding/decoding process.

Encoding: Each transmit signal is comprised of a common message of rate T_i and a private message of rate S_i . The common message codewords $W_i^n(j)$, $j = 1, 2, \dots, 2^{nT_i}$ of length n are generated according to the probability distribution $\Pi_{i=1}^n p(w_i|q)$, where $q \sim p(q)$ serves as the time-sharing random variable. Based on the common message codewords, user i generates codewords $X_i^n(j, k)$, $k = 1, 2, \dots, 2^{nS_i}$ of length n following the conditional distribution $\Pi_{i=1}^n p(x_i|w_i, q)$. Each input message $\theta_i \in [1, 2, \dots, 2^{S_i+T_i}]$, $i = 1, 2$ is mapped to a message pair $(s_i, t_i) \in [1, \dots, 2^{S_i}] \times [1, \dots, 2^{T_i}]$, then sent to the destinations as $X_i^n(s_i, t_i)$. At the relay, the quantization codebook is generated according to the probability distribution $p(\hat{y}_{R1}, \hat{y}_{R2}|y_R, q)$. After receiving Y_R^n , the relay quantizes Y_R^n into \hat{Y}_{R1}^n and \hat{Y}_{R2}^n , then bins \hat{Y}_{R1}^n to 2^{nC_1} bins, and bins \hat{Y}_{R2}^n to 2^{nC_2} bins, and sends the bin indices to the receivers through the digital links.

Decoding: The decoding process follows the Han-Kobayashi framework: X_1^n and W_2^n are decoded by receiver 1 with the help of the index of the relayed message \hat{Y}_{R1}^n ; X_2^n and W_1^n are decoded by receiver 2 with the help of the index of the relayed message \hat{Y}_{R2}^n . To decode, receiver 1 first constructs a list of candidates for the relayed message \hat{Y}_{R1}^n , then jointly decodes X_1^n , W_2^n and \hat{Y}_{R1}^n using typicality decoding. Similarly, receiver 2 jointly decodes X_2^n , W_1^n and \hat{Y}_{R2}^n . Following the error probability analysis in [23, Theorem 2], the rate tuple (S_1, T_1, S_2, T_2) satisfying the following constraints is achievable:

Constraints at receiver 1:

$$S_1 \leq \min\{I(X_1; Y_1|W_1, W_2, Q) + (C_1 - \xi_1)^+, I(X_1; Y_1, \hat{Y}_{R1}|W_1, W_2, Q)\} \quad (46)$$

$$S_1 + T_1 \leq \min\{I(X_1; Y_1|W_2, Q) + (C_1 - \xi_1)^+, I(X_1; Y_1, \hat{Y}_{R1}|W_2, Q)\} \quad (47)$$

$$S_1 + T_2 \leq \min\{I(X_1; W_2; Y_1|W_1, Q) + (C_1 - \xi_1)^+, I(X_1; W_2; Y_1, \hat{Y}_{R1}|W_1, Q)\} \quad (48)$$

$$S_1 + T_1 + T_2 \leq \min\{I(X_1; W_2; Y_1|Q) + (C_1 - \xi_1)^+, I(X_1; W_2; Y_1, \hat{Y}_{R1}|Q)\} \quad (49)$$

Constraints at receiver 2:

$$S_2 \leq \min\{I(X_2; Y_2|W_1, W_2, Q) + (C_2 - \xi_2)^+, I(X_2; Y_2, \hat{Y}_{R2}|W_1, W_2, Q)\} \quad (50)$$

$$S_2 + T_2 \leq \min\{I(X_2; Y_2|W_1, Q) + (C_2 - \xi_2)^+, I(X_2; Y_2, \hat{Y}_{R2}|W_1, Q)\} \quad (51)$$

$$S_2 + T_1 \leq \min\{I(X_2; W_1; Y_2|W_2, Q) + (C_2 - \xi_2)^+, I(X_2; W_1; Y_2, \hat{Y}_{R2}|W_2, Q)\} \quad (52)$$

$$S_2 + T_2 + T_1 \leq \min\{I(X_2; W_1; Y_2|Q) + (C_2 - \xi_2)^+, I(X_2; W_1; Y_2, \hat{Y}_{R2}|Q)\} \quad (53)$$

The achievable rate region consists of all rate pairs (R_1, R_2) such that $R_1 = S_1 + T_1$ and $R_2 = S_2 + T_2$. Applying the Fourier-Motzkin elimination procedure [37] gives the achievable rate region (29)-(35). ■

We remark here that although both Theorem 1 and Theorem 2 are stated for the digital noise-free relay-destination links, it can be easily verified that both results continue to hold when the digital links are replaced by analog additive Gaussian noise channels. The fact that the achievable rate region for the analog channel is at least as large as the rate region for the digital channel is obvious since one can always digitize the analog link. The fact that the outer bound continues to hold can be verified by going through the proof of that converse in Appendix A. The outer bounds in the converse involve terms like $I(X_1^n; Y_1^n, V_1^n)$, which is in turn upper bounded by $I(X_1^n; Y_1^n) + nC_1$. It is easy to show that when the digital link C_1 is replaced by an analog link with input X_{a1} and output Y_{a1} , the mutual information term is upper bounded by $I(X_1^n; Y_1^n) + I(X_{a1}^n; Y_{a1}^n)$. As a result, all the outer bounds in Theorem 1 continue to hold in the case of the analog relay link with C_1 replaced by $I(X_{a1}; Y_{a1})$ and C_2 replaced by $I(X_{a2}; Y_{a2})$.

C. Constant Gap in the Weak-Relay Regime

We now specialize to the Gaussian case, and show that under the weak-relay condition (1), the achievable rate region and the outer bounds of the Gaussian relay-interference channel with independent relay links can be made to be within a constant gap to each other. The relaying strategy that achieves this capacity to within a constant gap turns out to be naturally suited for the Gaussian relay-interference channel with a degraded broadcasting relay, thus establishing the constant-gap result for the broadcasting-relay case as well.

Assuming Gaussian codebooks and a Gaussian quantization scheme, the key design parameters are the choice of common-private power splitting ratio at the transmitters and the quantization level at the relay. Our choice of design parameters is inspired by that of Wang and Tse [6], where the capacity region of a Gaussian interference channel with rate-limited receiver cooperation is characterized to within a constant gap. Two key observations are made in [6]. First, the Etkin-Tse-Wang strategy [1] of setting the private power to be at the noise level at the opposite receiver is used. Second, the relay quantizes its observation at the private signal level in order to preserve all the information of interest to the destinations. At

the destinations, a joint decoding (see [22], [30], [38], [39]) is performed to recover the source messages.

Consider now the optimal power splitting in a Gaussian relay-interference channel with independent relay links. The Etkin-Tse-Wang strategy, i.e., setting private powers P_{ip} as

$$P_{1p} = \min\{1, h_{12}^{-2}\}, \quad P_{2p} = \min\{1, h_{21}^{-2}\}. \quad (54)$$

is near optimal for the Gaussian interference channel with conferencing receivers, but is not necessarily so for relay-interference channel shown in Fig. 2 in its most general form. Consider an extreme scenario of $C_1, C_2 \rightarrow \infty$. In this case, the relay fully cooperates with both receivers, so the relay-interference channel becomes a single-input multiple-output (SIMO) interference channel with two antennas at the receivers. Thus, the private powers at the transmitters must be set at the effective noise level for the two-antenna output in order to achieve capacity to within constant bits [34] [40], i.e.,

$$P_{1p} = \min\{1, (g_1^2 + h_{12}^2)^{-1}\}, \quad P_{2p} = \min\{1, (g_2^2 + h_{21}^2)^{-1}\}. \quad (55)$$

When C_1 and C_2 are finite, the optimal power splitting strategy is expected to be a function of not only h_{12} and h_{21} but also g_1, g_2, C_1 and C_2 , lying somewhere between (54) and (55).

This complication can be avoided, however, if we focus on the weak-relay regime (1), namely $|g_1| \leq \sqrt{\rho}|h_{12}|$ and $|g_2| \leq \sqrt{\rho}|h_{21}|$ for some finite constant ρ . In this case, the power splittings (54) and (55) differ by at most a constant factor. The main result of this section shows that in this weak-relay regime, the Etkin-Tse-Wang's original power splitting (54) is sufficient for achieving the capacity of the Gaussian relay-interference channel to within a constant gap (which is a function of ρ).

Consider next the optimization of the quantization level. Applying the insight of [6] to the Gaussian relay-interference channel with independent relay links shown in Fig. 2, the quantized messages for two receivers can be expressed as

$$\hat{Y}_{R1} = g_1 U_1 + g_1 W_1 + g_2 W_2 + \overbrace{g_2 U_2 + Z_R}^{\text{of no interest at } Y_1} + \eta_1 \quad (56)$$

$$\hat{Y}_{R2} = g_1 W_1 + g_2 U_2 + g_2 W_2 + \overbrace{g_1 U_1 + Z_R}^{\text{of no interest at } Y_2} + \eta_2 \quad (57)$$

where W_i and U_i are common message and private message respectively, and $\eta_i \sim \mathcal{N}(0, q_i)$ is the quantization noise, $i = 1, 2$. Therefore, a reasonable choice of the quantization levels for receiver 1 and receiver 2 is

$$q_1 = 1 + g_2^2 P_{2p}, \quad q_2 = 1 + g_1^2 P_{1p}. \quad (58)$$

Now observe that in the weak-relay regime, i.e., $|g_1| \leq \sqrt{\rho}|h_{12}|, |g_2| \leq \sqrt{\rho}|h_{21}|$, the above quantization levels (with Etkin-Tse-Wang power splitting) are between 1 and the constant $\rho + 1$. Thus, we can choose the quantization levels to be a constant and optimize it between 1 and $\rho + 1$.

Theorem 3 (Constant Gap in the Weak-Relay Regime). *For the Gaussian relay-interference channel with independent relay links as depicted in Fig. 2, in the weak-relay regime, using the generalized hash-and-forward relaying scheme with quantization levels $q_1 = q_2 = \frac{\sqrt{\rho^2 + 16\rho + 16} - \rho}{4}$, where ρ is defined*

in (6), and using the Han-Kobayashi scheme with Etkin-Tse-Wang power splitting strategy, $X_i = U_i + W_i, i = 1, 2$, where U_i and W_i are both Gaussian distributed with the powers of U_1 and U_2 set according to $P_{1p} = \min\{1, h_{12}^{-2}\}$ and $P_{2p} = \min\{1, h_{21}^{-2}\}$, respectively, the achievable rate region given in Theorem 2 is within

$$\delta = \frac{1}{2} \log \left(2 + \frac{\rho + \sqrt{\rho^2 + 16\rho + 16}}{2} \right) \quad (59)$$

bits of the capacity region outer bound in Theorem 1.

Proof: The main step is to show that using superposition coding $X_i = U_i + W_i, i = 1, 2$, where $U_i \sim \mathcal{N}(0, P_{ip})$ and $W_i \sim \mathcal{N}(0, P_{ic})$ with $P_{ip} + P_{ic} = 1, P_{1p} = \min\{1, h_{12}^{-2}\}$, and $P_{2p} = \min\{1, h_{21}^{-2}\}$, each of the achievable rate constraints in (29)-(35) is within a finite gap to the corresponding upper bound in (7)-(26). Specifically, it is shown in Appendix C that

(i) Individual rate (29) is within

$$\delta_{R1} = \max\{\alpha(q_1), \beta(q_1)\} \quad (60)$$

bits of the upper bound (7), where $\alpha(\cdot)$ and $\beta(\cdot)$ are as defined in (5);

(ii) Individual rate (30) is within

$$\delta_{R2} = \max\{\alpha(q_2), \beta(q_2)\} \quad (61)$$

bits of the upper bound (8);

(iii) Sum rates (31), (32), and (33) are within

$$\begin{aligned} \delta_{R1+R2} = \max\{ & \alpha(q_1) + \alpha(q_2), \alpha(q_1) + \beta(q_2), \\ & \beta(q_1) + \alpha(q_2), \beta(q_1) + \beta(q_2) \} \end{aligned} \quad (62)$$

bits of the upper bounds (9)-(20);

(iv) $2R_1 + R_2$ rate (34) is within

$$\begin{aligned} \delta_{2R1+R2} = \max\{ & 2\alpha(q_1) + \alpha(q_2), 2\beta(q_1) + \alpha(q_2), \\ & \alpha(q_1) + \beta(q_1) + \alpha(q_2), \\ & 2\alpha(q_1) + \beta(q_2), 2\beta(q_1) + \beta(q_2) \\ & \alpha(q_1) + \beta(q_1) + \beta(q_2) \} \end{aligned} \quad (63)$$

bits of the upper bounds (21)-(26);

(v) $R_1 + 2R_2$ rate (35) is within

$$\begin{aligned} \delta_{R1+2R2} = \max\{ & \alpha(q_1) + 2\alpha(q_2), \alpha(q_1) + 2\beta(q_2), \\ & \alpha(q_1) + \beta(q_2) + \alpha(q_2), \\ & \beta(q_1) + 2\alpha(q_2), \beta(q_1) + 2\beta(q_2), \\ & \beta(q_1) + \beta(q_2) + \alpha(q_2) \} \end{aligned} \quad (64)$$

bits of the upper bounds not shown explicitly in Theorem 1 but can be obtained by switching the indices 1 and 2 of (21)-(26).

