Asymptotically Optimal Distributed Channel Allocation: a Competitive Game-Theoretic Approach

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***Abstract*—In this paper we consider the problem of distributed channel allocation in large networks under the frequency-selective interference channel. Performance is measured by the weighted sum of achievable rates. First we present a natural non-cooperative game theoretic formulation for this problem. It is shown that, when interference is sufficiently strong, this game has a pure price of anarchy approaching infinity with high probability, and there is an asymptotically increasing number of equilibria with the worst performance. Then we propose a novel non-cooperative M Frequency-Selective Interference Game (M-FSIG), where users limit their utility such that it is greater than zero only for their M best channels, and equal for them. We show that the M-FSIG exhibits, with high probability, an increasing number of optimal pure Nash equilibria and no bad equilibria. Consequently, the pure price of anarchy converges to one in probability in any interference regime. In order to exploit these results algorithmically we propose a modified Fictitious Play algorithm that can be implemented distributedly. We carry out simulations that show its fast convergence to the proven pure Nash equilibria.**

I. Introduction

Channel allocation, the problem of assigning frequency bands to users, is a fundamental element in wireless networks. Channel allocation is necessary when channel access is through frequency division techniques such as FDMA or the more recent bandwidth efficient technique OFDMA. Other approaches, based on iterative water filling (IWF, see [1]), allow users to allocate their power over the spectrum as a whole. It is well-known that IWF leads to a FDMA solution for strong interference, and hence is more suitable for weak interference and is generally considered more complex. When splitting the channel into sub-channels, the question of how to assign these sub-channels to users arises. In the frequency-selective interference channel, different users experience different conditions in each channel due to fading and interference, so different allocations will result in varying levels of performance.

At first glance, it may seem that channel allocation is a special case of resource allocation and as such can be solved as an optimization problem. If N is the number of users and resources, the optimal permutation between them can be found with a complexity of $O(N^3)$, using the famous Hungarian Algorithm [2]. The basic problem with this approach is the information required to compute the optimal solution. To do so, some network entity (the base station, access point, etc) needs to know the preferences for all nodes. This entity should compute the optimal solution and transmit it back to the nodes. In a wireless environment, these preferences are not constant so this central knowledge involves significant communication overhead on the network. As networks grow larger, this requirement becomes less reasonable. Furthermore, future networks (like ad-hoc networks and cognitive radio) are envisioned to be more distributed in nature and less dependent on central entities. This leads to the need for a distributed channel allocation algorithm.

Recently, it has been shown that the optimal solution to the channel allocation problem can be achieved using a distributed algorithm [3], [4], [5]. This algorithm is a distributed version of the auction algorithm [6] and relies on a CSMA protocol. Although it has a very slow convergence rate, this result serves as a proof of concept and suggests that other approaches may achieve close to optimal performance in a distributed fashion. In [7], the authors designed an algorithm based on the stable matching concept that also uses a CSMA protocol. This algorithm has a much faster convergence rate and a good sum-rate performance. Due to their dependence on CSMA, both algorithms are vulnerable to the hidden terminal and exposed terminal problems. In order to avoid these problems, a RTS/CTS mechanism has to be implemented. Besides causing delays, RTS/CTS implementation requires some central network entities, and thus negatively impact the network scalability. Additionally, both algorithms have strong user synchronization requirements. Last, but not least, these algorithms ignore the inherent possibility of sharing channels between users.

There has been a considerable amount of work designed to apply game theory as a framework for distributed channel allocation algorithms (see [8], [9]). While game theory addresses the distribution requirement naturally, it does not guarantee good global performance. For example, it is well-known that the fixed points for the IWF algorithm are the Nash equilibrium points of the Gaussian interference game. For a two-user Gaussian interference game, a prisonerâĂŹs dilemma may occur which leads to a suboptimal solution [10]. To overcome this obstacle, some form of cooperation can be added using different game theoretic concepts. In

[11] the authors proposed a potential game theoretic formulation that requires each user to know the interference he causes to other users. In [12] the authors used the Nash bargaining solution and coalitions to enhance the fairness of the allocation at the price of a centralized architecture. In [13] and [14], a more stable algorithm to obtain the Nash bargaining solution was proposed, based on convex optimization techniques. Although cooperation can indeed enhance performance, it may be extremely complicated to achieve cooperative game-theoretic solution concepts in the general case without communication between users, which limits the distributed nature of the network.

The rest of this paper is organized as follows. In section II we formulate our wireless network scenario and present our approach. In section III we present a natural game formulation for this problem and show that it suffers from major drawbacks. In section IV we propose an enhanced game and provide its equilibria analysis. Section V suggests an algorithm each user can implement in a distributed fashion to converge to these equilibria. In section VI we present simulations of our proposed algorithm that show fast convergence to the proven equilibria. Finally, we draw conclusions in section VII.

II. Problem Formulation

Consider a wireless network consisting of N transmitter-receiver pairs (users) and K frequency bands (channels). Each user forms a link between his transmitter and receiver using a single frequency band. The channel between each transmitter and receiver is Gaussian frequency-selective and we assume that each frequency band is smaller than the coherence bandwidth of the channel. We also assume that the coherence time is large enough so that the channel gains can be considered static for a sufficiently long time.

The channel gains (fading coefficients) are modeled as N^2K independent random variables - one for each channel, each transmitter and each receiver. The coefficient between user ns transmitter and user ms receiver in channel k is denoted $h_{n,m,k}$. We also assume that $h_{m,n,1},...,h_{m,n,K}$ are identically distributed for each $m,n=1,...,N$.

