# Concentration to Zero Bit-Error Probability for Regular LDPC Codes on the Binary Symmetric Channel: Proof by Loop Calculus 

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#### Abstract

In this paper we consider regular low-density parity-check codes over a binary-symmetric channel in the decoding regime. We prove that up to a certain noise threshold the bit-error probability of the bit-sampling decoder converges in mean to zero over the code ensemble and the channel realizations. To arrive at this result we show that the bit-error probability of the sampling decoder is equal to the derivative of a Bethe free entropy. The method that we developed is new and is based on convexity of the free entropy and loop calculus. Convexity is needed to exchange limit and derivative and the loop series enables us to express the difference between the biterror probability and the Bethe free entropy. We control the loop series using combinatorial techniques and a first moment method. We stress that our method is versatile and we believe that it can be generalized for LDPC codes with general degree distributions and for asymmetric channels.


## I. Introduction

In 1968 Gallager [1] introduced error-correcting codes based on low-density parity-check (LDPC) matrices. Since then LDPC codes have been proven to be of great practical and theoretical relevance. LDPC codes perform very well under iterative decoding on a broad class of symmetric memoryless channels (BMS) [2], [3] and provably achieve capacity on the binary erasure channel (BEC) [4]. Since 1996 they have been integrated into many industrial standards from wireless communications to computer chips.

An important performance measure of an LDPC code and its associated decoder is the bit-error probability. It is the fraction of bits that are on average incorrectly reconstructed. The bit-error probability of LDPC codes under belief-propagation (BP) decoding is well-understood on BMS channels using the method of density evolution [5]. However it is a more challenging task to control the bit-error probability of the bit maximum a posteriori (MAP) decoder.

Lower and upper bounds on the noise threshold for vanishing bit-MAP error probability have already been derived in Gallager's thesis [1] for a class of BMS channels. These bounds have been improved and generalized for every BMS channels by Shamai and Sason [6].

In an attempt to locate exactly the noise threshold, most of the attention has been focused on the conditional entropy per bit. Its derivative with respect to the channel noise is the so-called generalized extrinsic information transfer (GEXIT) curve [7]. The GEXIT curve is proportional to the

[^0]"magnetization" or bit-error probability of the bit-sampling decoder for the BEC and the binary-input additive white Gaussian-noise channel (BAWGNC) [8]. The magnetization is an upper-bound on the bit-MAP error probability. Hence for these two channels GEXIT curves and bit-MAP error probabilities vanish in the same noise regime.
Surprisingly the conditional entropy and its derivative are related to the BP algorithm and its associated Bethe free entropy. It has been first proven in [9], [10] on the BEC channel that the conditional entropy is equal to an averaged form of the Bethe free entropy over the code ensemble. Bounds between the averaged Bethe free entropy and the conditional entropy are derived in [11], [12], [13] based on the interpolation method of Guerra and Toninelli [14], [15]. Equality has been proven on the binary-symmetric channel (BSC) using cluster expansions in a low-noise regime [16] and in a high-noise regime [17]. More recently equality between the GEXIT curve and the derivative of the average Bethe free entropy has been generalized to all BMS channels [18] combining the interpolation method and spatiallycoupled codes [19].

Although the conditional entropy and its threshold are completely characterized for BMS channels, its exact relation to the bit-MAP error probability remains unclear in general and in particular for the BSC channel. Due to Fano's inequality the conditional entropy is always a lower-bound on the bit-error probability. However inspired by previous results this inequality is conjectured to be tight for LDPC codes on a wide class of channels.
In this paper we prove that for regular LDPC codes over a BSC channel the "magnetization" or bit-error probability of the bit-sampling decoder vanishes up to a certain threshold. This result also shows that the posterior measure of LDPC codes concentrates over the LDPC ensemble and the noise realizations. To achieve this result we show that the magnetization is asymptotically equal to a perturbed version of the Bethe free entropy. The technique that we present is new and is based on loop calculus or loop series derived by Chertkov and Chernyak [20]. The loop series expresses the difference between a quantity and its Bethe counterpart as a sum over subgraphs. Proving that the loop series vanishes is tantamount to controlling a purely combinatorial object that depends solely on the LDPC graph ensemble. Suboptimal bounds on this object are obtained using McKay's estimates [21] following an idea developed in [17], [22].

[^1]The technique that we present has the advantage to be simple and versatile. To emphasize this point we also show that our results can be easily transposed to the BEC. Moreover we stress that our proofs do not rely explicitly on properties of the channel. Hence, we believe that this technique can be use to analyze LDPC codes over channels that are not symmetric.

In Section [II we give a precise definition of the bitsampling decoder and its associated bit-error probability and we present our main theorems. In Section III we derive the relation between the Bethe free entropy and the bit-error probability and we express the difference using loop calculus. In Section III we reduce the loop series to a counting problem that we control with a first moment method. Finally we discuss about future works and possible improvements in Section $\nabla$

## II. Main Results

## A. Regular LDPC codes on BMS channels

LDPC codes are defined by a regular bipartite graph $\Gamma=$ ( $V, C, E$ ) where $V$ is the set of variable nodes, $C$ is the set of check nodes and $E=V \times C$ is the set of undirected edges. There are $n=|V|$ variable nodes and $m=|C|$ check nodes.

We consider regular LDPC codes with variable-node degrees $l \geq 3$ and check-node degrees $r>l$. The design rate of the code is by definition $R_{\text {des }}=1-l / r$.

An LDPC code is generated randomly. The graph $\Gamma$ is drawn uniformly at random from the ensemble of $(l, r)$ regular bipartite graphs. Throughout the paper we write $\mathbb{E}_{\Gamma}[\cdot]$ the expectation with respect to the ensemble of regular $(l, r)$ bipartite graphs with uniform probability.

Denote the neighbors of a variable node $i \in V$ (resp. a check node $a \in C$ ) by $\partial i=\{a \in C \mid(i, a) \in E\}$ (resp. by $\partial a=\{i \in V \mid(i, a) \in E\}$ ). A codeword is a sequence $\mathbb{D}^{2}$ $\underline{\sigma}=\left\{\sigma_{i}\right\}_{i=1}^{n} \in\{-1,1\}^{n}$ that satisfies the parity-check sum

$$
\begin{equation*}
\prod_{i \in \partial a} \sigma_{i}=1 \tag{1}
\end{equation*}
$$

for all check nodes $a \in C$.
We transmit a codeword with uniform prior over a BMS channel with transition probability $q\left(s_{i} \mid \sigma_{i}\right)$, where the output of the channel could take any real value $s_{i} \in \mathbb{R}$. The symmetry property of the channel is expressed through the simple relation

$$
\begin{equation*}
q\left(s_{i} \mid \sigma_{i}\right)=q\left(-s_{i} \mid-\sigma_{i}\right) . \tag{2}
\end{equation*}
$$

We assume without loss of generality that the all-zero codeword ${ }^{3}$ is transmitted. Hence the output of the channel $\underline{s}=\left\{s_{i}\right\}_{i=1}^{n} \in \mathbb{R}^{n}$ is i.i.d. with distribution $q\left(s_{i} \mid+1\right)$. The posterior probability that the codeword $\underline{\sigma}$ is sent given that

[^2]$\underline{s}$ is transmitted reads
$\mu_{\Gamma}(\underline{\sigma} \mid \underline{s})=\frac{1}{Z(\Gamma, \underline{s})} \prod_{a \in C} \frac{1}{2}\left(1+\prod_{i \in \partial a} \sigma_{i}\right) \prod_{i \in V} q\left(s_{i} \mid \sigma_{i}\right)$,
where the normalization factor $Z(\Gamma, \underline{s})$ in Equation (3) is the partition function
\[

$$
\begin{equation*}
Z(\Gamma, \underline{s}):=\sum_{\underline{\sigma} \in\{-1,1\}^{n}} \prod_{a \in C} \frac{1}{2}\left(1+\prod_{i \in \partial a} \sigma_{i}\right) \prod_{i \in V} q\left(s_{i} \mid \sigma_{i}\right) . \tag{4}
\end{equation*}
$$

\]

## B. Concentration of the Bit-Error Probability for the Sampling Decoder

We are interested in the performance of regular LDPC codes with respect to the average bit-error probability of decoding. We consider the bit-sampling decoder

$$
\begin{equation*}
\widehat{\sigma}_{i}^{\text {sampling }}(\underline{s}):=\text { sample } \sigma_{i} \text { according to } \sum_{\underline{\sigma} \backslash \sigma_{i}} \mu(\underline{\sigma} \mid \underline{s}), \tag{5}
\end{equation*}
$$

where $\underline{\sigma} \backslash \sigma_{i}$ denotes the sequence of variables $\underline{\sigma}$ with the $i^{\text {th }}$ component removed.

