

Distributed Storage for Intermittent Energy Sources: Control Design and Performance Limits

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Abstract

One of the most important challenges in the integration of renewable energy sources into the power grid lies in their ‘intermittent’ nature. The power output of sources like wind and solar varies with time and location due to factors that cannot be controlled by the provider. Two strategies have been proposed to hedge against this variability: 1) use energy storage systems to effectively average the produced power over time; 2) exploit distributed generation to effectively average production over location. We introduce a network model to study the optimal use of storage and transmission resources in the presence of random energy sources. We propose a Linear-Quadratic based methodology to design control strategies, and we show that these strategies are asymptotically optimal for some simple network topologies. For these topologies, the dependence of optimal performance on storage and transmission capacity is explicitly quantified.

1 Introduction

It is widely advocated that future power grids should facilitate the integration of a significant amount of renewable energy sources. Prominent examples of renewable sources are wind and solar. These differ substantially from traditional sources in terms of two important qualitative features: **Intrinsically distributed.** The power generated by these sources is typically proportional to the surface occupied by the corresponding generators. For instance, the solar power reaching ground is of the order of 2 kWh per day per square meter. The wind power at ground level is of the order of 0.1 kWh per day per square meter [1]. These constraints on renewable power generation have important engineering implications. If a significant part of energy generation is to be covered by renewables, generation is argued to be distributed over large geographical areas.

Intermittent. The output of renewable sources varies with time and locations because of exogenous factors. For instance, in the case of wind and solar energy, the power output is ultimately determined by meteorological conditions. One can roughly distinguish two sources of variability: *predictable variability*, e.g. related to the day-night cycle, or to seasonal differences; *unpredictable variability*, which is most conveniently modeled as a random process.

Several ideas have been put forth to meet the challenges posed by intermittent production. The first one is to leverage geographically distributed production. The output of distinct generators is likely to be independent or weakly dependent over large distances and therefore the total production

of a large number of well separated generators should stay approximately constant, by a law-of-large-number effect. This averaging effect should be enhanced by the integration of different types of generators.

The second approach is to use energy storage to take advantage of over-production at favorable times, and cope with shortages at unfavorable times. Finally, a third idea is ‘demand response’, which aims at scheduling in optimal ways some time-insensitive energy demands. In several cases, this can be abstracted as some special form of energy storage (for instance, when energy is demanded for interior heating, deferring a demand is equivalent to exploiting the energy stored as hot air inside the building).

These approaches hedge against the energy source variability by averaging over location, or by averaging over time. Each of them requires specific infrastructures: a power grid with sufficient transmission capacity in the first case, and sufficient energy storage infrastructure in the second one. Further, these two directions are in fact intimately related. With current technologies, it is unlikely that centralized energy storage can provide effective time averaging of –say– wind power production, in a renewables-dominated scenario. In a more realistic scheme, storage is distributed at the consumer level, for instance leveraging electric car batteries (a scenario known as vehicle-to-grid or V2G). Distributed storage implies, in turn, substantial changes of the demand on the transmission system.

The use of storage devices to average out intermittent renewables production is well established. A substantial research effort has been devoted to its design, analysis and optimization (see, for instance, [2, 3, 4, 5, 6, 7]). In this line of work, a large renewable power generator is typically coupled with a storage system in order to average out its power production. Proper sizing, response time, and efficiency of the storage system are the key concerns.

If, however, we assume that storage will be mainly distributed, the key design questions change. It is easy to understand that both storage and transmission capacity will have a significant effect on the ability of the network to average out the energy source variability. For example, shortfalls at a node can be compensated by either withdrawals from local storage or extracting power from the rest of the network, or a combination of both. The main goal of this paper is to understand the optimal way of utilizing simultaneously these two resources and to quantify the impact of these two resources on performance. Our contributions are:

- a simple model capturing key features of the problem;
- a Linear-Quadratic(LQ) based methodology for the systematic design of control strategies;
- a proof of optimality of the LQ control strategies in simple network topologies such as the 1-D and 2-D grids and in certain asymptotic regimes.
- a quantification of how the performance depends on key parameters such as storage and transmission capacities.

The reader interested in getting an overview of the conclusions without the technical details can read Sections 2 and 4 only. Some details are omitted and deferred to the journal version of this paper.

2 Model and Problem Formulation

The power grid is modeled as a weighted graph G with vertices (buses or nodes) V , edges (lines) E . Time is slotted and will be indexed by $t \in \{0, 1, 2, \dots\}$. In slot t , at each node $i \in V$ a quantity of energy $E_{p,i}(t)$ is generated from a renewable source, and a demand of a quantity $E_{d,i}(t)$ is received. For our purposes, these quantities only enter the analysis through the net generation $\mathcal{Z}_i(t) = E_{p,i}(t) - E_{d,i}(t)$. Let $\mathcal{Z}(t)$ be the vector of $\mathcal{Z}_i(t)$'s. We will assume that $\{\mathcal{Z}(t)\}$ is a stationary process.

In order to average the variability in the energy supply, the system makes use of storage and transmission. Storage is fully distributed: each node $i \in V$ has a device that can store energy, with capacity S_i . We assume that stored energy can be fully recovered when needed (i.e., no losses). At each time slot t , one can transfer an amount of energy $\mathcal{Y}_i(t)$ to storage at node i . If we denote by $\mathcal{B}_i(t)$ the amount of stored energy at node i just before the beginning of time slot t , then:

$$\mathcal{B}_i(0) = 0, \quad \mathcal{B}_i(t+1) = \mathcal{B}_i(t) + \mathcal{Y}_i(t) \quad (1)$$

where $\mathcal{Y}_i(t)$ is chosen under the constraint that $\mathcal{B}_i(t+1) \in [0, S_i]$.

We will also assume the availability at each node of a fast generation source (such as a spinning reserve or backup generator) which allows covering up of shortfalls. Let $\mathcal{W}_i(t)$ be the energy obtained from such a source at node i at time slot t . We will use the convention that $\mathcal{W}_i(t)$ negative means that energy is consumed from the fast generation source, and positive means energy is dumped. The cost of using fast generation energy sources is reflected in the steady-state performance measure:

$$\varepsilon_{\mathcal{W}} \equiv \lim_{t \rightarrow \infty} \frac{1}{|V|} \sum_{i \in V} \mathbb{E}\{(\mathcal{W}_i(t))_-\} \quad (2)$$

The net amount of energy injection at node i at time slot t is:

$$\mathcal{Z}_i(t) - \mathcal{Y}_i(t) - \mathcal{W}_i(t).$$

These injections have to be distributed across the transmission network, and the ability of the network to distribute the injections and hence to average the random energy sources over space is limited by the transmission capacity of the network. To understand this constraint, we need to relate the injections to the power flows on the transmission lines. To this end, we adopt a ‘DC power flow’ approximation model [8].¹

Each edge in the network corresponds to a transmission line which is purely inductive, i.e. with susceptance $-jb_e$, where $b_e \in \mathbb{R}_+$. Hence, the network is lossless. Node $i \in V$ is at voltage $V_i(t)$, with all the voltages assumed to have the same magnitude, taken to be 1 (by an appropriate choice of units). Let $V_i(t) = e^{j\phi_i(t)}$ denote the (complex) voltage at node i in time slot t . If $I_{i,k}(t) = -jb_{ik}(V_i(t) - V_k(t))$ is the electric current from i to k , the corresponding power flow is then $\mathcal{F}_{i,k}(t) = \text{Re}[V_i(t)I_{i,k}(t)^*] = \text{Re}[jb_{ik}(1 - e^{j(\phi_i(t) - \phi_k(t))})] = b_{ik} \sin(\phi_i(t) - \phi_k(t))$, where $\text{Re}[\cdot]$ denotes the real part of a complex number.

The DC flow approximation replaces $\sin(\phi_i(t) - \phi_k(t))$ by $\phi_i(t) - \phi_k(t)$ in the above expression. This is usually a good approximation since the phase angles at neighboring nodes are typically maintained close to each other to ensure that the generators at the two ends remain in step. This

¹Despite the name, ‘DC flow’ is an approximation to the AC flow

leads to the following relation between angles and power flow $\mathcal{F}_{i,k}(t) = b_{ik}(\phi_i(t) - \phi_k(t))$. In matrix notation, we have

$$\mathcal{F}(t) = \nabla \phi(t), \quad (3)$$

where $\mathcal{F}(t)$ is the vector of all power flows, $\phi(t) = (\phi_1(t), \dots, \phi_n(t))$ and ∇ is a $|E| \times |V|$ matrix. $\nabla_{e,i} = b_e$ if $e = (i, k)$ for some k , $\nabla_{e,i} = -b_e$ if $e = (k, i)$ for some k , and $\nabla_{e,i} = 0$ otherwise.

Energy conservation at node i also yields

$$\mathcal{Z}_i(t) - \mathcal{Y}_i(t) - \mathcal{W}_i(t) = \sum_k \mathcal{F}_{(i,k)}(t) = (\nabla^T \mathbf{b}^{-1} \mathcal{F}(t))_i,$$

where $\mathbf{b} = \text{diag}(b_e)$ is an $|E| \times |E|$ diagonal matrix. Expressing $\mathcal{F}(t)$ in terms of $\phi(t)$, we get

$$\mathcal{Z}(t) - \mathcal{Y}(t) - \mathcal{W}(t) = -\Delta \phi(t), \quad (4)$$

where $\Delta = -\nabla^T \mathbf{b}^{-1} \nabla$ is a $|V| \times |V|$ symmetric matrix where $\Delta_{i,k} = -\sum_{l:(i,l) \in E} b_{il}$ if $i = k$, $\Delta_{i,k} = b_{ik}$ if $(i, k) \in E$ and 0 otherwise. In graph theory, Δ is called the graph Laplacian matrix. In power engineering, it is simply the imaginary part of the bus admittance matrix of the network. Note that if $b_e \geq 0$ for all edges e , then $-\Delta \succcurlyeq 0$ is positive semidefinite. If the network is connected (which we assume throughout), it has only one eigenvector with eigenvalue 0, namely the vector $\varphi_v = 1$ everywhere (hereafter we will denote this as the vector $\mathbf{1}$). This fits the physical fact that if all phases are rotated by the same amount, the powers in the network are not changed.

With an abuse of notation, we denote by Δ^{-1} the matrix such that $\Delta^{-1} \mathbf{1} = -M \mathbf{1}$, and Δ^{-1} is equal to the inverse of Δ on the subspace orthogonal to $\mathbf{1}$. Here $M > 0$ is arbitrary, and all of our results are independent of this choice (conceptually, one can think of M as very large). Explicitly, let $\Delta = -\mathbf{V} \alpha^2 \mathbf{V}^T$ be the eigenvalue decomposition of Δ , where α is a diagonal matrix with non-negative entries. Define $\alpha^\dagger$ to be the diagonal matrix with $\alpha_{ii}^\dagger = M$ if $\alpha_{ii} = 0$ and $\alpha_{ii}^\dagger = \alpha_{ii}^{-1}$ otherwise. Then $\Delta^{-1} = -\mathbf{V}(\alpha^\dagger)^2 \mathbf{V}^T$.