Note that NK of these coefficients serve as channel coefficients between a transmitter and receiver pair and are denoted for convenience by $h_{n,k}$ for user n in channel k. The other $NK(N-1)$ coefficients serve as interference coefficients between transmitters and unintended receivers. In this paper we assume $N=K$ for simplicity.

Each user has some preferred order of the K channels. Due to the independence of the channel coefficients between users, these preference lists are different and independent between users. Note that this preference order considers only the absolute value of the channel coefficient and not the interference (which indeed affects the achievable rate). We denote by $h_{n,(N-i+1)}$ the i-th best channel coefficient for user n (so $h_{n,(1)}$ is the worst channel).

We assume that each user has perfect channel state information (CSI) of all his K channel coefficients, which he can achieve using standard estimation techniques. In addition we assume that each user can sense the exact interference he experiences in each channel. Nevertheless, users do not have any knowledge about the channel coefficients of other users or about any of the interference coefficients. There is no central entity of any sort that knows the channel coefficients of all users. Note that, in contrast to [4] and [7], we do not prohibit two or more users in the same channel.

Our global performance metric is the weighted sum of achievable rates while treating interference as noise. Denote by $\mathbf{a}$ the allocation vector (soon to be called the strategy profile), s.t. $a_n=k$ if user n is using channel k. We want to maximize the following performance function over all possible allocations

$$W(\mathbf{a})=\sum_{n=1}^{N} w_n \log_2\left(1+\frac{P_n|h_{n,a_n}|^2}{N_0+I_{n,a_n}\left(\mathbf{a}_{-n}\right)}\right)$$

where N_0 is the Gaussian noise variance which is assumed to be the same for all users, P_n is user ns transmission power and $I_{n,k}\left(\mathbf{a}_{-n}\right)=\sum\limits_{m|a_m=k}|h_{m,n,k}|^2P_m$ is the interference user n experiences in channel k. We assume that the weights satisfy $w_{min}\leq w_n\leq w_{max}$ for some $w_{min},w_{max}>0$, for all n.

We want to find a fully distributed way to achieve close to optimal solutions for our channel allocation problem. Hence we need to analyze the interaction that results from N independent decision makers and ensure that the outcome is desirable. The natural way to tackle this problem is by applying game theory.

Definition 1. A normal-form game is defined as the tuple

$$G=<\mathcal{N},\{A_n\}_{n\in\mathcal{N}},\{u_n\}_{n\in\mathcal{N}}>$$

where $\mathcal{N}$ is the set of players, A_n is the set of pure strategies of player n and $u_n:A_1\times...\times A_N\rightarrow\mathbb{R}$ is the utility function of player n.

Game theory aims at analyzing the possible outcomes of a given interaction using solution concepts. The best known solution concept is the celebrated Nash Equilibrium (NE).

Definition 2. A strategy profile $(a_n^*,\mathbf{a}_{-n}^*)\in A_1\times...\times A_N$ is called a pure Nash equilibrium (PNE) if $u_n(a_n^*,\mathbf{a}_{-n}^*)\geq u_n(a_n,\mathbf{a}_{-\mathbf{n}}^*)$ for all $a_n\in A_n$ and all $n\in\mathcal{N}$.

This means that for each player n, if the other players act according to the equilibrium, player n can not improve his utility with another strategy. A game may exhibit multiple pure NE or none at all.

A more general notion of an equilibrium is the mixed Nash equilibrium, which is a probability assignment on the pure strategies set. It is well known that in any game with a finite number of players and finite strategy spaces, there exists a mixed NE [15]. We choose to avoid the notion of mixed NE due to its lack of practical meaning as a solution for the channel allocation problem.

In our case, the players are users (through their receiver) and the set of strategies for each player is the set of channels. The choice of the utility function is a more delicate issue.

One of our goals in this work is to show that this degree of freedom in the choice of the utility function can be exploited to achieve better global performance without inducing coordination between the users. Thus we distinguish between the global performance metric and the utility function each user aims to maximize, and we view the dynamic of the game solely as an algorithmic tool to converge to the desired steady state point (NE) in terms of global performance.

Unfortunately, not every game formulation has nice equilibria in terms of both tractability and performance. The notion of NE helps us predict the outcome of the resulting interaction between programmed distributed agents. The problem of tuning the dynamics to a desired equilibrium among all existing NE (equilibrium selection) is generally difficult and may require some coordination between the users. For this reason, a game formulation that results in a simple and robust equilibrium is desirable. The cost of this uncertainty on the resulting NE is often measured by the price of anarchy, defined as follows.

Definition 3. The pure price of anarchy (PPoA) of a game $G =< \mathcal{N}, \{A_n\}_{n\in\mathcal{N}}, \{u_n\}_{n\in\mathcal{N}} >$ with the performance function $W : A_1 \times ... \times A_N \to \mathbb{R}$ is $\frac{\max\limits_{\mathbf{a}\in A_1\times...\times A_N} W(\mathbf{a})}{\min\limits_{\mathbf{a}\in E_p} W(\mathbf{a})}$, where E_p is the set of PNE.

It is not hard to think of special cases of interference networks that have bad equilibria or no pure equilibria at all. We are interested in the vast majority of networks as dictated by the fading distribution, especially in large networks. Therefore, our approach is probabilistic and asymptotic in the number of users N (i.e. will produce results in the "with high probability" sense).

