Fiftieth Annual Allerton Conference
Allerton House, UIUC, Illinois, USA
October 1 - 5, 2012

On Robust Phase Retrieval for Sparse Signals

Kishore Jaganathan Samet Oymak Babak Hassibi

Department of Electrical Engineering, Caltech
Pasadena, CA-91125

***Abstract*—Recovering signals from their Fourier transform magnitudes is a classical problem referred to as phase retrieval and has been around for decades. In general, the Fourier transform magnitudes do not carry enough information to uniquely identify the signal and therefore additional prior information is required. In this paper, we shall assume that the underlying signal is sparse, which is true in many applications such as X-ray crystallography, astronomical imaging, etc. Recently, several techniques involving semidefinite relaxations have been proposed for this problem, however very little analysis has been performed.**

The phase retrieval problem can be decomposed into two tasks - (i) identifying the support of the sparse signal from the Fourier transform magnitudes, and (ii) recovering the signal using the support information. In earlier work [13], we developed algorithms for (i) which provably recovered the support for sparsities upto $O(n^{1/3-\epsilon})$. Simulations suggest that support recovery is possible upto sparsity $O(n^{1/2-\epsilon})$. In this paper, we focus on (ii) and propose an algorithm based on semidefinite relaxation, which provably recovers the signal from its Fourier transform magnitude and support knowledge with high probability if the support size is $O(n^{1/2-\epsilon})$.

***Index Terms*—Phase Retrieval, Semidefinite Relaxation, Sparse Signals, Autocorrelation**

I. Introduction

Physical measurement systems in many cases can only output the squared modulus of the Fourier transform. Phase information is either lost or unreliable in these systems. This is a fundamental problem in many areas of engineering and applied physics, including optics [1], X-ray crystallography [2], astronomical imaging [3], speech processing [4], particle scattering and electron microscopy.

This work was supported in part by the National Science Foundation under grants CCF-0729203, CNS-0932428 and CCF-1018927, by the Office of Naval Research under the MURI grant N00014-08-1-0747, and by Caltech's Lee Center for Advanced Networking.

Recovering a signal from its Fourier transform magnitude, or equivalently its autocorrelation, is known as phase retrieval. This problem has generated a lot of interest over the last few decades and a wide range of techniques have been proposed. The Gerchberg-Saxton algorithm [5] was the first popular method to solve this problem. Fienup, in his seminal paper [7], proposed a broad framework for iterative algorithms. Error reduction, Basic Input-Output (BIO) and Hybrid Input-Output (HIO) algorithms were presented. [6] provides a theoretical framework to understand the algorithms.

In many applications of phase retrieval, the signals encountered are naturally sparse. For example, astronomical imaging deals with the locations of stars in the sky, electron microscopy deals with the density of electrons and so on. Recently, attempts have been made to exploit the sparse nature of signals. [8] proposes an iterative algorithm based on alternating projections. Semidefinite relaxation based algorithms are explored in [9], [10], [11], [12] and [13].

In our earlier work [12], [13], we divided the phase retrieval problem into two tasks to exploit sparsity - (i) support recovery using the Fourier transform magnitudes and (ii) signal recovery using the signal support. In [13] we proposed an algorithm that provably recovers the support with high probability if the support size is $O(n^{1/3-\epsilon})$, simulations suggest it does so for support sizes upto $O(n^{1/2-\epsilon})$. In this paper, we focus on task (ii). In other words, analyze the convex program obtained using semidefinite relaxations on the phase retrieval problem with apriori knowledge of support of the signal, i.e., the locations where the signal has non-zero values. We discuss certain sufficient conditions for unique mapping between the signal and its autocorrelation, and show that for signals upto sparsity $O(n^{1/2-\epsilon})$, the convex program uniquely recovers them from their Fourier transform magnitude with very high probability if their support is known apriori.