Since the total power injection in the network adds up to zero (which must be true by energy conservation), we can invert (4) and obtain

$$\phi(t) = -\Delta^{-1}(\mathcal{Z}(t) - \mathcal{Y}(t) - \mathcal{W}(t)). \quad (5)$$

Plugging this into (3), we have

$$\mathcal{F}(t) = -\nabla \Delta^{-1}(\mathcal{Z}(t) - \mathcal{Y}(t) - \mathcal{W}(t)). \quad (6)$$

There is a capacity limit C_e on the power flow along each edge e ; this capacity limit depends on the voltage magnitudes and the maximum allowable phase differences between adjacent nodes, as well as possible thermal line limits. We will measure violations of this limit by defining

$$\varepsilon_{\mathcal{F}} \equiv \lim_{t \rightarrow \infty} \frac{1}{|E|} \sum_{e \in E} \mathbb{E}\{(\mathcal{F}_e(t) - C_e)_+ + (-C_e - \mathcal{F}_e(t))_+\}. \quad (7)$$

We are now ready to state the design problem:

For the dynamic system defined by equations (1) and (6), design a control strategy which, given the past and present random renewable supplies and the storage states,

$$\{(\mathcal{Z}(t), \mathcal{B}(t)); (\mathcal{Z}(t-1), \mathcal{B}(t-1)), \dots, \}$$

choose the vector of energies $\mathcal{Y}(t)$ to put in storage and the vector of fast generations $\mathcal{W}(t)$ such that the sum $\varepsilon_{\text{tot}} \equiv \varepsilon_{\mathcal{F}} + \varepsilon_{\mathcal{W}}$, cf. Eq. (2) and (7), is minimized.

3 Linear-Quadratic Design

In this section, we propose a design methodology that is based on Linear-Quadratic (LQ) control theory.

3.1 The Surrogate LQ Problem

The difficulty of the control problem defined above stems from both the nonlinearity of the dynamics due to the hard storage limits and the piecewise linearity of the cost functions giving rise to the performance parameters. Instead of attacking the problem directly, we consider a surrogate LQ problem where the hard storage limits are removed and the cost functions are quadratic:

$$B_i(0) = \frac{-S_i}{2}, \quad B_i(t+1) = B_i(t) + Y_i(t), \quad (8)$$

$$F(t) = -\nabla \Delta^{-1}(Z(t) - Y(t) - W(t)), \quad (9)$$

with performance parameters:

$$\varepsilon_{W_i}^{\text{surrogate}} = \lim_{t \rightarrow \infty} \mathbb{E}\{(W_i(t))^2\}, \quad i \in V, \quad (10)$$

$$\varepsilon_{F_e}^{\text{surrogate}} = \lim_{t \rightarrow \infty} \mathbb{E}\{(F_e(t))^2\}, \quad e \in E, \quad (11)$$

$$\varepsilon_{B_i}^{\text{surrogate}} = \lim_{t \rightarrow \infty} \mathbb{E}\{(B_i(t))^2\}. \quad (12)$$

The process $B_i(t)$ can be interpreted as the deviation of a virtual storage level process from the midpoint $S_i/2$, where the virtual storage level process is no longer hard-limited but evolves linearly. Instead, we penalize the deviation through a quadratic cost function in the additional performance parameters $\varepsilon_{B_i}^{\text{surrogate}}$.

The virtual processes $B(t)$, $F(t)$, $W(t)$, $Y(t)$ and $Z(t)$ are connected to the actual processes $\mathcal{B}(t)$, $\mathcal{F}(t)$, $\mathcal{W}(t)$, $\mathcal{Y}(t)$ and $\mathcal{Z}(t)$ via the mapping (where $[x]_a^b := \max(\min(x, b), a)$ for $a \leq b$):

$$\mathcal{Z}_i(t) = Z_i(t), \quad \mathcal{F}_e(t) = F_e(t), \quad (13)$$

$$\mathcal{B}_i(t) = [B_i(t) + S_i/2]_0^{S_i}, \quad (14)$$

$$\mathcal{Y}_i(t) = \mathcal{B}_i(t+1) - \mathcal{B}_i(t), \quad (15)$$

$$\mathcal{W}_i(t) = W_i(t) + Y_i(t) - \mathcal{Y}_i(t). \quad (16)$$

In particular, once we solve for the optimal control in the surrogate LQ problem, (15) and (16) tell us what control to use in the actual system. Notice that the actual fast generation control provides the fast generation in the virtual system plus an additional term that keeps the actual storage level process within the hard limit. Note also

$$\mathcal{W}_i(t) \geq W_i(t) - (B_i(t) - S_i/2)_+ - (-B_i(t) - S_i/2)_+. \quad (17)$$

Hence the performance parameters $\varepsilon_{\mathcal{F}}$, $\varepsilon_{\mathcal{W}}$ can be estimated from the corresponding ones for the virtual processes.

Now we turn to solving the surrogate LQ problem. First we formulate it in standard state-space form. For simplicity, we will assume $\{Z(t)\}_{t \geq 0}$ is an i.i.d. process (over time).² Hence

²The case of a process $\{Z(t)\}_{t \geq 0}$ with memory can be in principle studied within the same framework, by introducing a linear state space model for $Z(t)$ and correspondingly augmenting the state space of the control problem.

$X(t) := [F(t-1)^T, B(t)^T]^T$ is the state of the system. Also, $U(t) := [Y(t)^T, W(t)^T]^T$ is the control and $R(t) := [X(t)^T, Z(t)^T]^T$ is the observation vector available to the controller. Then

$$X(t+1) = \mathbf{A}X(t) + \mathbf{D}U(t) + \mathbf{E}Z(t), \quad (18)$$

$$R(t) = \mathbf{C}X(t) + \zeta(t), \quad (19)$$

where

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{bmatrix}, \quad \mathbf{D} \equiv \begin{bmatrix} -\nabla\Delta^{-1} & -\nabla\Delta^{-1} \\ \mathbf{I} & 0 \end{bmatrix}, \quad \mathbf{E} \equiv \begin{bmatrix} \nabla\Delta^{-1} \\ 0 \end{bmatrix}.$$

and $\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{bmatrix}$ and $\zeta(t) = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} Z(t)$. We are interested in trading off between the performance parameters $\varepsilon_{F_e}, \varepsilon_{W_i}$ and ε_{B_i} 's. Therefore we introduce weights γ_e 's, ξ_i 's, η_i 's and define the Lagrangian

$$\begin{aligned} \mathcal{L}(t) &\equiv \sum_{e=1}^{|E|} \gamma_e \mathbb{E}\{F_e(t)^2\} + \sum_{i=1}^{|V|} \xi_i \mathbb{E}\{B_i(t)^2\} + \sum_{i=1}^{|V|} \eta_i \mathbb{E}\{W_i(t)^2\} \\ &= \mathbb{E}\{X(t)^T \mathbf{Q}_1 X(t) + U(t)^T \mathbf{Q}_2 U(t)\}, \end{aligned} \quad (20)$$

where $\mathbf{Q}_1 = \text{diag}(\gamma_1, \dots, \gamma_{|E|}, \xi_1, \dots, \xi_{|V|})$ and $\mathbf{Q}_2 = \text{diag}(0, \dots, 0, \eta_1, \dots, \eta_{|E|})$.

We will let $\mathbb{E}\{Z(t)\} = \bar{Z}$. We will also assume that $\Sigma_Z \equiv \mathbb{E}[Z(t)^T Z(t)] = \mathbf{I}$, since if not, then we can define $\mathbf{E} = [\nabla\Delta^{-1}\sqrt{\Sigma_Z}^{-1}0]^T$, where $\sqrt{\Sigma_Z}$ is the symmetrical square root of Σ_Z .

An *admissible control policy* is a mapping $\{R(t), R(t-1), \dots, R(0)\} \mapsto U(t)$. The surrogate LQ problem is defined as the problem of finding the mapping that minimizes the stationary cost $\mathcal{L} \equiv \lim_{t \rightarrow \infty} \mathcal{L}(t)$.

Notice that the energy production-minus-consumption $Z(t)$ plays the role both in the evolution equation (18) and the observation (19). The case of correlated noise has been considered and solved for general correlation structure in [9]. Let $\mathbf{G} = \mathbb{E}[\zeta(t)Z(t)^T] = [\mathbf{I}0]^T$, $R_1(t) = [(Z(t) - \bar{Z})^T, 0]^T$ and $R_2(t) = [0, B(t)^T]^T$. Adapting the general result in [9] to our special case, we have

Lemma 3.1. *The optimal linear controller for the system in (18) and (19) and the cost function in (20) is given by*

$$U(t) = -(\mathbf{L}R_1(t) + \mathbf{K}^{-1}\mathbf{D}^T\mathbf{S}\mathbf{E}\mathbf{G}^T\mathbf{M}^{-1}R_2(t)) + \bar{U}, \quad (21)$$

where, letting $\eta \equiv \text{diag}(\eta_1, \dots, \eta_{|V|})$ and $\gamma \equiv \text{diag}(\gamma_1, \dots, \gamma_{|V|})$:

$$\begin{aligned} \bar{U} &= \begin{bmatrix} \bar{Y} \\ \bar{W} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ [I - \Delta(\nabla^T\gamma\nabla)^{-1}\Delta\eta]^{-1}\bar{Z} \end{bmatrix}, \end{aligned} \quad (22)$$

and $\mathbf{S}$ is given by the algebraic Riccati equation

$$\mathbf{S} = \mathbf{A}^T\mathbf{S}\mathbf{A} + \mathbf{Q}_1^T\mathbf{Q}_1 - \mathbf{L}^T\mathbf{K}\mathbf{L}, \quad (23)$$

where $\mathbf{K} = \mathbf{D}^T \mathbf{S} \mathbf{D} + \mathbf{Q}_2^T \mathbf{Q}_2$, $\mathbf{L} = \mathbf{K}^{-1}(\mathbf{D}^T \mathbf{S} \mathbf{A} + \mathbf{Q}_2^T \mathbf{Q}_1)$, and

$$\mathbf{M} = \mathbf{C} \mathbf{J} \mathbf{C}^T + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}, \quad (24)$$

where $\mathbf{J}$ satisfies the algebraic Riccati equation $\mathbf{J} = \mathbf{A} \mathbf{J} \mathbf{A}^T + \mathbf{E} \mathbf{E}^T - \mathbf{O} \mathbf{M} \mathbf{O}^T$, and $\mathbf{O} = (\mathbf{A} \mathbf{J} \mathbf{C}^T + \mathbf{E} \mathbf{G}^T) \mathbf{M}^{-1}$.