III. The Naive Frequency-Selective Interference Game

Given our performance metric, a natural choice for the utility of each user is his achievable rate. This choice makes the weighted sum-rate the weighted social welfare of the game. This means that in this game we do not exploit the degree of freedom when choosing the utility function and hence we call this the "naive game". This naivetÃl' can be interpreted as selfishness of the users and we will show that it may lead to poor global performance.

Definition 4. The Naive Frequency-Selective Interference Game (Naive-FSIG) is a normal-form game with N users as players, where each has $A_n = \{1, 2, ..., K\}$ as a strategy space. The utility function for player n is

$$u_n(\mathbf{a}) = \log_2\left(1 + \frac{P_n |h_{n,a_n}|^2}{N_0 + I_{n,a_n}(\mathbf{a}_{-n})}\right)$$

In this section we analyze the PNE of the Naive-FSIG for strong interference and evaluate the PPoA. Trivially, a user who obtained his best channel without interference cannot improve his utility. On the other hand, a user who is not in his best channel (with the best channel coefficient) cannot necessarily improve his utility if there are users in his more preferable channels. The influence of other users in the channel of user n on his utility is caused by the interference. Consequently, the strength of the interference has a crucial effect on the identity of the NE.

If the interference is strong enough, users in the same channel achieve negligible utility and the interference game becomes a "collision game".

Lemma 5. *If* $\frac{1}{N_0}\min\limits_{k} |h_{n,k}|^2 > \max\limits_{l} \frac{|h_{n,l}|^2}{N_0 + \min\limits_{m}\left(|h_{m,n,l}|^2 P_m\right)}$ *for each* n*, then the set of PNE of the Naive-FSIG is the set of permutations between users and channels, with cardinality* $N!$.

Proof: The inequality condition means that for every strategy profile that is a permutation of users to channels, a user who deviates gets lower utility. Consequently, every permutation is an equilibrium. Conversely, every pure equilibrium must be a permutation because all users prefer an empty channel over a shared one (i.e. with positive interference). ∎

The lemma above implies that in strong enough interference, a PNE of the Naive-FSIG may assign some users a bad channel. The next lemma shows that a bad channel can be asymptotically useless.

Lemma 6. *Assume that* $|h_{n,1}|, ..., |h_{n,N}|$ *are i.i.d for each* n*, with continuous distribution* $F_n(x)$*, s.t.* $F_n(x) > 0$ *for all* $x > 0$*. Let* M_N *be a sequence s.t.* $\lim\limits_{N\to\infty} \frac{M_N}{N} = 0$*. If* $m \le M_N$ *then* $\max\limits_{n} \left|h_{n,(m)}\right| \to 0$ *in probability as* $N \to \infty$*.*

Proof: Let $\varepsilon > 0$. Due to the i.i.d assumption, the number $N_{\varepsilon,n}$ of r.v. from $|h_{n,1}|, ..., |h_{n,N}|$ that are smaller than ε has a binomial distribution with $p_n = \Pr(|h_{n,1}| < \varepsilon)$. We use the Chernoff-Hoeffding Theorem [16] as a tail bound for $\frac{M_N}{N} < p_n$. By the assumption on M_N, $\frac{M_N}{N} < p_n$ holds for all $N > N_1$ for some large enough N_1, and so

$$\Pr(N_{\varepsilon,n} \le M_N) \le \exp\left(-ND\left(\frac{M_N}{N}\|p_n\right)\right)$$

where $D(q\|p) = q\ln\frac{q}{p} + (1-q)\ln\frac{1-q}{1-p}$, and in our case

$$D\left(\frac{M_N}{N}\|p_n\right) = \frac{M_N}{N}\ln\frac{M_N}{Np_n} + \left(1 - \frac{M_N}{N}\right)\ln\frac{1 - \frac{M_N}{N}}{1 - p_n}$$

for which

$$\begin{aligned}\lim_{N\to\infty} D\left(\frac{M_N}{N}\|p_n\right) &= -\lim_{N\to\infty}\frac{\ln\frac{N}{M_N}}{\frac{N}{M_N}} \\ &- \ln p_n \lim_{N\to\infty}\frac{M_N}{N} + \ln\left(\frac{1}{1-p_n}\right)\lim_{N\to\infty}\left(1 - \frac{M_N}{N}\right) \\ &+ \lim_{N\to\infty}\left(1 - \frac{M_N}{N}\right)\ln\left(1 - \frac{M_N}{N}\right) = \ln\left(\frac{1}{1-p_n}\right)\end{aligned}$$

So for large enough N the inequality $D\left(\frac{M_N}{N}\|p_n\right) \ge \ln\left(\frac{1}{1-p_n}\right) - \ln\left(\frac{1}{1-p_n^2}\right)$ holds and hence we get the following upper bound

$$\Pr\left(N_{\varepsilon,n} \le M_N\right) \le \exp\left(-ND\left(\frac{M_N}{N}\|p_n\right)\right) \le \left(1-p_n\right)^N\left(\frac{1}{1-p_n^2}\right)^N = \left(\frac{1}{1+p_n}\right)^N \to 0$$

Clearly, if there are at least M_N successes then the M_N smallest variables among $|h_{n,1}|$, ..., $|h_{n,N}|$ are smaller than ε. Consequently, using the union bound we get

$$\lim_{N\to\infty} \Pr\left(\max_n \left|h_{n,(m)}\right| > \varepsilon\right) = \lim_{N\to\infty} \Pr\left(\bigcup_{n=1}^{N}\left\{\left|h_{n,(m)}\right| > \varepsilon\right\}\right) \le \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{\left(1+p_n\right)^N} = 0$$

for each $m \le M_N$ and each $\varepsilon > 0$, and we reached our conclusion. ■

