# Source-Channel Secrecy for Shannon Cipher System 

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#### Abstract

Recently, a secrecy measure based on listreconstruction has been proposed [2], in which a wiretapper is allowed to produce a list of $2^{m R_{L}}$ reconstruction sequences and the secrecy is measured by the minimum distortion over the entire list. In this paper, we show that this list secrecy problem is equivalent to the one with secrecy measured by a new quantity lossy-equivocation, which is proven to be the minimum optimistic 1 -achievable source coding rate (the minimum coding rate needed to reconstruct the source within target distortion with positive probability for infinitely many blocklengths) of the source with the wiretapped signal as two-sided information, and also can be seen as a lossy extension of conventional equivocation. Upon this (or list) secrecy measure, we study source-channel secrecy problem in the discrete memoryless Shannon cipher system with noisy wiretap channel. Two inner bounds and an outer bound on the achievable region of secret key rate, list rate, wiretapper distortion, and distortion of legitimate user are given. The inner bounds are derived by using uncoded scheme and (operationally) separate scheme, respectively. Thanks to the equivalence between lossy-equivocation secrecy and list secrecy, information spectrum method is leveraged to prove the outer bound. As special cases, the admissible region for the case of degraded wiretap channel or lossless communication for legitimate user has been characterized completely. For both these two cases, separate scheme is proven to be optimal. Interestingly, however, separation indeed suffers performance loss for other certain cases. Besides, we also extend our results to characterize the achievable region for Gaussian communication case. As a side product optimistic lossy source coding has also been addressed.


Index Terms-Shannon cipher system, source-channel secrecy, lossy-equivocation, wiretap channel, optimistic coding, information spectrum method.

## I. Introduction

Shannon cipher system (the one with noisy channel depicted in Fig. 1] was first investigated by Shannon [3], in which a sender A communicates with a legitimate receiver B secretly by exploiting a secret key that is shared by them. Shannon showed that the perfect secrecy for this system is achievable if and only if the rate of secret key is not smaller than the entropy of the source. However in practice, the amount of key

[^0]may be insufficient, and the wiretapper might only want to reconstruct a lossy version of the source. Recently, Schieler et al. [2] proposed a distortion-based secrecy measure around the assumption that the wiretapper has ability to conduct list decoding with fixed list size, and the induced distortion is set to the minimum distortion over the entire list. They studied it in the Shannon cipher system with noiseless channel, and characterized the optimal tradeoff of secret key rate, list rate, wiretapper distortion, and distortion of legitimate user. For this secrecy measure, the wiretapper can be seen as a "guessing wiretapper" who produces a list of guesses to reconstruct the source within target distortion (suppose some testing mechanism is available at the wiretapper). From the perspective of computational security, the list size indicates the computational complexity that the wiretapper needs to reconstruct the source within target distortion (different from [4] the number of guesses for list secrecy is fixed during the guessing process). On the other hand, from the perspective of information-theoretic security, the minimum list rate also indicates the minimum additional information rate (received from an omniscient helper, named henchman) for the wiretapper to reconstruct the source within target distortion [2]. Besides, this secrecy measure is also compatible with the conventional equivocation, and actually it can be characterized by lossy-equivocation, a lossy extension of the conventional equivocation (this point will be shown in this paper). Furthermore, this kind of measure could apply to any secrecy system (not restricted to lossy communication system or the secrecy system with testing mechanism at the wiretapper), such as secure multimedia communication, communication of personal verification information (password or bank account), and communication of any other classified database that consists of sensitive information. This is because on one hand, the results in [2] still hold for lossless communication case; and on the other hand, the interpretation from the perspective of information-theoretic security is always valid for any secrecy system.
The secrecy measure in [2] is defined in sense of strong secrecy, which requires that there exists no sequence of codes achieving target distortion for all sufficiently large blocklengths. It implies the supremum of achievable list rate equals the infimum of all $R$ for which there exists a sequence of $R$-rate codes achieving target distortion for infinitely many blocklengths. This kind of codes is related to optimistic coding, which requires that good codes exist for infinitely many blocklengths as opposed to for all sufficiently large blocklengths, required by the (pessimistic) source coding or channel coding. Optimistic source coding and optimistic channel coding are
originated from the study of the source-channel separation theorem and strong converse for general sources and channels [5], [6], and then systematically investigated by Chen et al. as a separate problem [7]. Recently, Tan et al. [18] applied the optimistic coding to the secure communication over wiretap channels, and studied the optimistic capacity of legitimate user under given secrecy constraint. However, in this paper, it is used to study the secrecy performance, instead of the communication performance of legitimate user.
Furthermore, some researchers investigated source-channel secrecy problem. Yamamoto [10] studied it in Shannon cipher system with secrecy measured by the minimum distortion that incurs in reconstructing the source for a wiretapper. A different formulation of the problem was considered in [12], where the authors assumed there is a fixed information leakage to the wiretapper and wish to minimize the distortion at the legitimate receiver, while at the same time providing a graceful distortion degradation when there is an SNR (Signal Noise Ratio) mismatch. They showed that, for a positive leakage, this can be achieved by combining vector quantization and scaling together. This scenario was extended to consider side information at the receiver in [13] or side information at the sender in [14]. Besides, joint source-channel secrecy with noncausal side information available at both the legitimate user and the eavesdropper has been studied in [15], and with causal source disclosure at the eavesdropper has been studied in [16].

In this paper, the list secrecy measure proposed in [2] is applied to the source-channel secrecy problem for Shannon cipher system with a discrete memoryless source transmitted over a discrete memoryless noisy wiretap channel (see Fig. 11, and then we investigate the the achievable region of secret key rate, list rate, wiretapper distortion, and distortion of legitimate user for this system. The secrecy of the system is obtained by exploiting both the secret key and the wiretap channel. By information spectrum analysis, we find that this problem (or an equivalent problem, henchman problem) is equivalent to the one with secrecy measured by a new quantity, lossyequivocation, which is defined as a conditional rate-distortion function obtained by extending conventional equivocation (conditional entropy) to the lossy case. From perspective of optimistic source coding, the lossy-equivocation is proven to be the minimum optimistic 1 -achievable source coding rate (the minimum coding rate to achieve target distortion with positive probability for infinitely many blocklengths) of the source with the wiretapped signal as two-sided information. Thanks to the equivalence between lossy-equivocation secrecy and list secrecy, it enables us to leverage the information spectrum method to analyze these problems and obtain a converse result. Note that this proof method is different from the one used in [2] which mainly relies on the method of types. Besides, two achievability schemes, uncoded scheme and (operationally) separate scheme, are analyzed. When specialized to lossless communication case or degraded wiretap channel case, the separate scheme is proven to be optimal. Hence for these cases, the admissible regions are characterized completely. Interestingly, however, we observe that separate scheme indeed loses the optimality for other certain cases. This implies separation is not optimal in general for the source-
channel secrecy problem.
Besides, we extend our result to characterize the achievable region for the Gaussian communication case. Since the standard discretization technique, usually used in proving the achievability for the continuous source or continuous channel, is invalid in bounding the probability of excess distortion, some other techniques including d -tilted information, weak typicality, and specified discretization, are exploited in our proof.

In our work, optimistic source coding plays a key role in building a bridge between the list secrecy problem and the lossy-equivocation secrecy problem. Optimistic lossless source coding was investigated by Chen et al. [7]. As an extension to lossy case, optimistic lossy source coding has been addressed in this paper as a side product.

The rest of this paper is organized as follows. Section II summarizes basic notations, preliminaries, and formulation of the problem. Section III and Section IV give the main results for the discrete memoryless systems of lossless communication and lossy communication, respectively. Section V extends the results to quadratic Gaussian communication scenario. Finally, Section VI gives the concluding remarks.

## II. Problem Formulation and Preliminaries

## A. Notation and Preliminaries

We use $P_{X}(x)$ to denote the probability distribution of random variable $X$, which is also shortly denoted as $P_{X}$ or $P(x)$. We also use $P_{X}$ and $Q_{X}$ to denote different probability distribution with common alphabet $\mathcal{X}$.
The total variation distance between two probability measures $P$ and $Q$ with common alphabet is defined by

$$
\begin{equation*}
\|P-Q\|_{T V} \triangleq \sup _{A \in \mathcal{F}}|P(A)-Q(A)| \tag{1}
\end{equation*}
$$

where $\mathcal{F}$ is the $\sigma$-algebra of the probability space. The following properties of total variation distance hold.

Property 1. $[2]$ Total variation distance satisfies:

1) If the support of $P$ and $Q$ is a countable set $\mathcal{X}$, then

$$
\begin{equation*}
\|P-Q\|_{T V}=\frac{1}{2} \sum_{x \in \mathcal{X}}|P(\{x\})-Q(\{x\})| . \tag{2}
\end{equation*}
$$

2) Let $\varepsilon>0$ and let $f(x)$ be a function with bounded range of width $b>0$. Then

$$
\begin{equation*}
\|P-Q\|_{T V}<\varepsilon \Longrightarrow\left|\mathbb{E}_{P} f(X)-\mathbb{E}_{Q} f(X)\right|<\varepsilon b \tag{3}
\end{equation*}
$$

where $\mathbb{E}_{P}$ indicates that the expectation is taken with respect to the distribution $P$.
3) Let $P_{X} P_{Y \mid X}$ and $Q_{X} P_{Y \mid X}$ be two joint distributions with common channel $P_{Y \mid X}$. Then

$$
\begin{equation*}
\left\|P_{X} P_{Y \mid X}-Q_{X} P_{Y \mid X}\right\|_{T V}=\left\|P_{X}-Q_{X}\right\|_{T V} \tag{4}
\end{equation*}
$$

Information spectrum analysis [11] will be used frequently in this paper. For a general sequence of random variables $\mathbf{B}=$ $\left\{B_{n}\right\}_{n \in \mathbb{N}}^{11}$ define

$$
\begin{equation*}
\mathrm{p}-\liminf _{n \rightarrow \infty} B_{n} \triangleq \sup \left\{r: \lim _{n \rightarrow \infty} \mathbb{P}\left(B_{n}<r\right)=0\right\} \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{p}-\limsup _{n \rightarrow \infty} B_{n} \triangleq \inf \left\{r: \lim _{n \rightarrow \infty} \mathbb{P}\left(B_{n}>r\right)=0\right\} \tag{6}
\end{equation*}
$$

For any general pair of random variables ( $\mathbf{U}, \mathbf{V}$ ) with joint distribution $P_{\mathbf{U V}} \triangleq\left\{P_{U^{n} V^{n}}\right\}_{n \in \mathbb{N}}$, define, for each $n$, the normalized information density random variables ${ }^{2}$

$$
\begin{equation*}
\imath_{n}\left(U^{n} ; V^{n}\right) \triangleq \frac{1}{n} \log \frac{P_{V^{n} \mid U^{n}}\left(V^{n} \mid U^{n}\right)}{P_{V^{n}}\left(V^{n}\right)} \tag{7}
\end{equation*}
$$

Given $\left\{\imath_{n}\left(U^{n} ; V^{n}\right)\right\}_{n \in \mathbb{N}}$, we may now define

$$
\begin{align*}
& \underline{I}(\mathbf{U} ; \mathbf{V}) \triangleq \mathrm{p}-\liminf _{n \rightarrow \infty} \imath_{n}\left(U^{n} ; V^{n}\right)  \tag{8}\\
& \bar{I}(\mathbf{U} ; \mathbf{V}) \triangleq \mathrm{p}-\limsup _{n \rightarrow \infty} \imath_{n}\left(U^{n} ; V^{n}\right) \tag{9}
\end{align*}
$$

In information spectrum analysis, $\underline{I}(\mathbf{U} ; \mathbf{V})$ and $\bar{I}(\mathbf{U} ; \mathbf{V})$ are termed the spectral inf- and sup-mutual information rates respectively. They are respectively the p-liminf and p-limsup of the sequence of random variables $\left\{\imath_{n}\left(U^{n} ; V^{n}\right)\right\}_{n \in \mathbb{N}}$. As special cases, spectral inf- and sup-entropy rates are defined as

$$
\begin{align*}
& \underline{H}(\mathbf{U}) \triangleq \mathrm{p}-\liminf _{n \rightarrow \infty} \imath_{n}\left(U^{n} ; U^{n}\right)  \tag{10}\\
& \bar{H}(\mathbf{U}) \triangleq \mathrm{p}-\limsup _{n \rightarrow \infty} \imath_{n}\left(U^{n} ; U^{n}\right) . \tag{11}
\end{align*}
$$

The conditional versions of these quantities can be defined similarly, which will be used in the latter sections as well. Furthermore, for any sequence of distortion functions $d_{n}\left(u^{n}, v^{n}\right)$, define

$$
\begin{align*}
& \underline{D}(\mathbf{U}, \mathbf{V}) \triangleq \mathrm{p}-\liminf _{n \rightarrow \infty} d_{n}\left(U^{n}, V^{n}\right)  \tag{12}\\
& \bar{D}(\mathbf{U}, \mathbf{V}) \triangleq \mathrm{p}-\limsup _{n \rightarrow \infty} d_{n}\left(U^{n}, V^{n}\right) . \tag{13}
\end{align*}
$$

## B. List Secrecy

Consider Shannon cipher system with noisy wiretap channel shown in Fig. 1. where a sender A and a legitimate receiver B share a secret key $K$ that is uniformly distributed over $\left.\left[2^{m R_{K}}\right]\right]^{3}$ and independent of a source $S^{m}$. The sender A observes the discrete memoryless (DM) source sequence $S^{m}$ with each element i.i.d. (independent and identically distributed) according to $P_{S}$, and then transmits it to the legitimate user B over a DM wiretap channel $P_{Y Z \mid X}$ confidentially by utilizing the secret key and the wiretap channel. Finally, the legitimate user B produces a source reconstruction $\hat{S}^{m}$ using the received sequence $Y^{n}$ and the secret key $K$.

[^1]

Fig. 1. Shannon cipher system with noisy wiretap channel.

Definition 1. An $\left(m, n, R_{K}\right)$ block code consists of

1) Stochastic encoder: $P_{X^{n} \mid S^{m} K}: \mathcal{S}^{m} \times\left[2^{m R_{K}}\right] \mapsto \mathcal{X}^{n}$;
2) Decoder: $\psi: \mathcal{Y}^{n} \times\left[2^{m R_{K}}\right] \mapsto \hat{\mathcal{S}}^{m}$.

Assume the source sample rate is $B_{s}$ and the channel sample rate is $B_{c}$. Define the bandwidth mismatch factor as $\gamma \triangleq \frac{B_{c}}{B_{s}}$. Then for any ( $m, n, R_{K}$ ) block code that can be implemented in the system with bandwidth mismatch factor $\gamma$, it must hold that $\frac{n}{m} \leq \gamma$. Note that any $\left(m, n^{\prime}, R_{K}\right)$ code with $\frac{n^{\prime}}{m} \leq \gamma$ can be seen as a special case of $\left(m, n, R_{K}\right)$ codes with $n=\lfloor m \gamma\rfloor^{5}$ in which for each block, only $n^{\prime}$ channel symbols are used. Hence for the system with bandwidth mismatch factor $\gamma$, we only need consider ( $m, n, R_{K}$ ) codes with $n=\lfloor m \gamma\rfloor$.

Another output $Z^{n}$ of the channel is accessed by a wiretapper Eve. Based on $Z^{n}$, the wiretapper produces a list $\mathcal{L}\left(Z^{n}\right) \subseteq \breve{\mathcal{S}}^{m}$ and the induced distortion is set to the minimum one over the entire list, i.e., $\min _{\breve{s}^{m} \in \mathcal{L}\left(Z^{n}\right)} d_{E}\left(S^{m}, \check{s}^{m}\right)$, where $d_{E}\left(s^{m}, \check{s}^{m}\right) \triangleq \frac{1}{m} \sum_{i=1}^{m} d_{E}\left(s_{i}, \check{s}_{i}\right)$ is a distortion measure for the wiretapper. For given distortion levels $D_{B}$ and $D_{E}$, by exploiting the secret key and the wiretap channel, Nodes A and B want to communicate the source within distortion $D_{B}$, while ensuring that the wiretapper's strategy always suffers distortion above $D_{E}$ with high probability.
Definition 2. The tuple $\left(R_{K}, R_{L}, D_{B}, D_{E}\right)$ is achievable if there exists a sequence of $\left(m, n, R_{K}\right)$ codes with $n=\lfloor m \gamma\rfloor$ such that $\forall \delta>0$,

1) Distortion constraint:

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{P}\left[d_{B}\left(S^{m}, \hat{S}^{m}\right) \leq D_{B}+\delta\right]=1 \tag{14}
\end{equation*}
$$

where $d_{B}\left(s^{m}, \hat{s}^{m}\right) \triangleq \frac{1}{m} \sum_{i=1}^{m} d_{B}\left(s_{i}, \hat{s}_{i}\right)$ is a distortion measure for the legitimate user B;
2) Secrecy constraint:

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \geq D_{E}-\delta\right]=1 \tag{15}
\end{equation*}
$$

for any sequence of lists $\left\{\mathcal{L}_{m}\right\}$ such that $\lim \sup _{m \rightarrow \infty} \frac{1}{m} \log \left|\mathcal{L}_{m}\right| \leq R_{L}$, where $\check{S}^{m}=$ $\arg \min _{\check{s}^{m} \in \mathcal{L}_{m}\left(Z^{n}\right)} d_{E}\left(S^{m}, \check{s}^{m}\right)$.

It is easy to verify that in Definition 2, the constraints 1) and 2) can be respectively replaced with $\overline{D_{B}}(\mathbf{S}, \hat{\mathbf{S}}) \leq D_{B}$, and $\underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \geq D_{E}$ for any sequence of lists such that $\lim \sup _{m \rightarrow \infty} \frac{1}{m} \log \left|\mathcal{L}_{m}\right| \leq R_{L}$.

[^2]

Fig. 2. Henchman problem, where $\limsup _{m \rightarrow \infty} R_{m} \leq R_{L}$.

Definition 3. The admissible region $\mathcal{R} \triangleq$ Closure $\left\{\text { Achievable }\left(R_{K}, R_{L}, D_{B}, D_{E}\right)\right\}^{6}$

We assume all the alphabets of the source and its reconstructions (at the legitimate user or wiretapper), as well as the alphabets of channel input and outputs, are finite.

## C. Henchman Problem

The problem above is equivalent to a henchman problem [2], in which a wiretapper reconstructs a single sequence with the help of a rate-limited henchman who can access to both the source $S^{m}$ and the wiretapped signal $Z^{n}$. As depicted in Fig. 2. the wiretapper receives the best possible $m R_{m}$ bits from the henchman to assist in producing a reconstruction sequence $\check{S}^{m}$.

Definition 4. The $R_{m}$-rate henchman code of an $\left(m, n, R_{K}\right)$ block code consists of

1) Encoder: $\varphi_{H}: \mathcal{S}^{m} \times \mathcal{Z}^{n} \mapsto\left[2^{m R_{m}}\right]$;
2) Decoder: $\psi_{H}:\left[2^{m R_{m}}\right] \times \mathcal{Z}^{n} \mapsto \check{\mathcal{S}}^{m}$.

We assume that the wiretapper and henchman are aware of the $\left(m, n, R_{K}\right)$ block code adopted by Nodes A and B, and they cooperate to design a henchman code based on the $\left(m, n, R_{K}\right)$ block code.
Definition 5. The tuple $\left(R_{K}, R_{L}, D_{B}, D_{E}\right)$ is achievable in the henchman problem if there exists a sequence of ( $m, n, R_{K}$ ) codes with $n=\lfloor m \gamma\rfloor$ such that

1) Distortion constraint: $\overline{D_{B}}(\mathbf{S}, \hat{\mathbf{S}}) \leq D_{B}$;
2) Secrecy constraint: $\underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \geq D_{E}$ for any sequence of henchman codes such that $\limsup \operatorname{sum}_{m \rightarrow \infty} R_{m} \leq R_{L}$, where $R_{m}$ 's are the rates of henchman codes.

## D. Lossy-Equivocation Secrecy

Besides, the list secrecy problem and the henchman problem also can be formulated as a secure communication problem with secrecy measured by lossy-equivocation.