This paper is organized as follows. In Section 2, we formulate the phase retrieval problem. In Section 3, we discuss some sufficient conditions for unique signal recovery from its Fourier transform magnitude and prove that sparse signals (upto $O(n^{1/2-\epsilon})$) can be uniquely recovered with very high probability. The semidefinite relaxation based technique is proposed and analyzed in Section 4. Section 5 presents the simulation results and concludes the paper.

II. Problem Setup

Let $\mathbf{x} = (x_0, x_1, ...x_{n-1})$ be a real-valued signal of length n and sparsity k, where sparsity is defined as the number of non-zero entries in the signal. Its autocorrelation, denoted by $\mathbf{a} = (a_0, a_1, ...a_{n-1})$, is defined as

$$a_i \stackrel{def}{=} \sum_j x_j x_{j+i} = (\mathbf{x} \star \tilde{\mathbf{x}})_i \tag{1}$$

where $\tilde{\mathbf{x}}$ is the time reversed version of $\mathbf{x}$. Note that cyclic indexing scheme is used. Let $\mathbf{u} = (u_0, u_1, ...u_{n-1})$ denote the support of the signal $\mathbf{x}$, defined as

$$u_i = \begin{cases} 1 & \text{if } x_i \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In this work, we will assume that $\mathbf{u}$ is known apriori and the k locations are chosen from the n available locations uniformly and randomly. The signal values in the support are chosen from a Gaussian distribution independently. We are interested in sparse $\mathbf{x}$, i.e., $k << n$ where $k = \sum_i u_i$. Let $\mathbf{y} = (y_0, y_1, ...y_{n-1})$ be the Fourier transform of $\mathbf{x}$, i.e,

$$\mathbf{y} = \mathbf{F}\mathbf{x} \tag{3}$$

where $\mathbf{F}$ is the $n \times n$ DFT matrix. Observe that power spectral density, denoted by $|\mathbf{y}|^2 = (|y_0|^2, |y_1|^2, ...|y_{n-1}|^2)$ and $\mathbf{a}$ are Fourier pairs, and hence the problem of signal recovery from the magnitudes of Fourier transform is equivalent to recovering the signal from its autocorrelation. The problem of signal recovery with known support can be formulated as

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && a_i = \sum_j x_j x_{j+i} \qquad 0 \le i \le n-1 \\ &&& x_i = 0 \quad \text{iff} \quad u_i = 0 \quad 0 \le i \le n-1 \end{aligned} \tag{4}$$

Define $\mathbf{M}_i = \mathbf{f}_i \mathbf{f}_i^T \quad \forall i = \{0, 1, ...n-1\}$, where $\mathbf{f}_i$ is the i^{th} column of the $n \times n$ DFT matrix. (4) can also be written as

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && |y_i|^2 = \mathbf{x}^T \mathbf{M}_i \mathbf{x} \qquad 0 \le i \le n-1 \\ &&& x_i = 0 \quad \text{iff} \quad u_i = 0 \quad 0 \le i \le n-1 \end{aligned} \tag{5}$$

III. Unique Recovery

Since the mapping from signals to Fourier transform magnitudes is not one-to-one, unique recovery is not possible in general. For any Fourier transform magnitude, every possible phase combination corresponds to a different signal and hence, additional prior information is required to guarantee uniqueness. In this section, we show that for signals upto $O(n^{1/2-\epsilon})$ sparsity, knowledge of the support is sufficient to guarantee unique mapping with very high probability.

A. Sufficient Criteria for Uniqueness

Suppose $S = (s_0, s_1, ...s_{k-1}) = \{i | u_i \neq 0\}$ be the set of all elements that belong to the support of the signal. Construct a weighted graph G with k vertices, denoted by $\{v_0, v_1, ...v_{k-1}\}$ such that

(i) There exists an edge between vertices v_i and v_j iff the following condition is satisfied

$$\forall s_g, s_h \in S, \quad s_g - s_h \neq s_i - s_j$$
$$\text{unless} \quad (i,j) = (g,h) \quad \text{or} \quad (i,j) = (h,g) \tag{6}$$

where the difference considered is modulo n.