Since $\frac{1}{N}\sum_n w_n u_n \le w_{max}\max_n u_n$, it follows from the lemma above that the users that are assigned one of their M_N worst channel coefficients have an average utility that converges to zero in probability. To evaluate the performance of the worst PNE of the Naive-FSIG, we need to know how many users can be assigned such a bad channel. Unfortunately, there is an M_N s.t. $\lim_{N\to\infty}\frac{M_N}{N} = 0$, for which there exists a permutation between users and channels such that each user gets one of his M_N worst channel coefficients. Even worse, there are actually many such permutations. Our result is based on the following theorem.

Theorem 7 (Frieze & Melsted [17]). *Let Γ be a bipartite graph chosen uniformly from the set of graphs with bipartition L, R, $|L| = n, |R| = m$ s.t. each vertex of L has degree $d \ge 3$ and each vertex of R has degree at least two. Then with high probability the maximum matching in Γ is with size $min\left\{m,n\right\}$.*

Lemma 8. *Assume that $\{h_{n,m,k}\}$ are independent and $h_{n,1}$, ..., $h_{n,N}$ are identically distributed for each n. Let $\mathcal{M}_n = \left\{k \mid |h_{n,k}| \le \left|h_{n,(M_N)}\right|\right\}$. If $M_N \ge (1+\varepsilon)\ln(N)$ for some $\varepsilon > 0$, then the probability that a perfect matching exists between users and channels s.t. each $n \in \mathcal{N}$ gets a channel from $\mathcal{M}_n$ approaches 1 as $N \to \infty$.*

Proof: The degree of each user node is exactly M_N. We want to bound the probability of the event that there exists a channel with degree zero or one. Due to the i.i.d channel coefficients of each user, the probability that user n is not connected to channel k is given by

$$\Pr\left(k \notin \mathcal{M}_n\right) = \frac{\begin{pmatrix} N-1 \\ M_N \end{pmatrix}}{\begin{pmatrix} N \\ M_N \end{pmatrix}} = \frac{\frac{(N-1)!}{(N-1-M_N)!}}{\frac{N!}{(N-M_N)!}} = 1 - \frac{M_N}{N}$$

Since channel coefficients of different users are independent, the probability that the degree of vertex k is less than two is given by the binomial distribution of the number of users who prefer channel k, with $p = \frac{M_N}{N}$

$$\Pr\left(\deg(k) < 2\right) = \left(1 - \frac{M_N}{N}\right)^N + N\frac{M_N}{N}\left(1 - \frac{M_N}{N}\right)^{N-1}$$

So by the union bound on the channel vertices and for $M_N \ge (1+\varepsilon)\ln(N)$ with some $\varepsilon > 0$ we get, for $\delta > 0$ and large enough N, that

$$\Pr\left(\min_k \deg(k) < 2\right) \le N\left(1-\frac{M_N}{N}\right)^N + NM_N\left(1-\frac{M_N}{N}\right)^{N-1} \le (1+\delta)\left(Ne^{-M_N} + NM_N e^{-M_N}\right) \le (1+\delta)\frac{(1+\varepsilon)\ln(N)+1}{N^{\varepsilon}}$$

which goes to zero as $N \to \infty$. We know from Theorem 7 that given $\min_k\left(\deg(k)\right) \ge 2$, the probability that a perfect matching exists approaches 1 as $N \to \infty$ so by combining these results we obtain our conclusion. ■

The condition $M_N \ge (1+\varepsilon)\ln(N)$ was necessary to ensure that with high probability, no channel node degree is smaller than two. This large user nodes degree has its own major effect on the equilibria as well.

Theorem 9 (Marshall Hall Jr [18, Theorem 2]). *Suppose that $A_1, A_2, ..., A_N$ are the finite sets of desirable resources, i.e. user n desires resource a if and only if $a \in A_n$. If there exists a perfect matching between users and resources and $|A_n| \ge M$ for $n = 1$, ..., N where $M < N$, then the number of perfect matchings is at least $M!$.*

Joining together Theorem 9, Lemma 5, Lemma 6 and Lemma 8 we arrive at the following theorem, by choosing $M_N = N^{\mu}$ for some $\mu < 1$.

Theorem 10. *Assume that $\{h_{n,m,k}\}$ are independent and $|h_{n,1}|$, ..., $|h_{n,N}|$ are identically distributed for each n, with continuous distribution $F_n(x)$, s.t. $F_n(x) > 0$ for all $x > 0$. Also assume that there exist $w_{min}, w_{max} > 0$, s.t. $w_{min} \le w_n \le w_{max}$ for all n. If $\frac{1}{N_0}\min_k |h_{n,k}|^2 > \max_l \frac{|h_{n,l}|^2}{N_0 + \min_m\left(|h_{m,n,l}|^2 P_m\right)}$ holds for each n, then for all $\mu < 1$, there are at least $(N^{\mu})!$ PNE s.t. $\frac{1}{N}\sum_n w_n u_n \to 0$ in probability as $N \to \infty$. Specifically, The PPoA of the Naive-FSIG approaches infinity in probability as $N \to \infty$.*

IV. The M Frequency-Selective Interference Game

In this section, we want to exploit the degrees of freedom in choosing the utility function. Inspired by the hazards demonstrated by the naive game, we propose a new game formulation. The greediness of the users is moderated by an a-priori agreement to limit the utility of each user to be greater than zero only for his M best channels, and equal for them.