The bit-error probability of the bit-sampling decoder $P_{\Gamma}^{\text {bit-sampling }}$ is directly related to the marginals of the posterior probability (3)

$$
\begin{equation*}
P_{\Gamma}^{\text {bit-sampling }}:=\frac{1}{2}\left(1-\mathbb{E}_{\underline{s}}\left[\frac{1}{n} \sum_{i=1}^{n}\left\langle\sigma_{i}\right\rangle_{\mid \underline{s}}\right]\right) \tag{6}
\end{equation*}
$$

where $\mathbb{E}_{\underline{s}}[\cdot]$ denotes the expectation with respect to the channel output distribution and $\langle\cdot\rangle_{\mid \underline{S}}$ denotes the average with respect to the posterior probability (3). The expected quantity in Equation (6) is sometimes referred as the averaged magnetization in the physics community.

An important question is to know when the bit-error probability is vanishing in the limit where the codeword length goes to infinity. In this paper we consider two families of symmetric channels, the BEC and the BSC. The BEC has an output alphabet $s_{i} \in\{-1,0,1\}$ and is characterized by transition probabilities

$$
\begin{equation*}
q^{\mathrm{BEC}}(1 \mid 1)=1-\epsilon, q^{\mathrm{BEC}}(0 \mid 1)=\epsilon, q^{\mathrm{BEC}}(-1 \mid 1)=0 \tag{7}
\end{equation*}
$$

where $\epsilon \in[0,1]$ is the erasure probability. The BSC has binary outputs $s_{i} \in\{-1,1\}$ and is characterized by the transition probabilities

$$
\begin{equation*}
q^{\mathrm{BSC}}(1 \mid 1)=1-p, q^{\mathrm{BSC}}(-1 \mid 1)=p \tag{8}
\end{equation*}
$$

where $p \in[0,1 / 2]$ is the flipping probability.
Before we state our theorems we need to introduce the domain

$$
\begin{align*}
D(\rho)= & \left\{\left(x_{0}, x_{c}, \underline{y}\right) \in[0,1]^{2+\lfloor r / 2\rfloor} \mid \sum_{t=1}^{\lfloor r / 2\rfloor} y_{t} \leq 1,\right. \\
& \left.\sum_{t=1}^{\lfloor r / 2\rfloor} \frac{2 t}{r} y_{t}=(1-\rho) x_{0}+\rho x_{c}\right\} . \tag{9}
\end{align*}
$$

We also need to introduce the auxiliary function $f: D(\rho) \times$ $[0,1] \rightarrow \mathbb{R}$ defined as follow 4

$$
\begin{align*}
f\left(x_{0}, x_{c}, \underline{y}, \rho\right)= & -l h_{2}\left((1-\rho) x_{0}+\rho x_{c}\right) \\
& +(1-\rho) h_{2}\left(x_{0}\right)+\rho h_{2}\left(x_{c}\right) \\
& -\frac{l}{r}\left(1-\sum_{t=1}^{r} y_{t}\right) \ln \left(1-\sum_{t=1}^{r} y_{t}\right) \\
& -\frac{l}{r} \sum_{t=1}^{r} y_{t} \ln y_{t} \\
& +\frac{l}{r} \sum_{t=1}^{\lfloor r / 2\rfloor} y_{t} \ln \binom{r}{2 t} \tag{10}
\end{align*}
$$

and the function $k:[0,1]^{4} \rightarrow \mathbb{R}$ that reads

$$
\begin{equation*}
k\left(x_{0}, x_{c}, \rho, p\right)=\left(\rho x_{c}-(1-\rho) x_{0}\right) \ln \left(\frac{1-p}{p}\right) \tag{11}
\end{equation*}
$$

Our main contribution are the two theorems stated below which give sufficient conditions on the channel parameters for concentration of the bit-error probability of the sampling decoder.

Theorem 1 (Concentration of the Bit-Error Probability for the BEC). Consider the ensemble of $(l, r)$ regular LDPC codes on a BEC with erasure probability $\epsilon$. If the following function achieves its maximum only at the point

$$
\underset{\left(0, x_{c}, \underline{y}\right) \in D(\epsilon)}{\operatorname{argmax}} f\left(0, x_{c}, \underline{y}, \epsilon\right)=\{(0,0,0)\}
$$

then the bit-error probability of the sampling decoder converges in mean to zero in the large codeword limit

$$
\lim _{n \rightarrow \infty} \mathbb{E}_{\Gamma, \underline{s}}\left[P_{\Gamma}^{\text {bit-sampling }}\right]=0
$$

The same theorem holds for the BSC with a similar condition.

Theorem 2 (Concentration of the Bit-Error Probability for the BSC). Consider the ensemble of $(l, r)$ regular LDPC codes on a BSC with flipping probability p. If the following function achieves its maximum only at the point
$\underset{\left(x_{0}, x_{c}, \underline{y}\right) \in D(p)}{\operatorname{argmax}} f\left(x_{0}, x_{c}, \underline{y}, p\right)+k\left(x_{0}, x_{c}, p, p\right)=\{(0,0,0)\}$,
then the bit-error probability of the sampling decoder converges in mean to zero in the large codeword limit

$$
\lim _{n \rightarrow \infty} \mathbb{E}_{\Gamma, \underline{s}}\left[P_{\Gamma}^{\text {bit-sampling }}\right]=0
$$

Remark 3. Knowing that $P_{\Gamma}^{\text {bit-sampling }}$ vanishes implies that with high probability the posterior measure (3) concentrates on configurations that are at a Hamming distance $o(n)$ from the all-zero codeword.

We perform the global optimization numerically and we find for a few cases the maximum value of noise $\epsilon_{\text {loop }}$ and $p_{\text {loop }}$ for which Theorem 1 and Theorem 2 hold. Critical

[^3]values of noise are displayed in Table $\square$ for the BEC and in Table $\square$ for the BSC.

| $l$ | $r$ | $R_{\text {des }}$ | $\epsilon_{\text {BP }}$ | $\epsilon_{\text {loop }}$ | $\epsilon_{\text {MAP }}$ | $\epsilon_{\text {Sh }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 4 | $1 / 4$ | 0.64743 | $0.7442(9)$ | 0.74601 | 0.75 |
| 3 | 5 | $2 / 5$ | 0.51757 | $0.5872(4)$ | 0.59098 | 0.6 |
| 3 | 6 | $1 / 2$ | 0.42944 | $0.4833(6)$ | 0.48815 | 0.5 |
| 4 | 6 | $1 / 3$ | 0.50613 | $0.5767(2)$ | 0.66565 | 0.66667 |
| TABLE I |  |  |  |  |  |  |

Thresholds for some regular LDPC code ensembles over the BEC with erasure probability $\epsilon$. The belief-propagation THRESHOLD IS $\epsilon_{\text {BP }}$, THE MAXIMUM A POSTERIORI THRESHOLD IS $\epsilon_{\text {MAP }}$, the Shannon threshold is $\epsilon_{\text {Sh }}$ and our threshold is $\epsilon_{\text {Loop }}$. Values of BP and MAP thresholds are from [23].

| $l$ | $r$ | $R_{\text {des }}$ | $p_{\text {BP }}$ | $p_{\text {loop }}$ | $p_{\text {MAP }}$ | $p_{\text {Sh }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 4 | $1 / 4$ | 0.16692 | $0.2014(2)$ | 0.21011 | 0.21450 |
| 3 | 5 | $2 / 5$ | 0.11382 | $0.1146(8)$ | 0.13841 | 0.14610 |
| 3 | 6 | $1 / 2$ | 0.08402 | $0.0678(9)$ | 0.10101 | 0.11003 |
| 4 | 6 | $1 / 3$ | 0.11692 | $0.1705(2)$ | 0.17261 | 0.17395 |
| TABLE II |  |  |  |  |  |  |

Thresholds for some regular LDPC code ensembles over the BSC with erasure probability $p$. The belief-propagation
THRESHOLD IS $p_{\mathrm{BP}}$, THE MAXIMUM A POSTERIORITHRESHOLD IS $p_{\mathrm{MAP}}$, the Shannon threshold is $p_{\text {Sh }}$ and our threshold is $p_{\text {Loop }}$. Values of BP and MAP thresholds are from [23].