Since $\alpha(\cdot)$ is a monotonically increasing function and $\beta(\cdot)$ is a monotonically decreasing function. In order to minimize the above gaps over q_1 and q_2 , the quantization levels should be set such that

$$\alpha(q_1^*) = \beta(q_1^*) = \alpha(q_2^*) = \beta(q_2^*), \quad (65)$$

which results in $q_1^* = q_2^* = \frac{\sqrt{\rho^2 + 16\rho + 16} - \rho}{4}$. Substituting q_1^* and q_2^* into the above gaps, we prove that the constant gap is δ bits per dimension, where δ is given in (59). ■

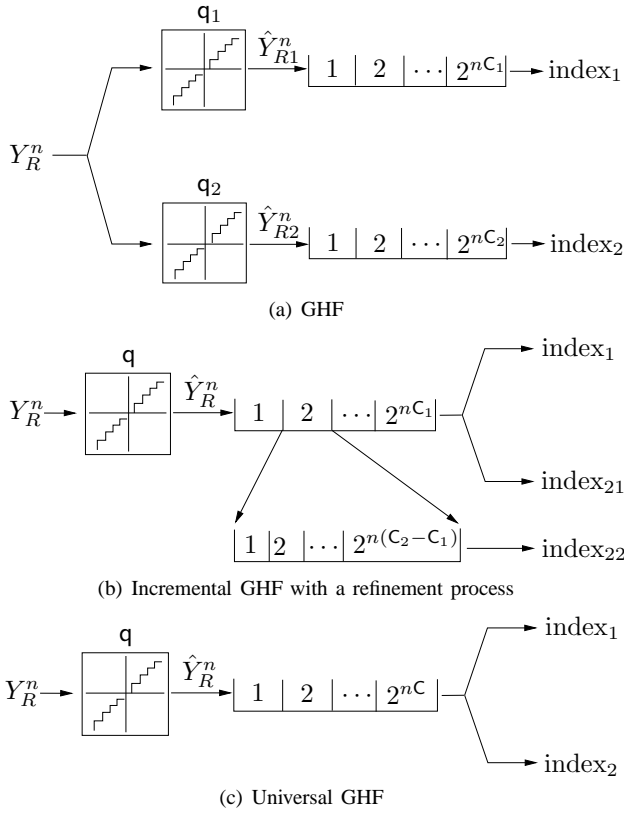


Fig. 3. Evolution of the generalized hash-and-forward relay scheme

Note that the finite capacity gap is an increasing function of ρ : smaller ρ results in a smaller gap. In the case that $\rho = 1$, i.e., $|g_1| \leq |h_{12}|$ and $|g_2| \leq |h_{21}|$, the optimal quantization levels are $q_1^* = q_2^* = \frac{\sqrt{33}-1}{4}$, and the gap to the capacity is given by $\frac{1}{2} \log \left(\frac{5+\sqrt{33}}{4} \right) = 1.2128$ bits.

D. Gaussian Relay-Interference Channel with a Broadcasting Relay

The GHF relaying scheme originally stated in Theorem 2 requires independent relay links. As shown in Fig. 3(a), the relay observation Y_R^n undergoes two separate quantization and binning processes to obtain the two messages for the two receivers. However, in the weak-relay regime, Theorem 3 shows that using an identical quantization level for the two receivers is without loss of approximate optimality, thus a common quantization process can be shared between the two receivers. Further, since the same \hat{Y}_R^n is binned into bins of sizes 2^{nC_1} and 2^{nC_2} , this is equivalent to first binning \hat{Y}_R^n into 2^{nC_1} bins (assuming $C_1 \leq C_2$) then further binning each bin into $2^{n(C_2-C_1)}$ sub-bins, as shown in Fig. 3(b). The message sent to receiver 2 can be thought of as the refinement of the message sent to receiver 1. This is exactly the incremental relaying strategy we seek for the Gaussian interference channel with a broadcasting relay, where the message to receiver 1 is a degraded version of the message to receiver 2. Finally, if $C_1 = C_2 = C$, the relay-interference channel reduces to the universal relaying scheme studied in [23], where a digital link is shared between the relay and the receivers, as shown

in Fig. 3(c). We note here that the outer bounds for the independent relay link case (Theorem 1) continues to hold for the degraded broadcast relay case.

Corollary 1. *The constant-gap-to-capacity result stated in Theorem 3 holds also for the Gaussian relay-interference channel with degraded broadcasting relay links, where (assuming $C_1 \leq C_2$) the message sent through the link with capacity C_1 must be a degraded version of the message sent through the link with capacity C_2 .*

E. Comments on the Strong-Relay Regime

The constant-gap result in this paper holds only in the weak-relay regime of $|g_1| \leq \sqrt{\rho}|h_{12}|$ and $|g_2| \leq \sqrt{\rho}|h_{21}|$, where ρ is finite. The main difficulty in extending this result to the general case is that both the choice of the Han-Kobayashi power splitting ratio and the GHF relay strategy are no longer optimal in the strong-relay regime. As mentioned earlier, the Etkin-Tse-Wang power splitting is not optimal when the relay links $g_i, i = 1, 2$ grow unboundedly stronger than the interference links h_{12} and h_{21} . Further, GHF may not be an appropriate relay strategy. To see this, assume a channel model with separate relay links and consider an extreme scenario where the relay links $g_i, i = 1, 2$ go to infinity, while all other channel parameters are kept constant. This special case is known as the cognitive relay-interference channel [41]. The capacity region outer bound of Theorem 1 for this case reduces to

$$R_1 \leq \frac{1}{2} \log(1 + \text{SNR}_1) + C_1 \quad (66)$$

$$R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2) + C_2 \quad (67)$$

$$R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) + C_1 + C_2 \quad (68)$$

$$R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + C_1 + C_2 \quad (69)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \text{INR}_2 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) + \frac{1}{2} \log \left(1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + C_1 + C_2 \quad (70)$$

$$2R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) + 2C_1 + C_2 \quad (71)$$

$$R_1 + 2R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log \left(1 + \text{INR}_2 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + C_1 + 2C_2, \quad (72)$$

which is in fact the outer bound of the underlying interference channel expanded by C_1 bits in the R_1 direction and C_2 in the R_2 direction. In this special case, a decode-and-forward strategy can easily achieve the capacity region to within a constant gap. This is because the relay is capable of decoding all the source messages, so it can simply forward the bin indices of the private messages to achieve $(R_1 + C_1, R_2 + C_2)$ for any achievable rate pair (R_1, R_2) in the absence of the relay. Etkin-Tse-Wang power splitting with decode-and-forward then achieves the outer bound to within a constant gap. In contrast, GHF cannot achieve the capacity region to within a constant gap in this case.

F. Generalized Degrees of Freedom

We can gain further insights into the effect of relaying on the Gaussian interference channel by analyzing the GDoF of the sum rate in the symmetric channel setting. Consider the case where $\text{INR}_1 = \text{INR}_2 = \text{INR}$, $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$, $\text{SNR}_{r1} = \text{SNR}_{r2} = \text{SNR}_r$, and $C_1 = C_2 = C$. In the high SNR regime, similar to [1], [6], define

$$\alpha := \lim_{\text{SNR} \rightarrow \infty} \frac{\log \text{INR}}{\log \text{SNR}}, \quad (73)$$

$$\beta := \lim_{\text{SNR} \rightarrow \infty} \frac{\log \text{SNR}_r}{\log \text{SNR}}, \quad (74)$$

$$\kappa := \lim_{\text{SNR} \rightarrow \infty} \frac{C}{\frac{1}{2} \log \text{SNR}}. \quad (75)$$

The GDoF of the sum capacity is defined as

$$d_{\text{sum}} = \lim_{\text{SNR} \rightarrow \infty} \frac{C_{\text{sum}}}{\frac{1}{2} \log \text{SNR}} \Big|_{\text{fixed } \alpha, \beta, \kappa} \quad (76)$$

As a direct consequence of the constant-gap result, d_{sum} can be characterized in the weak-relay regime as follows.

Corollary 2. *For the symmetric Gaussian relay-interference channel in the weak-relay regime (i.e., $\beta \leq \alpha$), the GDoF of the sum capacity is given by the following. When $0 \leq \alpha < 1$*

$$d_{\text{sum}} = \min \{ 2 - \alpha + \min \{ \beta, \kappa \}, 2 \max \{ \alpha, 1 - \alpha \} + 2\kappa, 2 \max \{ \alpha, 1 + \beta - \alpha \} \}. \quad (77)$$

When $\alpha \geq 1$

$$d_{\text{sum}} = \min \{ \alpha + \kappa, \alpha + \beta, 2(1 + \kappa), 2 \max \{ 1, \beta \} \}. \quad (78)$$

Note that when $\alpha = 1$, the GDoF of the sum capacity is in fact not well defined. This is because both $\text{INR} = \gamma \text{SNR}$ (where $\gamma \neq 1$ is finite) and $\text{INR} = \text{SNR}$ result in the same $\alpha = 1$. However, in the case of $\text{INR} = \text{SNR}$, the channel becomes ill conditioned, i.e. $\phi_1 = \phi_2 = 0$, which results in a d_{sum} other than the one in (78). In other words, multiple values of d_{sum} correspond to the same $\alpha = 1$. This is similar to the situation of [6, Theorem 7.3]. Applying the similar argument that the event $\{\text{INR} = \text{SNR}\}$ is of zero measure, we have the GDoF of the sum capacity as shown in (78) almost surely.

When the relay links and the interference links share the same channel gain, i.e. $\alpha = \beta$, the GDoF of the sum capacity reduces to

$$d_{\text{sum}} = \min \{ 2 + \kappa - \alpha, 2 \max \{ \alpha, 1 - \alpha \} + 2\kappa, 2 \} \quad (79)$$

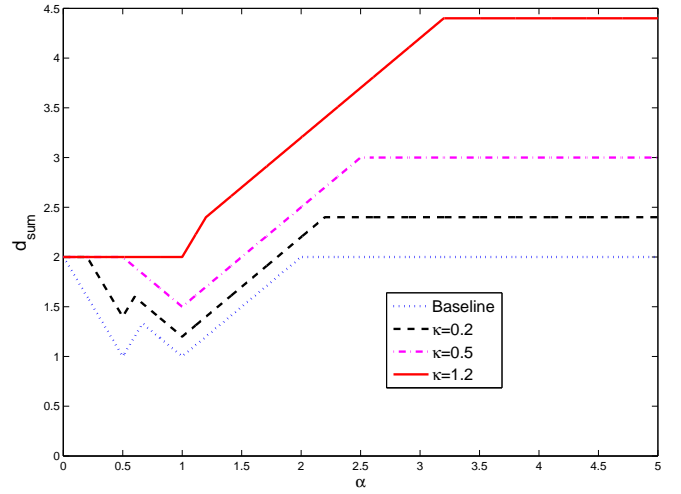


Fig. 4. The GDoF gain due to the relay in a symmetric Gaussian relay-interference channel for the $\alpha = \beta$ case

for $0 \leq \alpha < 1$, and

$$d_{\text{sum}} = \min \{ \alpha + \kappa, 2(1 + \kappa), 2\alpha \}, \quad (80)$$

for $\alpha \geq 1$. Interestingly, this is the same as the sum capacity (in GDoF) of the Gaussian interference channel with rate-limited receiver cooperation [6]. Therefore, the same sum capacity GDoF gain can be achieved with either receiver cooperation or with an independent in-band-reception and out-of-band-transmission relay assuming that the source-relay links are the same as the interfering links of the underlying interference channel (i.e. $\alpha = \beta$).