(ii) If there exists an edge between v_i and v_j, its weight, denoted by w_{ij} is given by

$$w_{ij} = a_{|s_i - s_j|} \tag{7}$$

The key idea is that if an edge exists between v_i and v_j, then $w_{ij} = x_{s_i}x_{s_j}$, i.e, if an edge exists between two vertices, then the product of the corresponding signal values is known, because no other term contributes to $a_{|s_i-s_j|}$.

Lemma III.1. *Suppose the graph G is connected and has a triangle, then the signal can be extracted uniquely from the autocorrelation upto a global sign.*

Proof: Without loss of generality, let the induced subgraph of $\{v_1, v_2, v_3\}$ be a triangle. We see that

$$\frac{w_{12}w_{13}}{w_{23}} = {x_{s_1}}^2, \quad x_{s_2} = \frac{w_{12}}{x_{s_1}} \quad \& \quad x_{s_3} = \frac{w_{13}}{x_{s_1}} \tag{8}$$

from which $(x_{s_1}, x_{s_2}, x_{s_3})$ can be recovered upto a global sign. Note that if there is an edge between v_i and v_j, and if one of x_{s_i} or x_{s_j} is known, the other can be recovered. Since the graph G is connected, with the knowledge of x_{s_1} upto a sign, all the entries can be recovered upto a global sign. ■

B. Probability of Success of Lemma III.1

In this part, we calculate the probability of the graph G being connected.

Lemma III.2. *Let p denote the probability that there is an edge between v_i and v_j for $(i, j) \in \{0, 1...k-1\}$. Then $p \geq 1 - \frac{k^2}{n}$.*

Proof: Consider any pair of vertices v_i and v_j. There will be no edge between them if there exists another pair of vertices v_g and v_h such that $s_g - s_h = s_i - s_j$ by construction. Since the support entries are chosen uniformly and randomly, we have

$$Pr\{\exists (g,h) \neq (i,j) \text{ or } (j,i) | s_g - s_h = s_i - s_j\}$$
$$\leq \sum_{q=1}^{q=k}\sum_{r=1}^{r=k} Pr\{s_q - s_r = s_i - s_j\} \leq \frac{k^2}{n}$$
$$p = 1 - Pr\{\exists (g,h) \neq (i,j), (j,i) | s_g - s_h = s_i - s_j\}$$
$$\geq 1 - \frac{k^2}{n} \tag{9}$$

All index additions and differences considered are modulo n. ■

Lemma III.3. *Suppose $\delta(G)$ denotes the minimum degree of a graph G, then $\delta(G) \geq k(1-1/t)$ with probability $q > 1 - \epsilon$ for any $\epsilon > 0$, $t > 0$ and $n > n(\epsilon)$ if $\frac{k^2 t}{n} < 1$.*

Proof: Consider a vertex v_i. Construct a graph G_i from G by removing all the edges which do not involve the vertex v_i. Let us consider the vertex exposure martingale [14] on this graph G_i with the graph function $d(v_i)$, where $d(v)$ denotes the degree of the vertex v. Let H_j be the induced subgraph of G_i formed by exposed vertices after j exposures. We define a martingale $X_0, X_1, ...X_k$ as follows

$$X_j = E[d(v_i)|H_j] \tag{10}$$

Refer to the Appendix for details. Observe that $X_0 = E[d(v_i)] \geq k(1 - \frac{k^2}{n})$ and $X_k = d(v_i)$, where $d(v_i)$ is the degree of the vertex v_i. Note that $|X_{j+1} - X_j| \leq 1 \quad \forall \quad 0 \leq j \leq m-1$. Azuma's inequality [14] gives us

$$Pr\{d(v_i) < E[d(v_i)] - \lambda\} \leq 2e^{-\lambda^2/2k} \tag{11}$$

for $\lambda > 0$. Choosing $\lambda = k(\frac{1}{t} - \frac{k^2}{n})$, which is greater than 0 when $\frac{k^2 t}{n} < 1$, we get

$$Pr\{d(v_i) < k(1 - \frac{1}{t})\} \leq 2e^{-\frac{k}{2}(\frac{1}{t} - \frac{k^2}{n})^2} \tag{12}$$