We would expect that the probability of error vanishes for $\epsilon<\epsilon_{\text {MAP }}$ and $p<p_{\text {MAP. }}$. Although the thresholds that we found are reasonably close to $\epsilon_{\mathrm{MAP}}$ and $p_{\mathrm{MAP}}$ for graphs with small degrees, they become worse in the limit of large degrees. A quick inspection of (10) and (11) shows that the functions $f / l$ and $k / l$ become independent of the noise parameter in the limit where $l$ and $r$ go to infinity with a fixed ratio $l / r$. It implies that $p_{\text {loop }}$ and $\epsilon_{\text {loop }}$ vanish. This behavior is in the opposite direction to what we can expect as in the limit of large degrees $p_{\mathrm{MAP}} \rightarrow p_{\mathrm{Sh}}$. In Section $\nabla$ we discuss about possible improvements in our analysis in order to make our thresholds tight.

The rest of the paper is organized as follows. In Section $\Pi I$ we show that the bit-error probability is related to the derivative of the so-called free entropy. Using the loop series, we express the free entropy as a combinatorial sum over subgraphs. In Section $I \mathbb{\text { we control the loop series with }}$ asymptotic estimates on subgraphs and Laplace's method. We prove Theorems 1 and 2 in this section. In Section $\nabla$ we discuss future directions and ways to improve and generalize our results.

## III. Free Entropy, Bethe Approximation and Loop SERIES

A. The Free Entropy and its Relation to the Bit-Error Probability

The bit-error probability (6) is related to a "perturbed" version of the partition function (4). Let $\eta \in \mathbb{R}$ be the perturbation parameter entering in the perturbed partition function

$$
\begin{align*}
Z(\Gamma, \underline{s}, \eta):= & \sum_{\underline{\sigma}} \prod_{a \in C} \frac{1}{2}\left(1+\prod_{i \in \partial a} \sigma_{i}\right) \\
& \times \prod_{i \in V} q\left(s_{i} \mid \sigma_{i}\right) e^{\eta\left(\sigma_{i}-1\right)} . \tag{12}
\end{align*}
$$

Note that $Z(\Gamma, \underline{s}, \eta)$ is a non-increasing function of $\eta$ and $Z(\Gamma, \underline{s}, 0)$ is the original partition function (4).

The free entropy is the (normalized) logarithm of the partition function (12)

$$
\begin{equation*}
\phi(\Gamma, \underline{s}, \eta):=\frac{1}{n} \ln Z(\Gamma, \underline{s}, \eta) \tag{13}
\end{equation*}
$$

A direct computation shows that the derivative of the free entropy with respect to its perturbation parameter reads

$$
\begin{equation*}
\left.\frac{\partial}{\partial \eta} \phi(\Gamma, \underline{s}, \eta)\right|_{\eta=0}=\frac{1}{n} \sum_{i=1}^{n}\left\langle\sigma_{i}\right\rangle_{\underline{s}}-1 \tag{14}
\end{equation*}
$$

Therefore the bit-error probability is related to the average entropy through the following relation

$$
\begin{equation*}
\left.\frac{\partial}{\partial \eta} \mathbb{E}_{\underline{s}}[\phi(\Gamma, \underline{s}, \eta)]\right|_{\eta=0}=-2 P_{\Gamma}^{\text {bit-sampling }} \tag{15}
\end{equation*}
$$

Since $Z(\Gamma, \underline{s}, \eta)$ is a non-increasing function of $\eta$, the free entropy is non-increasing as well. Moreover the free entropy is a convex function of $\eta$ as it can easily be verified by taking twice the derivative with respect to $\eta$. It implies that in order to show concentration of the bit-error probability it is sufficient to prove that there exists $\widetilde{\eta}<0$ independent of $n$ such that $\mathbb{E}_{\Gamma, \underline{s}}[\phi(\Gamma, \underline{s}, \widetilde{\eta})] \rightarrow 0$. If this condition is true then, thanks to monotonicity, the limit is also equal to zero for all $\eta \in[\widetilde{\eta}, \infty[$. Finally convexity of the free entropy enables us to exchange limit and derivative (see [24, p. 203]).

In order to prove that the free entropy vanishes we decompose it into two contributions: the Bethe free entropy that can be computed explicitly and the so-called loop series that is a sum over subgraphs of $\Gamma$. Using a first moment method and combinatorial tools from graph theory, we show that with high probability the loop series vanishes in the large codeword limit. The last statement implies that the free entropy is equal to the Bethe free entropy.

## B. The Bethe Approximation

The Bethe free entropy is an approximation of the free entropy (13). It is defined as a functional over "messages" that are probability distributions $\nu_{i \rightarrow a}\left(\sigma_{i}\right), \widehat{\nu}_{a \rightarrow i}\left(\sigma_{i}\right)$ associated with the directed edges $i \rightarrow a, a \rightarrow i$ of the graph. The messages satisfy the so-called belief-propagation (BP)
equations. For the free entropy (13) the BP equations take the following form

$$
\begin{align*}
& \widehat{\nu}_{a \rightarrow i}\left(\sigma_{i}\right) \propto \sum_{\underline{\sigma}_{\partial a} \backslash \sigma_{i}} \frac{1}{2}\left(1+\prod_{i \in \partial a} \sigma_{i}\right) \prod_{j \in \partial a \backslash i} \nu_{j \rightarrow a}\left(\sigma_{i}\right) \\
& \nu_{i \rightarrow a}\left(\sigma_{i}\right) \propto e^{\eta\left(\sigma_{i}-1\right)} q\left(s_{i} \mid \sigma_{i}\right) \prod_{b \in \partial i \backslash a} \widehat{\nu}_{b \rightarrow i}\left(\sigma_{i}\right), \tag{16}
\end{align*}
$$

where the symbol $\propto$ denotes equality up to a normalization factor and $\underline{\sigma}_{\partial a}:=\left\{\sigma_{j} \mid j \in \partial a\right\}$.

The Bethe free entropy evaluated at a fixed point of the BP equations is a sum of local contributions from nodes and edges of the graph $\Gamma=(V, C, E)$

$$
\begin{equation*}
\phi_{(\nu, \overline{\underline{\underline{L}}})}^{\mathrm{Bethe}}(\Gamma, \underline{s}, \eta):=\frac{1}{n} \sum_{a \in C} F_{a}+\frac{1}{n} \sum_{i \in V} F_{i}-\frac{1}{n} \sum_{(i, a) \in E} F_{i a}, \tag{17}
\end{equation*}
$$

where

$$
\begin{align*}
F_{a} & =\ln \left(\sum_{\underline{\sigma}_{\partial a}} \frac{1}{2}\left(1+\prod_{i \in \partial a} \sigma_{i}\right) \prod_{j \in \partial a} \nu_{j \rightarrow a}\left(\sigma_{i}\right)\right) \\
F_{i} & =\ln \left(\sum_{\sigma_{i}} e^{\eta\left(\sigma_{i}-1\right)} q\left(s_{i} \mid \sigma_{i}\right) \prod_{b \in \partial i} \widehat{\nu}_{b \rightarrow i}\left(\sigma_{i}\right)\right) \\
F_{i a} & =\ln \left(\sum_{\sigma_{i}} \nu_{i \rightarrow a}\left(\sigma_{i}\right) \widehat{\nu}_{a \rightarrow i}\left(\sigma_{i}\right)\right) \tag{18}
\end{align*}
$$

Note that once a fixed-point of the BP equations (16) is found, computing the Bethe free entropy 17 is a computationally easy task.

## C. Corrections to the Bethe Free Entropy: the Loop Series

The difference between the free entropy and the Bethe free entropy can be expressed with the so-called loop series derived by Chertkov and Chernyak [20]. It takes the form of the logarithm of a weighted sum over subgraphs of $\Gamma$. These subgraphs are called "loops" for they have no dangling edges. Note that if $\Gamma$ is a tree no such subgraph exists and we recover the well-known result that the Bethe free entropy is exact on trees.