Fig. 4 shows the GDoF gain due to the relay for the $\alpha = \beta$ case. There are several interesting features. When $\kappa = 0.2$, the GDoF curve remains the “W” shape for the conventional Gaussian interference channel [1]. The sum-capacity gain is 2κ in the very and moderately weak interference regimes (when $0.2 \leq \alpha \leq 0.6$) or the very strong interference regime ($\alpha \geq 2.2$), and is κ in other regimes ($\frac{2}{3} \leq \alpha \leq 2$). As κ gets larger, the left “V” branch of the “W” curve becomes smaller, and it disappears completely at the critical point of $\kappa = 0.5$. As κ keeps increasing, the right “V” of the “W” curve also eventually disappears. The detailed sum-capacity gains for different values of α are listed in Table I.

G. Interpretation via the Deterministic Relay Channel

In the Han-Kobayashi framework, each input signal of the interference channel consists of both a common message and a private message. The sum-capacity gain due to the relay in the relay-interference channel therefore in general includes improvements in both the common and the private message rates. This section illustrates that in the asymptotic high SNR regime, the rate improvement can be interpreted as either a private rate gain alone, or a common rate gain alone. Further, the one-bit-per-relay-bit or the two-bits-per-relay-bits GDoF improvement shown in the previous section can be interpreted using a deterministic relay model. The rest of this section illustrates this point for the symmetric Gaussian

TABLE I

SUM-CAPACITY GDoF GAIN DUE TO THE RELAY FOR THE SYMMETRIC GAUSSIAN RELAY-INTERFERENCE CHANNEL FOR THE $\alpha = \beta$ AND $\kappa \leq \frac{1}{2}$ CASE

Range of α	$\alpha \leq \kappa$	$\kappa \leq \alpha \leq \frac{2-\kappa}{3}$	$\frac{2-\kappa}{3} \leq \alpha \leq \frac{2}{3}$	$\frac{2}{3} \leq \alpha \leq 2$	$2 \leq \alpha \leq 2 + \kappa$	$\alpha \geq 2 + \kappa$
Gain	2α	2κ	$2 - 3\alpha + \kappa$	κ	$\alpha + \kappa - 2$	2κ

relay-interference channel in the $\alpha = \beta$ and $\kappa \leq \frac{1}{2}$ case as an example.

1) *Very Weak Interference Regime*: For the symmetric Gaussian interference channel, in the very weak interference regime of $0 \leq \alpha \leq \frac{1}{2}$, common messages do not carry any information (although it can be assigned nonzero powers as in the Etkin-Tse-Wang power splitting strategy). Setting X_1 and X_2 to be private messages only is capacity achieving in terms of GDoF ([1], [42]–[44]).

Assigning X_1 and X_2 to be private only is also optimal for GDoF for the symmetric Gaussian relay-interference channel in the very weak interference regime. This is because when X_1 and X_2 are both private messages and are treated as noises at Y_2 and Y_1 respectively, the relay-interference channel asymptotically becomes two deterministic relay channels in the high SNR regime. Consider the relay operation for Y_1 as illustrated in Fig. 5(a). When noise variances of Z_1 and Z_R go down to zero, the observation at the relay becomes $Y_R = gX_1 + gX_2$ and the received signal at receiver 1 becomes $Y_1 = h_dX_1 + h_cX_2$. In this case, the relay's observation is a deterministic function of X_1 and Y_1 , i.e. $Y_R = gX_1 + \frac{g}{h_c}(Y_1 - h_dX_1)$. Thus X_1 and Y_1 , along with the relay Y_R form a deterministic relay channel of the type studied in [45]. According to [45], the achievable rate of user 1 is given by

$$\begin{aligned} R_1 &= \min \{I(X_1; Y_1, Y_R), I(X_1; Y_1) + C\} \\ &= \min \left\{ \frac{1}{2} \log(1 + h_d^2), \frac{1}{2} \log \left(1 + \frac{h_d^2}{h_c^2} \right) + C \right\} \\ &\rightarrow \min \{1, 1 - \alpha + \kappa\}, \end{aligned} \quad (81)$$

resulting in one-bit improvement for each relay bit in the regime $\kappa \leq \alpha \leq \frac{1}{2}$. Similarly, as illustrated in Fig. 5(b), X_2 , Y_2 , and Y_R form another deterministic relay channel with X_2 as the input, Y_2 as the output, and Y_R as the relay. Thus, the achievable rate of user 2 is the same as user 1, resulting in the same one-bit-per-relay-bit improvement. Further, as shown in [45], a hash-and-forward relay strategy achieves the capacity for deterministic relay channels. As the hashing operation is the same for both case, the same relay bit can therefore benefit both receivers at the same time, resulting in two-bit increase in sum capacity for one relay bit, as first pointed out in [22].

2) *Moderately Weak and Strong Interference Regimes*: The above interpretation, which states that the GDoF improvement in the very weak interference regime comes solely from the private rate gain, is not the only possible interpretation. The rate gain can also be interpreted as improvement in common information rate — an interpretation that applies not only to the very weak interference regime, but in fact to all regimes (for the symmetric rate with symmetric channels). In the following, we illustrate this point by focusing on a two-stage Han-Kobayashi strategy, where common messages are decoded first, then the private messages. This is the same

two-stage Han-Kobayashi scheme used in [1] for the Gaussian interference channel without the relay.

Specifically, the relay uses the same GHF relaying strategy as in Theorem 3, but it is now designed to help the common messages only. Here, both common messages W_1^n and W_2^n are decoded and subtracted at both receivers with the help of the GHF relay first (while treating private messages as noise), the private messages are then decoded at each receiver treating each other as noise. The decoding of the private message at the second stage results in

$$\begin{aligned} R_u &= \frac{1}{2} \log \left(1 + \frac{\text{SNR}_p}{1 + \text{INR}_p} \right) \\ &\rightarrow \max \{0, 1 - \alpha\}, \end{aligned} \quad (82)$$

Note that the relay does not help the private rate.

In the common-message decoding stage, W_1^n and W_2^n are jointly decoded at both receiver 1 and receiver 2 with the help of the GHF relay. As a result, $(W_1^n, W_2^n, Y_1^n, Y_R^n)$ forms a multiple-access relay channel at receiver 1 with W_1^n, W_2^n as the inputs, Y_1^n as the output and Y_R^n as the relay. The achievable rate region of such a multiple-access channel with a GHF relay is given by

$$\begin{aligned} R_{w1} &\leq I(W_1; Y_1 | W_2) \\ &\quad + \min \{ (C - \xi)^+, I(W_1; \hat{Y}_R | Y_1, W_2) \} \\ R_{w2} &\leq I(W_2; Y_1 | W_1) \\ &\quad + \min \{ (C - \xi)^+, I(W_2; \hat{Y}_R | Y_1, W_1) \} \\ R_{w1} + R_{w2} &\leq I(W_1, W_2; Y_1) \\ &\quad + \min \{ (C - \xi)^+, I(W_1, W_2; \hat{Y}_R | Y_1) \}. \end{aligned}$$

With the Etkin-Tse-Wang input strategy (i.e. $P_{1p} = \min\{1, h_{12}^{-2}\}$, $P_{2p} = \min\{1, h_{21}^{-2}\}$) and the GHF relaying scheme with $q_1 = q_2 = \frac{\sqrt{\rho^2 + 16\rho + 16} - \rho}{4}$, it can be shown that the common-message rate region for the receiver 1 in the high SNR regime in term of GDoF is given as follows. When $0 \leq \alpha \leq 1$

$$\begin{aligned} R_{w1} &\leq \alpha \\ R_{w2} &\leq \min\{\alpha, \kappa + \max\{2\alpha - 1, 0\}\} \\ R_{w1} + R_{w2} &\leq \alpha + \min\{\alpha, \kappa\}. \end{aligned}$$

When $\alpha \geq 1$

$$\begin{aligned} R_{w1} &\leq \min\{\alpha, 1 + \kappa\} \\ R_{w2} &\leq \alpha \\ R_{w1} + R_{w2} &\leq \alpha + \kappa. \end{aligned}$$

Due to symmetry, the rate region for the multiple-access relay channel at receiver 2 can be obtained by switching the indices 1 and 2.

Note that in suitable interference regimes, both the individual rate and the sum rate can potentially be increased by

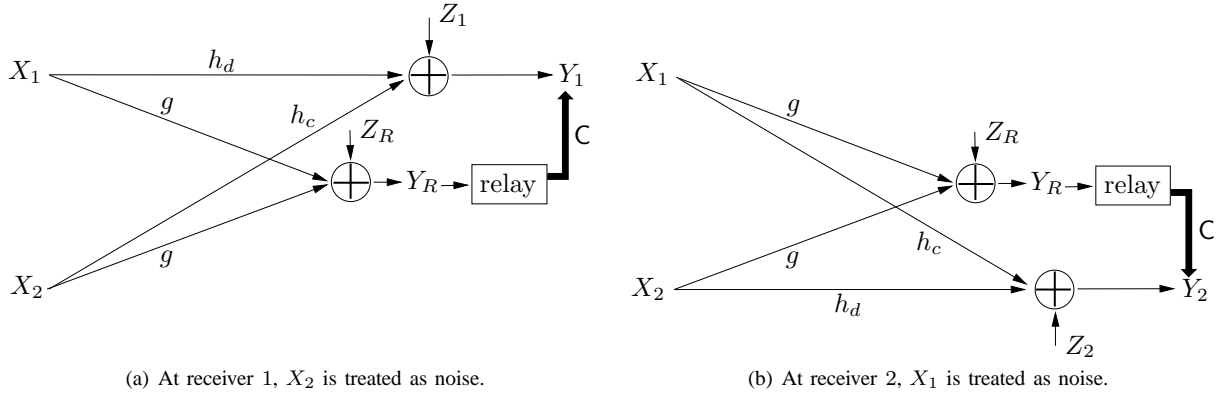


Fig. 5. Asymptotic deterministic relay channels in the very weak interference regime $\kappa \leq \alpha \leq \frac{1}{2}$.

one bit for each relay bit. This is again a consequence of the fact that the relay operation has a deterministic relay channel interpretation in the high SNR regime. For example, in the strong interference regime where $1 \leq \alpha \leq 2 + \kappa$, the sum rate of the multiple-access relay channel benefits by one bit for each relay bit in the high SNR regime as shown in Fig. 6(a). In the very strong interference regime, the interference can be decoded, subtracted or can serve as side information, therefore the individual rate increases by one bit for each relay bit as shown in Fig. 6(b).

Now, the achievable rates of common messages can be obtained by intersecting the two rate regions. Taking the achievable rates of private messages in (82) into account, it is easy to verify that this two-stage Han-Kobayashi scheme achieves the sum capacity in (79) and (80). As depicted in Fig. 4, the sum-capacity gain due to the relay can be one-bit-per-bit or two-bits-per-bit. In the following, we demonstrate in Fig. 7 how these gains are obtained by pictorially showing the intersection of the two common-message rate regions for different values of α .

- When $\alpha \leq \kappa$, as can be seen from Fig. 7(a), the two rate regions are identical and are both given by $\{(R_{w1}, R_{w2}) : R_{w1} \leq \alpha, R_{w2} \leq \alpha\}$. The intersection of the two is the same rectangle with the top-right corner located at (α, α) . This gives a 2α -bit gain over the baseline, which is located at the origin.
- As α increases to $\kappa \leq \alpha \leq \frac{1}{2}$, the baseline rate pair is still at the origin. With the help of the relay, the two common-message rate regions become rectangles $\{(R_{w1}, R_{w2}) : R_{w1} \leq \alpha, R_{w2} \leq \kappa\}$ and $\{(R_{w1}, R_{w2}) : R_{w1} \leq \kappa, R_{w2} \leq \alpha\}$ respectively. As shown in Fig. 7(b), the intersection of the two gives a square with the top-right corner located at (κ, κ) . As a result, the sum-capacity gain is 2κ bits.
- As α increases to $\frac{1}{2} \leq \alpha \leq 1$, the common-message rate regions at receivers 1 and 2 become pentagons. However, depending on the value of α , the sum rate improves by different amounts. When $\alpha \leq \frac{2-\kappa}{3}$, as shown in Fig. 7(c), the intersection of the two pentagon regions gives a square shape with the top-right corner located at $(2\alpha - 1 + \kappa, 2\alpha - 1 + \kappa)$. Compared with $(2\alpha - 1, 2\alpha - 1)$ achieved without the relay, a sum-capacity gain of 2κ

bits is obtained. However, when $\alpha \geq \frac{2-\kappa}{3}$, as depicted in Fig. 7(d), the intersection of the two rate regions is still a pentagon with the sum-capacity limited by $R_{w1} + R_{w2} \leq 2 - \alpha + \kappa$. In this case, depending on the value of α , the sum-rate gain is $2 - 3\alpha + \kappa$ bits when $\frac{2-\kappa}{3} \leq \alpha \leq \frac{2}{3}$, and is κ bits when $\frac{2}{3} \leq \alpha \leq 1$. (The latter case is shown in Fig. 7(d).)