Definition 6. For a general source $\mathbf{S}$, general two-sided information $\mathbf{Z}$, and a general distortion measure $d_{E}\left(s^{m}, \check{s}^{m}\right)$,

[^3]the (spectral inf-) lossy-equivocation (or rate-distortion based spectral inf-equivocation) $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)$ is defined as
\[

$$
\begin{equation*}
\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right) \triangleq \inf _{P_{\tilde{\mathbf{S}} \mid \mathbf{S Z}}: \underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}} \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{Z}) \tag{16}
\end{equation*}
$$

\]

for $D_{E} \geq D_{E, \min } \triangleq \inf _{\mathbf{g}(\mathbf{s})} \underline{D_{E}}(\mathbf{S}, \mathbf{g}(\mathbf{S}))$, where $\mathbf{g}(\mathbf{s})$ denotes a sequence of functions $\left\{g^{m}\left(s^{m}\right)\right\}_{m \in \mathbb{N}}$; and $\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right) \triangleq+\infty$ for $D_{E}<D_{E, \text { min }}$.

In the lossy-equivocation secrecy problem, a sender and a legitimate user jointly design an $\left(m, n, R_{K}\right)$ block code to maximize the lossy-equivocation.
Definition 7. The tuple $\left(R_{K}, R_{L}, D_{B}, D_{E}\right)$ is achievable if there exists a sequence of $\left(m, n, R_{K}\right)$ codes with $n=\lfloor m \gamma\rfloor$ such that

1) Distortion constraint: $\overline{D_{B}}(\mathbf{S}, \hat{\mathbf{S}}) \leq D_{B}$;
2) Secrecy constraint: $R_{L} \leq R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)$.

Besides, by setting $d_{B}\left(s^{m}, \hat{s}^{m}\right)=1\left\{s^{m} \neq \hat{s}^{m}\right\}$ and $D_{B}=$ 0 , the achievable tuples and admissible region for lossless communication can be defined similarly ${ }^{7}$.

Definition 8. The tuple $\left(R_{K}, R_{L}, D_{E}\right)$ is achievable for lossless communication if there exists a sequence of $\left(m, n, R_{K}\right)$ codes with $n=\lfloor m \gamma\rfloor$ such that $\lim _{m \rightarrow \infty} \mathbb{P}\left(S^{m} \neq \hat{S}^{m}\right)=0$, and the secrecy constraint in Definition 2, 5 or 7 holds. The admissible region $\mathcal{R} \triangleq$ Closure $\left\{\right.$ Achievable $\left.\left(R_{K}, R_{L}, D_{E}\right)\right\}$ for this case.

Observe that the secrecy constraint in Definition 5 requires that $D_{E}$ cannot be achieved (with any positive probability) by any sequence of $R_{L}$-rate henchman codes for all sufficiently large blocklengths. By contrapositive, the supremum of all $R_{L}$ satisfying the secrecy constraint equals the infimum of all $R$ for which there exists a sequence of $R$-rate codes achieving $D_{E}$ (with some positive probability) for infinitely many blocklengths. This is related to optimistic source coding, which requires that good codes exist for infinitely many blocklengths as opposed to for all sufficiently large blocklengths, required by the (pessimistic) source coding. Hence the supremum of $R_{L}$ also equals the infimum of all $R$ for which there exists a sequence of $R$-rate optimistic source codes achieving $D_{E}$. This enables us to apply optimistic source coding to analyzing the henchman problem.

The information spectrum characterization of optimistic lossy source coding is given in the following theorem, the proof of which is presented in Appendix A. To state our results concisely, for a general sequence of random variables $\mathbf{U}, \mathbf{V}$ and a sequence of distortion measures $d_{m}\left(u^{m}, v^{m}\right)$, we define

$$
\underline{D^{(\varepsilon)}}(\mathbf{U}, \mathbf{V}) \triangleq \inf \left\{d: \liminf _{m \rightarrow \infty} \mathbb{P}\left[d_{m}\left(U^{m}, V^{m}\right) \geq d\right]<\varepsilon\right\}
$$

Note that $\underline{D^{(\varepsilon)}}(\mathbf{U}, \mathbf{V})$ is consistent with $\underline{D}(\mathbf{U}, \mathbf{V})$ when $\varepsilon=1$, i.e., $\underline{D^{(1)}}(\mathbf{U}, \mathbf{V})=\underline{D}(\mathbf{U}, \mathbf{V})$.

[^4]Theorem 1 (Optimistic $\varepsilon$-Achievable Source Coding). Given a general source $\mathbf{S}$ and general two-sided information $\mathbf{Z}$, and a general distortion measure $d_{E}\left(s^{m}, s^{m}\right)$, there exists a sequence of $R_{m}$-rate fixed-length source codes $\left(f_{m}, g_{m}\right)_{m=1}^{\infty}$ such that

$$
\begin{align*}
& \limsup _{m \rightarrow \infty} R_{m} \leq R  \tag{18}\\
& \underline{D_{E}^{(\varepsilon)}}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E} \tag{19}
\end{align*}
$$

(these codes are named optimistic $\varepsilon$-achievable source codes; see [7]) with $\check{S}^{m} \triangleq g_{m}\left(f_{m}\left(S^{m}, Z^{n}\right), Z^{n}\right)$ denoting the reconstruction, if and only if

$$
\begin{equation*}
R \geq \underline{R_{\mathbf{S} \mid \mathbf{Z}}^{(\varepsilon)}}\left(D_{E}\right) \triangleq \inf _{P_{\tilde{\mathbf{s}} \mid \mathbf{S Z}}: \underline{D_{E}^{(\varepsilon)}}(\mathbf{S}, \widetilde{\mathbf{S}}) \leq D_{E}} \bar{I}(\mathbf{S} ; \widetilde{\mathbf{S}} \mid \mathbf{Z}) \tag{20}
\end{equation*}
$$

Remark 1. When $\varepsilon=1, D_{E}^{(1)}(\mathbf{S}, \check{\mathbf{S}})=\underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}})$. Hence $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)=R_{\mathbf{S} \mid \mathbf{Z}}^{(1)}\left(D_{E}\right)$, i.e., $\underline{R}_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)$ is the minimum optimistic 1-achievable source coding rate (the minimum coding rate such that $\left.\underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}\right)$.

Chen et al. have shown that $\underline{H}(\mathbf{S} \mid \mathbf{Z})$ is the minimum optimistic 1-achievable source coding rate for lossless case (minimum source coding rate for the source $\mathbf{S}$ and two-sided information $\mathbf{Z}$ such that $\left.\liminf _{m \rightarrow \infty} \mathbb{P}\left(S^{m} \neq \breve{S}^{m}\right)<1\right)$ [7]. By Remark 1. and the operational definitions of optimistic 1achievability for both lossless and lossy cases, we have the following proposition. Besides, to understand this proposition more intuitively, a direct proof is also given in Appendix B.
Proposition 1. For a general source S , general twosided information $\mathbf{Z}$, and a general distortion measure $d_{E}\left(s^{m}, \check{s}^{m}\right), R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z})$ for $D_{E} \geq D_{E, \text { min }}^{\prime} \triangleq$ $\inf _{\mathbf{g}(\mathbf{s})} \overline{D_{E}}(\mathbf{S}, \overline{\mathbf{g}(\mathbf{S})})$. Moreover, $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)=\underline{H}(\mathbf{S} \mid \mathbf{Z})$ for $d_{E}\left(s^{m}, \check{s}^{m}\right)=1\left\{s^{m} \neq \check{s}^{m}\right\}$ and $\overline{D_{E}}=0$.
Remark 2. For a DM source $\mathbf{S}$ with finite alphabet and an additive distortion measure $d_{E}\left(s^{m}, \check{s}^{m}\right)=\frac{1}{m} \sum_{i=1}^{m} d_{E}\left(s_{i}, \check{s}_{i}\right)$, we have $D_{E, \text { min }}=D_{E, \text { min }}^{\prime}=\inf _{\check{s}(s)} \mathbb{E} d_{E}(S, \check{s}(S))$, where $D_{E, \text { min }}$ is given in Definition 6 Hence for this case, $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z})$ for any $D_{E} \geq D_{E, \min }$.

It seems somewhat counterintuitive that as the optimistic 1-achievable source coding rate, the lossy equivocation $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)$ (and also $R_{\mathbf{S} \mid \mathbf{Z}}^{(\varepsilon)}\left(D_{E}\right)$ ), is defined via supconditional mutual information rate, but in [7], the lossless optimistic 1-achievable rate $\underline{H}(\mathbf{S} \mid \mathbf{Z})$ is characterized by infentropy rate (instead of sup-entropy rate). For the case of $d_{E}\left(s^{m}, \check{s}^{m}\right)=1\left\{s^{m} \neq \check{s}^{m}\right\}$ and $D_{E}=0, D_{E}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}$ is equivalent to $\liminf _{m \rightarrow \infty} \mathbb{P}\left(S^{m} \neq \breve{S}^{m}\right)<1$. Hence for this case, Proposition 1 implies $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)=\underline{H}(\mathbf{S} \mid \mathbf{Z})=$ $\inf _{P_{\tilde{\mathbf{S}} \mid \mathbf{S Z}}: \lim \inf _{m \rightarrow \infty} \mathbb{P}\left(S^{m} \neq \check{S}^{m}\right)<1} \bar{I}(\overline{\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{Z}) \text {. That is, } \underline{H}(\mathbf{S} \mid \mathbf{Z})}$ can be characterized by sup-conditional mutual information rate as well.

Due to Proposition 1, the quantity $\underline{H}(\mathbf{S} \mid \mathbf{Z})$ can be also named as spectral inf-equivocation (or conditional spectral inf-entropy). This term is closely related to the conventional equivocation $\lim \inf _{m \rightarrow \infty} H\left(S^{m} \mid Z^{n}\right)$. Both of them denote conditional entropies, but the former is defined in sense of limit inferior in probability, and the latter is defined in sense
of expectation. Moreover, $\underline{H}(\mathbf{S} \mid \mathbf{Z}) \leq \liminf _{m \rightarrow \infty} H\left(S^{m} \mid Z^{n}\right)$ holds in general.

Comparing the secrecy constraint of Definition 5 with (18) and (19) tells us that given source $S^{m}$ and two-sided information $Z^{n}$, the minimum optimistic 1 -achievable source coding rate equals the supremum of all $R_{L}$ satisfying the secrecy constraint. Hence Theorem 1 combined with Remark 1 implies the equivalence between the lossy-equivocation secrecy problem and the henchman problem.On the other hand, the equivalence between the list secrecy problem and the henchman problem has been proven by Schieler and Cuff [2, Prop. 1]. Hence these three problems are equivalent to each other, as stated in the following proposition.

Proposition 2. The tuple $\left(R_{K}, R_{L}+\delta_{1}, D_{B}, D_{E}+\delta_{1}\right)$ for some $\delta_{1}>0$ is achievable for lossy communication (or $\left(R_{K}, R_{L}+\delta_{1}, D_{E}+\delta_{1}\right)$ for lossless communication) in the list secrecy problem if and only if $\left(R_{K}, R_{L}+\delta_{2}, D_{B}, D_{E}+\delta_{2}\right)$ for some $\delta_{2}>0$ is achievable in the henchman problem, and also if and only if $\left(R_{K}, R_{L}+\delta_{3}, D_{B}, D_{E}+\delta_{3}\right)$ for some $\delta_{3}>0$ is achievable in the lossy-equivocation secrecy problem. In other words, the admissible region $\mathcal{R}$ remains the same for these three problems.

Proof: Schieler and Cuff [2, Prop. 1] have shown that for any tuple ( $R_{K}, R_{L}, D_{B}, D_{E}$ ), the achievability is equivalent for the list secrecy problem and the henchman problem. Hence we only need to show that ( $R_{K}, R_{L}+\delta_{2}, D_{B}, D_{E}+\delta_{2}$ ) for some $\delta_{2}>0$ is achievable in the henchman problem, if and only if $\left(R_{K}, R_{L}+\delta_{3}, D_{B}, D_{E}+\delta_{3}\right)$ for some $\delta_{3}>0$ is achievable in the lossy-equivocation secrecy problem.

Suppose that the tuple $\left(R_{K}, R_{L}+\delta_{2}, D_{B}, D_{E}+\delta_{2}\right)$ satisfies the secrecy constraint: $\underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \geq D_{E}+\delta_{2}$ for any sequence of henchman codes such that $\limsup _{m \rightarrow \infty} R_{m} \leq$ $R_{L}+\delta_{2}$. It immediately gives us $\underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}})>D_{E}+\frac{\delta_{2}}{2}$ for any sequence of codes such that $\limsup _{m \rightarrow \infty} R_{m} \leq R_{L}+\delta_{2}$. By Theorem 1 together with Remark 1, we have $R_{L}+\delta_{2}<$ $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}+\frac{\delta_{2}}{2}\right)$. Hence $\left(R_{K}, R_{L}+\delta_{3}, D_{B}, D_{E}+\delta_{3}\right)$ with $\overline{\delta_{3}=} \frac{\delta_{2}}{2}>0$ is achievable in the lossy-equivocation secrecy problem.

Suppose that the tuple $\left(R_{K}, R_{L}+\delta_{3}, D_{B}, D_{E}+\delta_{3}\right)$ satisfies the secrecy constraint: $R_{L}+\delta_{3} \leq R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}+\delta_{3}\right)$. It immediately gives us $R_{L}+\frac{\delta_{3}}{2}<R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E} \overline{\left.+\delta_{3}\right)}\right.$. Again, by Theorem 1 together with Remark 1, we have $\underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \geq D_{E}+\delta_{3}$ for any sequence of codes such that $\lim \sup _{m \rightarrow \infty} R_{m} \leq R_{L}+\frac{\delta_{3}}{2}$. Hence $\left(R_{K}, R_{L}+\delta_{2}, D_{B}, D_{E}+\delta_{2}\right)$ with $\delta_{2}=\frac{\delta_{3}}{2}>0$ is achievable in the henchman problem.

Therefore, the achievability of $\left(R_{K}, R_{L}+\delta_{2}, D_{B}, D_{E}+\delta_{2}\right)$ for the henchman problem is equivalent to the achievability of $\left(R_{K}, R_{L}+\delta_{3}, D_{B}, D_{E}+\delta_{3}\right)$ for the lossy-equivocation secrecy problem.

So far, we have shown that the achievability of the interior points of $\mathcal{R}$ is equivalent for these three problems. On the other hand, $\mathcal{R}$ is defined as a closed set. So the admissible region $\mathcal{R}$ remains the same for these three problems.
From this proposition and Definition 7, the admissible re-
gion for these problems can be characterized using information spectrum quantities as

$$
\mathcal{R}=\text { Closure } \bigcup_{P_{\mathbf{X} \mid \mathbf{S K}}, P_{\mathbf{\mathbf { S }} \mid \mathbf{Y K}}}\left\{\begin{array}{l}
\left(R_{K}, R_{L}, D_{B}, D_{E}\right):  \tag{21}\\
\overline{D_{B}}(\mathbf{S}, \mathbf{\mathbf { S }}) \leq D_{B} \\
R_{L} \leq \underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right)
\end{array}\right\}
$$

In addition to the DM system, we also consider the Shannon cipher system of communicating Gaussian source over power-constrained Gaussian wiretap channel. For this case, the channel input cost constraint

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}\left[\rho\left(X^{n}\right) \leq P+\delta\right]=1, \forall \delta>0 \tag{22}
\end{equation*}
$$

should be added to Definitions 2, 5, and 7, where $\rho\left(x^{n}\right) \triangleq$ $\frac{1}{n} \sum_{i=1}^{n} \rho\left(x_{i}\right)$ is cost function of the wiretap channel, and for power-constrained Gaussian wiretap channel $\rho(x)=x^{2}$. For the system involving channel cost constraint, Proposition 2 still holds.

## III. Lossless Communication

When lossless communication is required for the legitimate user, the admissible region $\mathcal{R}$ is given by the following theorem.

Theorem 2 (Lossless Communication). For the lossless DM Shannon cipher system, we have

$$
\mathcal{R}=\left\{\begin{array}{l}
\left(R_{K}, R_{L}, D_{E}\right): \gamma C_{B} \geq H(S)  \tag{23}\\
R_{L} \leq \min \left\{R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma} H(S)\right), R_{S}\left(D_{E}\right)\right\}
\end{array}\right\}
$$

where

$$
\begin{equation*}
C_{B}=\max _{P_{X}} I(X ; Y) \tag{24}
\end{equation*}
$$

denotes the capacity of the channel $P_{Y \mid X}$,

$$
\begin{equation*}
\Gamma_{1}(R) \triangleq \max _{\substack{P_{X} P_{V \mid X} P_{U \mid V}: \\ I(V ; Y) \geq R, I(U ; Y) \leq I(U ; Z)}}[I(V ; Y \mid U)-I(V ; Z \mid U)] \tag{25}
\end{equation*}
$$

denotes the equivocation-capacity function of the wiretap channel which was first defined by Csiszár et al. [9] and

$$
\begin{equation*}
R_{S}(D)=\min _{P_{\check{S} \mid S}: \mathbb{E} d_{E}(S, \check{S}) \leq D} I(S ; \check{S}) \tag{26}
\end{equation*}
$$

denotes the rate-distortion function of $S$.
Remark 3. More precisely, the equivocation-capacity function of the wiretap channel [9, Cor. 2] was defined as

$$
\begin{equation*}
\Gamma_{1}^{\prime}(R)=\min \left\{R, \Gamma_{1}(R)\right\} \tag{27}
\end{equation*}
$$

However, for our problem, it does not affect the admissible region $\mathcal{R}$ by replacing $\Gamma_{1}(R)$ with $\Gamma_{1}^{\prime}(R)$, since if $\Gamma_{1}(R)>R$ then $\min \left\{R_{K}+\gamma \Gamma_{1}^{\prime}\left(\frac{1}{\gamma} H(S)\right), R_{S}\left(D_{E}\right)\right\}=$ $\min \left\{R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma} H(S)\right), R_{S}\left(D_{E}\right)\right\}=R_{S}\left(D_{E}\right)$.
Remark 4. If the secrecy measure is replaced with the conventional equivocation $\lim \inf _{m \rightarrow \infty} H\left(S^{m} \mid Z^{n}\right)$, then the admissible region becomes the $\mathcal{R}$ in (23) with the term $R_{S}\left(D_{E}\right)$ replaced by $H(S)$. The achievability part of this claim follows from the fact $\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z}) \leq$
$\liminf _{m \rightarrow \infty} H\left(S^{m} \mid Z^{n}\right)$ and Theorem 2 with the setting $d_{E}\left(s^{m}, \check{s}^{m}\right)=\frac{1}{m} \sum_{i=1}^{m} 1\left\{s_{i} \neq \check{s}_{i}\right\}$ and $D_{E}=0$; the converse part of this claim can be shown by following similar steps to the proof of the converse part of Theorem [2] Note that this claim with $R_{K}=0$ is a restatement of the result of [9] with common message rate $R_{0}=0$, or the result of [8].

Proof: The proof of converse part is given in Appendix C As for the achievability part, lossless communication can be roughly considered as a special case of lossy communication, hence the achievability part can be obtained by following similar steps to the proof of the achievability part of Theorem (3) which is stated in Section IV

Note that the result for the case with no henchman (or only a single reconstruction allowed) is obtained by setting $R_{L}=0$ in the region of Theorem 2 For this case, any positive rate of secret key or any positive $\Gamma_{1}\left(\frac{1}{\gamma} H(S)\right)$ results in the maximum expected distortion that can occur. This observation coincides with that for noiseless wiretap channel case [2].

The first constraint of $\mathcal{R}$ is consistent with the sourcechannel separation theorem, and the second constraint of $\mathcal{R}$, roughly speaking, follows from the following argument. On one hand, the henchman and the wiretapper can always ignore the signal $Z^{n}$ and use a point-to-point code to achieve $R_{S}\left(D_{E}\right)$. On the other hand, the optimal strategy of the sender and legitimate user is an operationally separate coding scheme, in which the source is first compressed by an optimal source code with rate $H(S)$, then a part ( $R_{K}$ rate) of the resulting bitstream is encrypted by one-time pad using the secret key, finally all the bitstream is transmitted losslessly to the legitimate user using an optimal secrecy-channel code [9]. The optimal secrecy-channel code consists of two layers $U^{n}$ and $V^{n}$, and the secrecy is obtained only from the second layer, i.e., $V^{n}$. For such optimal strategy, upon $Z^{n}$ the wiretapper is able to reconstruct the first layer $U^{n}$ directly, and further reconstruct $V^{n}$ upon both $Z^{n}$ and $U^{n}$ by using rate $\gamma(I(V ; Y \mid U)-I(V ; Z \mid U))$. Then the wiretapper uses $R_{K}$ rate to decrypt the secret key, and finally reconstructs the source losslessly by using the secret key and the messages $\left(U^{n}, V^{n}\right)$ as the legitimate user does.