Definition 11. The M Frequency-Selective Interference Game (M-FSIG) is a normal-form game with parameter

$M > 0$, N users as players, where each has $A_n = \{1,2,...,K\}$ as a strategy space. The utility function for player n is

$$u_n(\mathbf{a}) = \begin{cases} \log_2\left(1+\frac{P_n|h_{n,(N-M+1)}|^2}{N_0+I_{n,a_n}(\mathbf{a}_{-n})}\right) & \frac{|h_{n,a_n}|}{|h_{n,(N-M+1)}|} \geq 1 \\ 0 & else \end{cases}$$

Define the set of indexes of the M best channel coefficients of user n by $\mathcal{M}_n = \left\{k \,|\, \frac{|h_{n,k}|}{|h_{n,(N-M+1)}|} \geq 1\right\}$. Note that because $|h_{n,(N-M+1)}| > 0$ for each $n \in \mathcal{N}$ with probability 1, user n will never choose a channel outside $\mathcal{M}_n$. Also note that due to the replacement of h_{n,a_n} by $h_{n,(N-M+1)}$ in the utility, maximizing u_n is equivalent to minimizing I_{n,a_n}. Hence in the M-FSIG each user $n \in \mathcal{N}$ accesses only channels in $\mathcal{M}_n$ and prefers those with smaller interference.

The identification of $\mathcal{M}_n$ is superior both in performance and practice over evaluating all the channels that are better than some threshold as was done in [3], [4]. If this threshold is constant with respect to N a significant rate loss may occur due to truncation, because the expected value of the best channel coefficients grows with N. If the threshold is dependent on N, this dependence is dictated by the fading distribution, which is not known to the users.

We will show that the M-FSIG has asymptotically only good PNE in any interference regime. For this reason, in this game, the convergence to some PNE is sufficient to provide good global performance. Furthermore, it only requires each user to track a small number of channels (e.g. $O(\ln N)$ instead of N).

A. Existence of an Asymptotically Optimal Permutation Equilibrium

In this subsection we establish that as $N \to \infty$ the probability that an asymptotically optimal PNE exists approaches 1.

The existence result in Lemma 8 (and Theorem 9) of permutations where each user n gets a channel from $\mathcal{M}_n$ is of course unaffected by how we choose the M members of the set $\mathcal{M}_n$. With $\mathcal{M}_n$ as defined for the M-FSIG we get the following corollary.

Corollary 12. *Assume that $\{h_{n,m,k}\}$ are independent and $h_{n,1}$, ..., $h_{n,N}$ are identically distributed for each n. If the M-FSIG parameter is chosen s.t. $M \geq (1+\varepsilon)\ln(N)$ for some $\varepsilon > 0$, then the probability that the M-FSIG has at least $M!$ PNE that are permutations of users to channels, s.t. user n gets a channel from $\mathcal{M}_n$ for each $n \in \mathcal{N}$, approaches 1 as $N \to \infty$.*

Finally, we state the result that shows that this set of permutation equilibria are indeed asymptotically optimal. This result depends on the statistics of the fading coefficients, and here we choose to present the common case of Rayleigh fading, although the same is true for a broad class of fading distributions. Details will be given in [19].

Theorem 13. *Assume that $\{h_{n,m,k}\}$ are independent and $|h_{n,1}|, ..., |h_{n,N}|$ are i.i.d Rayleigh distributed variables for each n. If $\lim_{N\to\infty} \frac{M}{N^\mu} = 0$ for $\mu < 1$ and $\lim_{N\to\infty} M = \infty$, then there exist at least $M!$ PNE for the M-FSIG for which $\min_{n\in\mathcal{N}} \frac{\log_2\left(1+\frac{P_n}{N_0}|h_{n,a_n^*}|^2\right)}{\log_2\left(1+\frac{P_n}{N_0}|h_{n,(N)}|^2\right)} \to 1$ in probability as $N \to \infty$.*

B. Non-Existence of Bad Equilibria

It turns out that the asymptotically optimal permutation PNE are typical equilibria for this game; in other words all other equilibria are almost a permutation and hence have the same asymptotic performance. This property eases the requirements for the dynamics and allows simpler convergence with good performance. We define a sharing user as a user who is in the same channel with at least one more user.

Theorem 14. *Assume that $\{h_{n,m,k}\}$ are independent and $h_{n,1}$, ..., $h_{n,N}$ are identically distributed for each n. Suppose that $M \geq (1+\varepsilon)\ln(N)$ for some $\varepsilon > 0$. If $\mathbf{a}^*$ is a PNE of the M-FSIG with N_c sharing users, then $\frac{N_c}{N} \to 0$ in probability as $N \to \infty$.*

Proof: The proof is based on the fact that the larger the number of sharing users, the larger the number of empty channels and hence the probability that none of these empty channels is in $\mathcal{M}_n$ for some n decreases with N. Details are omitted due to page constraints and will be given in [19]. ■

The weighted sum-rate of the non-sharing users approaches the optimal one, and almost all users are non-sharing users. Nevertheless, the sharing users do not necessarily suffer from poor conditions - they are in their minimal interference channel out of an increasing (with N) amount of good channels.

This result, aided by the existence of an asymptotically good permutation PNE from the last section, leads to the following satisfactory property of the M-FSIG that holds for any interference regime.