- When $1 \leq \alpha \leq 2 + \kappa$, the common-message rate regions are again pentagons and the interpretation is similar to the $\frac{2-\kappa}{3} \leq \alpha \leq 1$ case. Fig. 7(e) shows an example of $1 \leq \alpha \leq 1 + \kappa$. In this case, the two rate regions are identical pentagons with the sum capacity limited by $R_{w1} + R_{w2} \leq \alpha + \kappa$. Compared with the baseline sum capacity, a κ -bits gain is obtained. When $1 + \kappa \leq \alpha \leq 2 + \kappa$, the intersection of the two common-message rate regions again gives a sum-capacity of $\alpha + \kappa$. However, since the baseline sum capacity becomes saturated when $\alpha \geq 2$ ([1], [46], [47]), the sum-capacity gain over the baseline is κ bits when $1 \leq \alpha \leq 2$, and is $\alpha + \kappa - 2$ bits when $2 \leq \alpha \leq 2 + \kappa$.
- Finally, $\alpha \geq 2 + \kappa$ falls into the very strong interference regime. The intersection of the two common-message rate regions is a rectangle with the top-right corner located at $(1 + \kappa, 1 + \kappa)$ as shown in Fig. 7(f). The sum-capacity gain is thus 2κ bits in the very strong interference regime.

III. GAUSSIAN RELAY-INTERFERENCE CHANNEL WITH A SINGLE DIGITAL LINK

The result of the previous section shows that for the symmetric channel, the sum-capacity improvement can be thought as coming solely from the improvement of the common message rate, or in a very weak interference regime as coming solely from the improvement of the private message rates. Thus, the function of the relay for the symmetric rate in symmetric channel is solely in forwarding useful signals. This interpretation does not necessarily hold for the asymmetric cases. In this section, we study a particular asymmetric channel to illustrate the composition of the sum-capacity gain. We are motivated by the fact that the relay's observation in a relay-interference channel is a linear combination of the intended signal and the interfering signal. Clearly, forwarding the intended signal and the interfering signal can both be

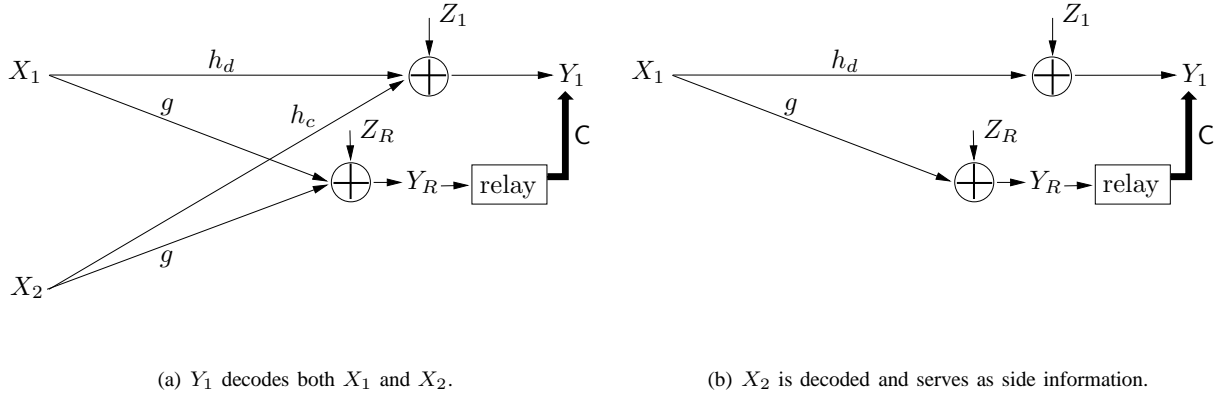
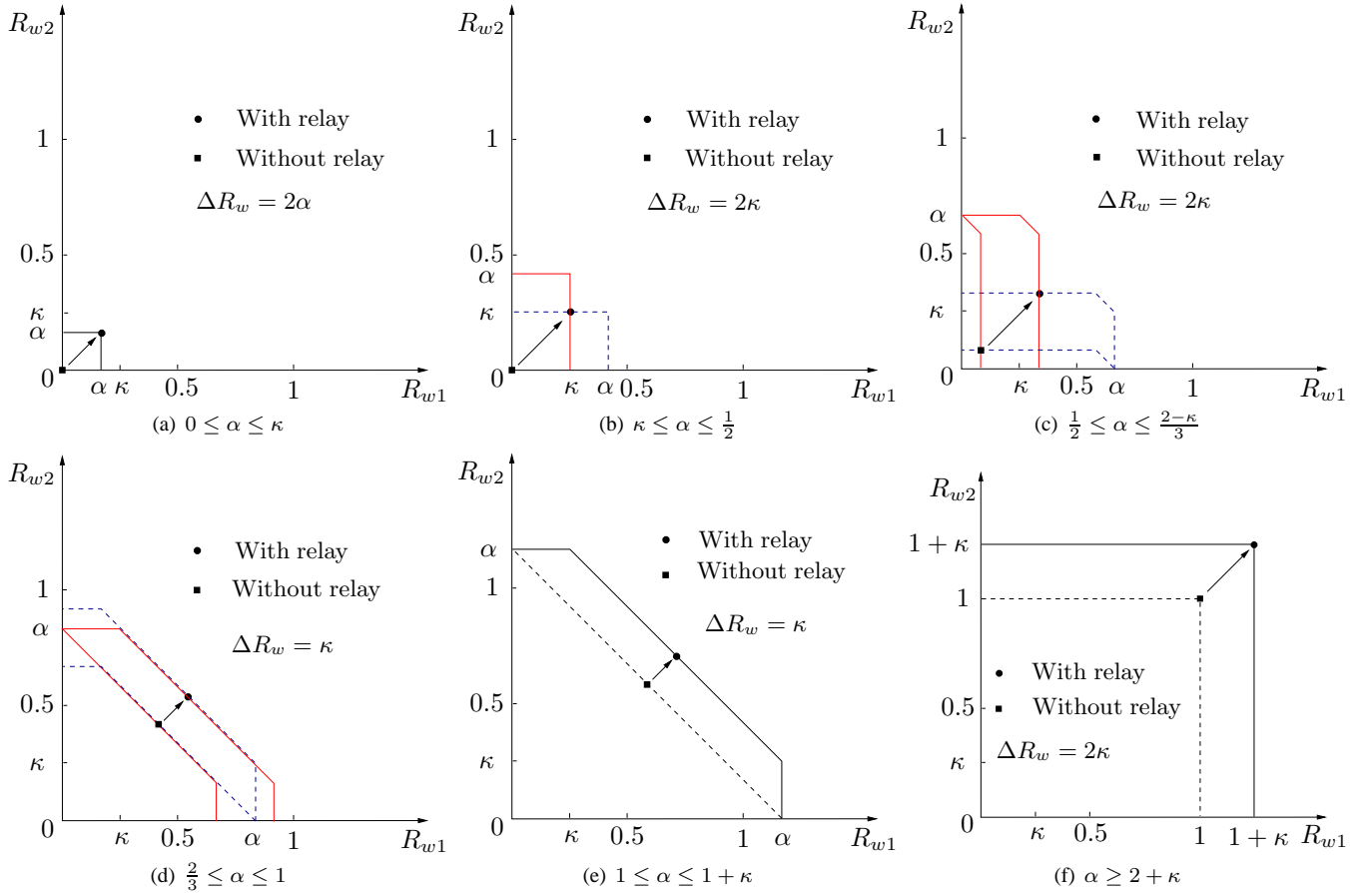


Fig. 6. Asymptotic deterministic relay channels in the strong and very strong interference regimes.

Fig. 7. Generalized-degree-of-freedom gain due to relaying is roughly κ or 2κ depending on how the two common-message multiple-access regions are intersected

beneficial (e.g. [12]). This section illustrates that depending on the different channel parameters, the sum-rate gain from forwarding both intended signal and interference signal happens to be the same as that of forwarding intended signal only or forwarding interference signal only.

Specifically, we focus on a particular asymmetric model as shown in Fig. 8, where the digital relay link exists only for receiver 1, and not for receiver 2, i.e., $C_2 = 0$. This section first derives a constant-gap-to-capacity result for this channel. Note that this channel is a special case of the general channel model studied in the previous section, but

the constant-gap-to-capacity result can be established in this special case for a broader set of channels. Unlike the weak-relay assumption $|g_1| \leq \sqrt{\rho}|h_{12}|$ and $|g_2| \leq \sqrt{\rho}|h_{21}|$ made in the previous section, this section assumes that $|g_2| \leq \sqrt{\rho}|h_{21}|$ only with no constraints on g_1 or h_{12} . Under this channel setup, it can be shown that in the high SNR regime, the sum capacity improvement can also be obtained as if only the intended signal is forwarded or only the interference signal is forwarded. Note that this conclusion applies to the case of a single relay-destination link only, and not necessarily to the general case with two relay-destination links.

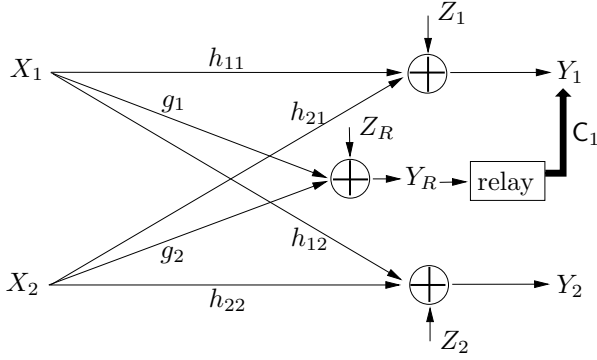


Fig. 8. Gaussian relay-interference channel with a single digital link

A. Capacity Region to within Constant Gap in the Weak-Relay Regime

Since the channel model studied in Fig. 8 is a special case of the general Gaussian relay-interference channel, we first simplify the achievable rate region in Theorem 2 to the following corollary by setting $C_2 = 0$. The only difference in the coding scheme is that instead of performing two quantizations as in the general relay-interference channel, the relay in Fig. 8 does one quantization of the received signal Y_R into \hat{Y}_{R1} and sends the bin index of \hat{Y}_{R1} to receiver 1 through the digital link C_1 .

Corollary 3. *For the Gaussian relay-interference channel with a single digital link as shown in Fig. 8, the following rate region is achievable:*

$$0 \leq R_1 \leq d_1 + \min \{(C_1 - \xi_1)^+, \Delta d_1\} \quad (83)$$

$$0 \leq R_2 \leq d_2 \quad (84)$$

$$R_1 + R_2 \leq a_1 + g_2 + \min \{(C_1 - \xi_1)^+, \Delta a_1\} \quad (85)$$

$$R_1 + R_2 \leq a_2 + g_1 + \min \{(C_1 - \xi_1)^+, \Delta g_1\} \quad (86)$$

$$R_1 + R_2 \leq e_1 + e_2 + \min \{(C_1 - \xi_1)^+, \Delta e_1\} \quad (87)$$

$$2R_1 + R_2 \leq a_1 + g_1 + e_2 + \min \{2(C_1 - \xi_1)^+, (C_1 - \xi_1)^+ + \Delta a_1, \Delta a_1 + \Delta g_1\} \quad (88)$$

$$R_1 + 2R_2 \leq a_2 + g_2 + e_1 + \min \{(C_1 - \xi_1)^+, \Delta e_1\}, \quad (89)$$

where all the parameters are as defined in Theorem 2.

The proof follows directly from Theorem 2. Note that in (88), we apply the fact that $\Delta a_1 \leq \Delta g_1$. Likewise, the capacity region outer bound in Theorem 1 also simplifies when $C_2 = 0$. We can now prove the following constant-gap theorem for the Gaussian relay-interference channel with a single digital link.

Theorem 4. *For the Gaussian relay-interference channel with a single digital link as depicted in Fig. 8, with the same signaling strategy as in Theorem 3, i.e. a combination of the Han-Kobayashi scheme with Etkin-Tse-Wang power splitting strategy and the GHF relaying scheme with the fixed quantization level $q_1 = \frac{\sqrt{\rho^2 + 16\rho + 16} - \rho}{4}$, in the weak-relay regime of $|g_2| \leq \sqrt{\rho}|h_{21}|$, the achievable rate region in Corollary 3 is within δ bits of the capacity region outer bound in Theorem 1 (with C_2 set to zero), where δ is defined in Theorem 3.*

Proof: Although the signalling scheme and the constant gap result resemble those of Theorem 3, Theorem 4 is not simply obtained by setting $C_2 = 0$ in Theorem 3, since the weak-relay condition has been relaxed. In the following, we prove the constant-gap result by directly comparing each achievable rate expression with its corresponding upper bound.