We recall that a subgraph $g=\left(V_{g}, C_{g}, E_{g}\right)$ of $\Gamma=$ $(V, C, E)$ is any graph with vertex set $V_{g} \subset V$, factor node set $C_{g} \subset C$ and edge set $E_{g} \subset\left(V_{g} \times C_{g}\right) \cap E$. For simplicity we denote the relation " $g$ is a subgraph of $\Gamma$ " with the inclusion symbol $g \subset \Gamma$. We also denote the induced neighborhood in $g$ of a variable node $i \in V_{g}$ (resp. check node $a \in C_{g}$ ) by $\partial_{g} i=\partial i \cap V_{g}$ (resp. by $\partial_{g} a=\partial a \cap C_{g}$ ).

The set of "loops" consists of any non-empty subgraphs, not necessarily connected, with no degree one variable-node and no degree one check-node
$\mathcal{L}_{\Gamma}:=\left\{g \subset \Gamma\left|\forall i \in V_{g},\left|\partial_{g} i\right| \geq 2\right.\right.$ and $\left.\forall a \in C_{g},\left|\partial_{g} a\right| \geq 2\right\}$.
The difference between the free entropy and the Bethe free entropy is related to the loop series through the following
equation

$$
\begin{equation*}
\phi(\Gamma, \underline{s}, \eta)-\phi_{(\underline{\nu}, \underline{\underline{\nu}})}^{\text {Bethe }}(\Gamma, \underline{s}, \eta)=\frac{1}{n} \ln \left(Z_{(\underline{\nu}, \underline{\hat{\nu}})}^{\text {loop }}\right), \tag{20}
\end{equation*}
$$

where the argument of the logarithm is a weighted sum over loops

$$
\begin{equation*}
Z_{(\underline{\nu}, \widehat{\underline{L}})}^{\mathrm{loop}}:=1+\sum_{g \in \mathcal{L}_{\Gamma}} K_{(\underline{\nu}, \widehat{\underline{\nu}})}(g) . \tag{21}
\end{equation*}
$$

The weight function over loops depends on the BP fixed point at which the Bethe free entropy is evaluated and can be expressed as a product over the nodes inside a loop

$$
\begin{equation*}
K_{(\nu, \widehat{\nu})}(g):=\prod_{i \in V_{g}} \kappa_{i} \prod_{a \in C_{g}} \kappa_{a} \tag{22}
\end{equation*}
$$

The factors $\kappa_{i}$ and $\kappa_{a}$ entering in (22) depend only on messages that are associated with edges neighboring the nodes $i \in V_{g}$ and $a \in C_{g}$

$$
\begin{align*}
\kappa_{i}:= & \left(\sum_{\sigma_{i}} q\left(s_{i} \mid \sigma_{i}\right) e^{\eta\left(\sigma_{i}-1\right)} \prod_{a \in \partial i} \widehat{\nu}_{a \rightarrow i}\left(\sigma_{i}\right)\right)^{-1} \\
\times & \left(\sum_{\sigma_{i}} q\left(s_{i} \mid \sigma_{i}\right) e^{\eta\left(\sigma_{i}-1\right)} \prod_{a \in \partial i \backslash \partial_{g} i} \widehat{\nu}_{a \rightarrow i}\left(\sigma_{i}\right)\right. \\
& \left.\times \prod_{a \in \partial_{g} i} \sigma_{i} \nu_{i \rightarrow a}\left(-\sigma_{i}\right)\right) \tag{23}
\end{align*}
$$

and

$$
\begin{align*}
\kappa_{a}:= & \left(\sum_{\underline{\sigma}_{\partial a}}\left(1+\prod_{i \in \partial a} \sigma_{i}\right) \prod_{i \in \partial a} \nu_{i \rightarrow a}\left(\sigma_{i}\right)\right)^{-1} \\
\times & \left(\sum_{\underline{\sigma}_{\partial a}}\left(1+\prod_{i \in \partial a} \sigma_{i}\right) \prod_{i \in \partial a \backslash \partial_{g} a} \nu_{i \rightarrow a}\left(\sigma_{i}\right)\right. \\
& \left.\times \prod_{i \in \partial_{g} a} \sigma_{i} \widehat{\nu}_{a \rightarrow i}\left(-\sigma_{i}\right)\right) . \tag{24}
\end{align*}
$$

For a complete derivation of the loop series for graphical models associated with linear codes, we refer the reader to [25].

## D. The Decoding Regime and its BP Fixed-Point

Note that the loop series, as well as the Bethe free entropy, are functions of fixed-points of the BP equations (16). The fixed-point associated with the decoding regime is the ferromagnetic fixed-point

$$
\begin{equation*}
\widehat{\nu}_{a \rightarrow i}^{+}\left(\sigma_{i}\right)=\nu_{i \rightarrow a}^{+}\left(\sigma_{i}\right)=\frac{1+\sigma_{i}}{2} \tag{25}
\end{equation*}
$$

One can easily see that ferromagnetic messages (25) satisfy the BP equations (16) regardless of the channel considered and of the value of the perturbation parameter $\eta \in \mathbb{R}$. The ferromagnetic fixed-point (25) describes a state for which the most likely configuration is the all-zero codeword i.e. $\sigma_{i}=+1$. This is the reason why this fixed-point is associated
with the decoding regime.
The Bethe free entropy (17) evaluated at the ferromagnetic fixed-point simply reads

$$
\begin{equation*}
\phi_{+}^{\text {Bethe }}(\Gamma, \underline{s}, \eta)=\frac{1}{n} \sum_{i \in V} \ln \left(q\left(s_{i} \mid+1\right)\right) . \tag{26}
\end{equation*}
$$

The factors entering in the weight function (22) are computed using Equations (24) for check nodes

$$
\kappa_{a}= \begin{cases}1 & \left|\partial_{g} a\right| \text { is even }  \tag{27}\\ 0 & \left|\partial_{g} a\right| \text { is odd }\end{cases}
$$

and Equation (23) for variable nodes

$$
\kappa_{i}= \begin{cases}(-1)^{l} e^{-2\left(\lambda\left(s_{i}\right)+\eta\right)} & \left|\partial_{g} i\right|=l  \tag{28}\\ 0 & \left|\partial_{g} i\right|<l\end{cases}
$$

where in the last expression we have used the half loglikelihood variables

$$
\begin{equation*}
\lambda\left(s_{i}\right):=\frac{1}{2} \ln \frac{q\left(s_{i} \mid+1\right)}{q\left(s_{i} \mid-1\right)} \tag{29}
\end{equation*}
$$

Based on the expression of the factors (27) and (28), the only subgraphs with a non-zero weight are those with an induced variable-node degree equal to $l$ and even induced check-node degree. This motivates the definition of the ferromagnetic loops ensemble

$$
\begin{equation*}
\mathcal{L}_{\Gamma}^{+}=\left\{g \in \mathcal{L}_{\Gamma}\left|\forall i, a \in g,\left|\partial_{g} i\right|=l \text { and }\right| \partial_{g} a \mid \text { is even }\right\} . \tag{30}
\end{equation*}
$$

A loop that is not an element of the ferromagnetic ensemble has a zero weight. Moreover the weight of a ferromagnetic loop is always non-negative

$$
\begin{equation*}
K_{+}(g)=\exp \left(-2 \eta\left|V_{g}\right|-2 \sum_{i \in V_{g}} \lambda\left(s_{i}\right)\right) \geq 0 \tag{31}
\end{equation*}
$$

In order to see that $K_{+}(g)$ is non-negative, notice that a sign is only associated with the factors $\kappa_{i}$ and is equal to $(-1)^{l}$. Therefore a loop can only have a negative weight if the product $l\left|V_{g}\right|$ is odd. Note that this product is the number of edges in a loop counted from the variable-node perspective. Therefore it should be equal to the number of edges counted from the check-node perspective

$$
\begin{equation*}
l\left|V_{g}\right|=\sum_{a \in C_{g}}\left|\partial_{g} a\right| \tag{32}
\end{equation*}
$$

Since for a ferromagnetic loop $\left|\partial_{g} a\right|$ is always even, $l\left|V_{g}\right|$ is also even and the weight of a loop is always non-negative.

Using Equations (20) and (26) we can express the average free entropy (13) in the simple form

$$
\begin{align*}
\mathbb{E}_{\Gamma, \underline{s}}[\phi(\Gamma, \underline{s}, \eta)]= & \mathbb{E}_{\Gamma, \underline{s}}\left[\frac{1}{n} \ln \left(1+\sum_{g \in \mathcal{L}_{\Gamma}^{+}} K_{+}(g)\right)\right] \\
& +\int d s q(s \mid 1) \ln (q(s \mid 1)) \tag{33}
\end{align*}
$$

Note that Equation (33) is valid for all BMS channels regardless of the noise parameter. However we can only expect that the ferromagnetic loop-series vanishes in the decoding regime.