Furthermore, if the legitimate user's channel is less noisy than the wiretapper's (i.e., for every $V \rightarrow X \rightarrow Y Z$, $I(V ; Y) \geq I(V ; Z)$ holds; see [17]), then Csiszár et al. [9. Thm. 3] showed

$$
\begin{equation*}
\Gamma_{1}(R)=\max _{P_{X}: I(X ; Y) \geq R}[I(X ; Y)-I(X ; Z)] \tag{28}
\end{equation*}
$$

## IV. Lossy Communication

Now, we consider lossy communication case. Define

$$
\begin{gathered}
\mathcal{R}_{\mathrm{unc}}^{(i)}=\bigcup_{P_{X^{n} \mid S^{m} K}}\left\{\begin{array}{l}
\left(R_{K}, R_{L}, D_{B}, D_{E}\right): \\
D_{B} \geq \min _{\hat{s}^{m}\left(y^{n}, k\right)} \mathbb{E} d_{B}\left(S^{m}, \hat{S}^{m}\right), \\
R_{L} \leq \frac{1}{m} R_{S^{m} \mid Z^{n}}\left(D_{E}\right)
\end{array}\right\}, \\
\mathcal{R}_{\mathrm{sep}}^{(i)}=\bigcup_{P_{\hat{S} \mid S}}\left\{\begin{array}{c}
\left(R_{K}, R_{L}, D_{B}, D_{E}\right): \gamma C_{B} \geq I(S ; \hat{S}) \\
D_{B} \geq \mathbb{E} d_{B}(S, \hat{S}) \\
R_{L} \leq \min \left\{R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma} I(S ; \hat{S})\right)\right. \\
\left.+R_{S \mid \hat{S}}\left(D_{E}\right), R_{S}\left(D_{E}\right)\right\}
\end{array}\right\},
\end{gathered}
$$

and

$$
\mathcal{R}^{(o)}=\bigcup_{P_{\hat{S} \mid S}}\left\{\begin{array}{c}
\left(R_{K}, R_{L}, D_{B}, D_{E}\right): \gamma C_{B} \geq I(S ; \hat{S}), \\
D_{B} \geq \mathbb{E} d_{B}(S, \hat{S}), \\
R_{L} \leq \min \left\{R_{K}+\gamma \Gamma_{2}\left(\frac{1}{\gamma} I(S ; \hat{S})\right)\right. \\
\left.\quad+R_{S \mid \hat{S}}\left(D_{E}\right), R_{S}\left(D_{E}\right)\right\}
\end{array}\right\}
$$

where $C_{B}, \Gamma_{1}(\cdot)$ and $R_{S}(\cdot)$ are respectively defined by 24, (25) and (26),

$$
\begin{equation*}
\Gamma_{2}(R) \triangleq \min _{\substack{Q_{Y Z \mid X}: Q_{Y \mid X}=P_{Y \mid X}, Q_{X}: I_{Q}(X ; Y) \geq R \\ Q_{Z \mid X}=P_{Z \mid X}}} \max _{Q}(X ; Y \mid Z) \tag{29}
\end{equation*}
$$

with $I_{Q}(\cdot)$ denoting the mutual information under distribution $Q_{X} Q_{Y Z \mid X}$, is another function specified by the wiretap channel, and

$$
\begin{equation*}
R_{S \mid \hat{S}}(D)=\min _{P_{\tilde{S} \mid S \hat{S}}: \mathbb{E} d_{E}(S, \check{S}) \leq D} I(S ; \check{S} \mid \hat{S}) \tag{30}
\end{equation*}
$$

denotes the rate-distortion function of $S$ with the two-sided information $\hat{S}$. Then we have the following theorem for lossy communication case.

Theorem 3 (Lossy Communication). For the lossy DM Shannon cipher system, we have

$$
\begin{equation*}
\mathcal{R}_{\text {unc }}^{(i)} \cup \mathcal{R}_{\text {sep }}^{(i)} \subseteq \mathcal{R} \subseteq \mathcal{R}^{(o)} \tag{31}
\end{equation*}
$$

Remark 5. As shown in the following, $\mathcal{R}_{\text {unc }}^{(i)}$ is achieved by an uncoded scheme. In this scheme, the secret key is used in a symbol-by-symbol way. However, applying the key to a sufficiently large blocklength will result in a better secrecy performance. Based on this idea, an improved version of this scheme is proposed in our another paper [20], which cascades a random permutation (according to the secret key) with a symbol-by-symbol mapping. We refer the interested reader to [20] for the details.
Remark 6. Similar to Remark (4), if the secrecy measure is replaced with the conventional equivocation $\liminf _{m \rightarrow \infty} H\left(S^{m} \mid Z^{n}\right)$, then by replacing the terms $\frac{1}{m} R_{S^{m} \mid Z^{n}}\left(D_{E}\right), R_{S \mid \hat{S}}\left(D_{E}\right)$ and $R_{S}\left(D_{E}\right)$ with $\frac{1}{m} H\left(S^{m} \mid Z^{n}\right)$, $H(S \mid \hat{S})$ and $H(S)$, Theorem 3 still holds. The proof of this claim follows similar argument to that of Remark (4). Note that this claim with $R_{K}=0$ is a lossy extension of the result of [9] with common message rate $R_{0}=0$ and the result of [8].

Proof: The proofs of the outer bound and the inner bound $\mathcal{R}_{\text {sep }}^{(i)}$ are given in Appendices D and E respectively. Now we give a proof of the inner bound $\mathcal{R}_{\text {unc }}^{(2)}$. For simplicity, we only consider the case of $m=n=1$. For given $P_{X \mid S K}$, suppose $\hat{s}(y, k)$ achieves $\min _{\hat{s}(y, k)} \mathbb{E} d_{B}(S, \hat{S})$. Then consider the following uncoded scheme.

Encoder: Upon $(S, K)$, the sender produces $X$ stochastically according to probability distribution $P_{X \mid S K}$.

Decoder: The legitimate user produces $\hat{S}=\hat{s}(Y, K)$.
Observe that $\left(S_{i}, K_{i}, X_{i}, Y_{i}, Z_{i}, \hat{S}_{i}\right)_{i=1}^{\infty}$ are i.i.d. Then by law of large numbers,

$$
\begin{equation*}
\mathbb{P}\left[d_{B}\left(S^{l}, \hat{S}^{l}\right) \leq \mathbb{E} d_{B}\left(S^{l}, \hat{S}^{l}\right)+\epsilon\right] \xrightarrow{l \rightarrow \infty} 1, \tag{32}
\end{equation*}
$$

for any $\epsilon>0$. Hence the distortion constraint for legitimate user is satisfied.
Next we prove the secrecy constraint is also satisfied if $R_{L}<R_{S \mid Z}\left(D_{E}\right)$. That is, if $R_{L}<R_{S \mid Z}\left(D_{E}\right)$ then $\lim _{l \rightarrow \infty} \mathbb{P}\left(d_{E}\left(S^{l}, \check{S}^{l}\right)<D_{E}-\epsilon\right)=0$ for any $\epsilon>0$. This is equivalent to the strong converse for lossy source coding with two-sided information.

Define $\mathcal{A} \triangleq\left\{\left(S^{l}, Z^{l}\right) \in \mathcal{T}_{\delta}^{l}\right\}$. From the property of typicality, we have $\mathbb{P}(\mathcal{A}) \rightarrow 1$ as $l \rightarrow \infty$. Denote the codebook of henchman code as $\mathcal{C}$. Given $Z^{l}, \mathcal{C}$ has $2^{l R_{l}}$ elements at most. Denote them as $\check{s}^{l}\left(i, z^{l}\right), i \in\left[2^{l R_{l}}\right]$. Consider

$$
\begin{align*}
& \mathbb{P}\left(d_{E}\left(S^{l}, \check{S}^{l}\right)<D_{E}-\epsilon\right) \\
\leq & \mathbb{P}\left(d_{E}\left(S^{l}, \check{S}^{l}\right)<D_{E}-\epsilon, \mathcal{A}\right)+\mathbb{P}\left(\mathcal{A}^{c}\right)  \tag{33}\\
\leq & \mathbb{P}\left(d_{E}\left(S^{l}, \check{S}^{l}\right)<D_{E}-\epsilon, \mathcal{A}\right)+\epsilon_{l}  \tag{34}\\
= & \sum_{z^{l}} P_{Z^{n}, \mathcal{C}}\left(z^{l}, c\right) \mathbb{P}\left(d_{E}\left(S^{l}, \check{S}^{l}\right)<D_{E}-\epsilon, \mathcal{A} \mid z^{l}, c\right)+\epsilon_{l}, \tag{35}
\end{align*}
$$

where $\epsilon_{l}$ is a term that vanishes as $l \rightarrow \infty$. Utilizing union bound, we have

$$
\begin{align*}
& \mathbb{P}\left(d_{E}\left(S^{l}, \check{S}^{l}\right)<D_{E}-\epsilon, \mathcal{A} \mid z^{l}, c\right) \\
\leq & \sum_{i \in\left[2^{l R_{l}}\right]} \mathbb{P}\left(d_{E}\left(S^{m}, \check{s}^{m}\left(i, z^{l}\right)\right)<D_{E}-\epsilon, \mathcal{A} \mid z^{l}, c\right) . \tag{36}
\end{align*}
$$

On the other hand, Lemma 10 (given in Appendix E implies $\mathbb{P}\left(d_{E}\left(S^{m}, \check{s}^{m}\left(i, z^{l}\right)\right)<D_{E}-\epsilon, \mathcal{A} \mid z^{l}, c\right) \leq$ $2^{-l\left(R_{S \mid Z}\left(D_{E}-\epsilon\right)-o(1)\right)} \leq 2^{-l\left(R_{S \mid Z}\left(D_{E}\right)-o(1)\right)}$. Then combining it with (35) and (36), we have

$$
\begin{gather*}
\quad \mathbb{P}\left(d_{E}\left(S^{l}, \check{S}^{l}\right)<D_{E}-\epsilon\right) \\
\leq 2^{-l\left(R_{S \mid Z}\left(D_{E}\right)-R_{l}-o(1)\right)}+\epsilon_{l} . \tag{37}
\end{gather*}
$$

If $R_{L}<R_{S \mid Z}\left(D_{E}\right)$, i.e., $\limsup _{l \rightarrow \infty} R_{l}<R_{S \mid Z}\left(D_{E}\right)$, then

$$
\begin{equation*}
\mathbb{P}\left(d_{E}\left(S^{l}, \breve{S}^{l}\right)<D_{E}-\epsilon\right) \xrightarrow{l \rightarrow \infty} 0 \tag{38}
\end{equation*}
$$

for any $\epsilon>0$. Hence $\left(R_{K}, R_{L}, D_{B}, D_{E}\right)$ is achievable.
Note that $\mathcal{R}_{\text {sep }}^{(i)}$ is a generalization of the achievability part of Theorem 2 to the lossy case, and obtained by an operationally separate scheme as well. $\mathcal{R}_{\text {unc }}^{(i)}$ is achieved by the uncoded scheme above in which both the encoder and decoder are symbol-by-symbol mappings. Different from the lossless case, for the lossy communication the source may be transmitted using uncoded scheme or other lossy joint source-channel secrecy code, and hence there may be no message (digital information) transmitted over the channel. For this case the wiretapper cannot decrypt the source through decrypting the digital information. This leads to the difficulty for proving the outer bound part. Here we leverage information spectrum method to derive the outer bound. Instead of reconstructing the source directly, an indirect decryption strategy is considered in our proof, which can be roughly considered as follows: the wiretapper first reconstructs $\hat{S}^{n}$ using rate $\gamma \Gamma_{2}\left(\frac{1}{\gamma} I(S ; \hat{S})\right)$, next decrypts the secret key using rate $R_{K}$, then upon $Y^{n}$ and secret key, produces the legitimate user's reconstruction $\hat{S}^{n}$, and finally upon $\hat{S}^{n}$ produces a final reconstruction $\check{S}^{n}$ using rate $R_{S \mid \hat{S}}\left(D_{E}\right)$. This leads to the outer bound $\mathcal{R}^{(o)}$.

Note that $\mathcal{R}^{(o)}$ and $\mathcal{R}_{\text {sep }}^{(i)}$ differ only in the gamma functions $\Gamma_{1}(\cdot)$ and $\Gamma_{2}(\cdot)$. Obviously, $\Gamma_{1}(\cdot)$ and $\Gamma_{2}(\cdot)$ both only depend on the margin distributions of the wiretap channel, and $\Gamma_{1}(R) \leq \Gamma_{2}(R)$, or equivalently, $\mathcal{R}_{\text {sep }}^{(i)} \subseteq \mathcal{R}^{(o)}$. Moreover for (stochastically) degraded wiretap channel, it is easy to verify that $\Gamma_{1}(R)=\Gamma_{2}(R)$. Hence for this case, $\mathcal{R}_{\text {sep }}^{(i)}$ and $\mathcal{R}^{(o)}$ coincide.

Theorem 4 (DM System with Degraded Wiretap Channel). For lossy DM Shannon cipher system with a degraded wiretap channel ( $X \rightarrow Y \rightarrow Z$ or $X \rightarrow Z \rightarrow Y$ ), we have

$$
\begin{equation*}
\mathcal{R}=\mathcal{R}_{\text {sep }}^{(i)}=\mathcal{R}^{(o)} . \tag{39}
\end{equation*}
$$

Theorem 4 implies the separate coding is optimal for the degraded channel case. However, this is not true in general. In fact, as illustrated by the following example, there exists some source-channel pair such that uncoded scheme strictly outperforms separate scheme. This interesting observation is very different from the case with no secrecy constraint, since without secrecy constraint separation is proven to be optimal for source-channel coding problem [17].

Example 1. (Uncoded scheme may strictly outperform separate scheme). Assume there is no secret key, i.e., $R_{K}=0$. Consider the case of transmitting a Bernoulli source $S \sim$ $\operatorname{Bern}\left(\frac{1}{2}\right)$ over a bandwidth-matched $(\gamma=1)$ wiretap channel $P_{Y Z \mid X}$ with $X \in\{0,1\}, Y \in\{0,1, e\}$, and $Z \in\{0,1\}$, where the channel from $X$ to $Y$ is a binary erasure channel $\operatorname{BEC}(\epsilon), \epsilon \in(0,1)$, and the channel from $X$ to $Z$ is a binary symmetric channel $\operatorname{BSC}(p), p \in\left(0, \frac{1}{2}\right)$. Assume $2 p<\epsilon \leq$ $4 p(1-p)$. Then $Y$ is less noisy than $Z$, but $Z$ is not a degraded version of $Y$ (see [17, Example 5.4]). Assume both the legitimate user's reconstruction and wiretapper's reconstruction satisfy $\hat{S}, \check{S} \in\{0,1, e\}$, and the distortion measures are set to the erasure distortion functions:

$$
d_{B}(s, \hat{s})=d_{E}(s, \hat{s})= \begin{cases}0, & \text { if }(s, \hat{s})=(0,0) \text { or }(1,1) \\ 1, & \text { if }(s, \hat{s})=(0, e) \text { or }(1, e) \\ \infty, & \text { if }(s, \hat{s})=(0,1) \text { or }(1,0)\end{cases}
$$

Observe that $C_{B}=R_{S}(\epsilon)$, hence $D_{B} \geq \epsilon$. Consider the case of $D_{B}=D_{E}=\epsilon$. Then to achieve capacity, $X \sim \operatorname{Bern}\left(\frac{1}{2}\right)$, and to achieve rate-distortion function, $P_{\hat{S} \mid S}$ forms a $\operatorname{BEC}(\epsilon)\}^{8}$ Hence $P_{S \hat{S}}=P_{X Y}$, and $R_{S \mid \hat{S}}(\epsilon)=0$. For less noisy wiretap channel, $\Gamma_{1}(R)$ is given by 28. Hence the maximum achievable $R_{L}$ by separate scheme is $R_{L, \text { sep }}=I(X ; Y)-I(X ; Z)=1-\epsilon-\left(1-H_{2}(p)\right)=$ $H_{2}(p)-\epsilon$, where $H_{2}$ denotes the binary entropy function, i.e., $H_{2}(p)=-p \log p-(1-p) \log (1-p)$. On the other hand, for uncoded scheme, the legitimate user's distortion $D_{B}=\epsilon$ can be achieved by setting $X=S$. Then the maximum achievable $R_{L}$ by uncoded scheme is $R_{L, \text { unc }}=R_{S \mid Z}(\epsilon)=$ $H_{2}(p)-\epsilon H_{2}\left(\frac{p}{\epsilon}\right)$. From the assumption $\frac{p}{\epsilon}<\frac{1}{2}$, we have

[^5]$H_{2}\left(\frac{p}{\epsilon}\right)<1$. It implies $R_{L, \text { unc }}>R_{L, \text { sep }}$, i.e., uncoded scheme provides stronger secrecy. Moreover, for this case,
\[

$$
\begin{align*}
\Gamma_{2}(R) & =\min _{Q_{X Y Z}: Q_{X}=P_{X}, Q_{Y \mid X}=P_{Y \mid X},} I_{Q}(X ; Y \mid Z)  \tag{40}\\
& =\min _{P_{U \mid X Z}=P_{Z \mid X}} I(X ; U \mid Z)  \tag{41}\\
& =H_{2}(p)-\epsilon H_{2 \mid X}\left(\frac{p}{\epsilon}\right) . \tag{42}
\end{align*}
$$
\]

Substitute this into $\mathcal{R}^{(o)}$, then we have the the upper bound $R_{L}^{(o)}$ of the achievable $R_{L}$ satisfies $R_{L}^{(o)}=R_{L, \text { unc }}>R_{L, \text { sep }}$. This implies that for the specified setting above, uncoded scheme is optimal, and meanwhile, separate coding is strictly suboptimal.

For the above example, if set $d_{E}(s, \hat{s})$ to the Hamming distortion, $D_{E}=0$, and remain other settings unchanged, then $R_{L}^{(o)}=R_{L, \text { unc }}>R_{L, \text { sep }}$ still holds. Therefore, from Remark 6 separate scheme is not optimal in general not only for the list secrecy problem, but also for the conventional equivocation secrecy problem. In [8], [9], Wyner, Csiszár and Körner have not found this interesting point, since on one hand, only the case of lossless communication for legitimate user was considered by them, and on the other hand, as shown by Theorem 2, for lossless communication case separate coding is optimal.

Proposition 3. Separate coding is neither optimal in general for the list secrecy (or lossy-equivocation secrecy) problem, nor optimal in general for the conventional equivocation secrecy problem.

Besides, when specialized to the communication over noiseless wiretap channel (with channel capacity $C_{B}$ ), i.e., $Y=$ $Z=X$ and $H(X)=C_{B}$, the problem turns into the one considered by Schieler and Cuff [2]. For this case, $\Gamma_{1}(R)=\Gamma_{2}(R)=0$ for any $R \geq 0$. Hence Theorem 3 recovers the admissible region given in [2, Thm.3], i.e.,

$$
\mathcal{R}=\bigcup_{P_{\hat{S} \mid S}}\left\{\begin{array}{l}
\left(R_{K}, R_{L}, D_{B}, D_{E}\right): \gamma C_{B} \geq I(S ; \hat{S}), \\
D_{B} \geq \mathbb{E} d_{B}(S, \hat{S}), \\
R_{L} \leq \min \left\{R_{K}+R_{S \mid \hat{S}}\left(D_{E}\right), R_{S}\left(D_{E}\right)\right\}
\end{array}\right\}
$$

## V. Gaussian Communication

The results given in previous section can be extended to Gaussian communication scenario. Consider the case of communicating a Gaussian source $S \sim \mathcal{N}\left(0, N_{S}\right)$ over a Gaussian wiretap channel, $Y=X+W_{B}, Z=Y+W_{E}^{\prime}$, where $W_{B} \sim \mathcal{N}\left(0, N_{B}\right)$ and $W_{E}^{\prime} \sim \mathcal{N}\left(0, N_{E}-N_{B}\right)$ are independent, and transmitting power is constrained as $\lim _{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} X_{i}^{2} \leq P+\delta\right]=1, \forall \delta>0$. Then we have the following theorem. The proofs of the converse part and the achievability part are given in Appendices $\square$ and $\square$. respectively.