Applying the inequalities of Lemma 1 and following along the same lines of the proof of Theorem 3 in Appendix C, it is easy to show that each of the achievable rates in (83)-(89) achieves to within a constant gap of its corresponding upper bound in Theorem 1 (with C_2 set to zero) in the weak-relay regime. The constant gaps are shown as follows:

(i) Individual rate (83) is within

$$\delta_{R_1} = \max \{\alpha(q_1), \beta(q_1)\} \quad (90)$$

bits of (7).

(ii) Individual rate (84) is within

$$\delta_{R_2} = \frac{1}{2} \quad (91)$$

bits of (8).

(iii) Sum rates (85), (86) and (87) are within

$$\delta_{R_1+R_2} = \frac{1}{2} + \max \{\alpha(q_1), \beta(q_1)\} \quad (92)$$

bits of their upper bounds (9), (16), (10), (15), (11), and (17). Specifically,

- The first term of (85) is within $\frac{1}{2} + \beta(q_1)$ bits of (9). The second term is within $\frac{1}{2} + \alpha(q_1)$ bits of (16).
- The first term of (86) is within $\frac{1}{2} + \beta(q_1)$ bits of (10). The second term is within $\frac{1}{2} + \alpha(q_1)$ bits of (15).
- The first term of (87) is within $\frac{1}{2} + \beta(q_1)$ bits of (11). The second term is within $\frac{1}{2} + \alpha(q_1)$ bits of (17).

Therefore, the achievable sum rate in (85)-(87) is within a constant gap of the sum-rate upper bound specified by (9)-(20) in the weak-relay regime.

(iv) $2R_1 + R_2$ rate (88) is within

$$\delta_{2R_1+R_2} = \frac{1}{2} + \max \{2\alpha(q_1), \alpha(q_1) + \beta(q_1), 2\beta(q_1)\} \quad (93)$$

bits of the upper bounds (21), (26), and (24). Specifically, the first term of (88) is within $\frac{1}{2} + 2\beta(q_1)$ bits of (21). The second term is within $\frac{1}{2} + \alpha(q_1) + \beta(q_1)$ bits of (26). The third term is within $\frac{1}{2} + 2\alpha(q_1)$ bits of (24).

(v) $R_1 + 2R_2$ rate (89) is within

$$\delta_{R_1+2R_2} = 1 + \max \{\alpha(q_1), \beta(q_1)\} \quad (94)$$

bits of the upper bounds

$$2R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log\left(1 + \text{INR}_2 + \frac{\text{SNR}_1}{1 + \text{INR}_1}\right) + \frac{1}{2} \log\left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2}\right) + C_1 \quad (95)$$

$$2R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log\left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2 + \text{SNR}_{r2}}\right) + \frac{1}{2} \log\left(1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1 + \text{INR}_2 + \text{SNR}_{r2}}\right), \quad (96)$$

which are not shown explicitly in Theorem 1 but can be obtained by switching the indices 1 and 2 of (21) and (25) followed by setting $C_2 = 0$.

Since $\alpha(\cdot)$ is an increasing function and $\beta(\cdot)$ is a decreasing function, to minimize the gaps above, we need

$$\alpha(q_1^*) = \beta(q_1^*), \quad (97)$$

which results in the quantization level $q_1^* = \frac{\sqrt{\rho^2 + 16\rho + 16} - \rho}{4}$. With this optimal quantization level applied to the gaps above, we prove that the achievable rate region (83)-(89) is within

$$\max \left\{ \frac{1}{2}, \frac{1}{2} \log \left(2 + \frac{\rho + \sqrt{\rho^2 + 16\rho + 16}}{2} \right) \right\} = \frac{1}{2} \log \left(2 + \frac{\rho + \sqrt{\rho^2 + 16\rho + 16}}{2} \right) \quad (98)$$

bits of the capacity region. ■

B. Generalized Degree of Freedom

We now derive the GDoF of the channel depicted in Fig. 8, for the case where the underlying interference channel is symmetric, i.e., $\text{INR}_1 = \text{INR}_2 = \text{INR}$ and $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$. In the high SNR regime, define

$$\beta_i := \lim_{\text{SNR} \rightarrow \infty} \frac{\log \text{SNR}_{ri}}{\log \text{SNR}}, \quad i = 1, 2, \quad (99)$$

$$\kappa_1 := \lim_{\text{SNR} \rightarrow \infty} \frac{C_1}{\frac{1}{2} \log \text{SNR}}, \quad (100)$$

Applying Theorem 4, we have the following result on the GDoF:

Corollary 4. *In the weak-relay regime where $\beta_2 \leq \alpha$, the GDoF sum capacity of the symmetric relay-interference channel with a single digital link is given by the following. For $0 \leq \alpha < 1$*

$$d_{\text{sum}} = \begin{cases} \min\{2 - \alpha, 2 \max(\alpha, 1 - \alpha) + \kappa_1, \max(\alpha, 1 - \alpha) + \max(\beta_1, 1 + \beta_2 - \alpha, \alpha)\}, & \beta_1 \leq 1 \\ \min\{2 - \alpha + \kappa_1, 2 \max(\alpha, 1 - \alpha) + \kappa_1, 1 + \beta_1 - \alpha\}, & \beta_1 \geq 1 \end{cases}$$

and for $\alpha \geq 1$

$$d_{\text{sum}} = \min\{\alpha, 2 + \kappa\}. \quad (101)$$

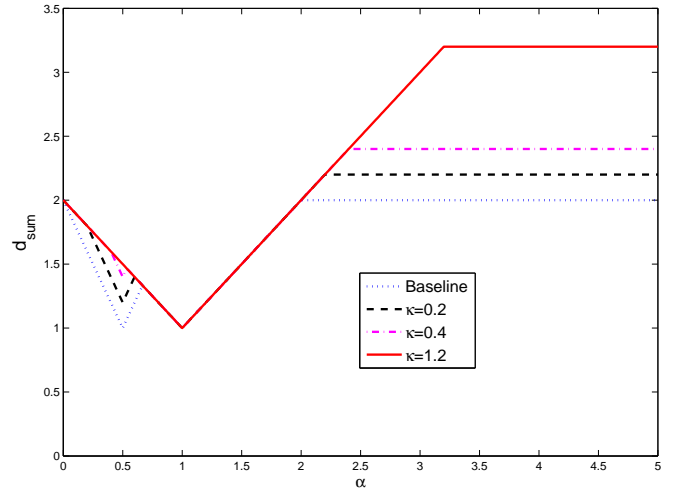


Fig. 9. Impact of the relay-destination link on sum capacity

Table II and Fig. 9 illustrate the GDoF gain due to the relay where the direct links, the interference links and the links to the relay are symmetric for both users, and where $\alpha = \beta_1 = \beta_2$. The main feature here is that there is no gain in sum capacity for $\frac{2}{3} \leq \alpha \leq 2$. In other regimes of α , the sum-capacity gain is roughly one bit per relay bit.

C. Signal Relaying vs. Interference Forwarding

In the relay-interference channel, the relay observes a corrupted version of the weighted sum of two source signals X_1 and X_2 , and forwards a description to the receiver. Intuitively, the observations about both source signals are helpful. For the receiver 1, the observation about X_1 helps receiver 1 reinforce the signal intended for it; the observation about X_2 helps receiver 1 mitigate the interference. The former can be thought of as signal relaying, the latter interference forwarding.

In this section, we show that the sum-capacity gain in a Gaussian interference channel due to a single relay link is equivalent to that achievable with signal relaying alone or with interference forwarding alone, depending on the channel parameters. Toward this end, we first set the source-relay link from X_2 to zero, i.e., $g_2 = 0$, and compute the GDoF of the sum capacity. In this case, the sum-capacity gain must be solely due to forwarding intended signal X_1 . Similarly, we can also set $g_1 = 0$, and compute the GDoF of the sum-capacity gain due solely to forwarding interference signal X_2 . By comparing these rates we show that interestingly when the relay link of user 1 is under certain threshold, i.e., $\beta_1 \leq 1 - \alpha + \beta_2$, the sum-capacity gain is equivalent to that achievable by interference forwarding. When $\beta_1 \geq 1 - \alpha + \beta_2$, the sum-capacity gain is equivalent to that achievable by signal relaying.

More specifically, with $g_2 = 0$, the sum-capacity can be computed as

$$d_{SR} = \begin{cases} \min\{2 - \alpha, 2 \max(\alpha, 1 - \alpha) + \kappa_1, \max(\alpha, 1 - \alpha) + \max(\beta_1, 1 - \alpha, \alpha)\}, & \beta_1 \leq 1 \\ \min\{2 - \alpha + \kappa_1, 2 \max(\alpha, 1 - \alpha) + \kappa_1, 1 + \beta_1 - \alpha\}. & \beta_1 \geq 1 \end{cases}$$

TABLE II
SUM-CAPACITY GDoF GAIN DUE TO THE RELAY FOR THE SYMMETRIC GAUSSIAN RELAY-INTERFERENCE CHANNEL WITH A SINGLE DIGITAL RELAY LINK FOR $\alpha = \beta_1 = \beta_2$

Range of α	$\alpha \leq \kappa$	$\kappa \leq \alpha \leq \frac{2-\kappa}{3}$	$\frac{2-\kappa}{3} \leq \alpha \leq \frac{2}{3}$	$\frac{2}{3} \leq \alpha \leq 2$	$2 \leq \alpha \leq 2 + \kappa$	$\alpha \geq 2 + \kappa$
Gain	α	κ	$2 - 3\alpha$	0	$\alpha - 2$	κ

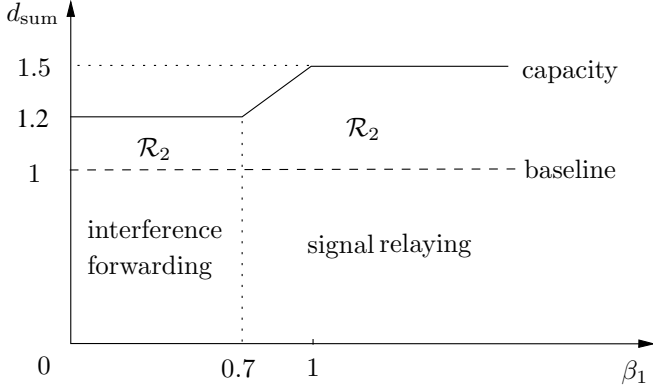


Fig. 10. Signal relaying vs. interference forwarding

Similarly, let $g_1 = 0$. The sum-capacity GDoF obtained by forwarding interference signal is

$$d_{IF} = \min\{2 - \alpha, 2 \max(\alpha, 1 - \alpha) + \kappa_1, \max(\alpha, 1 - \alpha) + \max(1 + \beta_2 - \alpha, \alpha)\}. \quad (102)$$

Comparing (101), (102), and (102), it is easy to verify that

$$d_{sum} = \begin{cases} d_{IF} & \text{when } \beta_1 \leq 1 + \beta_2 - \alpha \\ d_{SR} & \text{when } \beta_1 \geq 1 + \beta_2 - \alpha \end{cases}. \quad (103)$$

Therefore, we observe the following threshold effects. When the relay link from user 1 is weak, the sum-capacity gain is equivalent to a channel with a single source-relay link from X_2 . As the source-relay link from X_1 grows stronger and crosses a threshold $\beta_1 \geq 1 + \beta_2 - \alpha \triangleq \beta_1^*$, the sum-capacity gain becomes equivalent to that of a single source-relay link from X_1 . Note that this is a GDoF phenomenon in the high SNR regime. In the general SNR regime, the sum-capacity gain contains contributions from both signal relaying and interference forwarding.

To visualize the interaction of signal relaying and interference forwarding, a numerical example is provided in Fig. 10. The channel parameters are set to $\alpha = 0.5$, $\beta_2 = 0.2$, and $\kappa_1 = 0.5$. The GDoF of the sum capacity is plotted as a function of β_1 . The sum capacity of the interference channel without the relay serves as the baseline:

$$d_{BL} = \min\{2 - \alpha, 2 \max(\alpha, 1 - \alpha)\}. \quad (104)$$

Fig. 10 shows the sum-capacity gain due to the relay. When $\beta_1 \leq \beta_1^* = 0.7$, the gain (labeled as \mathcal{R}_1) is equivalent to that by forwarding interference signal only. When $\beta \geq 0.7$, the gain (labeled as \mathcal{R}_2) is equivalent to that by forwarding intended signal only.