Using union bound to accomodate all the vertices v_i for $i = \{0, 1...k-1\}$, we get

$$Pr\{\exists i | d(v_i) < k - \frac{k}{t}\} \leq \sum_{i=1}^{i=k} Pr\{d(v_i) < k - \frac{k}{t}\} \tag{13}$$

$$\leq 2ke^{-\frac{k}{2}(\frac{1}{t} - \frac{k^2}{n})^2} < \epsilon \quad \text{for} \quad n > n(\epsilon) \tag{14}$$

■

Theorem III.1. *The graph G is connected and has a triangle with probability $q > 1 - \epsilon$ for any $\epsilon > 0$ if $k = O(n^{1/2-\epsilon})$ and $n > n(\epsilon)$*

Proof: Suppose $k = O(n^{1/2-\epsilon})$ and $t = 2$. We see that all the conditions of Lemma (III.3) are met, and hence every vertex in the graph has a degree at least $\frac{k}{2}$ with very high probability. Dirac's theorem [15] states that such graphs have a Hamiltonian cycle, which shows that the graph is connected. The probability that there doesn't exist a triangle between any three vertices chosen can be upper bounded by $3\frac{k^2}{n}$ using union bounds. Hence, we have

$$q \geq 1-(2ke^{-\frac{k}{2}(\frac{1}{2}-\frac{k^2}{n})^2}+3\frac{k^2}{n}) > 1-\epsilon \quad (15)$$

if $n > n(\epsilon)$. ■

Hence, we see that signals with sparsity $O(n^{1/2-\epsilon})$ can be uniquely recovered from their Fourier transform magnitude with very high probability if their support is known apriori.

IV. Convex Algorithm for Robust Recovery

Note that (5) is a non-convex problem as the autocorrelation constraints are non-convex. While the ideas discussed in Section 3 can be used to develop an algorithm to recover the signal from their autocorrelation with known support information [13], we see that the algorithm would be very sensitive to noise for obvious reasons. Convex programs are known to be robust to noise, with the performance degrading gracefully with noise. In this section, we introduce a series of relaxations to convert (5) into a convex problem, and analyze its performance.

A. Semidefinite Relaxation

The semidefinite relaxation technique has shown great promise in solving many non-convex quadratically constrained programs. Suppose we define $\mathbf{X} = \mathbf{x}_S\mathbf{x}_S^T$, where $\mathbf{x}_S$ is the vector containing all the elements belonging to the support. This technique is also known as lifting in the popular literature. (5) can be rewritten as

$$\begin{aligned} &\text{find} && \mathbf{X} \\ &\text{subject to} && |y_i|^2 = trace(\mathbf{M}_i(S,S)\mathbf{X}) \\ &&& rank(X) = 1 \end{aligned} \quad (16)$$

where $\mathbf{M}_i(S,S)$ is the $k \times k$ submatrix of $\mathbf{M}_i$ with the indices corresponding to the support for $0 \leq i \leq k-1$. [16] shows that the best convex relaxation of (16) is

$$\begin{aligned} &\text{minimize} && trace(\mathbf{X}) \\ &\text{subject to} && |y_i|^2 = trace(\mathbf{M}_i(S,S)\mathbf{X}) \\ &&& X \succcurlyeq 0 \end{aligned} \quad (17)$$

(17) is hence the best convex relaxation of the phase retrieval problem with known support. If the output of the program is a matrix of rank 1, the signal can be extracted successfully by a simple decomposition.

B. Performance Analysis of the Convex Program

The signal can be successfully recovered by (17) if the output of the program is a rank 1 matrix. Analysis of semidefinite relaxation based programs to guarantee a rank 1 solution is a difficult task. In this section, we will consider a further relaxation of (17) and provide lower bounds for guarantees of rank 1 output with very high probability.