Corollary 15. *Assume that $\{h_{n,m,k}\}$ are independent and $|h_{n,1}|, ..., |h_{n,N}|$ are i.i.d Rayleigh r.v. for each n, and that $w_{min} \leq w_n \leq w_{max}$ with some $w_{min}, w_{max} > 0$, for each n. If $M = (1+\varepsilon)\ln(N)$ for some $\varepsilon > 0$, then each PNE $\mathbf{a}^*$satisfies*

$$\frac{\sum_{n=1}^N w_n \log_2\left(1+\frac{P_n\left|h_{n,a_n^*}\right|^2}{N_0+I_{n,a_n^*}}\right)}{\sum_{n=1}^N w_n \log_2\left(1+\frac{P_n}{N_0}\left|h_{n,(N)}\right|^2\right)} \to 1$$

in probability as $N \to \infty$. Consequently, the PPoA of the M-FSIG converges to 1 in probability as $N \to \infty$.

Note that because for $M = (1+\varepsilon)\ln(N)$ some of the PNE of the M-FSIG are permutations, the above corollary suggests that the ratio of the weighted sum-rate of the optimal permutation to $\sum_{n=1}^N w_n \log_2\left(1+\frac{P_n}{N_0}\left|h_{n,(N)}\right|^2\right)$ converges to 1 in probability as $N \to \infty$.

V. Modified Fictitious Play

In order to apply game theory algorithmically, an equilibrium analysis, although necessary, is not enough. A learning algorithm that each user can implement that leads to convergence to equilibrium is a crucial element. It should be emphasized that the performance of such an algorithm is already known from the equilibrium analysis. Therefore the algorithm that we are looking for is not tailored to our specific problem but rather has general properties of convergence to NE. One of the best known candidates for this task is the Fictitious Play (FP) algorithm [20]. In FP, each player keeps the empirical mean vectors of the frequencies each other player has played his actions, and plays his best response to this fictitious strategy. Alternatively a player can keep one empirical mean vector of the frequencies each strategy profile of its rivals has been played (joint strategy fictitious play). The simple relation of FP convergence to NE is summarized in the following proposition.

Proposition 16 ([21])**.** *Let N players play according to the FP algorithm.*

1) *If a PNE is attained at t_0 it will be played for all $t > t_0$ and the frequency vectors will converge to it.*
2) *If FP frequency vectors converge, they must converge to some NE (maybe mixed).*
3) *If $\mathbf{a} \in A_1 \times ... \times A_N$ is played for every $t > t_1$ then $\mathbf{a}$ is a PNE.*

Although its strong connection to NE, FP is not guaranteed to converge at all. Convergence has been proven only for some special games that do not include our game. Even if FP converges, it may be to a mixed NE and this is undesirable as was mentioned above. Furthermore, a common problem with implementing FP is the information it requires. In a wireless network, not only does a user have hardly any information about the previous action of each other user, but he also barely knows how many users there are. Fortunately, in our game the effect of the other users on the utility can be measured by measuring the interference.

To adjust the FP to the wireless environment we propose to modify it such that each user keeps track of a fictitious utility vector instead of the empirical mean vector of the rivals strategy profiles. We denote the fictitious utility for user n in channel k at time t by $\overline{U}_{n,k}(t)$. The fictitious utility is updated according to the following rule

$$\overline{U}_{n,k}(t) = (1-\alpha)\overline{U}_{n,k}(t-1) + \alpha u_{n,k}(t)$$

with some step size $0 < \alpha \leq 1$. To prevent mixed NE we suggest a constant step size instead of the common $\alpha = \frac{1}{t+1}$ that makes $\overline{U}_{n,k}$ the empirical mean utility. Note that $\alpha = 1$ corresponds to the best-response dynamics.

Additionally, we provide a simple mechanism to improve the chances of convergence to a PNE. The strategy profile determines the interference, but knowing the interference will not reveal the strategy profile. Nevertheless, the continuity of the random channel gains suggests that for each user, the interference vector is different for different strategy profiles with probability 1. Hence users can detect that two strategy profiles are different based on their measured interference. If a PNE is reached it is played repeatedly, so we can exploit this fact and let the users check for convergence to a PNE after enough time, and set their fictitious utility to zero if a PNE has not been reached.

The Modified FP is described in detail in the Algorithm 1 table, and its properties are summarized in the next proposition.

Proposition 17. *Let N players play according to the Modified FP algorithm.*

1) *If $\alpha = \frac{1}{t+1}$, then the dynamics of the joint strategy FP where each player has perfect information are identical to those of the Modified FP.*
2) *Assume a constant α. If a PNE is attained at t_0 then it will be played for all $t > t_0$ and if the fictitious utility vectors converge, then the resulting strategy profile is a PNE.*