Fig. 3. The region in Theorem 5 for $\gamma=1, R_{K}=0.5, N_{S}=1, P=$ $1, N_{B}=0.2$ and $D_{B}=\frac{N_{S} N_{B}}{P+N_{B}}$.

Theorem 5 (Gaussian Communication). For the Gaussian Shannon cipher system, the admissible region is

$$
\mathcal{R}=\left\{\begin{array}{c}
\left(R_{K}, R_{L}, D_{B}, D_{E}\right): D_{B} \geq \frac{N_{S}}{\left(1+P / N_{B}\right)^{\gamma}} \\
R_{L} \leq \min \left\{R_{K}+\frac{\gamma}{2} \log ^{+}\left(\frac{1+P / N_{B}}{1+P / N_{E}}\right)\right. \\
\left.+\frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}}, \frac{1}{2} \log ^{+} \frac{N_{S}}{D_{E}}\right\}
\end{array}\right\}
$$

where $\log ^{+} x \triangleq \max \{\log x, 0\}$.
The region in Theorem 5 is illustrated in Fig. 3 If the wiretapper's channel satisfies $N_{E} \leq N_{B}$, then $R_{L} \leq$ $\min \left\{R_{K}+\frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}}, \frac{1}{2} \log ^{+} \frac{N_{S}}{D_{E}}\right\}$, i.e., no secrecy can be obtained from exploiting the wiretap channel. On the contrary, if the wiretapper's channel satisfies $N_{E}>N_{B}$, then $R_{L} \leq$ $\min \left\{R_{K}+\frac{\gamma}{2} \log ^{+}\left(\frac{1+P / N_{B}}{1+P / N_{E}}\right)+\frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}}, \frac{1}{2} \log ^{+} \frac{N_{S}}{D_{E}}\right\}$, i.e., the secrecy of the system is obtained from exploiting the secret key and the wiretap channel at the same time. Moreover, if the wiretapper's channel is bad enough such that $N_{E} \geq \frac{P}{2^{2 R_{K-1}}}$, then $R_{L} \leq \frac{1}{2} \log ^{+} \frac{N_{S}}{D_{E}}$, i.e., the received signal $Z^{n}$ does not help the wiretapper to produce a better reconstruction.

The achievability part of Theorem 5 is proven by using a separate coding scheme. Apart from separate coding, two uncoded schemes, permutation based scheme and orthogonaltransform based scheme, have been proposed in [20] for secure Gaussian broadcast communication (two or more legitimate users are involved in the system) with matched bandwidth (i.e., $\gamma=1$ ). In these two uncoded schemes, the source is transmitted after random permutation or random orthogonal transform (according to the secret key) and (symbol-bysymbol) scaling operation. As shown in [20], they achieve a same region. Compared with the case of no secrecy constraint, they do not incur any performance loss in aspect of legitimate users' distortions; and meanwhile, achieve the optimal secrecy performance under some certain regimes. However, the separate coding proposed in this paper cannot achieve the
optimal distortion performance for broadcast scenarios. Hence for certain cases, the uncoded schemes in [20] will outperform the separate coding of this paper for the bandwidth-matched Gaussian broadcast communication.

When specialized to point-to-point communication (the system considered in this paper), the performance of such uncoded schemes is characterized by the following theorem.

Theorem 6 (Uncoded Schemes). [20] The uncoded schemes above could achieve the region

$$
\mathcal{R}_{\text {unc }}^{(i)}=\left\{\begin{array}{l}
\left(R_{K}, R_{L}, D_{B}, D_{E}\right): 0 \leq P^{\prime} \leq P \\
D_{B} \geq \frac{N_{S} N_{B}}{P^{\prime}+N_{B}}, \\
R_{L} \leq \min \left\{R_{K}+\frac{1}{2} \log ^{+}\left(\frac{N_{S} N_{E}}{D_{E}\left(P^{\prime}+N_{E}\right)}\right),\right. \\
\left.\frac{1}{2} \log ^{+} \frac{N_{S}}{D_{E}}\right\}
\end{array}\right\} .
$$

Obviously, $\mathcal{R}_{\text {unc }}^{(i)} \subseteq \mathcal{R}$. Moreover, it can be verified that under condition that $N_{E} \leq N_{B}, D_{B} \leq D_{E}$ or $N_{E} \geq$ $N_{B}, D_{B}=\frac{\lambda N_{B}}{P+N_{B}} \geq D_{E}$, i.e., the wiretapper has a better channel and wants to produce a worse reconstruction, or the legitimate user's distortion is restricted to be the Shannon limit and meanwhile the wiretapper has a worse channel and wants to produce a better reconstruction, the uncoded schemes above are optimal as well.

## VI. Concluding Remarks

In this paper, we investigated the source-channel secrecy problem with list secrecy measure for Shannon cipher system. By associating it with optimistic source coding, we established the equivalence between this problem and the one with secrecy measured by a new quantity, lossy-equivocation. Utilizing information spectrum method to bound the lossy-equivocation, we obtained several converse results for the systems with various classes of source-channel pairs. Some special cases including lossless communication, communication over degraded wiretap channel, and Gaussian communication, were completely resolved. For these cases, separate coding was proven to be optimal. But this does not hold in general.

The equivalence between the henchman problem and the lossy-equivocation secrecy problem implies that the quantity, lossy-equivocation, indicates the minimum additional information rate needed for the wiretapper to reconstruct the source within a target distortion. On the other hand, the lossyequivocation can be considered as a lossy extension of the conventional equivocation. Therefore, an explicit meaning of the conventional equivocation is obtained: roughly speaking, the conventional equivocation indicates the minimum additional rate needed for the wiretapper to reconstruct the source losslessly. From this perspective, the conventional equivocation, however, is not suitable for lossy communication systems, especially for the communication of continuous sources, since for these systems the wiretapper usually only want to reconstruct a lossy version of the source. Consequently, measuring secrecy by the lossy-equivocation is more reasonable for these systems. Besides, as a generation of the conventional equivocation, the lossy-equivocation also applies to lossless communication systems. For these reasons, the
lossy-equivocation secrecy is expected to have widespread applications in general secure network communications.

## Appendix A <br> Proof of Theorem 1

## A. Sufficiency

Suppose $\widetilde{\mathbf{S}}=\left\{\widetilde{S}^{m}\right\}_{m=1}^{\infty}$ achieves $R_{\mathbf{S} \mid \mathbf{Z}}^{(\varepsilon)}\left(D_{E}\right)$, and for $\delta>0$, denote

$$
\begin{align*}
R & =\bar{I}(\mathbf{S} ; \widetilde{\mathbf{S}} \mid \mathbf{Z})+\delta  \tag{43}\\
D_{E} & =\underline{D_{E}^{(\varepsilon)}}(\mathbf{S}, \widetilde{\mathbf{S}})+\delta \tag{44}
\end{align*}
$$

Then consider the following coding scheme.
Codebook Generation: For each $z^{n}$, we independently generate sequences $\widetilde{s}^{m}\left(i, z^{n}\right), i \in\left[2^{m R}\right]$ according to $P_{\widetilde{S}^{m} \mid Z^{n}=z^{n}}$. Denote the subcodebook as $\mathcal{C}_{z^{n}}$, and the whole codebook as $\mathcal{C}=\left\{\mathcal{C}_{z^{n}}\right\}_{z^{n} \in \mathcal{Z}^{n}}$.

Encoder: Upon $\left(S^{m}, Z^{n}\right)$, the encoder $f_{m}\left(S^{m}, Z^{n}\right)=M$, where $M$ is determined by $m=\min _{i \in\left[2^{m R}\right]} d_{E}\left(s^{m}, \widetilde{s}^{m}\left(i, z^{n}\right)\right)$.

Decoder: Upon $\left(M, Z^{n}\right)$, the decoder $g_{m}\left(M, Z^{n}\right)=$ $\widetilde{s}^{m}\left(M, Z^{n}\right)$.

Denote

$$
\begin{align*}
& \mathcal{T}_{1}^{m}=\left\{(\mathbf{s}, \widetilde{\mathbf{s}}, \mathbf{z}): \frac{1}{m} \log \frac{P_{\widetilde{S}^{m} \mid S^{m} Z^{n}}(\widetilde{\mathbf{s}} \mid \mathbf{s}, \mathbf{z})}{P_{\widetilde{S}^{m} \mid Z^{n}}(\widetilde{\mathbf{s}} \mid \mathbf{z})}<R\right\},  \tag{45}\\
& \mathcal{T}_{2}^{m}=\left\{(\mathbf{s}, \widetilde{\mathbf{s}}): \frac{1}{m} d_{E}(\mathbf{s}, \widetilde{\mathbf{s}})<D_{E}\right\} \times \mathcal{Z}^{n} \tag{46}
\end{align*}
$$

and $\mathcal{T}^{m}=\mathcal{T}_{1}^{m} \cap \mathcal{T}_{2}^{m}$. If define

$$
\begin{equation*}
P_{e}^{m}=\mathbb{P}\left(d_{E}\left(S^{m}, g_{m}\left(f_{m}\left(S^{m}, Z^{n}\right), Z^{n}\right)\right)>D_{E}\right) \tag{47}
\end{equation*}
$$

then following from the argument in [11, proof 1) of Thm. 5.2.1], we have the following lemma. The proof is omitted here.
Lemma 1.

$$
\begin{equation*}
P_{e}^{m} \leq \mathbb{P}\left(\left(S^{m}, \widetilde{S}^{m}, Z^{n}\right) \notin \mathcal{T}^{m}\right)+e^{e^{-m \delta}} \tag{48}
\end{equation*}
$$

On the other hand, according to the definitions of $\bar{I}(\cdot)$ and $\underline{D_{E}^{(\varepsilon)}}(\cdot)$, and from (43) and (44), we have

$$
\begin{align*}
\lim _{m \rightarrow \infty} \mathbb{P}\left(\left(S^{m}, \widetilde{S}^{m}, Z^{n}\right) \notin \mathcal{T}_{1}^{m}\right) & =0  \tag{49}\\
\liminf _{m \rightarrow \infty} \mathbb{P}\left(\left(S^{m}, \widetilde{S}^{m}, Z^{n}\right) \notin \mathcal{T}_{2}^{m}\right) & <\varepsilon \tag{50}
\end{align*}
$$

Hence $\lim \inf _{m \rightarrow \infty} \mathbb{P}\left(\left(S^{m}, \widetilde{S}^{m}, Z^{n}\right) \notin \mathcal{T}^{m}\right)<\varepsilon$. Combining this with Lemma 1 gives us $\lim \inf _{m \rightarrow \infty} P_{e}^{m}<\varepsilon$. Therefore, $\left(R, D_{E}\right)$ is optimistically $\varepsilon$-achievable. Letting $\delta \rightarrow 0$ completes the proof of the sufficiency.

## B. Necessity

Assume there exists a sequence of $R_{m}$-rate fixed-length codes $\left(f_{m}, g_{m}\right)_{m=1}^{\infty}$ with reconstructions $\check{\mathbf{S}}=\left\{\check{S}^{m}\right\}_{m=1}^{\infty}$ such that

$$
\begin{align*}
& \limsup _{m \rightarrow \infty} R_{m} \leq R  \tag{51}\\
& \underline{D_{E}^{(\varepsilon)}}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E} \tag{52}
\end{align*}
$$

Set $\widetilde{\mathbf{S}}=\check{\mathbf{S}}$, then 52 immediately yields

$$
\begin{equation*}
\underline{D_{E}^{(\varepsilon)}}(\mathbf{S}, \widetilde{\mathbf{S}}) \leq D_{E} \tag{53}
\end{equation*}
$$

Hence we only need prove $\bar{I}(\mathbf{S} ; \widetilde{\mathbf{S}} \mid \mathbf{Z}) \leq R$ or $\bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{Z}) \leq R$. Notice that given $\left.Z^{n}=z^{n}, \check{S}^{m}=g_{m}\left(f_{m}\left(S^{m}, z^{n}\right), z^{n}\right)\right)$ cannot take more than $2^{m R_{m}}$ values. Then the following lemma holds.

Lemma 2. [11] Lem. 2.6.2]

$$
\begin{equation*}
\mathbb{P}\left\{\frac{1}{m} \log \frac{1}{P_{\check{S}^{m} \mid Z^{n}}\left(\check{S}^{m} \mid Z^{n}\right)} \geq R_{m}+\delta\right\} \leq e^{-m \delta} \tag{54}
\end{equation*}
$$

On the other hand,

$$
\begin{equation*}
\frac{1}{m} \log \frac{P_{\check{S}^{m} \mid S^{m} Z^{n}}\left(\check{S}^{m} \mid S^{m}, Z^{n}\right)}{P_{\check{S}^{m} \mid Z^{n}}\left(\check{S}^{m} \mid Z^{n}\right)} \leq \frac{1}{m} \log \frac{1}{P_{\check{S}^{m} \mid Z^{n}}\left(\check{S}^{m} \mid Z^{n}\right)} \tag{55}
\end{equation*}
$$

and $\limsup \sup _{m \rightarrow \infty} R_{m} \leq R$ for any $z^{n}$. Hence it holds that

$$
\begin{equation*}
\mathbb{P}\left\{\frac{1}{m} \log \frac{P_{\check{S}^{m} \mid S^{m} Z^{n}}\left(\check{S}^{m} \mid S^{m}, Z^{n}\right)}{P_{\check{S}^{m} \mid Z^{n}}\left(\check{S}^{m} \mid Z^{n}\right)} \geq R+2 \delta\right\} \leq e^{-m \delta} \tag{56}
\end{equation*}
$$

which further implies $\bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{Z}) \leq R+2 \delta$. This means $\bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{Z}) \leq R$ because $\delta>0$ is arbitrary. Combining it with (53) completes the proof of $\underline{R_{\mathbf{S} \mid \mathbf{Z}}^{(\varepsilon)}}\left(D_{E}\right) \leq R$.

## Appendix B <br> Proof of Proposition 1

Actually, $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)$ equals the minimum source coding rate for the source $\mathbf{S}$ and two-sided information $\mathbf{Z}$ such that $D_{E}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}$ (see Theorem 1 and Remark 11; while $\underline{H(\mathbf{S} \mid \mathbf{Z}) \text { equals the minimum source coding rate }}$ for the source $\mathbf{S}$ and two-sided information $\mathbf{Z}$ such that $\limsup _{m \rightarrow \infty} \mathbb{P}\left(S^{m}=\check{S}^{m}\right)>0$ [7]. Therefore, there exists a source code with rate $\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta, \delta>0$, such that $\limsup _{m \rightarrow \infty} \mathbb{P}\left(S^{m}=\check{S}^{m}\right)>0$ which further implies $D_{E}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}$ for any $D_{E} \geq D_{E, \text { min }}^{\prime}$ or for $d_{E}\left(s^{m}, s^{m}\right)=$ $\left.\overline{1\{s} s^{m} \neq \check{s}^{m}\right\}$ and $D_{E}=0$. Then we have $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right) \leq$ $\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta$, since $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right)$ equals the minimum coding rate satisfying $\underline{D_{E}}(\mathbf{S}, \mathbf{S}) \leq D_{E}$. Observe that $\delta>0$ is arbitrary, hence $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z})$. Similarly, we can also have $R_{\mathbf{S} \mid \mathbf{Z}}\left(\overline{D_{E}}\right) \geq \underline{H}(\mathbf{S} \mid \mathbf{Z})$ for $d_{E}\left(s^{m}, \check{s}^{m}\right)=1\left\{s^{m} \neq \check{s}^{m}\right\}$ and $\overline{D_{E}}=0$.

The argument above is rather superficial. To understand the proposition more intuitively, we provide a direct proof in the following. Note that the following proof is essentially the same as the argument above.

For $\delta>0$, denote

$$
\begin{equation*}
\mathcal{T}^{m}=\left\{(\mathbf{s}, \mathbf{z}): \frac{1}{m} \log \frac{1}{P_{S^{m} \mid Z^{n}}(\mathbf{s} \mid \mathbf{z})}<\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta\right\} \tag{57}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{T}^{m}(\mathbf{z})=\left\{\mathbf{s}:(\mathbf{s}, \mathbf{z}) \in \mathcal{T}^{m}\right\} \tag{58}
\end{equation*}
$$

Then, we have $P_{S^{m} \mid Z^{n}}(\mathbf{s} \mid \mathbf{z})>2^{-m(\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta)}$ for $(\mathbf{s}, \mathbf{z}) \in$ $\mathcal{T}^{m}$. Since it holds that

$$
\begin{align*}
1 & \geq \sum_{\mathbf{s} \in \mathcal{T}^{m}(\mathbf{z})} P_{S^{m} \mid Z^{n}(\mathbf{s} \mid \mathbf{z})}  \tag{59}\\
& >\sum_{\mathbf{s} \in \mathcal{T}^{m}(\mathbf{z})} 2^{-m(\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta)}  \tag{60}\\
& \geq\left|\mathcal{T}^{m}(\mathbf{z})\right| 2^{-m(\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta)} \tag{61}
\end{align*}
$$

we have $\left|\mathcal{T}^{m}(\mathbf{z})\right|<2^{m(\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta)}$ for any $\mathbf{z}$.
For $\delta^{\prime}>0$, suppose $\mathbf{g}(\mathbf{s})$ is a sequence of functions such that $\overline{D_{E}}(\mathbf{S}, \mathbf{g}(\mathbf{S}))=D_{E, \text { min }}^{\prime}+\delta^{\prime}$. Set $\check{S}^{m}=$ $\left\{\begin{array}{l}S^{m}, \quad \text { if }\left(S^{m}, Z^{n}\right) \in \mathcal{T}^{m} \\ s_{0}^{m}, \quad \text { otherwise }\end{array}\right.$ and $\check{S}^{* m}=g^{m}\left(\check{S}^{m}\right)$, where $s_{0}^{m} \in \mathcal{S}^{m}$ is a constant vector. Then it immediately yields

$$
\begin{align*}
\bar{I}\left(\mathbf{S} ; \check{\mathbf{S}}^{*} \mid \mathbf{Z}\right) & \leq \bar{I}\left(\mathbf{S} ; \check{\mathbf{S}}^{*} \check{\mathbf{S}} \mid \mathbf{Z}\right)  \tag{62}\\
& =\bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{Z})  \tag{63}\\
& \leq \bar{H}(\check{\mathbf{S}} \mid \mathbf{Z}) \tag{64}
\end{align*}
$$

where 63) follows from that $\check{S}^{* m}$ is a function of $\check{S}^{m}$.
It has been shown that given $Z^{n}=z^{n}, \check{S}^{m}$ cannot take more than $2^{m(\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta)}$ values. Hence we have the following lemma.
Lemma 3. 11] Lem. 2.6.2] For any $\delta^{\prime \prime}>0$,
$\mathbb{P}\left\{\frac{1}{m} \log \frac{1}{P_{\check{S}^{m} \mid Z^{n}}\left(\check{S}^{m} \mid Z^{n}\right)} \geq \underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta+\delta^{\prime \prime}\right\} \leq e^{-m \delta^{\prime \prime}}$.