Lemma IV.1. *If there is an edge between vertices v_i and v_j in the graph G, then X_{ij} can be deduced from the autocorrelation.*

Proof: By construction, if there is an edge between v_i and v_j, then $a_{|s_i-s_j|} = x_{s_i}x_{s_j}$. By definition, $X_{ij} = x_{s_i}x_{s_j}$ and hence X_{ij} can be calculated from the autocorrelation. ■

A further relaxation of (17) can be obtained by using only the values of $\mathbf{X}$ which can be directly calculated from the autocorrelation, i.e,

$$\begin{aligned} &\text{minimize} && trace(\mathbf{X}) \\ &\text{subject to} && \mathbf{X}_{ij} = a_{|s_i-s_j|} \quad \text{if } v_i \leftrightarrow v_j \\ &&& X \succcurlyeq 0 \end{aligned} \quad (18)$$

where $v_i \leftrightarrow v_j$ implies there is an edge between v_i and v_j.

As a slight detour, let $\mathbf{Z}$ be a positive semidefinite $t \times t$ matrix with all the off-diagonal entries given. Suppose the off-diagonal entries are such that $Z_{ij} = z_iz_j$, where z is a $t \times 1$ vector. Since we have a positive semidefinite condition on $\mathbf{Z}$, any 2×2 submatrix of $\mathbf{Z}$ is also positive semidefinite, i.e.,

$$Z_{ii}Z_{jj} \geq (z_iz_j)^2 \quad \forall \text{ distinct } (i,j) \quad (19)$$

Consider the convex program

$$\begin{aligned} &\text{minimize} && trace(\mathbf{Z}) \\ &\text{subject to} && Z_{ii}Z_{jj} \geq (z_iz_j)^2 \quad \forall \text{ distinct } (i,j) \end{aligned} \quad (20)$$

Lemma IV.2. *(20) gives a rank 1 solution with a very high probability for sufficiently large t.*

Proof: Suppose the rank 1 completion $\mathbf{Z} = zz^T$ is not the minimizer of (20). Then there exists atleast one Z_{ii} which is strictly lesser than z_i^2, say $(1-\epsilon)z_i^2$ for $\epsilon > 0$. The constraints require all other diagonal entries to be greater than or equal to $\frac{1}{1-\epsilon}z_j^2 = (1+\frac{\epsilon}{1-\epsilon})z_j^2$. The objective function can be written as

$$trace(\mathbf{Z}) = \sum_{i=1}^{i=t} Z_{ii} \leq (1-\epsilon)z_i^2 + \sum_{j\neq i}(1+\frac{\epsilon}{1-\epsilon})z_j^2$$

$$= \sum_j {z_j}^2 + \frac{\epsilon}{1-\epsilon}(\sum_{j\neq i} z_j^2 - (1-\epsilon)z_i^2) \quad (21)$$

If we can guarantee that $(\sum_{j\neq i} z_j^2 - (1-\epsilon)z_i^2) > 0$ for all i, then we are through. [18] provides an exponentially decreasing probability in t for failure of the required condition. ∎

Theorem IV.1. *The convex program (18) extracts the signal from the magnitude of its Fourier transform with very high probability if $k = O(n^{1/2-\epsilon})$.*

Proof: Lemma III.3 shows that $\delta(G) > k(1-\frac{1}{t})$ if $\frac{k^2 t}{n} < 1$. Hajnal-Szemeredi theorem on disjoint cliques [17] states that if $\delta(G) > k(1-\frac{1}{t})$, there exists a subgraph which consists of $\frac{k}{t}$ vertex disjoint union of complete graphs of size t. Suppose we choose $t = log(n)$. Lemma IV.2 applies to each of the $\frac{k}{t}$ complete graphs and hence using union bound, we see that trace minimization gives us all the diagonal entries corresponding to the rank 1 solution with very high probability for $k = O(n^{1/2-\epsilon})$. Since the graph G is connected, we know both the diagonal and the principal off-diagonal entries. They come from a rank 1 matrix, hence the rank 1 completion is the only possible positive semidefinite completion, and hence the unique minimizer of the convex program. Hence the signal can be extracted uniquely by a simple decomposition as long as $k = O(n^{1/2-\epsilon})$.