IV. CONCLUSION

This paper investigates GHF as an incremental relay strategy for a Gaussian interference channel augmented with an out-of-band broadcasting relay, in which the relay message to

one receiver is a degraded version of the message to the other receiver. We focus on a weak-relay regime, where the transmitter-to-relay links are not unboundedly stronger than the interfering links of the interference channel, and show that GHF achieves to within a constant gap to the capacity region in the weak-relay regime. Further, in a symmetric setting, each common relay bit can be worth either one or two bits in the sum capacity gain, illustrating the potential for a cell-edge relay in improving the system throughput of a wireless cellular network.

Furthermore, the Gaussian relay-interference channels with a single relay link is also studied. The capacity region is characterized to within a constant gap for a larger range of channel parameters. It is shown that in the high SNR regime, the sum-capacity improvement is equivalent either to that of a single source-relay link from user 1 or that of a single source-relay link from user 2.

APPENDIX

A. Proof of Theorem 1

Define V_1^n as the output of the digital link C_1 , and V_2^n as the output of the digital link C_2 . The outer bounds are proved as follows:

(i) Individual-rate bounds: First, the first term of (7) is the simple cut-set upper bound for R_1 . For the second term, starting from Fano's inequality, we have

$$\begin{aligned} n(R_1 - \epsilon_n) &\leq I(X_1^n; Y_1^n, V_1^n) \\ &\leq I(X_1^n; Y_1^n, Y_R^n, X_2^n) \\ &\leq \frac{n}{2} \log(1 + \text{SNR}_1 + \text{SNR}_{r1}). \end{aligned} \quad (105)$$

The outer bound of R_2 in (8) can be proved in the same way.

(ii) Sum-rate bounds:

- First, (9)-(11) are obtained from Fano's inequalities, i.e.,

$$\begin{aligned} n(R_1 + R_2 - \epsilon_n) & \\ &\leq I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\ &= I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + I(X_1^n; V_1^n | Y_1^n) \\ &\quad + I(X_2^n; V_2^n | Y_2^n) \\ &\leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + h(V_1^n) + h(V_2^n) \\ &\leq nC_{sum}(0) + nC_1 + nC_2, \end{aligned} \quad (106)$$

where $C_{sum}(0)$ is the sum capacity of the interference channel without relay. Clearly, the sum-rate gain due to the digital relay is upper bounded by the rates of digital links. Although the sum-rate capacity $C_{sum}(0)$ is not known in general, its upper bound has been studied in literature [1], [34], [42]–[44], [48]. Applying the sum-rate outer bounds in [34], we obtain (9)-(11).

- Second, (12)-(14) can be obtained by the following steps:

$$\begin{aligned} n(R_1 + R_2 - \epsilon_n) & \quad (107) \\ & \leq I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\ & \stackrel{(a)}{\leq} I(X_1^n; Y_1^n) + h(V_1^n) + I(X_2^n; Y_2^n, Y_R^n), \end{aligned}$$

where in (a) we give genie Y_R^n to receiver 2 and apply the fact that \hat{Y}_R is a function of Y_R . Note that $I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n, Y_R^n)$ is upper bounded by the sum capacity of the SIMO interference channel with X_1^n and X_2^n as the input, and Y_1^n and (Y_2^n, Y_R^n) as the output. The sum-rate outer bound of such a SIMO interference channel has been studied in [34], which along with $h(V_1^n) \leq nC_1$ gives the outer bounds of (12)-(14).

- Third, (15)-(17) can be similarly derived following the same steps of (12)-(14) with indices 1 and 2 switched.
- Fourth, (18)-(20) can be obtained by giving Y_R^n as a genie to both receivers, i.e.,

$$\begin{aligned} n(R_1 + R_2 - \epsilon_n) & \quad (108) \\ & \leq I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\ & \leq I(X_1^n; Y_1^n, Y_R^n) + I(X_2^n; Y_2^n, Y_R^n), \end{aligned}$$

which is upper bounded by the sum capacity of the SIMO interference channel with X_1^n and X_2^n as input, and (Y_1^n, Y_R^n) and (Y_2^n, Y_R^n) as output. Applying the result in [34], we have (18)-(20).

(iii) $2R_1 + R_2$ bounds: Six upper bounds on $2R_1 + R_2$.

- First, (21) is simply the cut-set bound, i.e.,

$$\begin{aligned} n(2R_1 + R_2 - \epsilon_n) & \quad (109) \\ & \leq 2I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\ & \leq 2I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + 2h(V_1^n) + h(V_2^n), \end{aligned}$$

where $2I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n)$ is upper bounded by the $2R_1 + R_2$ bound of the interference channel with X_1^n and X_2^n as the input, and Y_1^n and Y_2^n as the output, which together with $h(V_1^n) \leq nC_1$ and $h(V_2^n) \leq nC_2$ gives the upper bound in (21).

- Second, (22) can be derived by giving genie Y_R^n to both receivers:

$$\begin{aligned} n(2R_1 + R_2 - \epsilon_n) & \quad (110) \\ & \leq 2I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\ & \leq 2I(X_2^n; Y_1^n, Y_R^n) + I(X_2^n; Y_2^n, Y_R^n), \end{aligned}$$

which is upper bounded by the $2R_1 + R_2$ bound of the SIMO interference channel with X_1^n and X_2^n as the input, and (Y_1^n, Y_R^n) and (Y_2^n, Y_R^n) as the output. Applying the result of [34], we obtain (22).

- Third, (23) can be obtained by giving genies (X_2^n, Y_R^n, S_1^n) to Y_1^n in one of the two R_1 expressions and (S_2^n, Y_R^n) to Y_2^n , where genies S_1^n and S_2^n are defined as

$$S_1^n = h_{12}X_1^n + Z_2, \quad S_2^n = h_{21}X_2^n + Z_1. \quad (111)$$

According to Fano's inequality, we have

$$\begin{aligned} n(2R_1 + R_2 - \epsilon_n) & \quad (112) \\ & \leq 2I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\ & \leq I(X_1^n; Y_1^n, Y_R^n, S_1^n, X_2^n) + I(X_1^n; Y_1^n) + h(V_1^n) \\ & \quad + I(X_2^n; Y_2^n, Y_R^n, S_2^n) \\ & \stackrel{(a)}{\leq} I(X_1^n; Y_1^n, Y_R^n, S_1^n | X_2^n) + h(Y_1^n) - h(S_2^n) + nC_1 \\ & \quad + I(X_2^n; S_2^n) + I(X_2^n; Y_2^n, Y_R^n | S_2^n) \\ & = I(X_1^n; S_1^n) + I(X_1^n; Y_1^n, Y_R^n | S_1^n, X_2^n) + h(Y_1^n) \\ & \quad - h(S_2^n) + nC_1 + h(S_2^n) - h(Z_1^n) \\ & \quad + h(Y_2^n, Y_R^n | S_2^n) - h(S_1^n) - h(Y_R^n | Y_2^n, X_2^n) \\ & = h(Y_1^n) - h(Z_1^n) + h(Y_1^n, Y_R^n | S_1^n, X_2^n) + nC_1 \\ & \quad - h(Z_1^n, Z_R^n) + h(Y_2^n, Y_R^n | S_2^n) - h(Z_2^n, Z_R^n) \\ & \quad - I(Y_R^n; X_1^n | X_2^n, Y_2^n) \\ & \leq h(Y_1^n) - h(Z_1^n) + h(Y_1^n, Y_R^n | S_1^n, X_2^n) + nC_1 \\ & \quad - h(Z_1^n, Z_R^n) + h(Y_2^n, Y_R^n | S_2^n) - h(Z_2^n, Z_R^n), \end{aligned}$$

where in (a) we use the fact that X_1^n is independent with X_2^n . Note that, the last inequality of (112) is maximized by Gaussian inputs X_1^n and X_2^n with i.i.d $\mathcal{N}(0, 1)$ entries, because

- $h(Y_1^n)$ is maximized by Gaussian distributions, and
- $h(Y_1^n, Y_R^n | S_1^n, X_2^n)$ and $h(Y_2^n, Y_R^n | S_2^n)$ are both maximized by Gaussian inputs since the conditional entropy under a power constraint is maximized by Gaussian distributions.

Applying Gaussian distributions to the last inequality of (112), we have (23).

- Fourth, (24) can be obtained by giving genie Y_R^n to Y_1^n , i.e.,

$$\begin{aligned} n(2R_1 + R_2 - \epsilon_n) & \quad (113) \\ & \leq 2I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\ & \leq 2I(X_1^n; Y_1^n, Y_R^n) + I(X_2^n; Y_2^n) + h(V_2^n), \end{aligned}$$

where $2I(X_1^n; Y_1^n, Y_R^n) + I(X_2^n; Y_2^n)$ is upper bounded by the $2R_1 + R_2$ bound of the SIMO interference channel with X_1^n and X_2^n as the input, and (Y_1^n, Y_R^n) and Y_2^n as the output. Applying the result of [34] and the fact that $h(V_2^n) \leq nC_2$, we obtain (24).

- Fifth, (25) can be obtained by giving genie Y_R^n to Y_2^n , i.e.,

$$\begin{aligned} n(2R_1 + R_2 - \epsilon_n) & \quad (114) \\ & \leq 2I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\ & \leq 2I(X_1^n; Y_1^n) + 2h(V_1^n) + I(X_2^n; Y_2^n, Y_R^n), \end{aligned}$$

where $2I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n, Y_R^n)$ is upper bounded by the $2R_1 + R_2$ bound of the SIMO interference channel with X_1^n and X_2^n as the input, and Y_1^n and (Y_2^n, Y_R^n) as the output. Applying the result of [34] and the fact that $h(V_1^n) \leq nC_1$, we obtain (25).

- Sixth, (26) can be obtained by giving genies (X_2^n, Y_R^n, S_1^n) to Y_1^n in one of the two R_1 expressions,

and S_2^n to Y_2^n , i.e.,

$$\begin{aligned}
n(2R_1 + R_2 - \epsilon_n) & \\
&\leq 2I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \\
&\leq I(X_1^n; Y_1^n, Y_R^n, S_1^n, X_2^n) + I(X_1^n; Y_1^n) + h(V_1^n) \\
&\quad + I(X_2^n; Y_2^n, S_2^n) + h(V_2^n) \\
&\leq I(X_1^n; S_1^n) + I(X_1^n; Y_1^n Y_R^n | S_1^n, X_2^n) + h(Y_1^n) \\
&\quad - h(S_2^n) + I(X_2^n; S_2^n) + I(X_2^n; Y_2^n | S_2^n) \\
&\quad + nC_1 + nC_2 \\
&\leq h(S_1^n) - h(Z_2^n) + h(Y_1^n, Y_R^n | S_1^n, X_2^n) \\
&\quad - h(Z_1^n, Z_R^n) + h(Y_1^n) - h(S_2^n) + h(S_2^n) - h(Z_1^n) \\
&\quad + h(Y_2^n | S_2^n) - h(S_1^n) + nC_1 + nC_2 \\
&= h(Y_1^n) - h(Z_1^n) + h(Y_1^n, Y_R^n | S_1^n, X_2^n) \\
&\quad - h(Z_1^n, Z_R^n) + h(Y_2^n | S_2^n) - h(Z_2^n) + nC_1 + nC_2,
\end{aligned} \tag{115}$$

which is maximized by Gaussian distributions of X_1^n and X_2^n with i.i.d entries following $\mathcal{N}(0, 1)$. Applying Gaussian distributions to (115), we obtain (26).

B. Useful Inequalities

This appendix provides several inequalities that are useful to prove the constant-gap theorems.

Lemma 1. For $\Delta a_i, a_i, \Delta d_i, d_i, \Delta e_i, e_i, \Delta g_i, g_i$ and $\xi_i, i = 1, 2$ as defined in (36)-(44), with Q set as a constant, when W_i, X_i are generated from a superposition coding of $X_i = U_i + W_i$ with $U_i \sim \mathcal{N}(0, P_{ip})$ and $W_i \sim \mathcal{N}(0, P_{ic})$, where $P_{ip} + P_{ic} = 1$ and $P_{1p} = \min\{1, h_{12}^{-2}\}, P_{2p} = \min\{1, h_{21}^{-2}\}$, and when the GHF quantization variables are set to $\hat{Y}_{R1} = Y_R + e_1, \hat{Y}_{R2} = Y_R + e_2$, where $e_1 \sim \mathcal{N}(0, q_1)$ and $e_2 \sim \mathcal{N}(0, q_2)$, in the weak-relay regime of $|g_1| \leq \sqrt{\rho}|h_{12}|, |g_2| \leq \sqrt{\rho}|h_{21}|$, the mutual information terms in (36)-(44) can be bounded as follows:

$$a_1 \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2}, \tag{116}$$

$$a_1 + \Delta a_1 \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1), \tag{117}$$

$$d_1 \geq \frac{1}{2} \log(1 + \text{SNR}_1) - \frac{1}{2}, \tag{118}$$

$$d_1 + \Delta d_1 \geq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{SNR}_{r1}) - \alpha(q_1), \tag{119}$$

$$e_1 \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} + \text{INR}_2 \right) - \frac{1}{2}, \tag{120}$$

$$e_1 + \Delta e_1 \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1} + \text{INR}_2 + \text{SNR}_{r2} \right) - \alpha(q_1), \tag{121}$$

$$g_1 \geq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} \tag{122}$$

$$g_1 + \Delta g_1 \geq \frac{1}{2} \log \left(1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{INR}_2 + \text{SNR}_{r2} \right) - \alpha(q_1), \tag{123}$$

$$\xi_1 \leq \frac{1}{2} \log \left(1 + \frac{1 + \rho}{q_1} \right) = \beta(q_1) - \frac{1}{2}, \tag{124}$$

and the lower bounds of $a_2, a_2 + \Delta a_2, d_2, d_2 + \Delta d_2, e_2, e_2 + \Delta e_2, g_2, g_2 + \Delta g_2$ and the upper bound of ξ_2 can be obtained by switching the indices of 1 and 2 in (116)-(124).