## IV. First Moment Method on the Loop Series

We use a first moment method to prove that the ferromagnetic loop-series in Equation (33) vanishes. In our case it is based on Jensen's inequality and consists of permuting the expectation over the graph ensemble and the logarithm in Equation (33).

Note that we cannot permute the expectation over the channel output realizations and the logarithm. It is easy to see that over the channel output realizations a loop has an expected weight (31) that increases exponentially fast for $\eta<0$

$$
\begin{equation*}
\mathbb{E}_{\underline{s}}\left[K_{+}(g)\right]=e^{-\eta\left|V_{g}\right|} \tag{34}
\end{equation*}
$$

This is because the loop series is dominated by events for which most of the bits are corrupted and have negative half log-likelihood (29). These events are rare but give rise to an exponentially large weight.

Therefore we estimate the expectation of the loop series over the ensemble of regular $(l, r)$ bipartite graphs for a fixed output realization of the channel.

## A. Probability Estimates on Graphs

For a given channel realization $\underline{s}$ of the BEC (resp. BSC) call $V_{c}$ the set of variable nodes with $s_{i}=0$ (resp. $s_{i}=-1$ ) and call $V_{0}$ the set of variable nodes $i \in V$ with $s_{i}=1$ (resp. $s_{i}=1$ ). The set $V_{0}$ contains bits that have been correctly transmitted and $V_{c}$ contains bits that have been corrupted. We denote the fraction of correctly transmitted bits by $(1-\rho)=$ $\left|V_{0}\right| / n$ and we denote the fraction of corrupted bits by $\rho=$ $\left|V_{c}\right| / n$. We recall that the total number of variable nodes is $n=|V|$ and the total number of check nodes is $m=|C|$.

We decompose the set of ferromagnetic loops (30) into subsets of loops having the same "type". The type of a loop $g \in \mathcal{L}_{\Gamma}^{+}$is the triplet $\left(x_{0}, x_{c}, \underline{y}\right) \in[0,1]^{2 \times\lfloor r / 2\rfloor}$ where $x_{0}=\left|V_{0} \cap V_{g}\right| / n$ is the fraction of correctly transmitted variable nodes in the loop, $x_{c}=\left|V_{c} \cap V_{g}\right| / n$ is the fraction of corrupted variable nodes in the loop and $y=\left\{y_{t}\right\}_{t=1}^{\lfloor r / 2\rfloor}$ is the fraction of check nodes with degree $2 t$. The set of loops of type $\left(x_{0}, x_{c}, \underline{y}\right)$ is denoted by $\Omega\left(x_{0}, x_{c}, \underline{y}\right)$.

Not all value of $\left(x_{0}, x_{c}, \underline{y}\right)$ are admissible loop types. The fraction of check nodes inside a loop is upper bounded by 1. Moreover counting edges from the variable-node perspective or from the check-node perspective obviously gives the same number. Therefore types that are admissible belong to the following set already introduced in Section (II Eq. (9)

$$
\begin{align*}
D(\rho)= & \left\{\left(x_{0}, x_{c}, \underline{y}\right) \in[0,1]^{2+\lfloor r / 2\rfloor} \mid \sum_{t=1}^{\lfloor r / 2\rfloor} y_{t} \leq 1\right. \\
& \left.\sum_{t=1}^{\lfloor r / 2\rfloor} \frac{2 t}{r} y_{t}=(1-\rho) x_{0}+\rho x_{c}\right\} . \tag{35}
\end{align*}
$$

The weight (31) of a loop $g \in \Omega\left(x_{0}, x_{c}, \underline{y}\right)$ is only a function of its type $K_{+}(g) \equiv K_{+}\left(x_{0}, x_{c}\right)$. Using the specific expression of the half log-likelihood (29) for each channels we find the explicit form of the weight function for the BEC

$$
K_{+}^{\mathrm{BEC}}\left(x_{0}, x_{c}\right)= \begin{cases}\exp \left(-2 n \eta x_{c} \rho\right) & x_{0}=0  \tag{36}\\ 0 & x_{0}>0\end{cases}
$$

and for the BSC

$$
\begin{align*}
K_{+}^{\mathrm{BSC}}\left(x_{0}, x_{c}\right)= & \exp \left(-2 n \eta\left(x_{0}(1-\rho)+x_{c} \rho\right)\right. \\
& \left.+n k\left(x_{0}, x_{c}, \rho, p\right)\right) \tag{37}
\end{align*}
$$

where $k\left(x_{0}, x_{c}, \rho, p\right)$ is the auxiliary function introduced in Section III Eq. (11)

$$
\begin{equation*}
k\left(x_{0}, x_{c}, \rho, p\right)=\left(\rho x_{c}-(1-\rho) x_{0}\right) \ln \left(\frac{1-p}{p}\right) . \tag{38}
\end{equation*}
$$

Therefore the expected value of the loop series over the graph ensemble can be expressed only through loop types

$$
\begin{align*}
\mathbb{E}_{\Gamma}\left[\sum_{g \in \mathcal{L}_{\Gamma}^{+}} K_{+}(g)\right]= & \sum_{\substack{\left(x_{0}, x_{c}, \underline{y}\right) \in D(\rho)}} K_{+}\left(x_{0}, x_{c}\right) \\
& \times \mathbb{E}_{\Gamma}\left[\left|\Omega\left(x_{0}, x_{c}, \underline{y}\right)\right|\right] . \tag{39}
\end{align*}
$$

The expected number of loops with prescribed type $\left(x_{0}, x_{c}, y\right)$ is upper bounded using McKay's combinatorial estimatt [21] for subgraphs with specified degrees

$$
\begin{align*}
\mathbb{E}_{\Gamma}\left[\left|\Omega\left(x_{0}, x_{c}, \underline{y}\right)\right|\right] \leq & n^{\delta_{l, r}}\binom{n l}{n l\left(x_{0}(1-\rho)+x_{c} \rho\right)}^{-1} \\
& \times\binom{ n(1-\rho)}{n x_{0}(1-\rho)}\binom{n \rho}{n x_{c} \rho} \\
& \times\binom{ m}{m y_{1}, \ldots, m y_{\lfloor r / 2\rfloor}} \\
& \times \prod_{t=1}^{\lfloor r / 2\rfloor}\binom{r}{2 t}^{m y_{t}} \tag{40}
\end{align*}
$$

where $\delta_{l, r}$ is a constant that depends only on $l$ and $r$. McKay's estimate has the advantage to have an asymptotically tight growth rate when $n$ goes to infinity.

It remains to prove that the average loop series (39) with the bound (40) vanishes in the large $n$ limit.

## B. Laplace's Method and Proof of Theorems

The loop series (39) is dominated by loop types that contribute to the sum with the biggest exponential growth. We apply Laplace's method in order to characterize the biggest exponent.

Using Stirling inequalities

$$
\begin{equation*}
e^{\frac{1}{12 n+1}} \leq \frac{n!}{\sqrt{2 \pi n} e^{-n} n^{n}} \leq e^{\frac{1}{12 n}} \tag{41}
\end{equation*}
$$

[^4]we find an asymptotically tight upper bound on the estimate (40)
\[

$$
\begin{equation*}
\mathbb{E}_{\Gamma}\left[\left|\Omega\left(x_{0}, x_{c}, \underline{y}\right)\right|\right] \leq C_{l, r} n^{\delta_{l, r}^{\prime}} \exp \left(n f\left(x_{0}, x_{c}, \underline{y}, \rho\right)\right), \tag{42}
\end{equation*}
$$

\]

where $C_{l, r}$ and $\delta_{l, r}^{\prime}$ are just numerical constants and $f\left(x_{0}, x_{c}, \underline{y}, \rho\right)$ is the auxiliary function introduced in Section (II) Eq. (1)

$$
\begin{align*}
f\left(x_{0}, x_{c}, \underline{y}, \rho\right)= & -l h_{2}\left((1-\rho) x_{0}+\rho x_{c}\right) \\
& +(1-\rho) h_{2}\left(x_{0}\right)+\rho h_{2}\left(x_{c}\right) \\
& -\frac{l}{r}\left(1-\sum_{t=1}^{r} y_{t}\right) \ln \left(1-\sum_{t=1}^{r} y_{t}\right) \\
& -\frac{l}{r} \sum_{t=1}^{r} y_{t} \ln y_{t} \\
& +\frac{l}{r} \sum_{t=1}^{\lfloor r / 2\rfloor} y_{t} \ln \binom{r}{2 t} . \tag{43}
\end{align*}
$$