Observe that the results derived in this section are for (18), which is a much relaxed version of (17). Hence, the sparsities derived act as lower bound guarantees and one can expect (17) to perform well for much higher sparsities. ∎

V. Simulation Results

In this section, we discuss the performance of the convex programs (17) and (18) in recovering signals from the magnitudes of their Fourier transform with apriori support knowledge. Since the program (17) is a tighter version of (18), it can be expected to perform atleast as good as (18), and hence (18) provides a lower bound to recovery success rate.

Simulations were performed for various choices of signal length n and their probabilities of successful recovery are plotted against various sparsities. For a given n and k, the support entries were chosen uniformly and randomly. The signal values in the support were chosen from a Gaussian distribution independent of each other.

Figures 1 and 2 show the success rate of programs SDR (17) and Lower Bound (18) for $n = 64$ and $n = 128$ respectively, for various choices of k. It can be observed that signals can be recovered with high probability if they have sparity $O(n^{1/2-\epsilon})$.

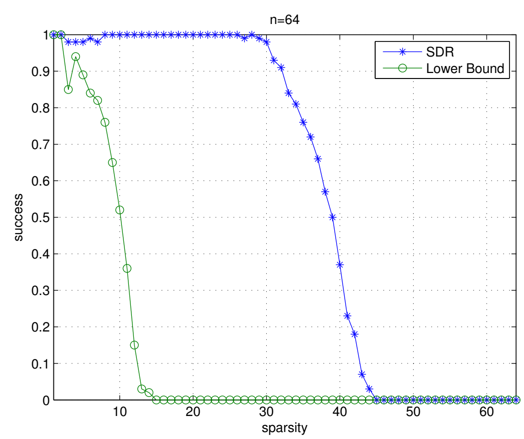

Fig. 1. Success rate of recovery of programs (17) and (18) for $n = 64$ and various sparsities

We observed that (17) recovers signals from their Fourier transform magnitudes for sparsities much higher than the lower bound $O(n^{1/2-\epsilon})$. As we can see from Figures 1 and 2, signals with sparsities of the order of $O(n)$ were recovered with very high probaiblity. This is due to the huge number of linear constraints in (17) in addition to fixing some entries in the matrix. Hence, if support information is available through other

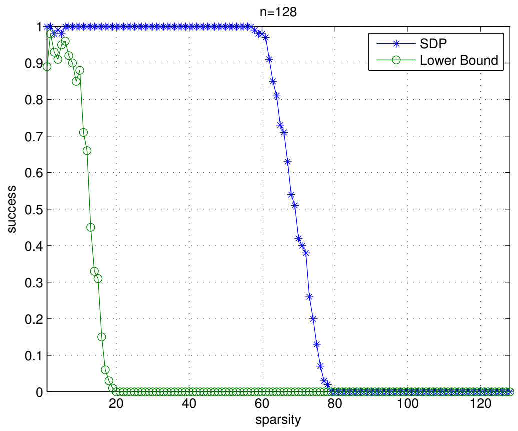

Fig. 2. Success rate of recovery of programs (17) and (18) for $n = 128$ and various sparsities

means apart from recovery algorithms, (17) is a very effective method to reproduce the signal values from their Fourier transform magnitudes even if they are not sparse.