Note that the optimal linear controller has a deterministic time-invariant component $\bar{\mathbf{U}} = [\bar{\mathbf{Y}}^T, \bar{\mathbf{W}}^T]^T$ and an observation-dependent control $-(\mathbf{L} \mathbf{R}_1(t) + \mathbf{K}^{-1} \mathbf{D}^T \mathbf{S} \mathbf{E} \mathbf{G}^T \mathbf{M}^{-1} \mathbf{R}_2(t))$. It is intuitive that $\bar{\mathbf{Y}} = 0$, since otherwise the storage process has a non-zero drift and will become unstable. The deterministic component $\bar{\mathbf{W}}$ for the fast generation can be seen to be the solution of a *static optimal power flow problem* with deterministic net renewable generation $\bar{\mathbf{Z}}$ and cost function given by:

$$\mathcal{L} = \sum_{e=1}^{|E|} \gamma_e \bar{F}_e^2 + \sum_{i=1}^{|V|} \eta_i \bar{W}_i^2.$$

On the other hand, the observation-dependent control is obtained by solving the LQ problem with the net generations shifted to zero-mean. Thus, the LQ design methodology naturally decomposes the control problem into a static optimal power flow problem and a dynamic problem of minimizing variances.

Notice that it might be also convenient to consider more general surrogate costs in which a generic quadratic function of the means $\bar{F}_e, \bar{W}_i$ is added to the second moment Lagrangian (20).

3.2 Transitive Networks

Lemma 3.1 gives an expression for the optimal linear controller. However, it is difficult in general to solve analytically the Riccati equation. To gain further insight, we consider the case of transitive networks.

An automorphism of a graph $G = (V, E)$ is a one-to-one mapping $f : V \rightarrow V$ such that for any edge $e = (u, v) \in E$, we have $e' = (f(u), f(v)) \in E$. A graph is called *transitive* if for any two vertices v_1 and v_2 , there is some automorphism $f : V \rightarrow V$ such that $f(v_1) = v_2$. Intuitively, a graph is transitive if it looks the same from the perspective of any of the vertices. Given an electric network, we say the network is transitive if it has a transitive graph structure, every bus has the same associated storage, every line has the same capacity and inductance, and $Z_i(t)$ is i.i.d. across the network. Without loss of generality, we will assume $S_i = S$, $C_e = C$, $b_e = 1$, $\mathbb{E}[Z_i(t)] = \mu$, $\text{Var}[Z_i(t)] = \sigma^2$. Since the graph is transitive, it is natural to take the cost matrices as $\mathbf{Q}_1 = \text{diag}(\gamma, \dots, \gamma, \xi, \dots, \xi)$ and $\mathbf{Q}_2 = \text{diag}(0, \dots, 0, 1, \dots, 1)$. Moreover, it can be seen from Eq. (22) that $\bar{\mathbf{U}} = 0$. Since the mean net production is the same at each node, the static optimal power flow problem is trivial with the mean flows being zero. We are left with the dynamic variance minimizing problem.

Recall that $\Delta = -\mathbf{V} \alpha^2 \mathbf{V}^T$ is the eigenvalue decomposition of Δ . Since $\Delta = -\nabla^T \nabla$, the singular value decomposition of ∇ is given by $\nabla = \mathbf{U} \alpha \mathbf{V}^T$ for some orthogonal matrix $\mathbf{U}$. The basic observation is that, with these choices of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, the Riccati equations diagonalize in the bases given by the columns of $\mathbf{V}$ (for vectors indexed by vertices) and columns of $\mathbf{U}$ (for vectors indexed by edges).

A full justification of the diagonal ansatz amounts to rewriting the Riccati equations in the new basis. For the sake of space we limit ourselves to deriving the optimal diagonal control. We rewrite the linear relation from $X(t)$ to $U(t)$ as

$$Y(t) = \mathbf{H}Z(t) - \mathbf{K}B(t), \quad (25)$$

$$W(t) = \mathbf{P}Z(t) + \mathbf{Q}B(t). \quad (26)$$

Substituting in Eq. (18), we get

$$B(t+1) = (\mathbf{I} - \mathbf{K})B(t) + \mathbf{H}Z(t), \quad (27)$$

$$F(t+1) = \nabla \Delta^{-1} \{ (\mathbf{I} - \mathbf{H} - \mathbf{P})Z(t) + (\mathbf{K} - \mathbf{Q})B(t) \}, \quad (28)$$

$$W(t) = \mathbf{P}Z(t) + \mathbf{Q}B(t). \quad (29)$$

Denoting as above by $\overline{B}$, $\overline{F}$, $\overline{W}$ the average quantities, it is easy to see that, in a transitive network, we can take $\overline{F} = 0$, $\overline{W} = \mu$ and hence $\overline{B} = 0$. In words, since all nodes are equivalent, there is no average power flow ($\overline{F} = 0$), the average overproduction is dumped locally ($\overline{W} = \mu$), and the average storage level is kept constant ($\overline{B} = 0$).

We work in the basis in which $\nabla = \mathbf{U}\alpha\mathbf{V}^T$ is diagonal. We will index singular values by $\theta \in \Theta$ hence $\alpha = \text{diag}(\{\alpha(\theta)\}_{\theta \in \Theta})$ (omitting hereafter the singular value $\alpha = 0$ since the relevant quantities have vanishing projection along this direction.) In the examples treated in the next sections, θ will be a Fourier variable. Since the optimal filter is diagonal in this basis, we write $\mathbf{K} = \text{diag}(k(\theta))$, $\mathbf{H} = \text{diag}(h(\theta))$ and $\mathbf{P} = \text{diag}(p(\theta))$, $\mathbf{Q} = \text{diag}(q(\theta))$.

We let $b_\theta(t)$, $z_\theta(t)$, $f_\theta(t)$, $w_\theta(t)$ denote the components of $B(t) - \overline{B}$, $Z(t) - \mu$, $F(t) - \overline{F}$, $W(t) - \overline{W}$ along in the same basis. From Eqs. (27) to (29), we get the scalar equations

$$b_\theta(t) = (1 - k(\theta))b_\theta(t-1) + h(\theta)z_\theta(t), \quad (30)$$

$$f_\theta(t) = -\alpha^{-1}(\theta) \{ (1 - h(\theta) - p(\theta))z_\theta(t) + (k(\theta) - q(\theta))b_\theta(t-1) \}, \quad (31)$$

$$w_\theta(t) = p(\theta)z_\theta(t) + q(\theta)b_\theta(t-1). \quad (32)$$

We will denote by $\sigma_B^2(\theta)$, $\sigma_F^2(\theta)$, $\sigma_W^2(\theta)$ the stationary variances of $b_\theta(t)$, $f_\theta(t)$, $w_\theta(t)$. From the above, we obtain

$$\sigma_B^2(\theta) = \frac{h^2}{1 - (1 - k)^2} \sigma^2, \quad (33)$$

$$\sigma_F^2(\theta) = \frac{1}{\alpha^2} \left[(1 - h - p)^2 + \frac{h^2(k - q)^2}{1 - (1 - k)^2} \right] \sigma^2, \quad (34)$$

$$\sigma_W^2(\theta) = \left[p^2 + \frac{h^2 q^2}{1 - (1 - k)^2} \right] \sigma^2. \quad (35)$$

(We omit here the argument θ on the right hand side.)

In order to find h, k, p, q , we minimize the Lagrangian (20). Using Parseval's identity, this decomposes over θ , and we can therefore separately minimize for each $\theta \in \Theta$

$$\mathcal{L}(\theta) = \sigma_W(\theta)^2 + \xi \sigma_B(\theta)^2 + \gamma \sigma_F(\theta)^2. \quad (36)$$

A lengthy but straightforward calculus exercise yields the following expressions.

	No transmission ($C = 0$)	No storage ($S = 0$)	Storage and Transmission
1-D	$\Theta(\frac{\sigma^2}{S})$ for $\mu S < \sigma^2$	$\Theta(\frac{\sigma^2}{C})$ for $\mu C < \sigma^2$ $\sigma \exp\{-\frac{\mu C}{\sigma^2}\}$ otherwise	$\sigma \exp\{-\sqrt{\frac{CS}{\sigma^2}}\}^\dagger$ for $\mu = \exp\{-\omega(\sqrt{\frac{CS}{\sigma^2}})\}$ $\sigma \exp\{-\frac{CS}{\sigma^2}\}$ for $\mu = \exp\{-o(\sqrt{\frac{CS}{\sigma^2}})\}$
2-D	$\sigma \exp\{-\frac{\mu S}{\sigma^2}\}$ otherwise	$\sigma \exp\{-\frac{C}{\sigma}\}^\dagger$ for $\mu = \exp\{-\omega(\frac{C}{\sigma})\}$ $\sigma \exp\{-\frac{C^2}{\sigma^2}\}$ for $\mu = \exp\{-o(\frac{C}{\sigma})\}$	$\sigma \exp\{-\frac{C \max(C, S)}{\sigma^2}\}$

Table 1: Asymptotically optimal $\varepsilon_{\mathcal{W}} + \varepsilon_{\mathcal{F}}$ in 1-D and 2-D grids. Logarithmic factors have been neglected (also in the exponent). $\dagger$ indicates the lower bound requires a conjecture in probability theory.

Theorem 1. *Consider a transitive network. The optimal linear control scheme is given, in Fourier domain $\theta \in \Theta$, by*

$$p(\theta) = q(\theta) = \xi \frac{\sqrt{4\beta(\theta) + 1} - 1}{2}, \quad (37)$$

$$h(\theta) = \frac{2\beta(\theta) + 1 - \sqrt{4\beta(\theta) + 1}}{2\beta(\theta)}, \quad (38)$$

$$k(\theta) = \frac{\sqrt{4\beta(\theta) + 1} - 1}{2\beta(\theta)}, \quad (39)$$

where $\beta(\theta)$ is given by

$$\beta(\theta) = \frac{\gamma}{\xi(\gamma + \alpha^2(\theta))}. \quad (40)$$

It is useful to point out a few analytical properties of these filters: (i) $\gamma/[\xi(\gamma + d_{\max})] \leq \beta \leq 1/\xi$ with $d_{\max}$ the maximum degree in G ; (ii) $0 \leq k \leq 1$ is monotone decreasing as a function of β , with $k = 1 - \beta + O(\beta^2)$ as $\beta \rightarrow 0$ and $k = 1/\sqrt{\beta} + O(1/\beta)$ as $\beta \rightarrow \infty$; (iii) $0 \leq h \leq 1$ is such that $h + k = 1$. In particular, it is monotone increasing as a function of β , with $h = \beta + O(\beta^2)$ as $\beta \rightarrow 0$ and $h = 1 - 1/\sqrt{\beta} + O(1/\beta)$ as $\beta \rightarrow \infty$; (iv) $p = q = \xi\beta k$.