Combining Equations (36), 37) and (42), we show that the leading exponent in Equation (39) is for the BEC

$$
\begin{equation*}
\alpha^{\mathrm{BEC}}(\rho, \eta)=\max _{\left(0, x_{c}, \underline{y}\right) \in D(\rho)} f\left(0, x_{c}, \underline{y}, \rho\right)-2 \eta x_{c} \rho, \tag{44}
\end{equation*}
$$

and is for the BSC

$$
\begin{align*}
\alpha^{\mathrm{BSC}}(\rho, \eta)= & \max _{\left(x_{0}, x_{c}, \underline{y}\right) \in D(\rho)}\left(-2 \eta\left(x_{0}(1-\rho)+x_{c} \rho\right)\right. \\
& \left.+f\left(x_{0}, x_{c}, \underline{y}, \rho\right)+k\left(x_{0}, x_{c}, \rho, p\right)\right) .(4 \tag{45}
\end{align*}
$$

Notice that for all $\rho$ and $\eta$ the exponent $\alpha^{\operatorname{BEC} / \mathrm{BSC}}(\rho, \eta)$ is non-negative. This is easily verified by evaluating the objective function at $\left(x_{0}, x_{c}, \underline{y}\right)=(0,0,0)$. Therefore the bit-error probability vanishes if $\alpha^{\operatorname{BEC/BSC}}(\rho, \eta)$ is equal to zero for all $\eta$ in a neighborhood of zero. The next Lemma shows that in fact only the maximization at $\eta=0$ is important.

Lemma 4. If the maximum of (44) (resp. (45)) is uniquely achieved in $\left(x_{0}, x_{c}, \underline{y}\right)=(0,0,0)$ for $\eta=0$, then there exists $\widetilde{\eta}<0$ such that $\alpha^{B E C}(\rho, \eta)=0$ (resp. $\alpha^{B S C}(\rho, \eta)=0$ ) for all $\eta \in] \widetilde{\eta}, \infty[$.

## Proof. See Appendix $\square$

In order to prove Theorems 1 and 2, we need to show that small variations around $\rho$ do not change $\alpha^{\operatorname{BEC}}(\rho, 0)$ and $\alpha^{\mathrm{BSC}}(\rho, 0)$. This is guaranteed by the following Lemma.
Lemma 5. For all $\rho \in[0,1]$, if $\alpha^{B E C}(\rho, 0)=0$ (resp. $\left.\alpha^{B S C}(\rho, 0)=0\right)$ and the maximum of (44) (resp. (45)) is uniquely achieved at $\left(x_{0}, x_{c}, \underline{y}\right)=(0,0,0)$, there exists $N$ sufficiently large such that
$\forall n \geq N, \forall \delta \in\left[-\sqrt{\frac{\ln n}{n}}, \sqrt{\frac{\ln n}{n}}\right], \alpha^{B E C / B S C}(\rho+\delta, 0)=0$
Proof. See Appendix [I]
We are now in position to prove our main theorems.

Proof of Theorem 11 Let $\epsilon$ be the probability of error of the BEC. First notice that the perturbed partition function (12) is trivially lower bounded by 1 and upper bounded by $2^{n} e^{2 n|\eta|}$. This implies that the free entropy (13) remains finite

$$
\begin{equation*}
0 \leq \phi(\Gamma, \underline{s}, \eta) \leq \ln 2+2|\eta| \tag{46}
\end{equation*}
$$

Therefore using Equation (33) and the fact that $K_{+}(g) \geq 0$ we see that the loop series remains finite as well

$$
\begin{align*}
2(\ln 2+|\eta|) \geq & \mid \mathbb{E}_{\underline{s}}[\phi(\Gamma, \underline{s}, \eta)] \\
& -\int d s q(s \mid 1) \ln (q(s \mid 1)) \mid \\
= & \mathbb{E}_{\underline{s}}\left[\frac{1}{n} \ln \left(1+\sum_{g \in \mathcal{L}_{\Gamma}^{+}} K_{+}(g)\right)\right] \tag{47}
\end{align*}
$$

Let $A$ be the following probabilistic event on the channel output realizations

$$
\begin{equation*}
A:=\left\{\left.\underline{s} \in\{-1,0,1\}^{n}| | \frac{1}{n} \sum_{i=1}^{n} s_{i}-(1-\epsilon) \right\rvert\, \leq \sqrt{\frac{\ln n}{n}}\right\} \tag{48}
\end{equation*}
$$

Output realizations in $A$ are close to the average output realization.

Using Hoeffding's inequality, we see that the probability of the complementary event $A^{c}$ vanishes

$$
\begin{equation*}
\mathbb{P}_{\underline{s}}\left[A^{c}\right] \leq \frac{2}{n^{-2}} \tag{49}
\end{equation*}
$$

Combining Jensen's inequality and the trivial bound 47) on the loop series we have the following estimate

$$
\begin{align*}
\mathbb{E}_{\Gamma, \underline{s}}[ & \left.\frac{1}{n} \ln \left(1+\sum_{g \in \mathcal{L}_{\Gamma}^{+}} K_{+}(g)\right)\right] \leq \frac{4}{n^{-2}}(\ln 2+|\eta|) \\
& +\mathbb{E}_{\underline{s}}\left[\left.\frac{1}{n} \ln \left(1+\mathbb{E}_{\Gamma}\left[\sum_{g \in \mathcal{L}_{\Gamma}^{+}} K_{+}(g)\right]\right) \right\rvert\, A\right] . \tag{50}
\end{align*}
$$

Since we have conditioned over channel output realizations that are in $A$, the fraction of corrupted bit is $|\rho-\epsilon| \leq$ $\sqrt{\ln n / n}$. Therefore combining Equation (42), Lemma 4 and Lemma 5 we have that if $\alpha^{\mathrm{BEC}}(\epsilon, 0)=0$ is uniquely achieved in $\left(x_{0}, x_{c}, \underline{y}\right)=(0,0,0)$ then for all $\left.\eta \in\right] \widetilde{\eta}, \infty[$ and $n$ sufficiently large,

$$
\begin{array}{r}
\mathbb{E}_{\underline{s}}\left[\left.\frac{1}{n} \ln \left(1+\mathbb{E}_{\Gamma}\left[\sum_{g \in \mathcal{L}_{\Gamma}^{+}} K_{+}(g)\right]\right) \right\rvert\, A\right] \\
\frac{1}{n} \ln \left(1+c_{3} n^{c_{4}}\right) \tag{51}
\end{array}
$$

where $c_{3}$ and $c_{4}$ are numerical constants independent of $n$.

We have proved that for all $\eta \in] \widetilde{\eta}, \infty[$ with $\widetilde{\eta}<0$ the average free entropy converges in expectation over the regular $(l, r)$ LDPC ensemble
$\lim _{n \rightarrow \infty} \mathbb{E}_{\Gamma}\left[\left|\mathbb{E}_{\underline{s}}[\phi(\Gamma, \underline{s}, \eta)]-\int d s q(s \mid 1) \ln (q(s \mid 1))\right| \mid\right]_{(52)}=0$.
In particular it implies that the average free entropy over the LDPC ensemble converges

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}_{\Gamma, \underline{s}}[\phi(\Gamma, \underline{s}, \eta)]=\int d s q(s \mid 1) \ln (q(s \mid 1)) \tag{53}
\end{equation*}
$$

Since $\mathbb{E}_{\Gamma, \underline{s}}[\phi(\Gamma, \underline{s}, \eta)]$ is a convex function of $\eta$ and converges pointwise in a neighborhood of zero, we can exchange the limit and the derivative

$$
\begin{align*}
0 & =\left.\frac{\partial}{\partial \eta} \lim _{n \rightarrow \infty} \mathbb{E}_{\Gamma, \underline{s}}[\phi(\Gamma, \underline{s}, \eta)]\right|_{\eta=0} \\
& =\left.\lim _{n \rightarrow \infty} \frac{\partial}{\partial \eta} \mathbb{E}_{\Gamma, \underline{s}}[\phi(\Gamma, \underline{s}, \eta)]\right|_{\eta=0} \\
& =\left.\lim _{n \rightarrow \infty} \mathbb{E}_{\Gamma}\left[\frac{\partial}{\partial \eta} \mathbb{E}_{\underline{s}}[\phi(\Gamma, \underline{s}, \eta)]\right]\right|_{\eta=0} \\
& =-2 \lim _{n \rightarrow \infty} \mathbb{E}_{\Gamma}\left[P_{\Gamma}^{\text {bit-sampling }}\right] \tag{54}
\end{align*}
$$

where in the last line we use Equation (15) that relates the free entropy to the bit-error probability.