VI. Appendix

A. Proof for vertex exposure martingale

We can see that $X_0, X_1, ...X_k$ form a martingale as follows

$$E[X_{j+1}|X_0, X_1, ...X_j]$$

$$= E[E[d(v_i)|H_{j+1}]|X_0, X_1, ...X_j]$$

$$= \sum_{H_{j+1}} p(H_{j+1}|X_0, X_1, ...X_j)E[d(v_i)|H_{j+1}]$$

$$= \sum_{H_{j+1}} p(H_{j+1}|H_j) \sum_{G'} p(G'|H_{j+1})d_{G'}(v_i)$$

where $d_{G'}(v_i)$ is the degree of v_i in the graph G' and the summation is done over all possible graphs G'.

$$= \sum_{G'} \sum_{H_{j+1}} p(G', H_{j+1}|H_j)d_{G'}(v_i)$$

$$= E[d(v_i)|H_j] = X_j \tag{22}$$

References

[1] A. Walther, ”The question of phase retrieval in optics,” Opt. Acta 10, 4149 (1963).
[2] R. P. Millane, ”Phase retrieval in crystallography and optics,” J. Opt. Soc. Am. A 7, 394-411 (1990)
[3] J.C. Dainty and J.R. Fienup,“Phase Retrieval and Image Reconstruction for Astronomy,” Chapter 7 in H. Stark, ed., Image Recovery: Theory and Application
[4] L. Rabiner, B.H. Juang, “Fundamentals of Speech Recognition,” Signal Proc. Series, Prentice Hall, 1993.
[5] R. W. Gerchberg, W. O. Saxton. “A practical algorithm for the determination of the phase from image and diffraction plane pictures”. Optik 35, 237 (1972).
[6] H.H. Bauschke, P.L. Combettes and D.R. Luke, ”Phase retrieval, error reduction algorithm, and Fienup variants: a view from convex optimization”. J. Opt. Soc. Am. A, 19(7): 1334-1345, 2002.
[7] J. R. Fienup, “Phase retrieval algorithms: a comparison”. Appl. Opt. 21, 2758–2769 (1982).
[8] Y.M. Lu and M. Vetterli. “Sparse spectral factorization: Unicity and reconstruction algorithms”. ICASSP 2011.
[9] E. J. Candes, Y. Eldar, T. Strohmer, and V. Voroninski. “Phase retrieval via matrix completion”. arXiv:1109.0573v2
[10] E. J. Candes, T. Strohmer and V. Voroninski. PhaseLift: exact and stable signal recovery from magnitude measurements via convex programming. To appear in Communications on Pure and Applied Mathematics
[11] Y. Shechtman, Y.C. Eldar, A. Szameit and M. Segev, ”Sparsity Based Sub-Wavelength Imaging with Partially Incoherent Light Via Quadratic Compressed Sensing”, Optics Express, vol. 19, Issue 16, pp. 14807-14822, Aug. 2011.
[12] K. Jaganathan, S. Oymak and B. Hassibi, “Phase Retrieval for Sparse Signals using Rank Minimization”. Acoustics, Speech and Signal Processing (ICASSP), 2012 IEEE International Conference.
[13] K. Jaganathan, S. Oymak and B. Hassibi, “Recovery of Sparse 1-D Signals from the Magnitudes of their Fourier Transform”. International Symposium on Information Theory (ISIT), 2012.
[14] N. Alon and J.H. Spencer, ”The Probabilistic Method”, Chapter 7.
[15] R.L. Graham, ”Handbook of Combinatorics”
[16] B.Recht, M.Fazel and P.Parrilo. ”Guaranteed Minimum-Rank Solutions of Linear Matrix Equations via Nuclear Norm Minimization”. SIAM Review, Vol 52, no 3, pages 471-501, 2010.
[17] Hajnal, A. and Szemeredi, E. ”Proof of a Conjecture of Erdos”. In Combinatorial Theory and Its Applications, Vol. 2 (Ed. P.Erdos, A.Renyi and V.T. Sos). Amsterdam, Netherlands: North-Holland, 1970.
[18] R. Vershynin, ”Introduction to the non-asymptotic analysis of random matrices”. arXiv: 1011.3027v7