It implies $\bar{H}(\check{\mathbf{S}} \mid \mathbf{Z}) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta$. Therefore,

$$
\begin{equation*}
\bar{I}\left(\mathbf{S} ; \check{\mathbf{S}}^{*} \mid \mathbf{Z}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta \tag{66}
\end{equation*}
$$

Next we will show $\underline{D_{E}}\left(\mathbf{S}, \check{\mathbf{S}}^{*}\right) \leq D_{E}$. First we have

$$
\begin{align*}
& \left\|P_{S^{m} Z^{n} \check{S}^{m}}-P_{S^{m} Z^{n} S^{m}}\right\|_{T V} \\
& \left.=\frac{1}{2} \sum_{\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \in \mathcal{S}^{m} \times \mathcal{Z}^{n} \times \mathcal{S}^{m}} \right\rvert\, P_{S^{m} Z^{n} \breve{S}^{m}}\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \\
& -P_{S^{m} Z^{n} S^{m}}\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \mid  \tag{67}\\
& \left.=\frac{1}{2} \sum_{\left(s_{1}^{m}, z^{n}\right) \in \mathcal{T}^{m}} \sum_{s_{2}^{m} \in \mathcal{S}^{m}} \right\rvert\, P_{S^{m} Z^{n} \breve{S}^{m}}\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \\
& -P_{S^{m} Z^{n} S^{m}}\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \mid \\
& \left.+\frac{1}{2} \sum_{\left(s_{1}^{m}, z^{n}\right) \notin \mathcal{T}^{m}} \sum_{s_{2}^{m} \in \mathcal{S}^{m}} \right\rvert\, P_{S^{m} Z^{n} \check{S}^{m}}\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \\
& -P_{S^{m} Z^{n} S^{m}}\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \mid  \tag{68}\\
& \left.=\frac{1}{2} \sum_{\left(s_{1}^{m}, z^{n}\right) \notin \mathcal{T}^{m}} \sum_{s_{2}^{m} \in \mathcal{S}^{m}} \right\rvert\, P_{S^{m}} Z^{n} \check{S}^{m}\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \\
& -P_{S^{m} Z^{n} S^{m}}\left(s_{1}^{m}, z^{n}, s_{2}^{m}\right) \mid  \tag{69}\\
& \leq \sum_{\left(s_{1}^{m}, z^{n}\right) \notin \mathcal{T}^{m}} P_{S^{m} Z^{n}}\left(s_{1}^{m}, z^{n}\right)  \tag{70}\\
& =\mathbb{P}\left[\left(S^{m}, Z^{n}\right) \notin \mathcal{T}^{m}\right] . \tag{71}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
& \mathbb{P}\left[\left(S^{m}, Z^{n}\right) \notin \mathcal{T}^{m}\right] \\
& =1-\mathbb{P}\left[\frac{1}{m} \log \frac{1}{P_{S^{m} \mid Z^{n}}\left(S^{m} \mid Z^{n}\right)}<\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta\right] \tag{72}
\end{align*}
$$

and according to the definition of $\underline{H}(\mathbf{S} \mid \mathbf{Z})$,

$$
\begin{equation*}
\limsup _{m \rightarrow \infty} \mathbb{P}\left[\frac{1}{m} \log \frac{1}{P_{S^{m} \mid Z^{n}}\left(S^{m} \mid Z^{n}\right)}<\underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta\right]>0 \tag{73}
\end{equation*}
$$

Combining (71), 72 and (73) gives us

$$
\begin{equation*}
\liminf _{m \rightarrow \infty}\left\|P_{S^{m}} Z^{n} \check{S}^{m}-P_{S^{m} Z^{n} S^{m}}\right\|_{T V}<1 \tag{74}
\end{equation*}
$$

which further yields

$$
\begin{equation*}
\liminf _{m \rightarrow \infty}\left\|P_{S^{m} \check{S}^{m}}-P_{S^{m} S^{m}}\right\|_{T V}<1 \tag{75}
\end{equation*}
$$

Furthermore, for any $D_{E} \geq D_{E, \text { min }}^{\prime}+$ $\delta^{\prime}$, we have $\overline{D_{E}}(\mathbf{S}, \mathbf{g}(\mathbf{S})) \quad \leq \quad D_{E}$, i.e., $\lim _{m \rightarrow \infty} \mathbb{P}\left[d_{E}\left(S^{m}, g^{m}\left(S^{m}\right)\right) \geq D_{E}+\delta^{\prime \prime}\right]=0, \forall \delta^{\prime \prime}>0$. Combining it with (75), and according to the definition of total variation distance, we have

$$
\begin{equation*}
\liminf _{m \rightarrow \infty} \mathbb{P}\left[d_{E}\left(S^{m}, g^{m}\left(\breve{S}^{m}\right)\right) \geq D_{E}+\delta^{\prime \prime}\right]<1 \tag{76}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
\limsup _{m \rightarrow \infty} \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{* m}\right)<D_{E}+\delta^{\prime \prime}\right]>0 \tag{77}
\end{equation*}
$$

Since $\delta^{\prime \prime}>0$ is arbitrary, it must hold that

$$
\begin{equation*}
\underline{D_{E}}\left(\mathbf{S}, \check{\mathbf{S}}^{*}\right) \leq D_{E} \tag{78}
\end{equation*}
$$

Combining 66 with 78) and by the definition of $\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right)$, we have

$$
\begin{equation*}
\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right) \leq \bar{I}\left(\mathbf{S} ; \check{\mathbf{S}}^{*} \mid \mathbf{Z}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z})+\delta \tag{79}
\end{equation*}
$$

Since $\delta>0$ is arbitrary, it holds that

$$
\begin{equation*}
\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z}) . \tag{80}
\end{equation*}
$$

Similarly, it can be shown that both $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z})$ and $R_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right) \geq \underline{H}(\mathbf{S} \mid \mathbf{Z})$ hold for the case of $d_{E}\left(s^{m}, s^{m}\right)=$


## Appendix C <br> Proof of Converse of Theorem 2

From source-channel coding theorem [17], we have $\gamma C_{B} \geq$ $H(S)$. Furthermore, if $R_{L}>R_{S}\left(D_{E}\right)$, then no matter what scheme Nodes A and B use, the henchman and the wiretapper can ignore $Z^{n}$ altogether and simply use a point-to-point ratedistortion code to describe $S^{m}$ within distortion $D_{E}$ (with probability 1). Hence, to prevent the wiretapper from achieving this, it should hold that $R_{L} \leq R_{S}\left(D_{E}\right)$. Furthermore, to show the converse part, we only need to prove if $\left(R_{K}, R_{L}, D_{E}\right)$ is achievable, then

$$
\begin{equation*}
R_{L} \leq R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma} H(S)\right) \tag{81}
\end{equation*}
$$

or equivalently to show

$$
\begin{equation*}
\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right) \leq R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma} H(S)\right) \tag{82}
\end{equation*}
$$

To that end, we borrow the information spectrum method [11]. By Proposition 1 together with Remark 2, we have

$$
\begin{equation*}
\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right) \leq \underline{H}(\mathbf{S} \mid \mathbf{Z}) . \tag{83}
\end{equation*}
$$

Next we prove $\underline{H}(\mathbf{S} \mid \mathbf{Z}) \leq R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma} H(S)\right)$.
Consider

$$
\begin{align*}
\underline{H}(\mathbf{S} \mid \mathbf{Z}) & \leq \underline{H}(\mathbf{S K} \mid \mathbf{Z})  \tag{84}\\
& \leq \bar{H}(\mathbf{K} \mid \mathbf{Z})+\underline{H}(\mathbf{S} \mid \mathbf{Z K})  \tag{85}\\
& \leq \bar{H}(\mathbf{K})+\underline{H}(\mathbf{S} \mid \mathbf{Z K})  \tag{86}\\
& =R_{K}+\underline{H}(\mathbf{S} \mid \mathbf{Z K}) . \tag{87}
\end{align*}
$$

To upper bound $\underline{H}(\mathbf{S} \mid \mathbf{Z K})$, the following lemma is needed.
Lemma 4. 11 Thm.1.7.2] For any $P_{\mathrm{KSZ}}$,

$$
\begin{equation*}
\underline{H}(\mathbf{S} \mid \mathbf{Z K}) \leq \liminf _{m \rightarrow \infty} \frac{1}{m} H\left(S^{m} \mid Z^{n} K\right) \tag{88}
\end{equation*}
$$

Observe that if $K=\emptyset$, the term $\frac{1}{m} H\left(S^{m} \mid Z^{n}\right)$ is just the conventional equivocation defined in [8], [9]. Csiszár et al. [9] proved that when there is no the secret key $K, \frac{1}{m} H\left(S^{m} \mid Z^{n}\right) \leq \gamma \Gamma_{1}\left(\frac{1}{\gamma} H(S)\right)$ holds. Here we follow similar steps to their proof, and show that for the case with secret key, $\frac{1}{m} H\left(S^{m} \mid Z^{n} K\right)$ is also upper bounded by $\gamma \Gamma_{1}\left(\frac{1}{\gamma} H(S)\right)$.

Denote $\epsilon_{m}=\mathbb{P}\left(S^{m} \neq \hat{S}^{m}\right)$, then $\lim _{m \rightarrow \infty} \epsilon_{m}=0$. By Fano's inequality,

$$
\begin{equation*}
H\left(S^{m} \mid Y^{n} K\right) \leq 1+m \epsilon_{m}|\mathcal{S}| \leq m \epsilon_{m}^{\prime} \tag{89}
\end{equation*}
$$

where $\epsilon_{m}^{\prime}$ is a term such that $\lim _{m \rightarrow \infty} \epsilon_{m}^{\prime}=0$. Then we have

$$
\begin{align*}
H\left(S^{m} \mid Z^{n} K\right) & \leq H\left(S^{m} \mid Z^{n} K\right)-H\left(S^{m} \mid Y^{n} K\right)+m \epsilon_{m}^{\prime} \\
& =I\left(S^{m} ; Y^{n} \mid K\right)-I\left(S^{m} ; Z^{n} \mid K\right)+m \epsilon_{m}^{\prime} \tag{91}
\end{align*}
$$

Observe that

$$
\begin{align*}
& I\left(S^{m} ; Y^{n} \mid K\right)=\sum_{i=1}^{n} I\left(S^{m} ; Y_{i} \mid Y^{i-1} Z_{i+1}^{n} K\right)+\Sigma_{1}-\Sigma_{2}  \tag{92}\\
& I\left(S^{m} ; Z^{n} \mid K\right)=\sum_{i=1}^{n} I\left(S^{m} ; Z_{i} \mid Y^{i-1} Z_{i+1}^{n} K\right)+\Sigma_{1}^{*}-\Sigma_{2}^{*} \tag{93}
\end{align*}
$$

where

$$
\begin{align*}
& \Sigma_{1}=\sum_{i=1}^{n} I\left(Z_{i+1}^{n} ; Y_{i} \mid Y^{i-1} K\right)  \tag{94}\\
& \Sigma_{1}^{*}=\sum_{i=1}^{n} I\left(Y^{i-1} ; Z_{i} \mid Z_{i+1}^{n} K\right) \tag{95}
\end{align*}
$$

and $\Sigma_{2}, \Sigma_{2}^{*}$ are the analogous sums with $S^{m} K$ instead of $K$. By Csiszár Sum Identity [9], [17], $\Sigma_{1}=\Sigma_{1}^{*}$ and $\Sigma_{2}=\Sigma_{2}^{*}$. Therefore,

$$
\begin{align*}
& \frac{1}{m} H\left(S^{m} \mid Z^{n} K\right) \\
& \leq \frac{1}{m} \sum_{i=1}^{n} I\left(S^{m} ; Y_{i} \mid Y^{i-1} Z_{i+1}^{n} K\right) \\
& \quad-\frac{1}{m} \sum_{i=1}^{n} I\left(S^{m} ; Z_{i} \mid Y^{i-1} Z_{i+1}^{n} K\right)+\epsilon_{m}^{\prime}  \tag{96}\\
& =\gamma I\left(S^{m} ; Y_{Q} \mid Y^{Q-1} Z_{Q+1}^{n} K Q\right) \\
& \quad \quad-\gamma I\left(S^{m} ; Z_{Q} \mid Y^{Q-1} Z_{Q+1}^{n} K Q\right)+\epsilon_{m}^{\prime}  \tag{97}\\
& =\gamma I(V ; Y \mid U)-\gamma I(V ; Z \mid U)+\epsilon_{m}^{\prime} \tag{98}
\end{align*}
$$

where $Q$ denotes a timesharing random variable uniformly distributed over $[n]$, and $U \triangleq Y^{Q-1} Z_{Q+1}^{n} K Q, V \triangleq S^{m} U, Y \triangleq$ $Y_{Q}, Z \triangleq Z_{Q}$.

Furthermore,

$$
\begin{align*}
H(S) & \leq \frac{1}{m} I\left(S^{m} ; Y^{n} \mid K\right)+\epsilon_{m}^{\prime}  \tag{99}\\
\frac{1}{m} I\left(S^{m} ; Y^{n} \mid K\right) & =\frac{1}{m} \sum_{i=1}^{n} I\left(S^{m} ; Y_{i} \mid Y^{i-1} K\right)  \tag{100}\\
& \leq \frac{1}{m} \sum_{i=1}^{n} I\left(S^{m} Y^{i-1} Z_{i+1}^{n} K ; Y_{i}\right)  \tag{101}\\
& =\gamma I(V ; Y) \tag{102}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{1}{m} I\left(S^{m} ; Y^{n} \mid K\right) \\
& =\frac{1}{m} \sum_{i=1}^{n} I\left(S^{m} ; Y_{i} \mid Y^{i-1} Z_{i+1}^{n} K\right)+\frac{1}{m} \Sigma_{1}-\frac{1}{m} \Sigma_{2} \\
& \leq \frac{1}{m} \sum_{i=1}^{n} I\left(S^{m} ; Y_{i} \mid Y^{i-1} Z_{i+1}^{n} K\right)+\frac{1}{m} \Sigma_{1} \\
& =\frac{1}{m} \sum_{i=1}^{n} I\left(S^{m} ; Y_{i} \mid Y^{i-1} Z_{i+1}^{n} K\right)+\frac{1}{m} \Sigma_{1}^{*} \\
& =\gamma I\left(S^{m} ; Y_{Q} \mid Y^{Q-1} Z_{Q+1}^{n} K Q\right)+\gamma I\left(Y^{Q-1} ; Z_{Q} \mid Z_{Q+1}^{n} K Q\right)  \tag{106}\\
& \leq \gamma I\left(S^{m} ; Y_{Q} \mid Y^{Q-1} Z_{Q+1}^{n} K Q\right)+\gamma I\left(Y^{Q-1} Z_{Q+1}^{n} K Q ; Z_{Q}\right) \tag{107}
\end{align*}
$$

Combining (83), (87), 88, (98), (99), (102) and (108), we have

$$
\begin{align*}
R_{L} & \leq R_{K}+\sup _{P_{\mathbf{X} \mid \mathrm{KS}}} \liminf _{m \rightarrow \infty}\left(\gamma \widetilde{\Gamma}_{1}\left(\frac{1}{\gamma} H(S)-\epsilon_{m}^{\prime}\right)+\epsilon_{m}^{\prime}\right) \\
& \leq R_{K}+\gamma \widetilde{\Gamma}_{1}\left(\frac{1}{\gamma} H(S)\right), \tag{109}
\end{align*}
$$

where

$$
\begin{equation*}
\widetilde{\Gamma}_{1}(R) \triangleq \max _{\substack{P_{X} P_{V \mid X} P_{U \mid V}: \\ I(V ; Y) \geq R, I(V ; Y \mid U)+I(U ; Z) \geq R}}[I(V ; Y \mid U)-I(V ; Z \mid U)] \tag{111}
\end{equation*}
$$

Actually $\widetilde{\Gamma}_{1}(R)=\Gamma_{1}(R)$ for any $R \geq 0$ (see [9, Proof of Cor. 2 ]), hence

$$
\begin{equation*}
R_{L} \leq R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma} H(S)\right) \tag{112}
\end{equation*}
$$

This completes the proof.

## Appendix D <br> Proof of Outer Bound of Theorem 3

Suppose $Q_{Y Z \mid X}$ is the pmf achieving the minimization in the definition of $\Gamma_{2}(R)$. Hence it satisfies $Q_{Y \mid X}=$ $P_{Y \mid X}, Q_{Z \mid X}=P_{Z \mid X}$. Furthermore, given the code adopted by Nodes A and B, the achievability of $\left(R_{K}, R_{L}, D_{B}, D_{E}\right)$ only depends on the margin distributions of the wiretap channel. Hence the admissible region $\mathcal{R}$ does not change if we replace the channel $P_{Y Z \mid X}$ with $Q_{Y Z \mid X}$. In the following, without loss of generality, we only need consider the case of $P_{Y Z \mid X}=Q_{Y Z \mid X}$, i.e., $P_{Y Z \mid X}$ achieves $\Gamma_{2}(R)$.

Following the argument for lossless communication case, it should hold that $R_{L} \leq R_{S}\left(D_{E}\right)$. Next, we prove that if ( $R_{K}, R_{L}, D_{B}, D_{E}$ ) is achievable, then there exists a conditional pmf $P_{\hat{S} \mid S}$ such that

$$
\begin{align*}
\gamma C_{B} & \geq I(S ; \hat{S})  \tag{113}\\
D_{B} & \geq \mathbb{E} d_{B}(S, \hat{S})  \tag{114}\\
\underline{R}_{\mathbf{S} \mid \mathbf{Z}}\left(D_{E}\right) & \leq R_{K}+\gamma \Gamma_{2}\left(\frac{1}{\gamma} I(S ; \hat{S})\right)+R_{S \mid \hat{S}}\left(D_{E}\right) \tag{115}
\end{align*}
$$

Since $\bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{Z}) \leq \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mathbf{Y} \mathbf{K} \mid \mathbf{Z})$, we have

$$
\begin{align*}
& \quad \underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right)  \tag{116}\\
& \leq \inf _{P_{\tilde{\mathbf{S}}_{\tilde{\mathbf{S Z Y K}}}}: D_{E}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}} \bar{I}(\mathbf{S} ; \check{\mathbf{S} \mathbf{Y K} \mid \mathbf{Z})} \\
& \leq \bar{I}(\mathbf{S} ; \mathbf{Y} \mid \mathbf{Z})+\bar{I}(\mathbf{S} ; \mathbf{K} \mid \mathbf{Y} \mathbf{Z})  \tag{117}\\
& \quad+\inf _{P_{\tilde{\mathbf{S}} \mid \mathbf{S Z Y K}}: \underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}} \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{K Y Z}) .
\end{align*}
$$

By data processing inequality,

$$
\begin{equation*}
\bar{I}(\mathbf{S} ; \mathbf{Y} \mid \mathbf{Z}) \leq \gamma \bar{I}(\mathbf{X} ; \mathbf{Y} \mid \mathbf{Z}) \tag{118}
\end{equation*}
$$

The second term in 117) is bounded by

$$
\begin{equation*}
\bar{I}(\mathbf{S} ; \mathbf{K} \mid \mathbf{Y} \mathbf{Z}) \leq \bar{H}(\mathbf{K} \mid \mathbf{Y} \mathbf{Z}) \leq \bar{H}(\mathbf{K})=R_{K} \tag{119}
\end{equation*}
$$

and the last term in 117) is bounded by

$$
\begin{align*}
& =\inf ^{P_{\tilde{\mathbf{S}} \mid \mathbf{S Z Y K}}: \underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}} \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{K Y Z})  \tag{120}\\
& \leq \inf _{P_{\tilde{\mathbf{S}} \mid \mathbf{S}}: \underline{D_{E}}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}} \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{K Y} \mathbf{Z} \hat{\mathbf{S}})  \tag{121}\\
& \leq \inf _{P_{\check{\mathbf{S}}| | \hat{\mathbf{s}}}: \underline{D E}(\mathbf{S}, \check{\mathbf{S}}) \leq D_{E}} \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \hat{\mathbf{S}})  \tag{122}\\
& \triangleq R_{\mathbf{S} \mid \hat{\mathbf{S}}}\left(D_{E}\right), \tag{123}
\end{align*}
$$

where $(120$ follows from $\mathbf{S}$ ŠZ $\rightarrow \mathbf{K Y} \rightarrow \hat{\mathbf{S}}$, and 122 follows from that for any $\check{\mathbf{S}}$ such that KYZ $\rightarrow \mathbf{S} \hat{\mathbf{S}} \rightarrow \check{\mathbf{S}}$,

$$
\begin{align*}
& \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \mathbf{K Y Z} \hat{\mathbf{S}}) \\
= & \mathrm{p}-\limsup _{m \rightarrow \infty} \frac{1}{m} \log \frac{P_{\check{S}^{m} \mid S^{m}} \hat{S}^{m}\left(\check{S}^{m} \mid S^{m} \hat{S}^{m}\right)}{P_{\check{S}^{m} \mid \hat{S}^{m} Y^{n} Z^{n} K}\left(\check{S}^{m} \mid S^{m} \hat{S}^{m} Y^{n} Z^{n} K\right)}  \tag{124}\\
= & \mathrm{p}-\limsup _{m \rightarrow \infty}\left(\frac{1}{m} \log \frac{P_{\check{S}^{m} \mid S^{m} \hat{S}^{m}}\left(\check{S}^{m} \mid S^{m} \hat{S}^{m}\right)}{P_{\check{S}^{m} \mid \hat{S}^{m}}\left(\check{S}^{m} \mid \hat{S}^{m}\right)}\right. \\
& \left.\quad-\frac{1}{m} \log \frac{P_{\check{S}^{m} \mid \hat{S}^{m} Y^{n} Z^{n} K}\left(\check{S}^{m} \mid S^{m} \hat{S}^{m} Y^{n} Z^{n} K\right)}{P_{\check{S}^{m} \mid \hat{S}^{m}}\left(\check{S}^{m} \mid \hat{S}^{m}\right)}\right)  \tag{125}\\
\leq & \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \hat{\mathbf{S}})-\underline{I}(\mathbf{K Y Z} ; \check{\mathbf{S}} \mid \hat{\mathbf{S}})  \tag{126}\\
\leq & \bar{I}(\mathbf{S} ; \check{\mathbf{S}} \mid \hat{\mathbf{S}}) . \tag{127}
\end{align*}
$$

The inequality (127) follows from $\underline{I}(\mathbf{K Y Z} ; \check{\mathbf{S}} \mid \hat{\mathbf{S}}) \geq 0$, which is a conditional version of [11, Eqn. (3.2.3)].