Theorem 2. *Consider a transitive network, and assume that the optimal LQ control is applied. The variances are given as follows in terms of $k(\theta)$, given in Eq. (38):*

$$\frac{\sigma_B^2(\theta)}{\sigma^2} = \frac{(1 - k(\theta))^2}{1 - (1 - k(\theta))^2}, \quad (41)$$

$$\frac{\sigma_F^2(\theta)}{\sigma^2} = \frac{\alpha^2(\theta)}{(\gamma + \alpha^2(\theta))^2} \frac{k^2(\theta)}{1 - (1 - k(\theta))^2}, \quad (42)$$

$$\frac{\sigma_W^2(\theta)}{\sigma^2} = \frac{\gamma^2}{(\gamma + \alpha^2(\theta))^2} \frac{k^2(\theta)}{1 - (1 - k(\theta))^2}. \quad (43)$$

4 1-D and 2-D Grids: Overview of Results

For the rest of the paper, we focus on two specific network topologies: the infinite one-dimensional grid (line network) and the infinite two-dimensional grid. We will assume that the net generations

are independent across time and position, with common expectation $\mathbb{E}Z_i(t) = \mu$, and we will place weak assumptions on the distributions (to be specified precisely later in Section 6.) We will focus on the regime when the achieved cost is small. In Section 5 we will evaluate the performance of the LQ scheme on these topologies. In Section 7. we will derive lower bounds on the performance of any schemes on these topologies to show that the LQ scheme is optimal in the small cost regime. As a result, we characterize explicitly the asymptotic performance in this regime. The results are summarized in Table 1.

Although the i.i.d. assumption simplifies significantly our derivations, we expect that the qualitative features of our results should not change for a significantly broader class of processes $\{Z_i(t)\}_t$. In particular, we expect our results to generalize under the weaker assumption that $Z_i(t)$ is stationary but close to independent beyond a time scale $T = O(1)$.

The parameter μ , the mean of the net generation at each node, can be thought of as a measure of the amount of *over-provisioning*. Let us first consider that case of a one-dimensional grid and assume that μ is vanishing or negligible. In other words, the average production balances the average load. Our results imply that a dramatic improvement is achieved by a joint use of storage and transmission resources. Consider first the case $C = 0$. The network then reduces to a collection of isolated nodes, each with storage S . It can be shown that the optimal cost decreases only slowly with the storage size S , namely as $1/S$. Similarly, when there is only transmission but no storage, the optimal cost decreases only slowly with transmission capacity C , like $1/C$. On the other hand, with both storage and transmission, the optimal costs decreases *exponentially* with $\sqrt{CS}$. Consider now positive over-provisioning $\mu > 0$. When there is no storage, the only way to drive the cost significantly down is at the expense of increasing the amount of over-provisioning beyond σ^2/C . The same performance can be achieved with a storage S equalling to this amount of over-provisioning and with the actual amount of over-provisioning exponentially smaller.

The 2-D grid provides significantly superior performance than the 1-D grid. For example, the cost exponentially decreases with the transmission capacity C even without over-provisioning and without storage. The increased connectivity in a 2-D grid allows much more spatial averaging of the random net generations than in the 1-D grid. In order to understand the fundamental reason for this difference, consider the case of vanishing over-provisioning $\mu = 0$ and vanishing storage $S = 0$ (also see Figure 1). Consider first a 1-D grid. The aggregate net generation inside a segment of l nodes has variance $l\sigma^2$ and hence this quantity is of the order of $\sqrt{l}\sigma$. This random fluctuation has to be compensated by power delivered from the rest of the grid, but this power can only be delivered through the two links, one at each end of the segment and each of capacity C . Hence,

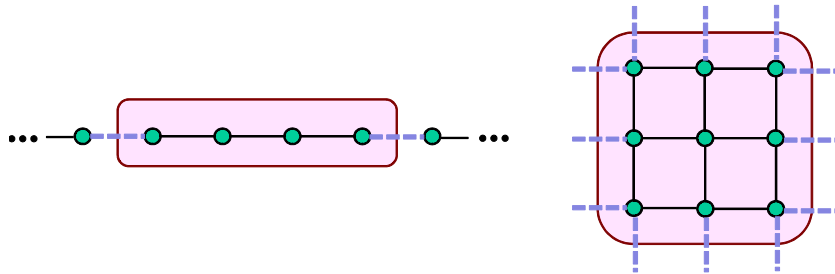


Figure 1: Boxes of side l in 1-D and in 2-D. The ratio boundary/ $\sqrt{\text{volume}}$ is $\Theta(1/\sqrt{l})$ in 1-D, and $\Theta(1)$ in 2-D.

successful compensation requires $l \lesssim C^2/\sigma^2$. One can think of $l_* := C^2/\sigma^2$ as the *spatial scale* over which averaging of the random generations is taking place. Beyond this spatial scale, the fluctuations will have to be compensated by fast generation. This fluctuation is of the order of $\sqrt{l_*}\sigma/l_* = \sigma^2/C$ per node. Note that a limit on the spatial scale of averaging translates to a large fast generation cost. In contrast, in the 2-D grid, (i) the net generation, and (ii) the total link capacity connecting an $l \times l$ box to the rest of the grid, both scale up linearly in l . This facilitates averaging over a very large spatial scale l , resulting in a much lower fast generation cost.

There is an interesting parallelism between the results for the 1-D grid with storage and the 2-D grid without storage. If we set $S = C$, the results are in fact identical. One can think of storage as providing an additional dimension for averaging: time (Section 7.2 formalizes this). Thus, *a one dimensional grid with storage behaves similarly to a two-dimensional grid without storage.*

5 Performance of LQ Scheme in Grids

In this section we evaluate the performances of the LQ scheme on the 1-D and 2-D grids. Both are examples of transitive graphs and hence we will follow the formulation in Section 3.2. For these two examples, the operator Δ is in fact invariant to spatial shifts so the θ -domain which diagonalizes the operator is simply the (spatial) Fourier domain.

For simplicity, in this section, we consider the case where $Z_i(t)$ are gaussian. In the next section we show how all our results immediately generalize to a much larger and more realistic class of distributions.

Suppose that $Z_i(t) \sim \mathcal{N}(\mu, \sigma^2)$ iid across nodes and time. It follows that B, F, W are Gaussian, and using Eq. (17), we get the following estimates

$$\varepsilon_{\mathcal{F}} \leq 2\sigma_F \mathcal{F}\left(\frac{C}{\sigma_F}\right), \quad \varepsilon_{\mathcal{W}} \leq \sigma_B \mathcal{F}\left(\frac{S}{2\sigma_B}\right) + \sigma_W \mathcal{F}\left(\frac{\mu}{\sigma_W}\right). \quad (44)$$

Here $\mathcal{F}$ is the tail of the Gaussian distribution $\mathcal{F}(z) \equiv \int_z^\infty \phi(x) dx = \Phi(-z)$, where $\phi(x) = \exp\{-x^2/2\}/\sqrt{2\pi}$ the Gaussian density and $\Phi(x) = \int_{-\infty}^x \phi(u) du$ is the Gaussian distribution.

In order to evaluate performances analytically and to obtain interpretable expressions, we will focus on two specific regimes. In the first one, no storage is available but large transmission capacity exists. In the second, large storage and transmission capacities are available.

5.1 No storage

In order to recover the performance when there is no storage, we let $\xi \rightarrow \infty$, implying $\sigma_B^2 \rightarrow 0$ by the definition of cost function (36). In this limit we have $\beta \rightarrow 0$, cf. Eq. (40). Using the explicit formulae for the various kernels, cf. Eqs. (37) to (39), we get:

$$p, q = \frac{\gamma}{\gamma + \alpha^2(\theta)} + O(1/\xi), \quad h = O(1/\xi), \quad k = 1 - O(1/\xi).$$

Substituting in Eqs. (27) to (28) we obtain the following prescription for the controlled variables (in matrix notation)

$$Y(t) = 0 \quad W(t) = \gamma(-\Delta + \gamma)^{-1}Z(t), \quad (45)$$

while the flow and storage satisfy

$$B(t) = 0, \quad F(t) = \nabla(-\Delta + \gamma)^{-1}Z(t), \quad (46)$$

The interpretation of these equations is quite clear. No storage is retained ($B = 0$) and hence no energy is transferred to storage. The matrix $\gamma(-\Delta + \gamma)^{-1}$ can be interpreted a low-pass filter and hence $\gamma(-\Delta + \gamma)^{-1}Z(t)$ is a smoothing of $Z(t)$ whereby the smoothing takes place on a length scale $\gamma^{-1/2}$. The wasted energy is obtained by averaging underproduction over regions of this size.

Finally, using Eqs. (42) and (43), we obtain the following results for the variances in Fourier space

$$\frac{\sigma_F(\theta)^2}{\sigma^2} = \frac{\alpha^2(\theta)}{(\gamma + \alpha^2(\theta))^2}, \quad \frac{\sigma_W(\theta)^2}{\sigma^2} = \frac{\gamma^2}{(\gamma + \alpha^2(\theta))^2}.$$

5.1.1 One-dimensional grid

In this case $\theta \in [-\pi, \pi]$, and $\alpha(\theta)^2 = 2 - 2\cos\theta$ (the Laplacian Δ is diagonalized via Fourier transform). The form of the optimal filter $\mathbf{P}$ is shown in Figure 2.

The Parseval integrals can be computed exactly but we shall limit ourselves to stating without proof their asymptotic behavior for small γ .

Lemma 5.1. *For the one-dimensional grid, in absence of storage, as $\gamma \rightarrow 0$, the optimal LQ control yields variances*

$$\sigma_F^2 = \sigma^2/4\sqrt{\gamma} \left\{ 1 + O(\gamma) \right\}, \quad \sigma_W^2 = \sigma^2\sqrt{\gamma}/4 \left\{ 1 + O(\gamma) \right\}.$$

Using these formulae and the equations (44) for the performance parameters, we get the following achievability result.

Theorem 3. *For the one-dimensional grid, in absence of storage, the optimal LQ control with Lagrange parameter $\gamma = \mu^2/C^2$ yields, in the limit $\mu/C \rightarrow 0$, $\mu C/\sigma^2 \rightarrow \infty$:*

$$\varepsilon_{\text{tot}} \leq \exp \left\{ -\frac{2\mu C}{\sigma^2} (1 + o(1)) \right\}. \quad (47)$$

The choice of γ given here is dictated by approximately minimizing the cost. In words, the cost is exponentially small in the product of the capacity, and overprovisioning μC . This is achieved by averaging over a length scale $\gamma^{-1/2} = C/\mu$ that grows only linearly in C and $1/\mu$. Note that the extent of averaging is limited by the transmission capacity C : the larger the extent of averaging, the larger the amount of power which has to be transported across the network. Optimal filters $\mathbf{P}$ for two different values of $\gamma^{-1/2}$ are displayed in Figure 2.