Proof: First, define the signal-to-noise and interference-to-noise ratios of the private messages as

$$\text{SNR}_{1p} = |h_{11}|^2 P_{1p}, \quad \text{SNR}_{2p} = |h_{22}|^2 P_{2p}, \tag{125}$$

$$\text{INR}_{1p} = |h_{12}|^2 P_{1p}, \quad \text{INR}_{2p} = |h_{21}|^2 P_{2p}, \tag{126}$$

$$\text{SNR}_{r1p} = |g_1|^2 P_{1p}, \quad \text{SNR}_{r2p} = |g_2|^2 P_{2p}, \tag{127}$$

which can be lower bounded or upper bounded as follows:

$$\begin{aligned}
\text{SNR}_{1p} &= |h_{11}|^2 P_{1p} \\
&= \min \left\{ |h_{11}|^2, \frac{|h_{11}|^2}{|h_{12}|^2} \right\} \\
&= \min \left\{ \text{SNR}_1, \frac{\text{SNR}_1}{\text{INR}_1} \right\} \\
&\geq \frac{\text{SNR}_1}{1 + \text{INR}_1},
\end{aligned} \tag{128}$$

and

$$0 \leq \text{INR}_{1p} = \min\{1, \text{INR}_1\} \leq 1, \tag{129}$$

and

$$\begin{aligned}
\text{SNR}_{r1p} &= |g_1|^2 P_{1p} \\
&= \min \left\{ |g_1|^2, \frac{|g_1|^2}{|h_{12}|^2} \right\} \\
&= \min \left\{ \text{SNR}_{r1}, \frac{\text{SNR}_{r1}}{\text{INR}_1} \right\} \\
&\geq \frac{\text{SNR}_{r1}}{1 + \text{INR}_1}.
\end{aligned} \tag{130}$$

Since $|g_1| \leq \sqrt{\rho}|h_{12}|$, SNR_{r1p} is upper bounded by ρ . Therefore

$$\rho \geq \text{SNR}_{r1p} \geq \frac{\text{SNR}_{r1}}{1 + \text{INR}_1}. \tag{131}$$

Switching the indices of 1 and 2, we have

$$\text{SNR}_{2p} \geq \frac{\text{SNR}_2}{1 + \text{INR}_2}, \tag{132}$$

$$1 \geq \text{INR}_{2p} \geq 0, \tag{133}$$

$$\rho \geq \text{SNR}_{r2p} \geq \frac{\text{SNR}_{r2}}{1 + \text{INR}_2}. \tag{134}$$

Now, starting from (116), we prove the inequalities one by one.

- First, (116) is lower bounded by

$$\begin{aligned}
a_1 &= I(X_1; Y_1 | W_1, W_2) \\
&= \frac{1}{2} \log \left(\frac{1 + \text{SNR}_{1p} + \text{INR}_{2p}}{1 + \text{INR}_{2p}} \right) \\
&\stackrel{(a)}{\geq} \frac{1}{2} \log(1 + \text{SNR}_{1p}) - \frac{1}{2} \\
&\stackrel{(b)}{\geq} \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2},
\end{aligned} \tag{135}$$

where (a) holds because $0 \leq \text{INR}_{2p} \leq 1$ and (b) is due to the fact that $\text{SNR}_{1p} \geq \frac{\text{SNR}_1}{1 + \text{INR}_1}$.

$$\begin{aligned}
a_1 + \Delta a_1 &= I(X_1; Y_1 | W_1, W_2) + I(X_1; \hat{Y}_{R1} | Y_1, W_1, W_2) \\
&= \frac{1}{2} \log \left(\frac{(q_1 + 1)(1 + \text{SNR}_{1p} + \text{INR}_{2p}) + \text{SNR}_{r1p} + \text{SNR}_{r2p}(1 + \phi_1^2 \text{SNR}_{1p})}{(q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p}} \right) \\
&\geq \frac{1}{2} \log(1 + \text{SNR}_{1p} + \text{SNR}_{r1p}) - \frac{1}{2} \log((q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p}) \\
&\stackrel{(a)}{\geq} \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1), \tag{136}
\end{aligned}$$

- Second, (117) is lower bounded by (136), where (a) holds because $\text{SNR}_{1p} \geq \frac{\text{SNR}_1}{1 + \text{INR}_1}$, $\text{SNR}_{r1p} \geq \frac{\text{SNR}_{r1}}{1 + \text{INR}_1}$, and

$$\begin{aligned}
&\frac{1}{2} \log((q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p}) \\
&\leq \frac{1}{2} \log((q_1 + 1)(1 + 1) + \rho) \\
&= \alpha(q_1). \tag{137}
\end{aligned}$$

- Third, (118) is lower bounded by

$$\begin{aligned}
d_1 &= I(X_1; Y_1 | W_2) \\
&= \frac{1}{2} \log \left(\frac{1 + \text{SNR}_1 + \text{INR}_{2p}}{1 + \text{INR}_{2p}} \right) \\
&\geq \frac{1}{2} \log(1 + \text{SNR}_1) - \frac{1}{2}. \tag{138}
\end{aligned}$$

- Fourth, (119) is lower bounded by (139).
- Fifth, (120) is lower bounded by

$$\begin{aligned}
e_1 &= I(X_1, W_2; Y_1 | W_1) \\
&= \frac{1}{2} \log \left(\frac{1 + \text{SNR}_{1p} + \text{INR}_2}{1 + \text{INR}_{2p}} \right) \\
&\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} + \text{INR}_2 \right) - \frac{1}{2}. \tag{140}
\end{aligned}$$

- Sixth, (121) is lower bounded by (141).
- Seventh, (122) is lower bounded by

$$\begin{aligned}
g_1 &= I(X_1, W_2; Y_1) \\
&= \frac{1}{2} \log \left(\frac{1 + \text{SNR}_1 + \text{INR}_2}{1 + \text{INR}_{2p}} \right) \\
&\geq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2}. \tag{142}
\end{aligned}$$

- Eighth, (123) is lower bounded by (143).
- Ninth, (124) is upper bounded by

$$\begin{aligned}
\xi_1 &= I(Y_R : \hat{Y}_{R1} | Y_1, X_1, W_2) \\
&= \frac{1}{2} \log \left(1 + \frac{1}{q_1} \left(1 + \frac{\text{SNR}_{r2p}}{1 + \text{INR}_{2p}} \right) \right) \\
&\leq \frac{1}{2} \log \left(1 + \frac{1 + \rho}{q_1} \right) \tag{144}
\end{aligned}$$

■

C. Proof of Theorem 3

In this appendix, we show that using the Han-Kobayashi power splitting strategy with the private message power set to $P_{1p} = \min\{1, h_{12}^{-2}\}$ and $P_{2p} = \min\{1, h_{21}^{-2}\}$, all the achievable rates in (29)-(35) are within constant bits of their

corresponding outer bounds in Theorem 1. Note that, in the following proof, inequalities in Appendix B are implicitly used without being mentioned.

(i) First, (29) is within constant bits of (7), and (30) is within constant bits of (8). To see this, the first term of (29) is lower bounded by

$$\begin{aligned}
&d_1 + (C_1 - \xi_1)^+ \\
&\geq \frac{1}{2} \log(1 + \text{SNR}_1) - \frac{1}{2} + C_1 - \xi_1 \\
&\geq \frac{1}{2} \log(1 + \text{SNR}_1) + C_1 - \left(\frac{1}{2} + \frac{1}{2} \log \left(1 + \frac{1 + \rho}{q_1} \right) \right) \tag{145}
\end{aligned}$$

which is within $\beta(q_1)$ bits of the first term of (7).

According to Lemma 1, the second term of (29) is lower bounded by

$$d_1 + \Delta d_1 \geq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{SNR}_{r1}) - \alpha(q_1), \tag{146}$$

which is within $\alpha(q_1)$ bits of the second term of (7). As a result, the gap between (29) and (7) is bounded by

$$\delta_{R_1} = \max \{ \alpha(q_1), \beta(q_1) \}. \tag{147}$$

Due to symmetry, (30) is within

$$\delta_{R_2} = \max \{ \alpha(q_2), \beta(q_2) \} \tag{148}$$

bits of the upper bound (8).

(ii) Second, (31)-(33) are within constant bits of their upper bounds (9)-(20). To see this, inspecting the expressions of the achievable sum rates, it is easy to see that each of (31)-(33) has four possible combinations: having both C_1 and C_2 , having C_1 only, having C_2 only, and having none of C_1 and C_2 . In the following, we show that, when specialized into the above four combinations, (31)-(33) are within constant gap to the upper bounds (9)-(20). The constant gaps are given by $\delta_{R_1+R_2}^{(C_1, C_2)}$, $\delta_{R_1+R_2}^{(C_1, 0)}$, $\delta_{R_1+R_2}^{(0, C_2)}$, and $\delta_{R_1+R_2}^{(0, 0)}$ (to be defined later) respectively, each corresponding to a specific combination.

- First, when having both C_1 and C_2 , (31)-(33) become

$$R_1 + R_2 \leq a_1 + g_2 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+, \tag{149}$$

$$R_1 + R_2 \leq a_2 + g_1 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+, \tag{150}$$

$$R_1 + R_2 \leq e_1 + e_2 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+, \tag{151}$$

$$\begin{aligned}
d_1 + \Delta d_1 &= I(X_1; Y_1 | W_2) + I(X_1; \hat{Y}_{R1} | Y_1, W_2) \\
&= \frac{1}{2} \log \left(\frac{(q_1 + 1)(1 + \text{SNR}_1 + \text{INR}_{2p}) + \text{SNR}_{r1} + \text{SNR}_{r2p}(1 + \phi_1^2 \text{SNR}_1)}{(q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p}} \right) \\
&\geq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{SNR}_{r1}) - \alpha(q_1).
\end{aligned} \tag{139}$$

$$\begin{aligned}
e_1 + \Delta e_1 &= I(X_1, W_2; Y_1 | W_1) + I(X_1, W_2; \hat{Y}_{R1} | Y_1, W_1) \\
&= \frac{1}{2} \log \left(\frac{(q_1 + 1)(1 + \text{SNR}_{1p} + \text{INR}_2) + \text{SNR}_{r1p} + \text{SNR}_{r2}(1 + \phi_1^2 \text{SNR}_{1p})}{(q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p}} \right) \\
&\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1} + \text{INR}_2 + \text{SNR}_{r2} \right) - \alpha(q_1).
\end{aligned} \tag{141}$$

$$\begin{aligned}
g_1 + \Delta g_1 &= I(X_1, W_2; Y_1) + I(X_1, W_2; \hat{Y}_{R1} | Y_1) \\
&= \frac{1}{2} \log \left(\frac{(q_1 + 1)(1 + \text{SNR}_1 + \text{INR}_2) + \text{SNR}_{r1} + \text{SNR}_{r2}(1 + \phi_1^2 \text{SNR}_1)}{(q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p}} \right) \\
&\geq \frac{1}{2} \log(1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{INR}_2 + \text{SNR}_{r2}) - \alpha(q_1).
\end{aligned} \tag{143}$$

which are within constant bits of (9)-(11) respectively. To show this, first, according to Lemma 1, (149) is lower bounded by

$$\begin{aligned}
&a_1 + g_2 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+ \\
&\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2} \\
&\quad + \frac{1}{2} \log(1 + \text{SNR}_2 + \text{INR}_1) - \frac{1}{2} \\
&\quad + C_1 - \xi_1 + C_2 - \xi_2,
\end{aligned} \tag{152}$$

which is within

$$\delta_{R_1+R_2}^{(C_1, C_2)} = \beta(q_1) + \beta(q_2) \tag{153}$$

bits of the upper bound (9). Due to symmetry, (150) is within $\delta_{R_1+R_2}^{(C_1, C_2)}$ bits of the upper bound (10) as well. Now applying Lemma 1, (151) is lower bounded by

$$\begin{aligned}
&e_1 + e_2 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+ \\
&\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} + \text{INR}_2 \right) - \frac{1}{2} \\
&\quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} + \text{INR}_1 \right) - \frac{1}{2} \\
&\quad + C_1 - \xi_1 + C_2 - \xi_2,
\end{aligned} \tag{154}$$

which is within $\delta_{R_1+R_2}^{(C_1, C_2)}$ bits of the upper bound (11). Therefore, when specialized to the form with both C_1 and C_2 as shown in (149)-(151), (31)-(33) have a gap of $\delta_{R_1+R_2}^{(C_1, C_2)}$ bits to their upper bounds (9)-(11).