Combining (117), (118), (119) and (123) gives us

$$
\begin{equation*}
\underline{R_{\mathbf{S} \mid \mathbf{Z}}}\left(D_{E}\right) \leq R_{K}+\gamma \bar{I}(\mathbf{X} ; \mathbf{Y} \mid \mathbf{Z})+\underline{R_{\mathbf{S} \mid \hat{\mathbf{S}}}}\left(D_{E}\right) \tag{128}
\end{equation*}
$$

Hence to show (113)-(115) we only need to prove there exists a conditional pmf $P_{\hat{S} \mid S}$ such that (113), 114), and

$$
\begin{align*}
\bar{I}(\mathbf{X} ; \mathbf{Y} \mid \mathbf{Z}) & \leq \Gamma_{2}\left(\frac{1}{\gamma} I(S ; \hat{S})\right)  \tag{129}\\
\underline{R_{\mathbf{S} \mid \hat{\mathbf{S}}}}\left(D_{E}\right) & \leq R_{S \mid \hat{S}}\left(D_{E}\right) \tag{130}
\end{align*}
$$

We first consider 129, and prove $\bar{I}(\mathbf{X} ; \mathbf{Y} \mid \mathbf{Z}) \leq$ $\Gamma_{2}\left(\frac{1}{\gamma} \bar{I}(\mathbf{S} ; \hat{\mathbf{S}})\right)$. Denote $X^{n}$, and $Y^{n}$ and $Z^{n}$ to be output variables of the channel corresponding to $X^{n}$. We also denote $\bar{X}^{n}$ to be a sequence of independent random variables with $\operatorname{pmf} P_{\bar{X}^{n}}=\prod_{i=1}^{n} P_{X_{i}}$, and $\bar{Y}^{n}$ and $\bar{Z}^{n}$ to be output variables of the channel corresponding to $\bar{X}^{n}$. Obviously, $P_{\bar{X}_{i} \bar{Y}_{i} \bar{Z}_{i}}=P_{X_{i} Y_{i} Z_{i}}$ for $1 \leq i \leq n$. Consider the sequence of random variables

$$
\begin{align*}
U_{n} & \triangleq \frac{1}{n} \log \frac{P_{Y^{n} \mid X^{n} Z^{n}}\left(Y^{n} \mid X^{n}, Z^{n}\right)}{P_{\bar{Y}^{n} \mid \bar{Z}^{n}}\left(Y^{n} \mid Z^{n}\right)}  \tag{131}\\
& =\frac{1}{n} \sum_{i=1}^{n} \log \frac{P_{Y_{i} Z_{i} \mid X_{i}}\left(Y_{i}, Z_{i} \mid X_{i}\right)}{P_{\bar{Y}_{i} \mid \bar{Z}_{i}}\left(Y_{i} \mid Z_{i}\right) P_{Z_{i} \mid X_{i}}\left(Z_{i} \mid X_{i}\right)}  \tag{132}\\
& =\frac{1}{n} \sum_{i=1}^{n} \log \frac{P_{Y_{i} Z_{i} \mid X_{i}}\left(Y_{i}, Z_{i} \mid X_{i}\right)}{P_{Y_{i} \mid Z_{i}}\left(Y_{i} \mid Z_{i}\right) P_{Z_{i} \mid X_{i}}\left(Z_{i} \mid X_{i}\right)} \tag{133}
\end{align*}
$$

where (132) follows from that $P_{\bar{Y}^{n}} \mid \bar{Z}^{n}, P_{Y^{n} Z^{n} \mid X^{n}}$ and $P_{Z^{n} \mid X^{n}}$ are memoryless.

From the sub-additivity of p-limsup [11, Sec. 1.3], and by introducing the product distribution $P_{\bar{Y}^{n} \mid \bar{Z}^{n}}$, we obtain

$$
\begin{align*}
& \bar{I}(\mathbf{Y} ; \mathbf{X} \mid \mathbf{Z})=\mathrm{p}-\limsup _{n \rightarrow \infty}\left(\frac{1}{n} \log \frac{P_{Y^{n} \mid X^{n} Z^{n}}\left(Y^{n} \mid X^{n}, Z^{n}\right)}{P_{\bar{Y}^{n} \mid Z^{n}}\left(Y^{n} \mid Z^{n}\right)}\right. \\
& \left.-\frac{1}{n} \log \frac{P_{Y^{n} \mid Z^{n}}\left(Y^{n} \mid Z^{n}\right)}{P_{\bar{Y}^{n} \mid \bar{Z}^{n}}\left(Y^{n} \mid Z^{n}\right)}\right)  \tag{134}\\
& \leq \mathrm{p}-\limsup _{n \rightarrow \infty} \frac{1}{n} \log \frac{P_{Y^{n} \mid X^{n} Z^{n}}\left(Y^{n} \mid X^{n}, Z^{n}\right)}{P_{\bar{Y}^{n} \mid \bar{Z}^{n}}\left(Y^{n} \mid Z^{n}\right)} \\
& -\mathrm{p}-\liminf _{n \rightarrow \infty} \frac{1}{n} \log \frac{P_{Y^{n}} \mid Z^{n}\left(Y^{n} \mid Z^{n}\right)}{P_{\bar{Y}^{n}} \mid \bar{Z}^{n}\left(Y^{n} \mid Z^{n}\right)} . \tag{135}
\end{align*}
$$

The final term is non-negative following [11, Lem. 3.2.1] and hence

$$
\begin{equation*}
\bar{I}(\mathbf{Y} ; \mathbf{X} \mid \mathbf{Z}) \leq \mathrm{p}-\limsup _{n \rightarrow \infty} U_{n} \tag{136}
\end{equation*}
$$

Now we bound p-limsup $\sup _{n \rightarrow \infty} U_{n}$. First define the information density random variables $L_{i}$ as

$$
\begin{equation*}
L_{i} \triangleq \log \frac{P_{Y_{i} Z_{i} \mid X_{i}}\left(Y_{i}, Z_{i} \mid X_{i}\right)}{P_{Y_{i} \mid Z_{i}}\left(Y_{i} \mid Z_{i}\right) P_{Z_{i} \mid X_{i}}\left(Z_{i} \mid X_{i}\right)} \tag{137}
\end{equation*}
$$

Then

$$
\begin{equation*}
U_{n}=\frac{1}{n} \sum_{i=1}^{n} L_{i} \tag{138}
\end{equation*}
$$

Obviously,

$$
\begin{equation*}
\mathbb{E}\left[L_{i}\right]=I\left(Y_{i} ; X_{i} \mid Z_{i}\right) \tag{139}
\end{equation*}
$$

Now define

$$
\begin{equation*}
P_{\tilde{X}}(x)=\frac{1}{n} \sum_{i=1}^{n} P_{X_{i}}(x) \tag{140}
\end{equation*}
$$

and the induced distribution

$$
\begin{equation*}
P_{\tilde{X} \tilde{Y} \tilde{Z}}(x, y, z)=P_{\tilde{X}}(x) P_{Y Z \mid X}(y, z \mid x) \tag{141}
\end{equation*}
$$

Since $I(Y ; X \mid Z)$ is a concave- $\cap$ function of the input probability distribution $P_{X}$ [19, Thm. 2], we have

$$
\begin{equation*}
\frac{1}{n} \sum_{i=1}^{n} I\left(Y_{i} ; X_{i} \mid Z_{i}\right) \leq I(\tilde{X} ; \tilde{Y} \mid \tilde{Z}) \tag{142}
\end{equation*}
$$

It means that

$$
\begin{equation*}
\mu \triangleq \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} L_{i}\right] \leq I(\tilde{X} ; \tilde{Y} \mid \tilde{Z}) \leq \min \{\log |\mathcal{X}|, \log |\mathcal{Y}|\} \tag{143}
\end{equation*}
$$

Furthermore, by [11, Rmk. 3.1.1]

$$
\begin{equation*}
\operatorname{Var}\left[L_{i}\right] \leq \log \frac{8 \min \{|\mathcal{X}|,|\mathcal{Y}|\}}{e^{2}} \triangleq \sigma_{0}^{2} \tag{144}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Var}\left[\frac{1}{n} \sum_{i=1}^{n} L_{i}\right]=\frac{1}{n^{2}} \sum_{i=1}^{n} \operatorname{Var}\left[L_{i}\right] \leq \frac{\sigma_{0}^{2}}{n} \tag{145}
\end{equation*}
$$

Hence by the Chebyshev inequality, for any $\alpha>0$,

$$
\begin{equation*}
\mathbb{P}\left[\left|\frac{1}{n} \sum_{i=1}^{n} L_{i}-\mu\right| \geq \alpha\right] \leq \frac{\sigma_{0}^{2}}{n \alpha^{2}} \tag{146}
\end{equation*}
$$

The upper bound $\frac{\sigma_{0}^{2}}{n \alpha^{2}}$ clearly tends to zero as $n \rightarrow \infty$. From the definition of p-limsup, we have for any $\alpha>0$,

$$
\begin{equation*}
\text { p- } \limsup _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} L_{i} \leq \limsup _{n \rightarrow \infty} \mu+\alpha \tag{147}
\end{equation*}
$$

Finally combine (136, 138, 143 and 147, then we have

$$
\begin{equation*}
\bar{I}(\mathbf{Y} ; \mathbf{X} \mid \mathbf{Z}) \leq I(\tilde{X} ; \tilde{Y} \mid \tilde{Z})+\alpha \tag{148}
\end{equation*}
$$

In the same way, we have

$$
\begin{equation*}
\frac{1}{\gamma} \bar{I}(\mathbf{S} ; \hat{\mathbf{S}}) \leq \bar{I}(\mathbf{X} ; \mathbf{Y}) \leq I(\tilde{X} ; \tilde{Y})+\alpha \leq C_{B}+\alpha \tag{149}
\end{equation*}
$$

Combining (148) and (149), we have

$$
\begin{align*}
\bar{I}(\mathbf{Y} ; \mathbf{X} \mid \mathbf{Z}) & \leq I(\tilde{X} ; \tilde{Y} \mid \tilde{Z})+\alpha  \tag{150}\\
& \leq \max _{P_{X}: I(X ; Y) \geq \frac{1}{\gamma} \bar{I}(\mathbf{S} ; \hat{\mathbf{S}})-\alpha} I(X ; Y \mid Z)+\alpha  \tag{151}\\
& =\Gamma_{2}\left(\frac{1}{\gamma} \bar{I}(\mathbf{S} ; \hat{\mathbf{S}})-\alpha\right)+\alpha \tag{152}
\end{align*}
$$

where 152 follows from the assumption that the considered channel $P_{Y Z \mid X}$ achieves $\Gamma_{2}(R)$.

Since 149) and 152 hold for any $\alpha>0$ and $\Gamma_{2}(R)$ is continuous in $R$ [8], by letting $\alpha \rightarrow 0$, we have

$$
\begin{align*}
\frac{1}{\gamma} \bar{I}(\mathbf{S} ; \hat{\mathbf{S}}) & \leq C_{B}  \tag{153}\\
\bar{I}(\mathbf{Y} ; \mathbf{X} \mid \mathbf{Z}) & \leq \Gamma_{2}\left(\frac{1}{\gamma} \bar{I}(\mathbf{S} ; \hat{\mathbf{S}})\right) \tag{154}
\end{align*}
$$

Furthermore, from the assumption, we have

$$
\begin{equation*}
\overline{D_{B}}(\mathbf{S}, \hat{\mathbf{S}}) \leq D_{B} \tag{155}
\end{equation*}
$$

From (153)-(155), and the fact that $\Gamma_{2}(R)$ is a decreasing function, we have that to complete the proof, we only need prove there exists a conditional pmf $P_{\hat{S} \mid S}$ such that

$$
\left.\begin{array}{rl}
\bar{I}(\mathbf{S} ; \hat{\mathbf{S}}) & \geq I(S ; \hat{S}) \\
\overline{D_{B}}(\mathbf{S}, \hat{\mathbf{S}}) & \geq \mathbb{E} d_{B}(S, \hat{S}), \\
R_{\mathbf{S} \mid \hat{\mathbf{S}}} & \left(D_{E}\right) \tag{158}
\end{array}\right) R_{S \mid \hat{S}}\left(D_{E}\right) .
$$

For $\left(S^{m}, \hat{S}^{m}\right)$, denote $\check{V}^{m}$ to be a sequence of conditionally independent random variables with pmf $P_{\check{V}^{m} \mid S^{m} \hat{S}^{m}}=$ $\prod_{i=1}^{m} P_{\tilde{V}_{i} \mid S_{i} \hat{S}_{i}}$. Then we have

$$
\begin{equation*}
\underline{R_{\mathbf{S} \mid \mathbf{\mathbf { S }}}}\left(D_{E}\right) \leq \inf _{P_{\tilde{\mathbf{V}} \mid \mathbf{S} \hat{\mathbf{s}}}: \underline{D_{E}}(\mathbf{S}, \check{\mathbf{V}}) \leq D_{E}} \bar{I}(\mathbf{S} ; \check{\mathbf{V}} \mid \hat{\mathbf{S}}) \tag{159}
\end{equation*}
$$

For $\left(S^{m}, \hat{S}^{m}, \check{V}^{m}\right)$, denote $\left(U^{m}, V^{m}\right)$ to be a sequence of independent random variables with pmf $P_{S^{m} U^{m} V^{m}}=$ $\prod P_{S_{i} U_{i} V_{i}}$, and $P_{S_{i} U_{i} V_{i}}=P_{S_{i} \hat{S}_{i} \check{V}_{i}}$ for $1 \leq i \leq m$. Following similar steps to (134)-147, we can get

$$
\begin{align*}
\bar{I}(\mathbf{S} ; \check{\mathbf{V}} \mid \hat{\mathbf{S}}) & \leq \limsup _{m \rightarrow \infty} \mathbb{E}\left[\frac{1}{m} \sum_{i=1}^{m} \log \frac{P_{V_{i} \mid S_{i} U_{i}}\left(\check{V}_{i} \mid S_{i} \hat{S}_{i}\right)}{P_{V_{i} \mid U_{i}}\left(\check{V}_{i} \mid \hat{S}_{i}\right)}\right]  \tag{160}\\
& =\limsup _{m \rightarrow \infty} I\left(S_{Q} ; V_{Q} \mid U_{Q} Q\right) \tag{161}
\end{align*}
$$

where $Q$ is a timesharing random variable and uniformly distributed over $[m]$, independent of other random variables.

By [11, Lem. 5.8.1] and [11, Lem. 5.8.2] we have

$$
\begin{equation*}
\overline{D_{B}}(\mathbf{S}, \hat{\mathbf{S}}) \geq \overline{D_{B}}(\mathbf{S}, \mathbf{U})=\limsup _{m \rightarrow \infty} \mathbb{E} d_{B}\left(S_{Q}, U_{Q}\right) \tag{162}
\end{equation*}
$$

and

$$
\begin{align*}
\bar{I}(\mathbf{S} ; \hat{\mathbf{S}}) & \geq \bar{I}(\mathbf{S} ; \mathbf{U})  \tag{163}\\
& =\limsup _{m \rightarrow \infty} I\left(S_{Q} ; U_{Q} \mid Q\right)  \tag{164}\\
& \geq \limsup _{m \rightarrow \infty} I\left(S_{Q} ; U_{Q}\right) \tag{165}
\end{align*}
$$

Following similar steps to the proof of the first inequality of [11, Thm. 1.7.2] and the proof of [11, Lem. 5.8.1], we have

$$
\begin{equation*}
\underline{D_{E}}(\mathbf{S}, \check{\mathbf{V}}) \leq \underline{D_{E}}(\mathbf{S}, \mathbf{V})=\liminf _{m \rightarrow \infty} \mathbb{E} d_{E}\left(S_{Q}, V_{Q}\right) . \tag{166}
\end{equation*}
$$

According to the definitions of liminf and limsup, for any arbitrarily small $\epsilon>0$, there always exists a sufficiently large $m_{0}$ such that

$$
\begin{equation*}
\limsup _{m \rightarrow \infty} I\left(S_{Q} ; V_{Q} \mid U_{Q} Q\right) \leq I\left(S_{Q} ; V_{Q} \mid U_{Q} Q\right)+\epsilon, \exists m \geq m_{0} \tag{167}
\end{equation*}
$$

$$
\begin{equation*}
\liminf _{m \rightarrow \infty} \mathbb{E} d_{E}\left(S_{Q}, V_{Q}\right) \leq \mathbb{E} d_{E}\left(S_{Q}, V_{Q}\right)+\epsilon, \forall m \geq m_{0} \tag{168}
\end{equation*}
$$

$$
\begin{equation*}
\limsup _{m \rightarrow \infty} \mathbb{E} d_{B}\left(S_{Q}, U_{Q}\right) \geq \mathbb{E} d_{B}\left(S_{Q}, U_{Q}\right)-\epsilon, \forall m \geq m_{0} \tag{169}
\end{equation*}
$$

$$
\begin{equation*}
\limsup _{m \rightarrow \infty} I\left(S_{Q} ; U_{Q}\right) \geq I\left(S_{Q} ; U_{Q}\right)-\epsilon, \forall m \geq m_{0} \tag{170}
\end{equation*}
$$

Hence there exists some $m$ (or equivalently there exists some $\left(V_{Q}, U_{Q}, Q\right)$ ) satisfying the inequalities (167)-170) simultaneously. Combining (159), 161), 166, 167) and (168) gives us (the subscript $Q$ is omitted)

$$
\begin{align*}
R_{\mathbf{S} \mid \hat{\mathbf{S}}}\left(D_{E}\right) & \leq \inf _{P_{V \mid S U Q}: E d_{E}(S, V)+\epsilon \leq D_{E}} I(S ; V \mid U Q)+\epsilon  \tag{171}\\
& =R_{S \mid U Q}\left(D_{E}-\epsilon\right)+\epsilon  \tag{172}\\
& \leq R_{S \mid U}\left(D_{E}-\epsilon\right)+\epsilon, \tag{173}
\end{align*}
$$

where 173 follows from the fact that introducing side information reduces the minimum rate of source coding. Combining (162), 165, 169) and 170 gives us (the subscript $Q$ is omitted)

$$
\begin{align*}
\overline{D_{B}}(\mathbf{S}, \hat{\mathbf{S}}) & \geq \mathbb{E} d_{B}(S, U)-\epsilon  \tag{174}\\
\bar{I}(\mathbf{S} ; \hat{\mathbf{S}}) & \geq I(S ; U)-\epsilon \tag{175}
\end{align*}
$$

Since $\epsilon>0$ is arbitrary and rate-distortion function $R_{S \mid U}(\cdot)$ is continuous, combining (173), (174) and (175) gives us

$$
\begin{align*}
\overline{D_{B}}(\mathbf{S}, \hat{\mathbf{S}}) & \geq \mathbb{E} d_{B}(S, U)  \tag{176}\\
\bar{I}(\mathbf{S} ; \hat{\mathbf{S}}) & \geq I(S ; U)  \tag{177}\\
R_{\mathbf{S} \mid \hat{\mathbf{S}}}\left(D_{E}\right) & \leq R_{S \mid U}\left(D_{E}\right) \tag{178}
\end{align*}
$$

Hence $P_{U \mid S}$ is the desired distribution. This completes the proof of the converse part.