Proof: Assume we are at turn $t = T$ and define $p_i = \sum_{t=1}^{T} \frac{I(\mathbf{a}_{-n}(t)=\mathbf{a}_{i,-n})}{T}$ for the rivals strategy profile $\mathbf{a}_{i,-n}$, where I is the indicator function. For $\alpha = \frac{1}{t+1}$ the equivalence of the algorithms follows immediately from the identity

$$\sum_i p_i u_n(a_n, \mathbf{a}_{i,-n}) = \sum_i p_i \log_2\left(1 + \frac{P_n |h_{n,(N-M+1)}|^2}{N_0 + I_{n,a_n}(\mathbf{a}_{i,-n})}\right) = \frac{1}{T}\sum_{t=1}^{T} \log_2\left(1 + \frac{P_n |h_{n,(N-M+1)}|^2}{N_0 + I_{n,a_n}\left(\mathbf{a}_{-n}\left(t\right)\right)}\right)$$

because $\sum_i p_i u_n(a_n, \mathbf{a}_{i,-n})$ is the mean empirical utility for a_n according to the fictitious rivals profile. Consider next the case of a constant α. If a PNE $\mathbf{a}^*$ is attained at t_0 then $a_n^*(t_0) = \arg\max_k u_{n,k}(t_0)$ and $a_n^*(t_0) = \arg\max_k \overline{U}_{n,k}(t_0 - 1)$ for each $n \in \mathcal{N}$. Considering the update rule and because $a_n^*(t_0) = \arg\max_k \overline{U}_{n,k}(t_0-1) = \arg\max_k u_{n,k}(t_0)$[1] we get

$$\arg\max_k (1-\alpha)\overline{U}_{n,k}(t_0-1) + \arg\max_k \alpha u_{n,k}(t_0) = \arg\max_k \overline{U}_{n,k}(t) = a_n^*(t_0+1)$$

and so on, for each $t > t_0$. If the fictitious utility vectors converge, then $\lim_{t\to\infty} \overline{U}_{n,k}(t)$ exists and is finite for each k and n. From the update rule we get $\alpha \lim_{t\to\infty} \overline{U}_{n,k}(t) = \alpha \lim_{t\to\infty} u_{n,k}(t)$ for each n, k which means $\lim_{t\to\infty} \overline{U}_{n,k}(t) = \lim_{t\to\infty} u_{n,k}(t)$ for constant α. Consequently, for all $t > t_1$ for some large enough t_1, $a_n(t) = \arg\max_k \overline{U}_{n,k}(t) = \arg\max_k u_{n,k}(t)$ for each $n \in \mathcal{N}$ and hence $\mathbf{a}$ is a PNE. ■

In the next section we show that in our case there is indeed convergence to PNE, and a very fast one.

[1]For the proof it is enough to break ties in $\overline{U}_{n,k}(t)$ by choosing the previous action if it is maximal, otherwise break ties arbitrarily. For $t = 0$ step (d) of the Modified FP suggests breaking ties at random.

Algorithm 1 Modified Fictitious Play

1) **Initialization** - Choose some $0 < \alpha \leq 1$ and $\tau > 0$. Each user initializes his fictitious utility - $\bar{U}_{n,k}(0) = 0$ for each $k \in \mathcal{M}_n$, where $\mathcal{M}_n$ is the set of his M best channels (interference free).

2) **For t=1,...T and for each user n=1,...N do**

   a) Choose a transmission channel

   $$a_n(t) = \arg\max_k \overline{U}_{n,k}(t-1)$$

   b) Sense the interference. For each $k \in \mathcal{M}_n$

   $$I_{n,k}(t) = \sum_{m|a_m(t)=k} |h_{m,n,k}|^2 P_m$$

   c) Update fictitious utilities. For each $k \in \mathcal{M}_n$

   $$\overline{U}_{n,k}(t) = (1-\alpha)\overline{U}_{n,k}(t-1) + \alpha u_{n,k}(t)$$

   where

   $$u_{n,k}(t) = \log_2\left(1 + \frac{P_n |h_{n,(N-M+1)}|^2}{N_0 + I_{n,k}(t)}\right)$$

   d) (optional) Check for convergence to a PNE. If $t = \tau$ and $I_{n,k}(t) \neq I_{n,k}(t-1)$ for some $k \in A_n$ then return to step 1, i.e. $t = 0$.

VI. Simulation Results

Our analysis is probabilistic and asymptotic with the number of users. Thus, we carried out some simulations to ensure that the asymptotic effects take place for reasonable N values. In fact, the situation for some finite N tends to be much more optimistic than the lower bounds we used in most of our proofs.

In our simulations we used a Rayleigh fading network; i.e. $\{|h_{m,n,k}|\}$ are i.i.d Rayleigh random variables. Hence $\left\{|h_{m,n,k}|^2\right\}$ are i.i.d exponential random variables with parameter λ, which is chosen to be $\lambda = 1$ so all the exponential variables have unit variance. Unless specified otherwise, the transmission powers were chosen such that the mean SNR for each link, in the absence of interference, is 20[dB]. Users play according to the Modified FP algorithm from last section including step (d) with $\tau = 60$ and $\alpha = 0.5$.

In Fig. 1 we present the convergence of the Modified FP in two different network realizations, for $N = K = 100$. We can see that convergence is very fast and occurs within 100 iterations. The upper figure is for $M = 9$, where the ratio of the sum of achievable rates to that of an optimal allocation is close to 1, and the ratio of the minimal achievable rate is a bit smaller. This corresponds to a convergence to one of the permutation PNE; hence, there are no sharing users. The lower figure shows another realization for both $M = 7, 14$. The sum-rate ratio to optimal is still close to 1, as predicted by the converging PPoA, but the minimal rate is significantly lower. The minimal rate is experienced by one of the four sharing users in this case. Nevertheless, choosing $M = 14$ results in a negligible reduction in the sum-rate but significantly improves the minimum rate due to the convergence to a permutation with no sharing users. We can also see the benefits of step (d) of the Modified FP, which leads to a detection of non-convergence by the users at $t = 60$. This indeed results in a convergence to a PNE after the users selected initial channels at random again.