5.1.2 Two-dimensional grid

In this case $\theta = (\theta_1, \theta_2) \in [-\pi, \pi]^2$, and $\alpha(\theta)^2 = 4 - 2\cos\theta_1 - 2\cos\theta_2$. Again, we evaluate Parseval's integral as $\gamma \rightarrow 0$, and present the result.

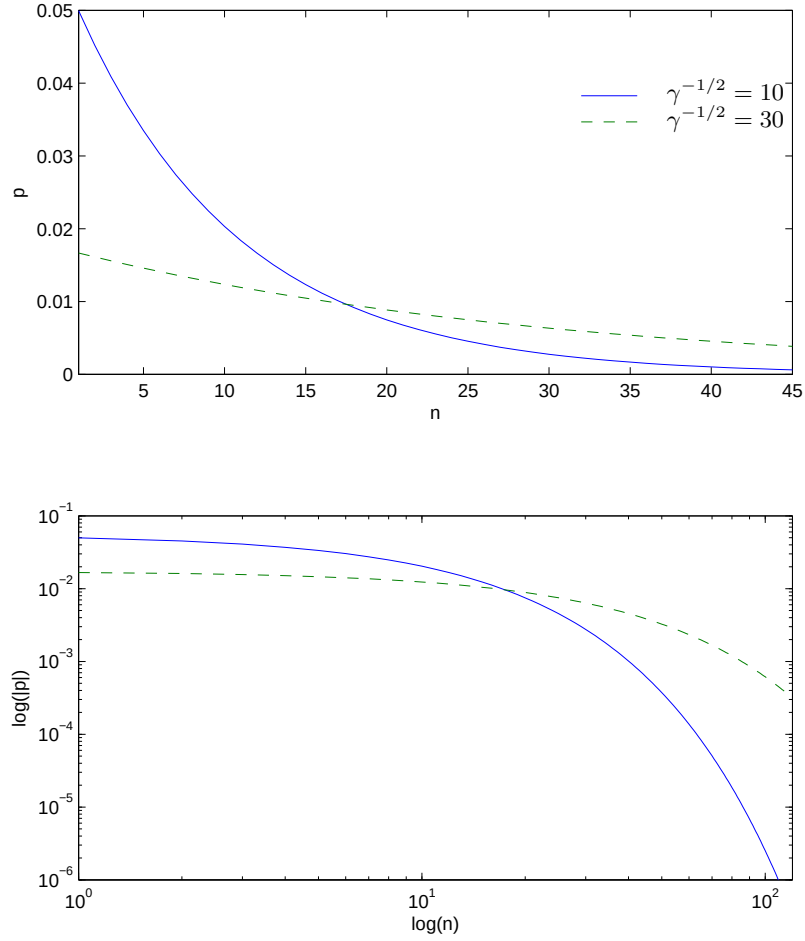


Figure 2: The filter $\mathbf{P}$ for a one-dimensional grid $\mathbb{Z}$ in the case in which no storage is available. Notice that by translation invariance $\mathbf{P}_{i,j} = P_{i-j}$ for any $i, j \in \mathbb{Z}$, and further $P_n = P_{-n}$. Here we plot P_n for two values of the effective length scale $1/\sqrt{\gamma}$.

Lemma 5.2. *For the two-dimensional grid, in absence of storage, as $\gamma \rightarrow 0$, the optimal LQ control yields variances*

$$\sigma_F^2 = \frac{\sigma^2}{4\pi} \left\{ \log \left(\frac{1}{e\gamma} \right) + O(\gamma) \right\}, \quad \sigma_W^2 = \frac{\sigma^2 \gamma}{4\pi} \left\{ 1 + O(\gamma) \right\}.$$

Using these formulae and the equations (44) for the performance parameters, and approximately optimizing over γ , we obtain the following achievability result.

Theorem 4. *For the two-dimensional grid, in absence of storage, the optimal LQ control with Lagrange parameter $\gamma = (\mu^2/C^2) \log(C^2/\mu^2 e)$ yields, in the limit $\mu/C \rightarrow 0$, $C^2/(\sigma^2 \log(C/\mu)) \equiv M \rightarrow \infty$:*

$$\varepsilon_{\text{tot}} \leq \exp \left\{ -\frac{2\pi C^2}{\sigma^2 \log(C^2/\mu^2 e)} (1 + o(1)) \right\}. \quad (48)$$

Notice the striking difference with respect to the one-dimensional case, cf. Theorem 3. The cost goes exponentially to 0, but now overprovisioning plays a significantly smaller role. For instance, if we fix the link capacity C to be the same, the exponents in Eq. (47) are matched if $\mu_{2d} \approx \exp(-\pi C/2\mu_{1d})$, i.e. an exponentially smaller overprovisioning is sufficient.

5.2 With Storage

In this section we consider the case in which storage is available. Again we focus on the regime where the optimal cost is small. Within our LQ formulation we want therefore to penalize σ_W much more than σ_B and σ_F . This corresponds to the asymptotics $\gamma \rightarrow 0$, $\xi \equiv \gamma/s \rightarrow 0$ (the ratio s need not to be fixed). It turns out that the relevant behavior is obtained by considering $\alpha^2 = \Theta(\gamma)$ and hence $\beta \rightarrow \infty$. The linear filters are given in this regime by

$$p(\theta) = q(\theta) = (\gamma/\sqrt{s}) (\gamma + \alpha(\theta)^2)^{-1/2}, \\ k(\theta) \approx (1/\sqrt{s}) (\gamma + \alpha(\theta)^2)^{1/2}, \quad h(\theta) \approx 1.$$

Using these filters we obtain

$$\begin{aligned} \frac{\sigma_B(\theta)^2}{\sigma^2} &\approx \frac{1}{2} \left(\frac{s}{\gamma + \alpha(\theta)^2} \right)^{1/2}, \\ \frac{\sigma_F(\theta)^2}{\sigma^2} &\approx \frac{\alpha^2(\theta)}{2\sqrt{s}} \left(\frac{1}{\gamma + \alpha(\theta)^2} \right)^{3/2}, \\ \frac{\sigma_W(\theta)^2}{\sigma^2} &\approx \frac{\gamma^2}{2\sqrt{s}} \left(\frac{1}{\gamma + \alpha(\theta)^2} \right)^{3/2}. \end{aligned}$$

5.2.1 One-dimensional grid

The variances are obtained by Parseval's identity, integrating $\sigma_{B,W,F}^2(\theta)$ over $\theta \in [-\pi, \pi]$. The form of the optimal filters $\mathbf{P}$ and $\mathbf{K}$ is presented in Figure 3. We obtain the following asymptotic results.

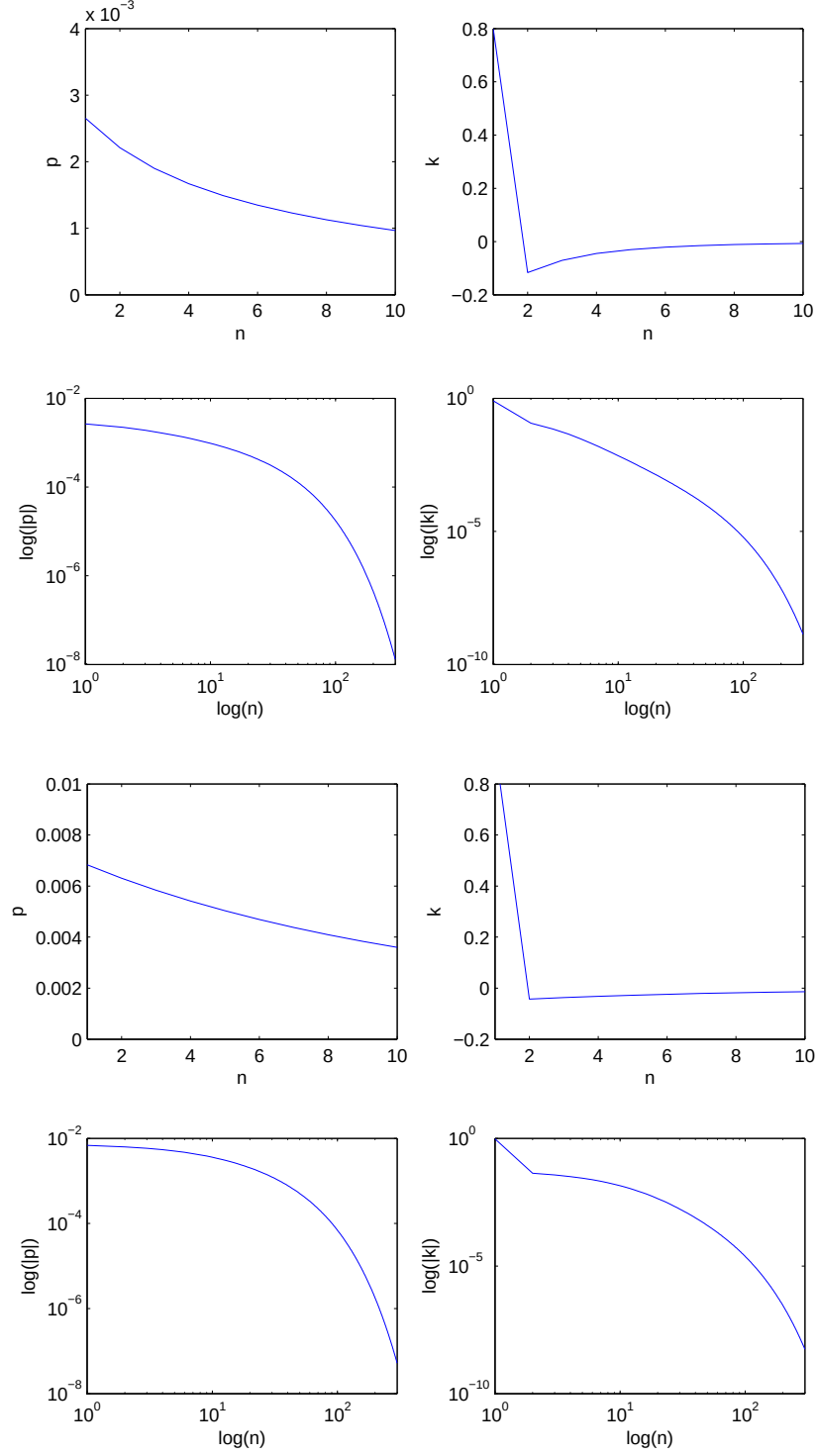


Figure 3: The filters $\mathbf{P}$, $\mathbf{K}$ for a one-dimensional grid $\mathbb{Z}$ in the case in which storage is available. Notice that by translation invariance $\mathbf{P}_{i,j} = P_{i-j}$, $\mathbf{K}_{i,j} = K_{i-j}$ for any $i, j \in \mathbb{Z}$, and further $P_n = P_{-n}$, $K_n = K_{-n}$. Here we plot the filters for effective length scale: $1/\sqrt{\gamma} = 30$ and $S = C$ (top panels) or $S = C/4$ (lower panels).