## Appendix E Proof of InNer Bound $\mathcal{R}_{\text {SEP }}^{(i)}$ OF THEOREM 3

It is hard to obtain any satisfactory inner bound through bounding the lossy-equivocation directly, but it is feasible by analyzing the henchman problem instead. Next we prove $\mathcal{R}_{\text {sep }}^{(i)}$ by following similar steps to the proof of the achievability part in [2]. It is worth noting that the complications introduced by a source-channel setting compared to [2] mainly comes from that, for the following equivalent subproblem, the chosen index is not uniformly or nearly-uniformly distributed in general (which is dependent on the codebook) given the wiretapper's received signal. This is different from noiseless channel case [2]. For that case, the chosen index naturally has a uniform conditional distribution. Hence the proof given in [2] cannot be applied to our case directly. To address this difficulty, we first need to find a high-probability set of codebooks on which the chosen index with a nearly-uniform conditional distribution (given the wiretapper's received signal) is generated. Then following similar steps to the proof given in [2], we show that for any codebook in this high-probability set, the probability that the wiretapper achieves the target distortion vanishes. This leads to our result. Furthermore, applying two-layered secrecychannel code is necessary to achieve the desired performance. This makes our problem more complicated.

Before proving $\mathcal{R}_{\text {sep }}^{(i)}$ of Theorem 33, we first consider the following equivalent problem (the proof of the equivalence will be given in Appendix E-B.

## A. Subproblem: lossy compression of a codeword drawn uniformly from a random codebook with side information

Let $\epsilon>0$. Assume $X$ and $Y$ are the variables achieving the maximum in the definition of $\Gamma_{1}(R)$ with $R=\frac{1}{\gamma}(I(S ; \hat{S})+3 \epsilon)$. Then $I(V ; Y \mid U)-I(V ; Z \mid U)=$ $\Gamma_{1}\left(\frac{1}{\gamma}(I(S ; \hat{S})+3 \epsilon)\right)$ and $I(V ; Y) \geq \frac{1}{\gamma}(I(S ; \hat{S})+3 \epsilon)$. Fix

$$
\begin{align*}
R_{0} & =\gamma I(U ; Y)-\epsilon,  \tag{179}\\
R_{1} & =\gamma I(V ; Y \mid U)-\epsilon,  \tag{180}\\
R_{1}^{\prime} & =\gamma I(V ; Z \mid U)+\epsilon, \tag{181}
\end{align*}
$$

and define ${ }^{9}$

$$
\begin{align*}
R_{t} & =R_{0}+R_{1}  \tag{182}\\
R_{c} & =R_{1}-R_{1}^{\prime}  \tag{183}\\
R_{p} & =R_{t}-R_{c}-R_{K}=R_{0}+R_{1}^{\prime}-R_{K} \tag{184}
\end{align*}
$$

Hence $R_{t} \geq I(S ; \hat{S})+\epsilon$. Besides, for simplicity we assume $m \gamma$ is an integer, hence $n=m \gamma$.

Codebook Generation: Randomly and independently generate sequences $\quad \hat{s}^{m}\left(j_{k}, j_{p}, j_{c}\right),\left(j_{k}, j_{p}, j_{c}\right) \quad \in$ $\left[2^{m R_{K}}\right]\left[2^{m R_{p}}\right]\left[2^{m R_{c}}\right]$ with each according to $\prod_{i=1}^{m} P_{\hat{S}}\left(\hat{s}_{i}\right)$. Randomly and independently generate sequences $u^{n}\left(m_{0}\right), m_{0} \in\left[2^{m R_{0}}\right]$ with each according to $\prod_{i=1}^{n} P_{U}\left(u_{i}\right)$, and for each message $m_{0} \in\left[2^{m R_{0}}\right]$, randomly and

[^6]independently generate sequences $v^{n}\left(m_{0}, m_{1}\right), m_{1} \in\left[2^{m R_{1}}\right]$ with each according to $\prod_{i=1}^{n} P_{V \mid U}\left(v_{i} \mid u_{i}\left(m_{0}\right)\right)$. The codebook
\[

$$
\begin{aligned}
\mathcal{C}= & \left\{\hat{s}^{m}\left(j_{k}, j_{p}, j_{c}\right),\left(j_{k}, j_{p}, j_{c}\right) \in\left[2^{m R_{K}}\right]\left[2^{m R_{p}}\right]\left[2^{m R_{c}}\right]\right. \\
& \left.\left(u^{n}\left(m_{0}\right), v^{n}\left(m_{0}, m_{1}\right)\right),\left(m_{0}, m_{1}\right) \in\left[2^{m R_{0}}\right]\left[2^{m R_{1}}\right]\right\},
\end{aligned}
$$
\]

is revealed to all parties including the wiretapper.
Subproblem: Denote $j=\left(j_{k}, j_{p}, j_{c}\right)^{10}$ then choose an index $J$ uniformly at random from $\left[2^{m R_{K}}\right]\left[2^{m R_{p}}\right]\left[2^{m R_{c}}\right]$ and generate $M_{k}=J_{k} \wedge K$, where $\wedge$ is the one-time pad operation. Map $\left(M_{k}, J_{p}\right)$ into $\left(M_{0}, M_{1}^{\prime}\right) \in\left[2^{m R_{0}}\right]\left[2^{m R_{1}^{\prime}}\right]$ through an arbitrary bijective function $\left(m_{0}, m_{1}^{\prime}\right)=g\left(m_{k}, j_{p}\right)$. Denote $M \triangleq\left(M_{0}, M_{1}\right), M_{1} \triangleq\left(M_{1}^{\prime}, M_{c}\right), M_{c} \triangleq J_{c}$. Given $K=k$, the mapping between $M$ and $J$, denoted as $j(m, k)$ or $m(j, k)$, is also bijective. Moreover, since $J$ follows a uniform distribution, $M$ follows a uniform distribution as well.

Based on the codebook above, on one hand, pass $\hat{s}^{m}(J)$ through a memoryless channel $\prod P_{S \mid \hat{S}}$ to generate a sequence $S^{m}$; on the other hand, pass $v^{n}(M)$ through a memoryless channel $\prod P_{Z \mid V}$ to generate a sequence $Z^{n}$ (this is equivalent to applying a stochastic channel encoder $P_{X^{n} \mid M, \mathcal{C}}=$ $\prod_{i=1}^{n} P_{X \mid V}\left(x_{i} \mid v_{i}(M)\right)$ for the channel $\left.\prod P_{Z \mid X}\right)$. Finally, transmit $S^{m}$ over a $R_{m}$ rate noiseless channel with the help of two-sided information $\left(Z^{n}, M_{0}\right)$. See Fig. 4 .

The distribution $Q_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}}$ incurred by the subproblem above can be expressed as

$$
\begin{align*}
& Q_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}} \\
\triangleq & P_{K} Q_{S^{m} J} P_{\hat{S}^{m} \mid J} P_{M \mid K J} P_{U^{n} V^{n} \mid M} P_{Y^{n} Z^{n} \mid V^{n}} \\
= & 2^{-m R_{K}} Q_{S^{m} J} 1\left\{\hat{s}^{m}=\hat{S}^{m}(j)\right\} P_{M \mid K J} \\
& 1\left\{u^{n}=U^{n}\left(m_{0}\right), v^{n}=V^{n}(m)\right\} \prod P_{Y Z \mid V} \tag{185}
\end{align*}
$$

where $P_{M \mid K J}$ is the distribution induced by the one-time pad operation, and

$$
\begin{equation*}
Q_{S^{m} J}\left(s^{m}, j\right) \triangleq 2^{-m R_{t}} \prod_{i=1}^{m} P_{S \mid \hat{S}}\left(s_{i} \mid \hat{s}_{i}(j)\right) \tag{186}
\end{equation*}
$$

It is easy to verify

$$
\begin{equation*}
\mathbb{E}_{\mathcal{C}}\left[Q_{S^{m} \hat{S}^{m}}\left(s^{m}, \hat{s}^{m}\right)\right]=\prod_{i=1}^{m} P_{S \hat{S}}\left(s_{i}, \hat{s}_{i}\right) \tag{187}
\end{equation*}
$$

and
$\mathbb{E}_{\mathcal{C}}\left[Q_{U^{n} V^{n} Y^{n} Z^{n}}\left(u^{n}, v^{n}, y^{n}, z^{n}\right)\right]=\prod_{i=1}^{n} P_{U V Y Z}\left(u_{i}, v_{i}, y_{i}, z_{i}\right)$.
For this subproblem, we have the following theorem.
Theorem 7. If

$$
\begin{align*}
\limsup _{m \rightarrow \infty} R_{m} \leq \min & \left\{R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma}(I(S ; \hat{S})+3 \epsilon)\right)\right. \\
& \left.+R_{S \mid \hat{S}}\left(D_{E}\right), R_{S}\left(D_{E}\right)\right\}-5 \epsilon \tag{189}
\end{align*}
$$

then
$\lim _{m \rightarrow \infty} \mathbb{E}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{R_{m} \text { codes }} \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right]\right]=0$.

[^7]

Fig. 4. Lossy compression of a codeword drawn uniformly from a random codebook with side information.

Proof of Theorem 7. To prove Theorem 7 we need the following lemma, the proof of which is given in Appendix $F$
Lemma 5. For any sequence of random variables $\left\{X_{n}\right\}$ and any sequence of events $\left\{\mathcal{A}_{n}\right\}, \lim _{n \rightarrow \infty} \mathbb{P}\left(\mathcal{A}_{n}\right)=0$, if and only if $\lim _{n \rightarrow \infty} \mathbb{P}\left[\mathbb{P}\left(\mathcal{A}_{n} \mid X_{n}\right)>\tau_{n}\right]=0$ for some sequence $\left\{\tau_{n}\right\}$ such that $\tau_{n}>0$ and $\lim _{n \rightarrow \infty} \tau_{n}=0$.

From Lemma 5, to prove Theorem 7 we only need to show that if $R_{m}$ satisfies 189), then
$\lim _{m \rightarrow \infty} \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{R_{m} \text { codes }} \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right]>\tau_{m}\right]$ $=0$,
for some sequence $\left\{\tau_{m}\right\}$ with $\tau_{m}>0$ and $\lim _{m \rightarrow \infty} \tau_{m}=0$. Next we prove this.

First define several events

$$
\left.\begin{array}{rl} 
& \mathcal{A}_{1} \triangleq\left\{\left(S^{m}, \hat{S}^{m}(J)\right) \in \mathcal{T}_{\delta}^{m}\right\} \\
& \mathcal{A}_{2} \triangleq\left\{\left(U^{n}\left(M_{0}\right), Z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}\right\} \\
& \mathcal{A}_{3} \triangleq\left\{\left(U^{n}\left(M_{0}\right), V^{n}\left(M_{0}, M_{1}\right), Z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}, \\
\mathcal{A}_{123} \triangleq \mathcal{A}_{1} \mathcal{A}_{2} \mathcal{A}_{3}, \\
& \mathcal{A}_{23} \triangleq \mathcal{A}_{2} \mathcal{A}_{3}, \\
\mathcal{B} \triangleq & \min \triangleq\left\{\begin{array}{l}
\left(m_{0}, m_{c}, z^{n}\right):\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n} \\
\max _{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right) \geq 1 \\
\left(m_{0}, m_{c}, z^{n}\right):\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}
\end{array} \phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right) \leq 2^{2 m \epsilon}\right\} \tag{197}
\end{array}\right\},
$$

where $0<\delta^{\prime}<\delta$,

$$
\begin{align*}
& \sum_{m_{1}^{\prime} \in\left[2^{m R_{1}^{\prime}}\right]}^{\phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right) \triangleq} 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}^{\prime}, m_{c}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}
\end{align*}
$$

and $\epsilon$ is the same to that in 189). Observe that if the codebook satisfies $\mathcal{B}$, then for any $\left(m_{0}, m_{c}, z^{n}\right) \in\left[2^{m R_{0}}\right]\left[2^{m R_{c}}\right] \mathcal{Z}^{n}$ such that $\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}$, it holds that

$$
\begin{align*}
2^{m R_{c}} & \leq \sum_{m_{1} \in\left[2^{m R_{1}}\right]} 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\} \\
& \leq 2^{m\left(R_{c}+2 \epsilon\right)} \tag{199}
\end{align*}
$$

The $\delta$-typical set is defined according to the notion of strong typicality, see [17]:

$$
\begin{equation*}
\mathcal{T}_{\delta}^{m}(S) \triangleq\left\{s^{m} \in \mathcal{S}^{m}:\left|T_{s^{m}}-P_{S}\right|<\delta P_{S}\right\} \tag{200}
\end{equation*}
$$

where $T_{s^{m}}$ denotes the type (or empirical distribution) of $s^{m}$. For simplicity, $\mathcal{T}_{\delta}^{m}(S)$ is also shortly denoted as $\mathcal{T}_{\delta}^{m}$.

Then we have the following lemmas.
Lemma 6. $\lim _{m \rightarrow \infty} \mathbb{P}\left[\mathcal{A}_{123}\right]=1$.
Lemma 7. $\lim _{m \rightarrow \infty} \mathbb{P}[\mathcal{B}]=1$.
Lemma 8. For any codebook c satisfying $\mathcal{B}$,

$$
\begin{aligned}
& \quad \mathbb{P}\left(M_{1}=m_{1}, \mathcal{A}_{3} \mid \mathcal{C}=c, Z^{n}=z^{n}, M_{0}=m_{0}\right) \\
& \leq 2^{-m\left(R_{c}-\epsilon_{\delta}\right)} 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}
\end{aligned}
$$

where $\epsilon_{\delta}$ is a term such that $\epsilon_{\delta} \rightarrow 0$ as $\delta \rightarrow 0$.
The proofs of Lemma 7 and Lemma 8 are given in Appendices $G$ and $H$, respectively. Furthermore, from 187) and (188), and the fact typical set has total probability close to one [17], we can easily establish Lemma 6

Consider that the optimal $R_{m}$-rate code that maximizes $\quad \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right]$ is adopted. Then we only need to show $\lim _{m \rightarrow} \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\mathbb{P}\left[d_{E}\left(S^{m}, \breve{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right]>\tau_{m}\right]=0$ $m \rightarrow \infty$
for this code. By utilizing Lemmas 5, 6 and 7, we have

$$
\begin{align*}
& \quad \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right]>\tau_{m}\right] \\
& \leq \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right]>\tau_{m}, \mathcal{B},\right. \\
& \left.\mathbb{P}\left[\mathcal{A}_{123}^{c} \mid \mathcal{C} Z^{n} M_{0}\right] \leq \epsilon_{m}\right]+\mathbb{P}\left[\mathcal{B}^{c}\right] \\
& \quad+\mathbb{P}\left[\mathbb{P}\left[\mathcal{A}_{123}^{c} \mid \mathcal{C} Z^{n} M_{0}\right]>\epsilon_{m}\right]  \tag{201}\\
& \leq \\
& \leq \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right]>\tau_{m}, \mathcal{B},\right.  \tag{202}\\
& \\
& \left.\quad \mathbb{P}\left[\mathcal{A}_{123}^{c} \mid \mathcal{C} Z^{n} M_{0}\right] \leq \epsilon_{m}\right]+\epsilon_{m}^{\prime},
\end{align*}
$$

for some $\epsilon_{m}$ and $\epsilon_{m}^{\prime}$ such that $\epsilon_{m} \rightarrow 0$ and $\epsilon_{m}^{\prime} \rightarrow 0$ as $m \rightarrow \infty$. Furthermore,

$$
\begin{align*}
& \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\leq & \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right] \\
& \quad+\mathbb{P}\left[\mathcal{A}_{123}^{c} \mid \mathcal{C} Z^{n} M_{0}\right]  \tag{203}\\
\leq & \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right]+\epsilon_{m} \tag{204}
\end{align*}
$$

where (204) follows from the event $\mathbb{P}\left[\mathcal{A}_{123}^{c} \mid \mathcal{C} Z^{n} M_{0}\right] \leq \epsilon_{m}$ in (202).

Owing to the rate constraint, given $\left(\mathcal{C}, Z^{n}, M_{0}\right)$, the reconstruction $\check{S}^{m}$ cannot take more than $R_{m}$ values. Denote the set of possible values as $c\left(\mathcal{C}, Z^{n}, M_{0}\right)$. Then

$$
\begin{align*}
& \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right] \\
= & \mathbb{P}\left[\min _{\check{s}^{m} \in c\left(\mathcal{C}, Z^{n}, M_{0}\right)} d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right] . \tag{205}
\end{align*}
$$

Now we apply a union bound to 205 and write

$$
\begin{align*}
& \mathbb{P}\left[\min _{\check{s}^{m} \in c\left(\mathcal{C}, Z^{n}, M_{0}\right)} d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right] \\
& \leq \sum_{\check{s}^{m} \in c\left(\mathcal{C}, Z^{n}, M_{0}\right)} \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right]  \tag{206}\\
& \leq 2^{m R_{m}} \max _{\tilde{s}^{m} \in c\left(\mathcal{C}, Z^{n}, M_{0}\right)} \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right] \\
& \leq 2^{m R_{m}} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right]  \tag{207}\\
& =2^{m R_{m}} \max _{\widetilde{s}^{m} \in \overline{\mathcal{S}}^{m}} \sum_{m_{1}=1}^{2^{m R_{1}}} \sum_{k=1}^{2^{m R_{K}}} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{23} \mid \mathcal{C} Z^{n} M_{0}\right]  \tag{208}\\
& \begin{array}{l}
=2^{m R_{m}} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{m_{1}=1} \sum_{k=1} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{23} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\quad \times \mathbb{P}\left[K=k \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M\right] \\
\quad \times \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M K\right] \\
\leq 2^{m\left(R_{m}-R_{K}\right)} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{m_{1}=1}^{2^{m R_{1}}} \sum_{k=1}^{2^{m R_{K}}} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{3} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\quad \times \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \hat{S}^{m}(j(m, k))\right],
\end{array}  \tag{209}\\
& \begin{array}{l}
=2^{m R_{m}} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{m_{1}=1} \sum_{k=1} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{23} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\quad \times \mathbb{P}\left[K=k \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M\right] \\
\quad \times \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M K\right] \\
\leq 2^{m\left(R_{m}-R_{K}\right)} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{m_{1}=1}^{2^{m R_{1}}} \sum_{k=1}^{2^{m R_{K}}} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{3} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\quad \times \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \hat{S}^{m}(j(m, k))\right],
\end{array} \\
& \begin{array}{l}
=2^{m R_{m}} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{m_{1}=1} \sum_{k=1} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{23} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\quad \times \mathbb{P}\left[K=k \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M\right] \\
\quad \times \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M K\right] \\
\leq 2^{m\left(R_{m}-R_{K}\right)} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{m_{1}=1}^{2^{m R_{1}}} \sum_{k=1}^{2^{m R_{K}}} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{3} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\quad \times \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \hat{S}^{m}(j(m, k))\right],
\end{array} \\
& \begin{array}{l}
=2^{m R_{m}} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{m_{1}=1} \sum_{k=1} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{23} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\quad \times \mathbb{P}\left[K=k \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M\right] \\
\quad \times \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M K\right] \\
\leq 2^{m\left(R_{m}-R_{K}\right)} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{m_{1}=1}^{2^{m R_{1}}} \sum_{k=1}^{2^{m R_{K}}} \mathbb{P}\left[M_{1}=m_{1}, \mathcal{A}_{3} \mid \mathcal{C} Z^{n} M_{0}\right] \\
\quad \times \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \hat{S}^{m}(j(m, k))\right],
\end{array}
\end{align*}
$$

where (206) follows from a union bound, and 210) follows from $\mathbb{P}\left[K=k \mid \mathcal{A}_{23} \mathcal{C} Z^{n} M\right]=\mathbb{P}[K=k \mid M]=2^{-m R_{K}}$ and $\mathcal{A}_{23} \mathcal{C} Z^{n} M K \rightarrow \hat{S}^{m}(J) \rightarrow S^{m} \mathcal{A}_{1}$.