Theorem 2 has a proof almost identical to that of Theorem 1 .

## V. Path Forward

We would like to stress that the techniques developed in this paper are quite general. In particular they do not rely on a special form of channels or on the regular-degree distribution of the LDPC ensemble. Therefore we plan to improve our results in the following ways.

## A. Generalization to Arbitrary Degree Distributions

The entire analysis can easily be extended to general degree distributions with bounded degrees. It will simply transform the function (43) that counts subgraphs into a more convoluted object. However extending our results to distributions with unbounded degrees, like for instance Poisson distributions, may be more complicated. One would have to derive an estimate for counting subgraphs in this particular case.

## B. Asymmetric Channels

The loop series and the Bethe free entropy for general channels are almost exactly similar than for symmetric channels. For general channels we can no longer assume that the all-zero codeword is transmitted. Instead we have to average the bit-error probability over all possible input codewords $\underline{\tau}$. In this case the weight of a loop remains similar than for symmetric channels. The weight is also non-negative and depends on the generalized half log-likelihood ratio

$$
\begin{equation*}
\lambda\left(s_{i} \mid \tau_{i}\right)=\frac{1}{2} \log \frac{q\left(s_{i} \mid \tau_{i}\right)}{q\left(s_{i} \mid-\tau_{i}\right)} \tag{55}
\end{equation*}
$$

where $\underline{s}$ denotes as usual the channel observations. In order to control the loop series, we will need to perform a conditioned
expectation in 50 over joint typical sequences of input codewords and noise realizations.

## C. Tight Thresholds

As described in Section III the thresholds that we obtain are not tight. In fact at fixed rate they become worse and converge to zero as the degrees of the graph become large. The reason why we obtain such loose bounds for large degrees comes from the function $f\left(x_{0}, x_{c}, \underline{y}, \rho\right)$ defined in (43). This function counts the growing rate of the average number of subgraphs with a prescribed type $\left(x_{0}, x_{c}, \underline{y}\right)$

$$
\begin{equation*}
f=\lim _{n \rightarrow \infty} \frac{1}{n} \ln \left(\mathbb{E}_{\Gamma}\left[\left|\Omega\left(x_{0}, x_{c}, \underline{y}\right)\right|\right]\right) \tag{56}
\end{equation*}
$$

One can verify that if instead of $f$ we use the function

$$
\begin{equation*}
\tilde{f}=\lim _{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\Gamma}\left[\ln \left(\left|\Omega\left(x_{0}, x_{c}, \underline{y}\right)\right|\right)\right] \tag{57}
\end{equation*}
$$

we obtain tight lower and upper bound on the threshold for vanishing bit-error probability.

The function $\tilde{f}$ only depends on the random graph ensembles that we consider and does not depend on a particular channel. Computing this function would provide a proof of the exact location of the MAP threshold for an extensive class of channels. However this computation could prove to be a very difficult task.

A way around the problem of computing (57) is to condition the expectation (56) on some rare events with respect to the random graph measure. Note that by Jensen's inequality $\widetilde{f}$ is always upper-bounded by $f$. This is because the expectation (56) is dominated by rare events that are associated with a large weight $\left|\Omega\left(x_{0}, x_{c}, \underline{y}\right)\right|$. Conditioning on these rare events will lead to better estimates of (57) and will provide tighter bounds at least in the limit of large degrees.

## Appendix I <br> Proof of Lemma 4

Proof. We prove Lemma 4 only for the BSC (the proof for the BEC is almost identical). For a given $\rho$ and $p$, let us define the following function

$$
\begin{align*}
g^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, \eta\right)= & f\left(x_{0}, x_{c}, \underline{y}, \rho\right)+k\left(x_{0}, x_{c}, \rho, p\right) \\
& -2 \eta\left(x_{0}(1-\rho)+x_{c} \rho\right) \tag{58}
\end{align*}
$$

The function $g^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, \eta\right)$ corresponds to the exponent of the loop series (39) associated with the loop type $\left(x_{0}, x_{c}, \underline{y}\right)$. In order to prove Lemma 4 we have to find $\widetilde{\eta}<0$ such that $g^{\mathrm{BSC}}$ is non-positive on $D(\rho) \times[\widetilde{\eta},+\infty[$.

We first show that for any $\widetilde{\eta}_{1}<0$, there exists a neighborhood $U$ of $\left(x_{o}, x_{c}, \underline{y}\right)=(0,0,0)$ such that $g^{\text {BSC }}$ is non-positive on $U \cap D(\rho) \times\left[\widetilde{\eta}_{1},+\infty\left[\right.\right.$. For a fixed $\widetilde{\eta}_{1}<0$ we construct a function $\bar{g}^{\mathrm{BSC}}$ that is an upper bound of $g^{\mathrm{BSC}}$. We restrict ourselves to the domain $V \cap D(\rho) \times\left[\widetilde{\eta}_{1},+\infty[\right.$, where $V=\mathbb{B}(0,1 / 3 r)$ is the ball of radius $1 / 3 r$ centered at ( $0,0,0$ ).
Let us explicitly write down the function (58) term by term

$$
\begin{align*}
g^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, \eta\right)= & -2 \eta\left(x_{0}(1-\rho)+x_{c} \rho\right) \\
& +\left(\rho x_{c}-(1-\rho) x_{0}\right) \ln \left(\frac{1-p}{p}\right) \\
& +\frac{l}{r} \sum_{t=1}^{\lfloor r / 2\rfloor} y_{t} \ln \binom{r}{2 t} \\
& -l h_{2}\left((1-\rho) x_{0}+\rho x_{c}\right) \\
& +(1-\rho) h_{2}\left(x_{0}\right)+\rho h_{2}\left(x_{c}\right) \\
& -\frac{l}{r}\left(1-\sum_{t=1}^{r} y_{t}\right) \ln \left(1-\sum_{t=1}^{r} y_{t}\right) \\
& -\frac{l}{r} \sum_{t=1}^{r} y_{t} \ln y_{t} . \tag{59}
\end{align*}
$$

We bound each term of 59) separately. Denote the fraction of variable nodes in the loop by $X=x_{0}(1-\rho)+x_{c} \rho$. The inequalities below trivially hold

$$
\begin{align*}
\left(\rho x_{c}-(1-\rho) x_{0}\right) \ln \left(\frac{1-p}{p}\right) & \leq 2 \ln \left(\frac{1-p}{p}\right) X \\
\frac{l}{r} \sum_{t=1}^{\lfloor r / 2\rfloor} y_{t} \ln \binom{r}{2 t} & \leq l \ln \binom{r}{2\lfloor r / 2\rfloor} X \\
-2 \eta\left(x_{0}(1-\rho)+x_{c} \rho\right) & \leq-2 \widetilde{\eta}_{1} X \tag{60}
\end{align*}
$$

As the entropy is a concave function, we have the following inequality

$$
\begin{equation*}
(1-\rho) h_{2}\left(x_{0}\right)+\rho h_{2}\left(x_{c}\right) \leq h_{2}(X) \tag{61}
\end{equation*}
$$

Concativty of $-x \ln x$ gives us

$$
\begin{align*}
-\sum_{t=1}^{\lfloor r / 2\rfloor} y_{t} \ln y_{t} \leq & -\left(\sum_{t=1}^{\lfloor r / 2\rfloor} y_{t}\right) \ln \left(\frac{1}{\lfloor r / 2\rfloor} \sum_{t=1}^{\lfloor r / 2\rfloor} y_{t}\right) \\
\leq & -\left(\sum_{t=1}^{\lfloor r / 2\rfloor} y_{t}\right) \ln \left(\sum_{t=1}^{\lfloor r / 2\rfloor} y_{t}\right) \\
& +r \ln (\lfloor r / 2\rfloor) X \tag{62}
\end{align*}
$$