- Second, when having C_1 only, (31)-(33) become

$$R_1 + R_2 \leq a_1 + g_2 + \Delta g_2 + (C_1 - \xi_1)^+, \tag{155}$$

$$R_1 + R_2 \leq a_2 + \Delta a_2 + g_1 + (C_1 - \xi_1)^+, \tag{156}$$

$$R_1 + R_2 \leq e_1 + e_2 + \Delta e_2 + (C_1 - \xi_1)^+, \tag{157}$$

where (155) is lower bounded by

$$\begin{aligned}
&a_1 + g_2 + \Delta g_2 + (C_1 - \xi_1)^+ \\
&\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2} + C_1 - \xi_1 \\
&\quad + \frac{1}{2} \log(1 + \text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2} \\
&\quad + \text{INR}_1 + \text{SNR}_{r1}) - \alpha(q_2),
\end{aligned} \tag{158}$$

which is within

$$\delta_{R_1+R_2}^{(C_1, 0)} = \alpha(q_2) + \beta(q_1) \tag{159}$$

bits of the upper bound (12), and (156) is lower bounded by

$$\begin{aligned}
&a_2 + \Delta a_2 + g_1 + (C_1 - \xi_1)^+ \\
&\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2 + \text{SNR}_{r2}}{1 + \text{INR}_2} \right) - \alpha(q_2) \\
&\quad + \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} + C_1 + \xi_1,
\end{aligned} \tag{160}$$

which is within $\delta_{R_1+R_2}^{(C_1, 0)}$ bits of the upper bound (13), and (157) can be lower bounded by

$$\begin{aligned}
&e_1 + e_2 + \Delta e_2 + (C_1 - \xi_1)^+ \\
&\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} + \text{INR}_2 \right) - \frac{1}{2} + C_1 - \xi_1 \\
&\quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} \right. \\
&\quad \left. + \text{INR}_1 + \text{SNR}_{r1} \right) - \alpha(q_2),
\end{aligned} \tag{161}$$

which is within $\delta_{R_1+R_2}^{(C_1, 0)}$ bits of the upper bound (14).

- Third, when having C_2 only, (31)-(33) become

$$R_1 + R_2 \leq a_1 + \Delta a_1 + g_2 + (C_2 - \xi_2)^+, (162)$$

$$R_1 + R_2 \leq a_2 + g_1 + \Delta g_1 + (C_2 - \xi_2)^+, (163)$$

$$R_1 + R_2 \leq e_1 + \Delta e_1 + e_2 + (C_2 - \xi_2)^+. (164)$$

Due to the symmetry between (162)-(164) and (155)-(157), and the symmetry between their upper bounds, we can see that (162), (163) and (164) are within

$$\delta_{R_1+R_2}^{(0,C_2)} = \alpha(q_1) + \beta(q_2) \quad (165)$$

bits of the upper bounds (15), (16), and (17) respectively.

- Fourth, when having none of C_1 and C_2 , (31)-(33) become

$$R_1 + R_2 \leq a_1 + \Delta a_1 + g_2 + \Delta g_2, \quad (166)$$

$$R_1 + R_2 \leq a_2 + \Delta a_2 + g_1 + \Delta g_1, \quad (167)$$

$$R_1 + R_2 \leq e_1 + \Delta e_1 + e_2 + \Delta e_2, \quad (168)$$

where (166) is lower bounded by

$$\begin{aligned} & a_1 + \Delta a_1 + g_2 + \Delta g_2 \\ & \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) \\ & \quad + \frac{1}{2} \log (1 + \text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2} \\ & \quad + \text{INR}_1 + \text{SNR}_{r1}) - \alpha(q_2), \end{aligned} \quad (169)$$

which is within

$$\delta_{R_1+R_2}^{(0,0)} = \alpha(q_1) + \alpha(q_2) \quad (170)$$

bits of the upper bound (18). Due to symmetry, (167) is within $\delta_{R_1+R_2}^{(0,0)}$ bits of the upper bound (19) as well. Further, (168) can be lower bounded by

$$\begin{aligned} & e_1 + \Delta e_1 + e_2 + \Delta e_2 \\ & \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1} \right. \\ & \quad \left. + \text{INR}_2 + \text{SNR}_{r2} \right) - \alpha(q_1) \\ & \quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} \right. \\ & \quad \left. + \text{INR}_1 + \text{SNR}_{r1} \right) - \alpha(q_2), \end{aligned} \quad (171)$$

which is within $\delta_{R_1+R_2}^{(0,0)}$ bits of the upper bound (20). Therefore, when specialized into the form with none of C_1 and C_2 , (31)-(33) is within $\delta_{R_1+R_2}^{(0,0)}$ bits of their upper bounds (18)-(20).

Overall, the gap between the achievable sum-rates (31)-(33) and the upper bounds in (9)-(20) is upper bounded as follows:

$$\delta_{R_1+R_2} = \max \left\{ \delta_{R_1+R_2}^{(C_1,C_2)}, \delta_{R_1+R_2}^{(C_1,0)}, \delta_{R_1+R_2}^{(0,C_2)}, \delta_{R_1+R_2}^{(0,0)} \right\}. \quad (172)$$

(iii) Third, the achievable rate (34) is within constant bits of upper bounds (21)-(26). To see this, note that (34) has 8

different forms as follows:

$$a_1 + (C_1 - \xi_1)^+ + g_1 + (C_1 - \xi_1)^+ + e_2 + (C_2 - \xi_2)^+, (173)$$

$$a_1 + \Delta a_1 + g_1 + \Delta g_1 + e_2 + \Delta e_2, (174)$$

$$a_1 + \Delta a_1 + g_1 + (C_1 - \xi_1)^+ + e_2 + \Delta e_2, (175)$$

$$a_1 + \Delta a_1 + g_1 + \Delta g_1 + e_2 + (C_2 - \xi_2)^+, (176)$$

$$a_1 + (C_1 - \xi_1)^+ + g_1 + (C_1 - \xi_1)^+ + e_2 + \Delta e_2, (177)$$

$$a_1 + \Delta a_1 + g_1 + (C_1 - \xi_1)^+ + e_2 + (C_2 - \xi_2)^+, (178)$$

$$a_1 + (C_1 - \xi_1)^+ + g_1 + \Delta g_1 + e_2 + \Delta e_2, (179)$$

$$a_1 + (C_1 - \xi_1)^+ + g_1 + \Delta g_1 + e_2 + (C_2 - \xi_2)^+, (180)$$

where (179) is redundant compared with (175) and (180) is redundant compared with (178) due to the fact that $\Delta g_1 \geq \Delta a_1$. Therefore, there are six active rate constraints in total. In the following, we prove that all active achievable rates of $2R_1 + R_2$ in (173)-(178) are within constant bits of their corresponding upper bounds in (21)-(26).

- First, (173) is lower bounded by

$$\begin{aligned} & a_1 + (C_1 - \xi_1)^+ + g_1 + (C_1 - \xi_1)^+ + e_2 + (C_2 - \xi_2)^+ \\ & \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2} + C_1 - \xi_1 \\ & \quad + \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} + C_1 - \xi_1 \\ & \quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} + \text{INR}_1 \right) - \frac{1}{2} + C_2 - \xi_2, \end{aligned} \quad (181)$$

which is within

$$\delta_{2R_1+R_2}^{(2C_1,C_2)} = 2\beta(q_1) + \beta(q_2) \quad (182)$$

bits of the upper bound (21).

- Second, (174) is lower bounded by

$$\begin{aligned} & a_1 + \Delta a_1 + g_1 + g_1 + \Delta g_1 + e_2 + \Delta e_2 \\ & \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) \\ & \quad + \frac{1}{2} \log (1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} \\ & \quad + \text{INR}_2 + \text{SNR}_{r2}) - \alpha(q_1) \\ & \quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} \right. \\ & \quad \left. + \text{INR}_1 + \text{SNR}_{r1} \right) - \alpha(q_2), \end{aligned} \quad (183)$$

which is within

$$\delta_{2R_1+R_2}^{(0,0)} = 2\alpha(q_1) + \alpha(q_2) \quad (184)$$

bits of the upper bound (22).

- Third, (175) is lower bounded by

$$\begin{aligned} & a_1 + \Delta a_1 + g_1 + (C_1 - \xi_1)^+ + e_2 + \Delta e_2 \\ & \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) \\ & \quad + \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} + C_1 - \xi_1 \\ & \quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} \right. \\ & \quad \left. + \text{INR}_1 + \text{SNR}_{r1} \right) - \alpha(q_2), \end{aligned} \quad (185)$$

which is within

$$\delta_{2R_1+R_2}^{(C_1,0)} = \alpha(q_1) + \alpha(q_2) + \beta(q_1) \quad (186)$$

bits of the upper bound (23).

- Fourth, (176) is lower bounded by

$$\begin{aligned} & a_1 + \Delta a_1 + g_1 + \Delta g_1 + e_2 + (C_2 - \xi_2)^+ \\ & \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) \\ & \quad + \frac{1}{2} \log (1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} \\ & \quad + \text{INR}_2 + \text{SNR}_{r2}) - \alpha(q_1) \\ & \quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} + \text{INR}_1 \right) - \frac{1}{2} + C_2 - \xi_2, \end{aligned} \quad (187)$$

which is within

$$\delta_{2R_1+R_2}^{(0,C_2)} = 2\alpha(q_1) + \beta(q_2) \quad (188)$$

bits of the upper bound (24).

- Fifth, (177) is lower bounded by

$$\begin{aligned} & a_1 + (C_1 - \xi_1)^+ + g_1 + (C_1 - \xi_1)^+ + e_2 + \Delta \tilde{e}_2 \\ & \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2} + C_1 - \xi_1 \\ & \quad + \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} + C_1 - \xi_1 \\ & \quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} \right. \\ & \quad \left. + \text{INR}_1 + \text{SNR}_{r1} \right) - \alpha(q_2), \end{aligned} \quad (189)$$

which is within

$$\delta_{2R_1+R_2}^{(2C_1,0)} = \alpha(q_2) + 2\beta(q_1) \quad (190)$$

bits of the upper bound (25).

- Sixth, (178) is lower bounded by

$$\begin{aligned} & a_1 + \Delta a_1 + g_1 + (C_1 - \xi_1)^+ + e_2 + (C_2 - \xi_2)^+ \\ & \geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) \\ & \quad + \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} + C_1 - \xi_1 \\ & \quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} + \text{INR}_1 \right) \\ & \quad - \frac{1}{2} + C_2 - \xi_2, \end{aligned} \quad (191)$$

which is within

$$\delta_{2R_1+R_2}^{(C_1,C_2)} = \alpha(q_1) + \beta(q_1) + \beta(q_2) \quad (192)$$

bits of the upper bound (26).

Therefore, the gap between the achievable rate (34) and the corresponding upper bounds (21)-(26) is bounded by the following constant

$$\delta_{2R_1+R_2} = \max \left\{ \delta_{2R_1+R_2}^{(2C_1,C_2)}, \delta_{2R_1+R_2}^{(0,0)}, \delta_{2R_1+R_2}^{(C_1,0)}, \delta_{2R_1+R_2}^{(0,C_2)}, \delta_{2R_1+R_2}^{(2C_1,0)}, \delta_{2R_1+R_2}^{(C_1,C_2)} \right\}. \quad (193)$$

Due to the symmetry between (35) and (34), and the symmetry between their corresponding upper bounds, it is easy to see that (35) is also within constant gap to the upper bounds. The constant gap $\delta_{R_1+2R_2}$ can be obtained by simply switching indices of 1 and 2 in $\delta_{2R_1+R_2}$.

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