Lemma 5.3. Consider a one-dimensional grid, subject to the LQ optimal control. For $\gamma \rightarrow 0$ and $\xi = \gamma/s \rightarrow 0$

$$\begin{aligned}\frac{\sigma_B^2}{\sigma^2} &= \frac{\sqrt{s}}{4\pi} \log \frac{1}{\gamma} + O(1), \quad \frac{\sigma_F^2}{\sigma^2} = \frac{1}{4\pi\sqrt{s}} \log \frac{1}{\gamma} + O(1, s^{-1}), \\ \frac{\sigma_W^2}{\sigma^2} &= \frac{\Omega_1}{2\sqrt{s}} \gamma + O(\gamma^2, \gamma^{3/2}/s),\end{aligned}$$

where Ω_d is the integral (here $d^d u \equiv du_1 \times \dots \times du_d$)

$$\Omega_d \equiv 1/(2\pi)^d \int_{\mathbb{R}^d} 1/(1 + \|u\|^2)^{3/2} d^d u. \quad (49)$$

Using Eqs. (44) to estimate the total cost ε_{tot} and minimizing it over γ we obtain the following.

Theorem 5. Consider a one-dimensional grid and assume $CS/\sigma^2 \rightarrow \infty$. The optimal LQ scheme achieves the following performance:

$$\begin{aligned}\mu = e^{-\omega\left(\sqrt{\frac{CS}{\sigma^2}}\right)} &\Rightarrow \varepsilon_{\text{tot}} \leq \exp\left\{-\sqrt{\frac{\pi CS}{2\sigma^2}}(1 + o(1))\right\}, \\ \mu = e^{-o\left(\sqrt{\frac{CS}{\sigma^2}}\right)} &\Rightarrow \varepsilon_{\text{tot}} \leq \exp\left\{-\frac{\pi CS(1 + o(1))}{2\sigma^2 \log C/\mu}\right\},\end{aligned}$$

under the further assumption $\sqrt{\pi CS/2\sigma^2} - \log(C/S) \rightarrow \infty$ (in the first case) and $\mu^2 \log(C/\mu)/\min(C, S)^2 \rightarrow 0$ (in the second). In the first case the claimed behavior is achieved by $s = S^2/4C^2$, and $\gamma = \exp\{-(2\pi CS/\sigma^2)^{1/2}\}$. In the second by letting $s = S^2/4C^2$, and $\gamma = \mu^2 \log(C/\mu)/(\pi\Omega_1 C^2)$.

This theorem points at a striking threshold phenomenon. If overprovisioning is extremely small, or vanishing, then the cost is exponentially small in $\sqrt{CS}$. On the other hand, even a modest overprovisioning changes this behavior leading to a decrease that is exponential in CS (barring exponential factors). Overprovisioning also reduces dramatically the effective averaging length scale $\gamma^{-1/2}$. It also instructive to compare the second case in Theorem 5 with its analogue in the case of no storage, cf. Eq. (47): storage seem to replace overprovisioning.

5.2.2 Two-dimensional grid

As done in the previous cases, the variances of B , F , W are obtained by integrating $\sigma_{B,F,W}^2(\theta)$ over $\theta = (\theta_1, \theta_2) \in [-\pi, \pi]^2$.

Lemma 5.4. Consider a two-dimensional grid, subject to the LQ optimal control. For $\gamma \rightarrow 0$ and $s = \Theta(1)$, we have

$$\begin{aligned}\frac{\sigma_B^2}{\sigma^2} &= G_B(s) + O(1, \sqrt{\gamma}), \quad \frac{\sigma_F^2}{\sigma^2} = G_F(s) + O(\sqrt{\gamma}), \\ \frac{\sigma_W^2}{\sigma^2} &= \frac{\Omega_2}{2\sqrt{s}} \gamma^{3/2} + O(\gamma^2),\end{aligned}$$

where Ω_2 is the constant defined as per Eq. (49), and $G_B(s)$, $G_F(s)$ are strictly positive and bounded for s bounded. Further, as $s \rightarrow \infty$ $G_B(s) = \mathcal{K}_2\sqrt{s}/2 + O(1)$, $G_F(s) = \mathcal{K}_2/(2\sqrt{s}) + O(1/s)$, where $\mathcal{K}_2 \equiv \int_{[-\pi, \pi]^2} \frac{1}{|\alpha(\theta)|} d\theta$.

Minimizing the total outage over s, γ , we obtain:

Theorem 6. *Assume $CS/\sigma^2 \rightarrow \infty$ and $C/S = \Theta(1)$. The optimal cost for scheme a memory-one linear scheme on the two-dimensional grid network then behaves as follows*

$$\varepsilon_{\text{tot}} \leq \exp \left\{ -\frac{CS}{2\sigma^2\Gamma(S/C)}(1 + o(1)) \right\}. \quad (50)$$

Here $u \mapsto \Gamma(u)$ is a function which is strictly positive and bounded for u bounded away from 0 and ∞ . In particular, $\Gamma(u) \rightarrow \mathcal{K}_2$ as $u \rightarrow \infty$, and $\Gamma(u) = \Gamma_0 u + o(u)$ as $u \rightarrow 0$ ($\Gamma_0 > 0$).

The claimed behavior is achieved by selecting $s = f(S/C)$, and γ as follows. If $\mu = \exp\{-o(CS/\sigma^2)\}$ then $\gamma = \tilde{f}(S/C)(\mu^2/CS)^{2/3}$. If instead $\mu = \exp\{-\omega(CS/\sigma^2)\}$, then $\gamma = \exp\{-2CS/(3\Gamma(S/C)\sigma^2)\}$, for suitable functions $f, \tilde{f}$ (In the first case, we also assume $\mu/C \rightarrow 0$.)

The functions $\Gamma, f, \tilde{f}$ in the last statement can be characterized analytically, but we omit such characterization for the sake of brevity. As seen by comparing with Theorem 5, the greater connectivity implied by a two dimensional grid leads to a faster decay of the cost.

6 Extension to a larger class of distributions

We find that our results from Section 5 immediately generalize to a much broader class of distributions for $Z_i(t)$ than Gaussian.

To define this class, first we provide the definition of *sub-Gaussian* random variables. (See, for instance, [10] for more details.)

Definition 6.1. *A random variable X is sub-Gaussian with tail parameter s^2 if, for any $\lambda \in \mathbb{R}$,*

$$\mathbb{E}\{e^{\lambda(X - \mathbb{E}X)}\} \leq e^{\lambda^2 s^2 / 2}. \quad (51)$$

Two important examples of sub-Gaussian random variables are:

1. Gaussian random variables with variance σ^2 are sub-Gaussian with tail parameter $s^2 = \sigma^2$.
2. Random variables with bounded support on $[a, b]$ are sub-Gaussian with tail parameter $s^2 = (b - a)^2 / 4$.

Notice that the tail parameter is always an upper bound on the variance σ^2 , namely $\sigma^2 \leq s^2$ (this follows by Taylor expansion of Eq. (51) for small λ). We will consider the class of distributions for the net production $Z_i(t)$ to be sub-Gaussian with tail parameter of the same order as the variance. More precisely:

Definition 6.2. *Fix constant $\kappa > 0$. Let $\mathcal{S}(\kappa)$ be the class of distributions such that the sub-Gaussian tail parameter s^2 and the standard deviation σ^2 satisfy:*

$$1 \leq \frac{s^2}{\sigma^2} < \kappa.$$

This is a natural class of distributions for modeling renewable power production. The power generated by wind turbines, for example, are bounded between 0 and a upper power limit, with significant probability that the power is near 0 or capped at the upper limit. Hence, it is of bounded support with the range comparable to the standard deviation. It is sub-Gaussian with tail parameter of the same order as the variance.

We will now argue that all the results we derived in Section 5 for Gaussian net productions extend to this class of distribution. The only fact we used to connect variances with the costs, where we used the Gaussianity assumption, is Eq. (44). This equation implies that $\varepsilon_{\mathcal{F}}$ decreases exponentially with $(C/\sigma_F)^2$, and $\varepsilon_{\mathcal{W}}$ decreases exponentially with $(S/\sigma_B)^2$ and with $(\mu/\sigma_W)^2$, which in turns leads to Theorems 3, 4, 5, 6. We will show that these exponential dependencies hold for distributions in $\mathcal{S}(\kappa)$ as well, and a similar versions of these theorems hold for these distributions.

First we need some elementary properties of sub-Gaussian random variables. The first property follows by elementary manipulations with moment generating functions.

Lemma 6.3. *Assume X_1 and X_2 to be independent random variables with tail parameters s_1^2 and s_2^2 . Then, for any $a_1, a_2 \in \mathbb{R}$, $X = a_1X_1 + a_2X_2$ is sub-Gaussian with tail parameter $(a_1^2s_1^2 + a_2^2s_2^2)$.*

Notice that by this lemma, the parameters of sub-Gaussian random variables behave exactly as variances (as far as linear operations are involved). In particular, it implies that the class $\mathcal{S}(\kappa)$ is closed under linear operations.

The second property is a well known consequence of Markov inequality, and shows that the tail of a sub-Gaussian random variable is dominated by the tail of a Gaussian with the same parameter.

Lemma 6.4. *If X is a sub-Gaussian random variable with parameter s^2 , then, for any $a \geq 0$ $\mathbb{P}\{X \geq a + \mathbb{E}X\}, \mathbb{P}\{X \leq -a + \mathbb{E}X\} \leq \exp\{-a^2/(2s^2)\}$.*

Now suppose the net productions $Z_i(t)$'s have distributions in $\mathcal{S}(\kappa)$. The LQ scheme developed in Section 3 implies that the controlled variables $B_i(t)$, $F_e(t)$, $W_i(t)$ are linear functions of the net productions $Z_i(t)$, and hence it follows that $B_i(t)$, $F_e(t)$, $W_i(t)$ are in $\mathcal{S}(\kappa)$. Now, if we let F_e be the flow at edge e at steady-state, with sub-Gaussian tail parameter s_F^2 , then

$$\begin{aligned} \varepsilon_{\mathcal{F}} &= \mathbb{E}\{(\mathcal{F}_e - C)_+ + (-C - \mathcal{F}_e)_+\} \\ &= \int_C^\infty \mathbb{P}\{F_e > a\} da + \int_{-\infty}^{-C} \mathbb{P}\{F_e < a\} da \\ &\leq \int_C^\infty \exp\{-a^2/(2s_F^2)\} da + \int_{-\infty}^{-C} \exp\{-a^2/(2s_F^2)\} da \\ &\leq 2 \int_C^\infty \exp\{-a^2/(2\kappa\sigma_F^2)\} da \\ &= 2\sqrt{2\pi}\sigma_F F\left(\frac{C}{\sqrt{\kappa}\sigma_F}\right). \end{aligned}$$

Here, as in Eq. (44), $F(\cdot)$ is the complementary cumulative distribution function of the standard Gaussian random variable: $F(x) = 1 - \Phi(x)$. Similarly, one can show that:

$$\varepsilon_{\mathcal{W}} \leq \sqrt{2\pi}\sigma_B F\left(\frac{S}{2\sqrt{\kappa}\sigma_B}\right) + \sqrt{2\pi}\sigma_W F\left(\frac{\mu}{\sqrt{\kappa}\sigma_W}\right).$$

Thus, $\varepsilon_{\mathcal{F}}$ decreases exponentially with $(C/\sigma_F)^2$, and $\varepsilon_{\mathcal{W}}$ decreases exponentially with $(S/\sigma_B)^2$ and with $(\mu/\sigma_W)^2$, as in the Gaussian case, except for an additional factor of $1/\kappa$ in the exponent. This means that analogues of Theorems 3, 4, 5, 6 also hold for distributions in $\mathcal{S}(\kappa)$ with an additional factor of $1/\kappa$ in the exponent of ε_{tot} . Note that all these exponents are proportional to $1/\sigma^2$, where σ^2 is the variance of the Gaussian distributed $Z_i(t)$. Therefore, equivalently, one can say that the performance under a sub-Gaussian distributed $Z_i(t)$ with tail parameter s^2 is at least as good as if $Z_i(t)$ were Gaussian with variance s^2 .