In Fig. 2 we show the effect of the number of users on the rates, with $M = \lceil 3\ln(N) \rceil$. We present the average and minimal achievable rates, compared to the sum-rate optimal permutation allocation and random permutation allocation average and minimal achievable rates. The benefit over a random permutation is significant, especially in terms of the minimal rate. The rates increase is due to the growing expected value of the best channel coefficients for each user. This phenomenon (multi-user diversity) of course does not take place for a random permutation. In a random permutation the average user gets his median channel coefficient, and the minimal allocated channel coefficient has a decreasing expectation. The standard deviations of the mean rates are small as expected from the similarity of all NE, and the standard deviations of the minimal rate are higher due to changing number of sharing users between different realizations.

In Fig. 3 we show the effect of the mean SNR on the gain of our algorithm over a random permutation allocation, compared to the optimal permutation allocation. This simulation is for $N = 200$ and $M = \lceil 3\ln(N) \rceil$. As the mean SNR grows larger, the logarithmic behavior of the achievable rate causes the rate difference between two given channel coefficients to be smaller. At the same time, the rate difference between an occupied channel and a free one grows and hence the number of sharing users drops (7.68 at $SNR = -10[dB]$ and 1.2 at $SNR = 25[dB]$).

VII. Conclusion

In this paper we analyzed, using asymptotic probabilistic tools, two game formulations for the distributed channel allocation problem in the frequency-selective interference channel. The performance metric was the weighted sum of achievable rates when treating interference as noise.

First we presented a naive non-cooperative game (Naive-FSIG) and showed that with strong enough interference it has $\Omega\left((N^{\mu})!\right)$, for all $\mu < 1$, bad pure NE, where N is the number of users.

We then proposed an enhanced non-cooperative game formulation (M-FSIG) based on an a-priori agreement between users to limit their utility to be greater than zero only for their M best channels, with the same value for those channels. We proved that for many fading distributions (including Rayleigh fading), our game has a pure price of anarchy that approaches 1 as $N \to \infty$ in any interference regime. For some fixed N the introduced parameter M can be chosen to compromise between sum-rate and fairness. This game enables a fully distributed implementation that achieves close to optimal performance without resorting to coordinated solutions.

Due to the almost completely orthogonal transmissions in equilibria our allocation algorithm is more suitable for the

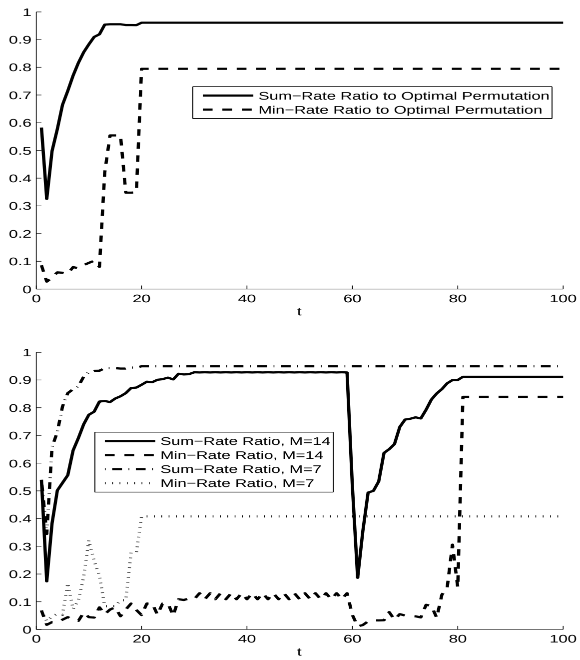

Figure 1. Sum-rate and min-rate compared to the optimal permutation allocation sum-rate for two different realizations

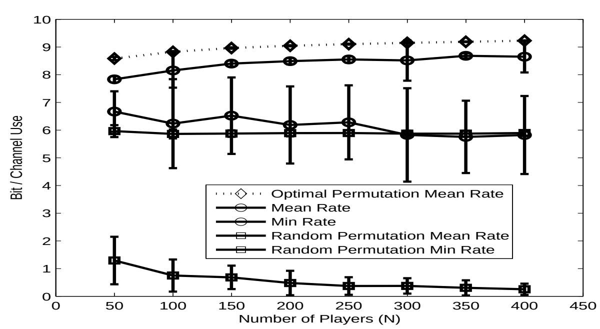

Figure 2. Rates as a function of N averaged over 50 realizations

medium-strong interference regime.

We also proposed a modified Fictitious Play algorithm and showed through simulations that it converges very fast to the proven pure NE. The fast convergence enables frequent runs of the algorithm in the network, which results in maintaining the multi-user diversity in a dynamic fading environment.

The simplicity and high performance of our algorithm make it an appealing base for a dynamic channel access protocol for distributed networks.

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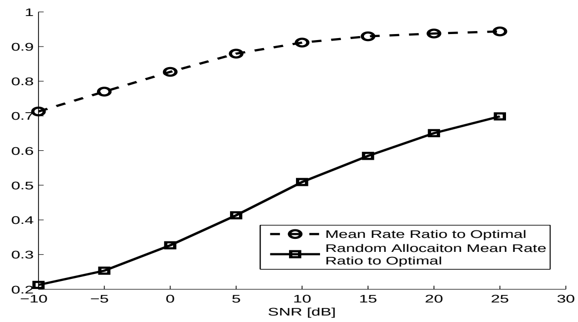

Figure 3. Mean rates as a function of the mean SNR, averaged over 50 realizations

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