Note that since the domain is restricted to types in a ball of radius $1 / 3 r$, the fraction of variable nodes in a loop is upper-bounded $X \leq 1 / 3 r$. In particular it implies that

$$
\begin{align*}
\sum_{t=1}^{\lfloor r / 2\rfloor} y_{t} & \leq \frac{r}{2}\left(\sum_{t=1}^{\lfloor r / 2\rfloor} \frac{2 t}{r} y_{t}\right) \\
& =\frac{r}{2} X \\
& \leq \frac{1}{6} \\
& \leq \frac{1}{e} \tag{63}
\end{align*}
$$

where $e$ is the Euler constant. Finally as the entropy is
increasing on $\left[0, \frac{1}{e}\right]$, we have

$$
\begin{equation*}
\frac{l}{r} h_{2}\left(\sum_{t=1}^{\lfloor r / 2\rfloor} y_{t}\right) \leq \frac{l}{r} h_{2}\left(\frac{r}{2} X\right) \tag{64}
\end{equation*}
$$

The upper bound on the function (59) is simply the sum of Inequalities 60, (61), 62) and depends only on the fraction of variable nodes in a loop i.e. $\bar{g}^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, \eta\right) \equiv \bar{g}^{\mathrm{BSC}}(X)$ and

$$
\begin{equation*}
\bar{g}^{\mathrm{BSC}}(X)=\frac{l}{r} h_{2}\left(\frac{r}{2} X\right)-(l-1) h_{2}(X)+M X, \tag{65}
\end{equation*}
$$

where $M$ is a constant independent of $\eta$ and $\rho$
$M=2 \ln \left(\frac{1-p}{p}\right)+l \ln \binom{r}{2\lfloor r / 2\rfloor}+l \ln (\lfloor r / 2\rfloor)-2 \widetilde{\eta}_{1}$.
Notice that $\bar{g}^{\mathrm{BSC}}(0)=0$ and that the derivative $\frac{d}{d X} \bar{g}^{\mathrm{BSC}}(X)$ behaves like $\left(\frac{l}{2}-1\right) \ln X$ in the neighborhood of 0 . Hence, for $l \geq 3$, there exists $\delta>0$ such that $\bar{g}^{\mathrm{BSC}}$ is negative on $] 0, \delta]$. Therefore for all types $\left(x_{o}, x_{c}, \underline{y}\right) \in D(\rho)$ in the domain $U=\mathbb{B}(0, \delta) \cap \mathbb{B}(0,1 / 3 r)$ and for all $\eta \in\left[\widetilde{\eta}_{1},+\infty[\right.$ we have

$$
\begin{align*}
g^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, \eta\right) & \leq \bar{g}^{\mathrm{BSC}}(X) \\
& \leq 0 \tag{67}
\end{align*}
$$

By hypothesis the maximum of (45) is uniquely achieved in $(0,0,0)$ for $\eta=0$. It implies that there exists $\lambda<0$ such that

$$
\begin{equation*}
\max _{\left(x_{0}, x_{c}, \underline{y}\right) \in D(\rho) \backslash U} f\left(x_{0}, x_{c}, \underline{y}, \rho\right)+k\left(x_{0}, x_{c}, \rho, p\right)=\lambda \tag{68}
\end{equation*}
$$

Therefore for $\eta>\widetilde{\eta}_{2}=\lambda / 2$

$$
\begin{equation*}
\max _{\left(x_{0}, x_{c}, \underline{y}\right) \in D(\rho) \backslash U} g^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, \eta\right) \leq \lambda-2 \widetilde{\eta}_{2}=0 \tag{69}
\end{equation*}
$$

We see that $\widetilde{\eta}=\max \left(\widetilde{\eta}_{1}, \widetilde{\eta}_{2}\right)<0$ satisfies by construction the condition of Lemma 4

## Appendix II

## Proof of Lemma 5

Proof. We prove Lemma 5 only for the BSC (the proof for the BEC is almost identical). For a given $\rho$ and $p$, we recall the function $g_{p, \rho}^{\mathrm{BSC}} \equiv g^{\mathrm{BSC}}$ and $\bar{g}_{p, \rho}^{\mathrm{BSC}} \equiv \bar{g}^{\mathrm{BSC}}$ as defined in Appendix [] We prove that for $n$ sufficiently large and for all $\delta \in\left[-\sqrt{n^{-1} \ln n}, \sqrt{n^{-1} \ln n}\right]$, the function $g_{p, \rho+\delta}^{\mathrm{BSC}}$ is still non-positive on $D(\rho)$.
First notice that the upper bound $\bar{g}_{p, \rho}^{\mathrm{BSC}}$ does not depend on $\rho$. Using the same argument as in Appendix [ there exists a neighborhood $U$ of $(0,0,0)$ such that for all type $\left(x_{0}, x_{c}, \underline{y}\right) \in U \cap D(\rho+\delta)$ and for all $\delta \in$ $\left[-\sqrt{n^{-1} \ln n}, \sqrt{n^{-1} \ln n}\right]$

$$
\begin{align*}
g_{p, \rho+\delta}^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, 0\right) & \leq \bar{g}_{p, \rho}^{\mathrm{BSC}}(X) \\
& \leq 0 \tag{70}
\end{align*}
$$

It remains to show that the variation of $g^{\mathrm{BSC}}$ on
$D(\rho+\delta) \backslash U$ is bounded. Let us make the change of variables $\left(x_{0}, x_{c}\right) \rightarrow\left(X, x_{c}\right)$ and $g_{p, \rho+\delta}^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, 0\right) \rightarrow$ $g_{p, \rho+\delta}^{\mathrm{BSC}}\left(X, x_{c}, \underline{y}, 0\right)$. The following inequality holds

$$
\begin{align*}
g_{p, \rho+\delta}^{\mathrm{BSC}}\left(X, x_{c}, \underline{y}, 0\right) \leq & 2 \sqrt{\frac{\ln n}{n}}\left(\ln 2+\ln \left(\frac{1-p}{p}\right)\right) \\
& +g_{p, \rho}^{\mathrm{BSC}}\left(X, x_{c}, \underline{y}, 0\right) \tag{71}
\end{align*}
$$

Hence we can bound the maximum of $g^{\mathrm{BSC}}$ on $D(\rho+\delta) \backslash U$ by

$$
\left.\left.\begin{array}{rl}
\max _{\left(X, x_{c}, \underline{y}\right) \in D(\rho+\delta) \backslash U} & g_{p, \rho+\delta}^{\mathrm{BSC}}\left(X, x_{c}, \underline{y}, 0\right)
\end{array}\right) \leq \begin{array}{ll}
\max _{\left(X, x_{c}, \underline{y}\right) \in D(\rho) \backslash U} & g_{p, \rho}^{\mathrm{BSC}}\left(X, x_{c}, \underline{y}, 0\right)
\end{array}\right)+c \sqrt{\frac{\ln n}{n}}(72) .
$$

The maximum of $g_{p, \rho}^{\mathrm{BSC}}\left(X, x_{c}, \underline{y}, 0\right)$ on $D(\rho+\delta) \backslash U$ is by hypothesis negative (see Equation 681). Therefore for $n$ sufficiently large we have that for all $\delta \in$ $\left[-\sqrt{n^{-1} \ln n}, \sqrt{n^{-1} \ln n}\right]$

$$
\begin{equation*}
\max _{\left(x_{0}, x_{c}, \underline{y}\right) \in D(\rho+\delta) \backslash U} g_{p, \rho+\delta}^{\mathrm{BSC}}\left(x_{0}, x_{c}, \underline{y}, 0\right) \leq 0 \tag{73}
\end{equation*}
$$

which concludes the proof.

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[^1]:    ${ }^{1}$ The bit-sampling decoder assigns random values to decoded bits based on their posterior marginal distribution.

[^2]:    ${ }^{2}$ As we use concepts from statistical physics it is more convenient to employ the binary alphabet $\{-1,1\}$ instead of the traditional $\{0,1\}$.
    ${ }^{3}$ In the binary alphabet $\{-1,1\}$, the all-zero codeword is the sequence $\{1, \ldots, 1\}$.

[^3]:    ${ }^{4}$ The binary entropy $h_{2}(p):=-(1-p) \ln (1-p)-p \ln p$ is computed in nat.

[^4]:    ${ }^{5}$ McKay's bound in its original form is only applicable for subgraphs of size less than $n-4 r^{2}$. We refer to [17] for a careful analysis.