7 Performance limits

In this section, we prove general lower bounds on the outage $\varepsilon_{\text{tot}} = \varepsilon_{\mathcal{W}} + \varepsilon_{\mathcal{F}}$ of *any scheme*, on the 1-D and 2-D grids. Our proofs use cutset type arguments. Throughout this section, we will assume the $Z_i(t)$ to be i.i.d. random variables, with $Z_i(t) \sim \mathcal{N}(\mu, \sigma^2)$, with the exception of the case of a one-dimensional grid without storage, cf. Theorem 7. In this case, we will make the weaker assumption that the $Z_i(t)$ are i.i.d. sub-Gaussian.

7.1 No storage

7.1.1 One-dimensional grid

Theorem 7. *Consider a one-dimensional grid without storage, and assume the net productions $Z_i(t)$ to be i.i.d. sub-Gaussian random variables in the class $\mathcal{S}(\kappa)$. (In particular this assumption holds if $Z_i(t) \sim \mathcal{N}(\mu, \sigma^2)$.)*

There exist finite constants $\kappa_0, \kappa_1, \kappa_3 > 0$ dependent only on κ , such that the following happens. For $\mu < \kappa_0\sigma$ and $\sigma < \kappa_0C$, we have

$$\varepsilon_{\text{tot}} \geq \begin{cases} \kappa_1\sigma^2/C & \text{if } \mu < \sigma^2/C, \\ \mu \exp\{-\kappa_2\mu C/\sigma^2\} & \text{otherwise.} \end{cases} \quad (52)$$

Proof. Consider a segment of length ℓ . Let $\mathbf{E}$ be the event that the segment has net demand at least $3C$. Then we have

$$\mathbb{P}[\mathbf{E}] \geq \kappa_3 \exp\left\{-\frac{\kappa_4(3C + \ell\mu)^2}{2\sigma^2\ell}\right\}, \quad (53)$$

for some $\kappa_3, \kappa_4 > 0$. (This inequality is immediate for $Z_i(t) \sim \mathcal{N}(\mu, \sigma^2)$ and follows from Lemma A.1 proved in the appendix for general random variables in $\mathcal{S}(\kappa)$.)

If $\mathbf{E}$ occurs at some time t , this leads to a shortfall of at least C in the segment of length ℓ . This shortfall contributes either to $\varepsilon_{\mathcal{W}}$ or to $2\varepsilon_{\mathcal{F}}$, yielding

$$2\varepsilon_{\text{tot}} \geq \varepsilon_{\mathcal{W}} + 2\varepsilon_{\mathcal{F}} \geq \frac{\kappa_3 C}{\ell} \exp\left\{-\frac{\kappa_4(3C + \ell\mu)^2}{\sigma^2\ell}\right\}. \quad (54)$$

Choosing $\ell = \min(C/\mu, C^2/\sigma^2)$, we obtain the result. $\square$

Note that the lower bound is tight both for $\mu \geq \sigma^2/C$ (by Theorem 3) and $\mu < \sigma^2/C$ (by a simple generalization of the same theorem that we omit).

7.1.2 Two-dimensional grid

We prove a lower bound almost matching the upper bound proved in Theorem 4.

Theorem 8. *There exists $\kappa < \infty$ such that, for $C \geq \min(\mu, \sigma)$,*

$$\varepsilon_{\text{tot}} \geq \sigma \exp \left\{ -\kappa C^2 / \sigma^2 \right\}.$$

Proof. Follows from a single node cutset bound. $\square$

We next make a conjecture in probability theory, which, if true, leads to a significantly stronger lower bound for small μ . For any set of vertices $\mathcal{A}$ of the two-dimensional grid, we denote by $\partial\mathcal{A}$ the *boundary* of $\mathcal{A}$, i.e., the set of edges in the grid that have one endpoint in $\mathcal{A}$ and the other in $\mathcal{A}^c$.

Conjecture 7.1. *There exists $\delta > 0$ such that the following occurs for all $\ell \in \mathbb{N}$. Let $(X_v)_{v \in \mathcal{S}}$ be a collection of i.i.d. $\mathcal{N}(0, 1)$ random variables indexed by $\mathcal{S} = \{1, \dots, \ell\} \times \{1, \dots, \ell\} \subseteq \mathbb{Z}^2$. Then*

$$\mathbb{E} \left[\max_{\substack{\mathcal{A} \subseteq \mathcal{S} \text{ s.t.} \\ |\partial\mathcal{A}| \leq 4\ell}} \sum_{v \in \mathcal{A}} X_v \right] \geq \delta \ell \log \ell. \quad (55)$$

It is not hard to see that this conjecture implies a tight lower bound.

Theorem 9. *Consider the two-dimensional grid without storage, and assume Conjecture 7.1. Then there exists $\kappa < \infty$ such that for any $\mu \leq \sigma \exp(-\kappa C / \sigma)$ and $C > \sigma$ we have*

$$\varepsilon_{\text{tot}} \geq \sigma \exp \left\{ -\kappa C / \sigma \right\}. \quad (56)$$

Proof. Consider a square of side ℓ . Conjecture 7.1 yields that we can find a subset of vertices in the square with a boundary capacity no more than $4C\ell$, but with a net demand of at least $\delta\sigma\ell \log \ell - \mu\ell^2$. This yields

$$2\varepsilon_{\text{tot}} \geq \varepsilon_{\mathcal{W}} + 2\varepsilon_{\mathcal{F}} \geq \frac{\delta\sigma\ell \log \ell - 4C\ell - \mu\ell^2}{\ell^2}. \quad (57)$$

Choosing $\ell = \exp(\kappa C / \sigma)$ with an appropriate choice of κ , we obtain the result. $\square$

Our conjecture was arrived at based on a heuristic divide-and-conquer argument. We validated our conjecture numerically as follows: We obtain a lower bound to the left hand side of Eq. (55), by maximizing over a restricted class of subsets $\mathbb{S}_{\text{op}}$, consisting of subsets that can be formed by dividing the square into two using an oriented path (each step on such a path is either upwards or to the right). It is easy to see that if $\mathcal{S} \in \mathbb{S}_{\text{op}}$, then $|\partial\mathcal{S}| \leq 4\ell$. Define

$$G(l) \equiv \max_{\mathcal{S} \in \mathbb{S}_{\text{op}}} \sum_{v \in \mathcal{S}} X_v, \quad (58)$$

where $\mathbb{S}_{\text{op}}$ is implicitly a function of l . The advantage of considering this quantity is that $G(l)$ can be computed using a simple dynamic program of quadratic complexity. Numerical evidence, plotted in Figure 7.1.2, suggests that $\mathbb{E}[G(l)] = \Omega(l \log l)$, which implies our conjecture.

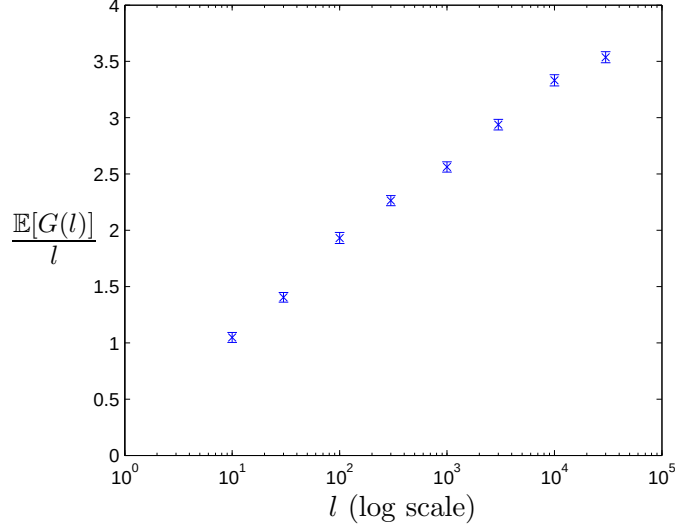


Figure 4: Numerical evidence for Conjecture 7.1: The approximate straight line in the plot (cf. definition of $G(l)$ in Eq. (58)) suggests the validity of our conjecture.

7.2 With storage

7.2.1 One-dimensional grid

Our approach involves mapping the time evolution of a control scheme in a one-dimensional grid, to a feasible (one-time) flow in a two-dimensional grid. One of the dimensions represents ‘space’ in the original grid, whereas the other dimension represents time.

Consider the one-dimensional grid, with vertex set $\mathbb{Z}$. We construct a two-dimensional ‘space-time’ grid $(\hat{V}, \hat{E})$ consisting of copies of each $v \in V$, one for each time $t \in \mathbb{Z}$: define $\hat{V} \equiv \{(v, t) : v \in \mathbb{Z}, t \in \mathbb{Z}\}$. The edge set $\hat{E}$ consists of ‘space-edges’ E^{sp} and ‘time-edges’ E^{t} .

$$\begin{aligned}\hat{E} &\equiv E^{\text{sp}} \cup E^{\text{t}} \\ E^{\text{sp}} &\equiv \{((v, t), (v + 1, t)) : v \in \mathbb{Z}, t \in \mathbb{Z}\} \\ E^{\text{t}} &\equiv \{((v, t), (v, t + 1)) : v \in \mathbb{Z}, t \in \mathbb{Z}\}\end{aligned}$$

Edges are undirected. Denote by $\hat{C}_e$ the capacity of $e \in \hat{E}$. We define $\hat{C}_e \equiv C$ for $e \in E^{\text{sp}}$ and $\hat{C}_e = S/2$ for $e \in E^{\text{t}}$.

Given a control scheme for the 1-D grid with storage, we define the flows in the space-time grid as

$$\begin{aligned}\hat{\mathcal{F}}_e &\equiv \mathcal{F}_{(v, v+1)}(t) && \text{for } e = ((v, t), (v + 1, t)) \in E^{\text{sp}} \\ \hat{\mathcal{F}}_e &\equiv \mathcal{B}_v(t + 1) - S/2 && \text{for } e = ((v, t), (v, t + 1)) \in E^{\text{t}}\end{aligned}$$

Notice that these flows are not subject to Kirchoff constraints, but the following energy balance equation is satisfied at each node $(v, t) \in \hat{V}$,

$$\mathcal{Z}_i(t) - \mathcal{W}_i(t) - \mathcal{Y}_i(t) = \sum_{(v', t') \in \partial(v, t)} \hat{\mathcal{F}}_{(v, t), (v', t')} \quad (59)$$

We use performance parameters as before (this definition applies to finite networks and must be suitably modified for infinite graphs):

$$\begin{aligned}\varepsilon_{\widehat{\mathcal{F}}} &\equiv \frac{1}{|\widehat{E}|} \sum_{e \in \widehat{E}} \mathbb{E}\{(\widehat{\mathcal{F}}_e(t) - \widehat{C}_e)_+ + (\widehat{C}_e - \widehat{\mathcal{F}}_e(t))_+\}, \\ \varepsilon_{\mathcal{W}} &\equiv \frac{1}{|\widehat{V}|} \sum_{(i,t) \in \widehat{V}} \mathbb{E}\{(W_i(t))_-\}.\end{aligned}$$

Notice that $\varepsilon_{\mathcal{W}}$ is unchanged, and $\varepsilon_{\widehat{\mathcal{F}}} = \varepsilon_{\mathcal{F}}$, in our mapping from the 1-D grid with storage to the 2-D space-time grid.

Our first theorem provides a rigorous lower bound which is almost tight for the case $\mu = e^{-o(\sqrt{CS/\sigma^2})}$ (cf. Theorem 5). It is proved by considering a rectangular region in the space-time grid of side $l = \max(C/S, 1)$ in space and $T = \max(1, S/C)$ in time.

Theorem 10. *Suppose $\mu \leq \min(C, S)$, $CS/\sigma^2 > \max(\log(\sigma/\min(C, S)), 1)$. There exists $\kappa < \infty$ such that*

$$\varepsilon_{\text{tot}} \geq \sigma \exp(-\kappa CS/\sigma^2). \quad (60)$$

Proof. Consider a segment of length $\ell = \max(C/S, 1)$ and a sequence of $T = \ell S/C$ consecutive time slots. (Rounding errors are easily dealt with.) The number of nodes in the corresponding region $\mathcal{R}$ in the space-time grid is

$$n \equiv \ell T = \max(C, S)/\min(C, S).$$

The cut, i.e., the connection between $\mathcal{R}$ and the rest of the grid, is of size $2(\ell S + TC) = 4 \max(C, S)$. The net generation inside $\mathcal{R}$ is $N(n\mu, \sigma^2 n)$. Now $\mu \leq C$ by assumption, implying $n\mu \leq \max(C, S)$. Let $\mathbf{E}$ be the event that the net generation inside $\mathcal{R}$ is at least $5 \max(C, S)$. We have

$$\mathbb{P}[\mathbf{E}] \geq \exp\left(-\frac{\kappa_1(\max(C, S))^2}{\sigma^2 n}\right) \geq \exp\left(-\frac{\kappa_1 CS}{\sigma^2}\right)$$

for some $\kappa_1 < \infty$. Moreover, $\mathbf{E}$ leads to a shortfall of at least $\max(C, S)$ over n nodes in the space-time grid. It follows that

$$\varepsilon_{\text{tot}} \geq (\max(C, S)/n) \exp\left(-\frac{\kappa_1 CS}{\sigma^2}\right) = \min(C, S) \exp\left(-\frac{\kappa_1 CS}{\sigma^2}\right),$$

which yields the result, using $CS/\sigma^2 > \log(\sigma/\min(C, S))$. □

Next we provide a sharp lower bound for small μ using Conjecture 7.1. Recall Theorem 9 and notice that its proof does not make any use of Kirchoff flow constraints (encoded in Eq. (3)). Thus, the same result holds for a 2-D space-time grid. We immediately obtain the following result, suggesting that the upper bound in Theorem 5 for small μ is tight.

Theorem 11. *There exists $\kappa < \infty$ such that the following occurs if we assume that Conjecture 7.1 is valid. Consider the one-dimensional grid with parameters $C = S > \sigma$, and $\mu \leq \exp(-\kappa C/\sigma)$. We have*

$$\varepsilon_{\text{tot}} \geq \sigma \exp \left\{ -\kappa \sqrt{CS/\sigma^2} \right\}. \quad (61)$$

We remark that the requirement $C = S$ can be relaxed if we assume a generalization of Conjecture 7.1 to rectangular regions in the two-dimensional grid.

7.2.2 Two-dimensional grid

Theorem 12. *There exists a constant $\kappa < \infty$ such that on the two-dimensional grid,*

$$\varepsilon_{\text{tot}} \geq \sigma \exp \left\{ -\frac{\kappa C \max(C, S)}{\sigma_i^2} \right\}. \quad (62)$$

The theorem is proved by considering a single node, using a cutset type argument, similar to the proof of Theorem 10. It implies that the upper bound in Theorem 6 is tight up to constants in the exponent.

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A A probabilistic lemma

Lemma A.1. *Let $\{X_1, X_2, \dots, X_n, \dots\}$ be a collection of i.i.d. sub-Gaussian random variables in $\mathcal{S}(\kappa)$ with $\mathbb{E}X_1 = 0$, $\mathbb{E}\{X_1^2\} = \sigma^2$.*

Then there exists finite constants $\kappa_1 = \kappa_1(\kappa) > 0$, $\kappa_2 = \kappa_2(\kappa) > 0$, $n_0 = n_0(\kappa)$ depending uniquely on κ such that, for all $n \geq n_0$, $0 \leq \gamma \leq \kappa_1 \sigma$, we have

$$\mathbb{P} \left\{ \sum_{i=1}^n X_i \geq \gamma n \right\} \geq \frac{1}{4} \exp \left\{ -\frac{n\gamma^2}{\kappa_2 \sigma^2} \right\}. \quad (63)$$

Proof. By scaling, we will assume, without loss of generality, $\sigma^2 = 1$. Throughout the proof $\kappa', \kappa'', \dots$ denote constants depending uniquely on κ . We will use the same symbol even if the constants have to be redefined in the course of the proof.

For any $\lambda \in \mathbb{R}$, let $\mathbb{P}_\lambda$, $\mathbb{E}_\lambda$ denote probability and expectation with respect to the measure defined implicitly by

$$\mathbb{E}_\lambda \{f(X_1, \dots, X_n)\} \equiv \frac{\mathbb{E} \{f(X_1, \dots, X_n) e^{\lambda \sum_{i=1}^n X_i}\}}{\mathbb{E} \{e^{\lambda \sum_{i=1}^n X_i}\}}, \quad (64)$$

for all measurable functions f . Notice that this measure is well defined for all λ by sub-Gaussianity.

Let $g(\lambda) \equiv \mathbb{E}_\lambda X_1$. Then $\lambda \mapsto g(\lambda)$ is continuous, monotone increasing with $g(0) = 0$, $g'(\lambda) = \text{Var}_\lambda(X_1)$, $g''(\lambda) = \mathbb{E}_\lambda X_1^3 - 3\mathbb{E}_\lambda X_1 \mathbb{E}_\lambda X_1^2$. Bounding these quantities by sub-Gaussianity, it follows that, for $0 \leq \lambda \leq \kappa'$, we have $1/\kappa'' \leq \text{Var}_\lambda(X_1) \leq \kappa''$, and hence

$$\frac{\lambda}{\kappa''} \leq g(\lambda) \leq \kappa'' \lambda. \quad (65)$$

Define $\gamma_{+/-} \equiv \lim_{\lambda \rightarrow \pm\infty} g(\lambda)$. Notice that $\gamma_- < 0 < \gamma_+$ and that g^{-1} (the inverse function of g) is well defined on the interval (γ_-, γ_+) .

Define

$$h(\lambda) \equiv \frac{(\mathbb{E}e^{\lambda X_1})^2}{\mathbb{E}(e^{2\lambda X_1})}. \quad (66)$$

By Taylor expansion, we get $h(\lambda) = 1 - \lambda^2 \mathbb{E}(X_1^2) + O(\lambda^3) = 1 - \lambda^2 + O(\lambda^3)$. Proceeding as above, it is not hard to prove that $h(\lambda) \geq 1 - \kappa'' \lambda^2$ for all $0 \leq \lambda \leq \kappa'$ for some finite constants $\kappa', \kappa'' > 0$ (eventually different from above). Finally, for $\gamma \in (\gamma_-, \gamma_+)$ we define

$$H(\gamma) \equiv h(g^{-1}(\gamma)). \quad (67)$$

Combining the above, we have $H(\gamma) = 1 - \kappa'' \gamma^2$ for all $\gamma \in [0, \kappa']$, and therefore

$$H(\gamma) \geq e^{-\gamma^2/\kappa_2} \quad \text{for all } \gamma \in [0, \kappa_1]. \quad (68)$$

Now, for $\gamma \in [0, \kappa_1]$, let $\mathbf{E} = \mathbf{E}(\gamma)$ be the event that $X_1 + \dots + X_n \geq n\gamma$. Take $\lambda = g^{-1}(\gamma)$ and define $Z(\lambda) \equiv \exp\{\lambda \sum_{i=1}^n X_i\}$. By Cauchy-Schwarz inequality

$$\begin{aligned} \mathbb{P}\{\mathbf{E}\} &\geq \frac{\mathbb{E}\{\mathbb{I}_{\mathbf{E}} Z(\lambda)\}^2}{\mathbb{E}\{Z(\lambda)^2\}} \\ &= \mathbb{P}_{\lambda}\{\mathbf{E}\}^2 \frac{\{\mathbb{E}Z(\lambda)\}^2}{\mathbb{E}\{Z(\lambda)^2\}} \\ &= \mathbb{P}_{\lambda}\{\mathbf{E}\}^2 H(\gamma)^n \\ &\geq \mathbb{P}_{\lambda}\{\mathbf{E}\}^2 \exp\left\{-\frac{n\gamma^2}{\kappa_2}\right\}. \end{aligned}$$

The proof is completed by noting that $\mathbb{P}_{\lambda}\{\mathbf{E}\}^2 \geq 1/4$ for all $n \geq n_0(\kappa)$, by Berry-Esseen central limit theorem (note indeed that, under $\mathbb{P}_{\lambda}$, $X_1, \dots, X_n$ have mean γ , variance lower bounded by $\text{Var}_{\lambda}(X_i) \geq \kappa' > 0$ and $\mathbb{E}_{\lambda}(|X_i|^3) \leq \kappa'' < \infty$). $\square$

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