Furthermore, for any codebook $c$ satisfying $\mathcal{B}$ and for any $\left(m_{0}, m_{c}, z^{n}\right) \in\left[2^{m R_{0}}\right]\left[2^{m R_{c}}\right] \mathcal{Z}^{n}$ such that $\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}$, there exists at most $2^{2 m \epsilon}$ codewords $V^{n}\left(m_{0}, m_{1}^{\prime}, m_{c}\right)$ such that $\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}^{\prime}, m_{c}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}$. Define
$\mathcal{A}_{m_{0}, m_{c}, z^{n}} \triangleq\left\{m_{1}^{\prime}:\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}^{\prime}, m_{c}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}$.

Then we can bound

$$
\begin{equation*}
\left|\mathcal{A}_{m_{0}, m_{c}, z^{n}}\right| \leq 2^{2 m \epsilon} \tag{211}
\end{equation*}
$$

for any $\left(m_{0}, m_{c}, z^{n}\right) \in\left[2^{m R_{0}}\right]\left[2^{m R_{c}}\right] \mathcal{Z}^{n}$ such that $\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}$.

Combining 205, 210 and Lemma 8, we have

$$
\begin{align*}
& \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E}, \mathcal{A}_{123} \mid \mathcal{C} Z^{n} M_{0}\right] \\
& \leq 2^{m \lambda} \max _{\check{s}^{m} \in \mathcal{S}^{m}} \sum_{k=1}^{2^{m R_{K}}} \sum_{m_{c}=1}^{2^{m R_{c}}} \sum_{m_{1}^{\prime} \in \mathcal{A}_{m_{0}, m_{c}, z^{n}}} \\
& \quad \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \hat{S}^{m}(j(m, k))\right] \tag{212}
\end{align*}
$$

where $\lambda=R_{m}-R_{K}-R_{c}+\epsilon_{\delta}$. Combining (202, (204) and
(212), we further have

$$
\begin{align*}
& \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E}\right]>\tau_{m}\right] \\
& \leq \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{\check{s}^{m} \in \tilde{\mathcal{S}}^{m}} \sum_{k=1}^{2^{m R_{K}}} \sum_{2_{c}=1}^{2^{m R_{c}}} \sum_{m_{1}^{\prime} \in \mathcal{A}_{m_{0}, m_{c}, z^{n}}}\right. \\
& \left.\eta_{m, k, \check{s}^{m}}>\tau_{m}^{\prime} 2^{-m \lambda}\right]+\epsilon_{m}^{\prime}  \tag{213}\\
& \leq \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{\check{s}^{m} \in \check{\mathcal{S}}^{m}, m_{1}^{\prime} \in\left[2^{\left.m R_{1}^{\prime}\right]}\right.} \sum_{k=1}^{2^{m R_{K}}} \sum_{2_{c}^{m R_{c}}} \sum_{m_{c}=1}^{\left.\eta_{m, k, \check{s}^{m}}>\tau_{m}^{\prime} 2^{-m \lambda^{\prime}}\right]+\epsilon_{m}^{\prime}}\right. \\
& \leq 2^{m R_{1}^{\prime}}|\check{\mathcal{S}}|^{m} \max _{\check{s}^{m} \in \check{\mathcal{S}}^{m}, m_{1}^{\prime} \in\left[2^{m R_{1}^{\prime}}\right]}  \tag{214}\\
& \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\sum_{k=1}^{2^{m R_{K}}} \sum_{2_{c}^{m R_{c}}}^{\sum_{m_{c}=1}} \eta_{m, k, \breve{s}^{m}}>\tau_{m}^{\prime} 2^{-m \lambda^{\prime}}\right]+\epsilon_{m}^{\prime},
\end{align*}
$$

where $\tau_{m}^{\prime}=\tau_{m}-\epsilon_{m}, \lambda^{\prime}=\lambda+2 \epsilon$ and

$$
\begin{align*}
& \eta_{m, k, \breve{s}^{m}} \triangleq \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \hat{S}^{m}(j(m, k))\right]  \tag{216}\\
& =\sum_{s^{m} \in \mathcal{S}^{m}} \prod_{i=1}^{m} P_{S \mid \hat{S}}\left(s_{i} \mid \hat{S}_{i}(j(m, k))\right) \\
& 1\left\{d_{E}\left(s^{m}, \check{s}^{m}\right) \leq D_{E},\left(s^{m}, \hat{S}^{m}(j(m, k))\right) \in \mathcal{T}_{\delta}^{n}\right\} \tag{217}
\end{align*}
$$

To guarantee $\tau_{m}^{\prime}>0$ for any $m$, we choose the sequence $\left\{\tau_{m}\right\}$ such that $\tau_{m}>\epsilon_{m}$.

If we can show that the probability in (215) decays doubly exponentially fast with $m$, then the proof will be complete. Now we prove this.

Note that $\eta_{m, k,,^{m}}$ is a quantity depending on $\hat{S}^{m}(j(m, k))$, and the one-time pad satisfies that given $m_{k}$, different $k$ 's correspond to different $j_{k}$ 's. Hence given $\left(m_{0}, m_{1}^{\prime}\right)$, for different $\left(k, m_{c}\right)$ 's, $j(m, k)$ 's are different as well. This guarantees that given $\left(\check{s}^{m}, m_{0}, m_{1}^{\prime}\right), \eta_{m, k, \breve{s}^{m}},\left(m_{c}, k\right) \in\left[2^{m R_{c}}\right]\left[2^{m R_{K}}\right]$ are i.i.d. random variables due to the nature of the random codebook, with mean

$$
\begin{align*}
& \mathbb{E}_{\mathcal{C}} \eta_{m, k, \check{s}^{m}} \\
& =\mathbb{E}_{\mathcal{C}} \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \mathcal{A}_{1} \mid \hat{S}^{m}(j(m, k))\right]  \tag{218}\\
& \leq \mathbb{E}_{\mathcal{C}} \mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, S^{m} \in \mathcal{T}_{\delta}^{m} \mid \hat{S}^{m}(j(m, k))\right]  \tag{219}\\
& =\mathbb{P}\left[d_{E}\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, S^{m} \in \mathcal{T}_{\delta}^{m}\right] \tag{220}
\end{align*}
$$

Now we need the following lemmas.
Lemma 9. [2] If $S^{m}$ is i.i.d. according to $P_{S}$, then for any $\check{s}^{m}$,

$$
\begin{equation*}
\mathbb{P}\left[d\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, S^{m} \in \mathcal{T}_{\delta}^{m}\right] \leq 2^{-m\left(R_{S}\left(D_{E}\right)-\epsilon_{m, \delta}\right)} \tag{221}
\end{equation*}
$$

where $\epsilon_{m, \delta}$ is a term that vanishes as $\delta \rightarrow 0$ and $m \rightarrow \infty$.

Lemma 10. [2] Fix $P_{S \hat{S}}$ and $\hat{s}^{m} \in \hat{\mathcal{S}}^{m}$. If $S^{m}$ is distributed according to $\prod_{i=1}^{m} P_{S \mid \hat{S}=\hat{s}_{i}}$, then for any $\check{s}^{m}$,

$$
\begin{align*}
& \mathbb{P}\left[d\left(S^{m}, \check{s}^{m}\right) \leq D_{E},\left(S^{m}, \hat{s}^{m}\right) \in \mathcal{T}_{\delta}^{m} \mid \hat{S}^{m}=\hat{s}^{m}\right] \\
& \quad \leq 2^{-m\left(R_{S \mid \hat{S}}\left(D_{E}\right)-\epsilon_{m, \delta}\right)} \tag{222}
\end{align*}
$$

where $\epsilon_{m, \delta}$ is a term that vanishes as $\delta \rightarrow 0$ and $m \rightarrow \infty$.
Lemma 11. [2] If $X^{l}$ is a sequence of i.i.d. random variables on the interval $[0, a]$ with $\mathbb{E}\left[X_{i}\right]=p$, then

$$
\begin{equation*}
\mathbb{P}\left[\sum_{i=1}^{l} X_{i}>k\right] \leq\left(\frac{e \cdot l \cdot p}{k}\right)^{k / a} \tag{223}
\end{equation*}
$$

From Lemmas 9 and 10, we see that

$$
\begin{align*}
\mathbb{E}_{\mathcal{C}} \eta_{m, k, \check{s}^{m}} & \leq 2^{-m\left(R_{S}\left(D_{E}\right)-\epsilon_{m, \delta}\right)}  \tag{224}\\
\eta_{m, k, \check{s}^{m}} & \leq 2^{-m\left(R_{S \mid \hat{S}}\left(D_{E}\right)-\epsilon_{m, \delta}\right)} \tag{225}
\end{align*}
$$

Using these bounds, we can apply Lemma 11 to the probability in 215) by identifying

$$
\begin{align*}
& l \leq 2^{m\left(R_{K}+R_{c}\right)}  \tag{226}\\
& a=2^{-m\left(R_{S \mid \hat{S}}\left(D_{E}\right)-\epsilon_{m, \delta}\right)}  \tag{227}\\
& p \leq 2^{-m\left(R_{S}\left(D_{E}\right)-\epsilon_{m, \delta}\right)}  \tag{228}\\
& k=\tau_{m}^{\prime} 2^{-m \lambda^{\prime}}=\tau_{m}^{\prime} 2^{-m\left(R_{m}-R_{K}-R_{c}+\epsilon_{\delta}+2 \epsilon\right)} \tag{229}
\end{align*}
$$

Set $\left\{\tau_{m}^{\prime}\right\}$ with $\tau_{m}^{\prime}>0$ to be a sub-exponential sequence (i.e., $\left.\tau_{m}^{\prime}=2^{-o(m)}\right)$ by choosing a proper $\left\{\tau_{m}\right\}$. Then we have

$$
\begin{equation*}
\mathbb{P}\left[\sum_{k=1}^{2^{m R_{K}}} \sum_{m_{c}=1}^{2^{m R_{c}}} \eta_{m, k, \check{s}^{m}}>\tau_{m}^{\prime} 2^{-m \lambda^{\prime}}\right] \leq 2^{-m \alpha 2^{m \beta}} \tag{230}
\end{equation*}
$$

where

$$
\begin{align*}
\alpha= & -\left(R_{K}+R_{c}\right)+R_{S}\left(D_{E}\right)-\epsilon_{m, \delta} \\
& \quad-\left(R_{m}-R_{K}-R_{c}+\epsilon_{\delta}+2 \epsilon\right) \\
= & R_{S}\left(D_{E}\right)-R_{m}-2 \epsilon-\epsilon_{\delta}-\epsilon_{m, \delta} \\
\geq & 3 \epsilon-\epsilon_{\delta}-\epsilon_{m, \delta}  \tag{231}\\
\beta= & R_{S \mid \hat{S}}\left(D_{E}\right)-\epsilon_{m, \delta}-\left(R_{m}-R_{K}-R_{c}+\epsilon_{\delta}+2 \epsilon\right) \\
= & R_{S \mid \hat{S}}\left(D_{E}\right)+R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma}(I(S ; \hat{S})+3 \epsilon)\right) \\
& \quad-R_{m}-\epsilon_{\delta}-4 \epsilon-\epsilon_{m, \delta} \\
\geq & \epsilon-\epsilon_{\delta}-\epsilon_{m, \delta} . \tag{232}
\end{align*}
$$

For any fixed $\epsilon$, large enough $m$ and small enough $\delta$, both $\alpha$ and $\beta$ are positive and bounded away from zero. Then (230) vanishes doubly exponentially fast, and it in turn implies 215, vanishes. This completes the proof of Theorem 7.

## B. Likelihood encoder

Consider the codebook defined in the above subproblem, and define a likelihood encoder by $P_{J \mid S^{m}}\left(j \mid s^{m}\right) \propto$ $\prod_{i=1}^{m} P_{S \mid \hat{S}}\left(s_{i} \mid \hat{s}_{i}(j)\right)$, where $\propto$ indicates that appropriate normalization is required. Now we consider Node A concatenates this likelihood encoder, the one-time pad $M_{k}=J_{k} \wedge K$, the bijective function $\left(m_{0}, m_{1}^{\prime}\right)=g\left(m_{k}, j_{p}\right)$, and the stochastic
channel encoder $P_{X^{n} \mid M, \mathcal{C}}=\prod_{i=1}^{n} P_{X \mid V}\left(x_{i} \mid v_{i}(M)\right)$ as described in the subproblem above. For such cascaded encoder, the induced overall distribution is

$$
\begin{align*}
& P_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}} \\
& =P_{K} P_{S^{m} J} P_{\hat{S}^{m} \mid J} P_{M \mid K J} P_{U^{n} V^{n} \mid M} P_{Y^{n} Z^{n} \mid V^{n}} \\
& \triangleq 2^{-m R_{K}} P_{S^{m} J} 1\left\{\hat{s}^{m}=\hat{S}^{m}(j)\right\} P_{M \mid K J} \\
& \quad 1\left\{u^{n}=U^{n}\left(m_{0}\right), v^{n}=V^{n}(m)\right\} \prod P_{Y Z \mid V} \tag{233}
\end{align*}
$$

Furthermore, $P_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}}$ is intimately related to the idealized distribution $Q_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}}$ which is defined in previous subsection.

Schieler and Cuff [2, Sec. VIII-B] showed that if $R_{t}>$ $I(S ; \hat{S})$, then

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{E}_{\mathcal{C}}\left\|P_{S^{m} J}-Q_{S^{m} J}\right\|_{T V}=0 \tag{234}
\end{equation*}
$$

where $R_{t}$ given in 182 denotes the exponent of the size of codebook. Using the property (4), we further have
$\lim _{m \rightarrow \infty} \mathbb{E}_{\mathcal{C}}\left\|P_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}}-Q_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}}\right\|_{T V^{\text {and }}}$ $=\lim _{m \rightarrow \infty} \mathbb{E}_{\mathcal{C}}\left\|P_{S^{m} J}-Q_{S^{m} J}\right\|_{T V}=0$.
Combine this with Theorem7, utilize the property (3), and let $\epsilon \rightarrow 0$, then we have if $R_{L} \leq \min \left\{R_{K}+\gamma \Gamma_{1}\left(\frac{1}{\gamma}(I(S ; \hat{S}))\right)+\right.$ $\left.R_{S \mid \hat{S}}\left(D_{E}\right), R_{S}\left(D_{E}\right)\right\}$, the cascaded encoder above satisfies

$$
\lim _{m \rightarrow \infty} \mathbb{E}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{R_{m} H \text { codes }} \mathbb{P}\left[d_{E}\left(S^{m}, \check{S}^{m}\right) \leq D_{E} \mid \mathcal{C} Z^{n} M_{0}\right]\right]
$$

$$
\begin{equation*}
=0 \tag{236}
\end{equation*}
$$

It implies that the wiretapper cannot decrypt source even when both $Z^{n}$ and $M_{0}$ are revealed to him. Hence as a weaker version, he cannot decrypt source as well when only $Z^{n}$ is revealed to him. This completes the proof of the secrecy constraint.

To complete the proof of the achievability part, we now show that the cascaded encoder above can achieve the distortion $\mathbb{E} d_{B}(S, \hat{S})$ for the legitimate receiver. Instead of considering the induced distribution $P_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}}$ directly, we prove this holds under the desired distribution $Q_{K S^{m} J M \hat{S}^{m} U^{n} V^{n} Y^{n} Z^{n}}$. It is easy to verify that $M$ is uniformly distributed under $Q$. Furthermore, the codebook is randomly generated, and given $M$, the signal $Y^{n}$ follows $\prod_{i=1}^{n} P_{Y \mid V}\left(y_{i} \mid v_{i}(M)\right)$. It is well known that a good channel decoder (e.g., joint typicality decoder) with respect to the memoryless channel $P_{Y \mid V}$ will drive the error probability to zero as $m$ goes to infinity, if $R_{t}<\gamma I(V ; Y)$. That is

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{E}_{\mathcal{C}} \mathbb{P}_{Q}[\hat{M} \neq M]=0 \tag{237}
\end{equation*}
$$

Then using the secret key, the legitimate user could recover $J$ with high probability. That is

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{E}_{\mathcal{C}} \mathbb{P}_{Q}[\hat{J} \neq J]=0 \tag{238}
\end{equation*}
$$

Furthermore, 187) implies $\left(S^{m}, \hat{S}^{m}(J)\right)$ is an i.i.d sequence under $Q$, hence by law of large numbers, we have for any $\tau>0$,

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{P}_{Q}\left[d_{B}\left(S^{m}, \hat{S}^{m}(J)\right) \leq \mathbb{E} d_{B}(S, \hat{S})+\tau\right]=1 \tag{239}
\end{equation*}
$$

This implies under distribution $Q, \mathbb{E} d_{B}(S, \hat{S})$ is achieved by the legitimate user. Further, since the total variance between $Q$ and $P$ vanishes as $m \rightarrow \infty, 239$ also holds under the distribution $P$. That is, the cascaded encoder above achieves distortion $\mathbb{E} d_{B}(S, \hat{S})$ for the legitimate user. This completes the proof of $\mathcal{R}_{\text {sep }}^{(i)}$.

## Appendix F

Proof of Lemma 5
Observe for any $\alpha>0$, any random variable $X$, and any event $\mathcal{A}$,

$$
\begin{align*}
\mathbb{P}(\mathcal{A}) & =\mathbb{E}_{X} \mathbb{P}(\mathcal{A} \mid X) \\
& \geq \mathbb{E}_{X}[\mathbb{P}(\mathcal{A} \mid X) 1\{\mathbb{P}(\mathcal{A} \mid X) \geq \alpha\}] \\
& \geq \alpha \mathbb{E}_{X}[1\{\mathbb{P}(\mathcal{A} \mid X) \geq \alpha\}] \\
& =\alpha \mathbb{P}[\mathbb{P}(\mathcal{A} \mid X) \geq \alpha] \tag{240}
\end{align*}
$$

$$
\begin{align*}
\mathbb{P}(\mathcal{A})= & \mathbb{E}_{X} \mathbb{P}(\mathcal{A} \mid X) \\
= & \mathbb{E}_{X}[\mathbb{P}(\mathcal{A} \mid X) 1\{\mathbb{P}(\mathcal{A} \mid X) \geq \alpha\}] \\
& \quad+\mathbb{E}_{X}[\mathbb{P}(\mathcal{A} \mid X) 1\{\mathbb{P}(\mathcal{A} \mid X)<\alpha\}] \\
\leq & \mathbb{E}_{X}[1\{\mathbb{P}(\mathcal{A} \mid X) \geq \alpha\}] \\
& \quad+\alpha \mathbb{E}_{X}[1\{\mathbb{P}(\mathcal{A} \mid X)<\alpha\}] \\
\leq & \mathbb{P}[\mathbb{P}(\mathcal{A} \mid X) \geq \alpha]+\alpha \tag{241}
\end{align*}
$$

Then we have the following lemma.
Lemma 12. For any $\alpha>0$, any random variable $X$, and any event $\mathcal{A}, \mathbb{P}(\mathcal{A})-\alpha \leq \mathbb{P}[\mathbb{P}(\mathcal{A} \mid X) \geq \alpha] \leq \frac{\mathbb{P}(\mathcal{A})}{\alpha}$.

Consider a sequence of random variables $\left\{X_{n}\right\}$, and a sequence of events $\left\{\mathcal{A}_{n}\right\}$. Applying the lemma above, we have

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{A}_{n}\right)-\tau_{n} \leq \mathbb{P}\left[\mathbb{P}\left(\mathcal{A}_{n} \mid X_{n}\right) \geq \tau_{n}\right] \leq \frac{\mathbb{P}\left(\mathcal{A}_{n}\right)}{\tau_{n}} \tag{242}
\end{equation*}
$$

for any positive sequence $\left\{\tau_{n}\right\}$. If $\lim _{n \rightarrow \infty} \mathbb{P}\left(\mathcal{A}_{n}\right)=0$, then from the fact that no a worst convergent series exists [26], we have there exists a sequence of real numbers $\left\{\tau_{n}\right\}$ such that $\lim _{n \rightarrow \infty} \tau_{n}=0$ and $\lim _{n \rightarrow \infty} \frac{\mathbb{P}\left(\mathcal{A}_{n}\right)}{\tau_{n}}=$ 0 . Hence $\lim _{n \rightarrow \infty} \mathbb{P}\left[\mathbb{P}\left(\mathcal{A}_{n} \mid X_{n}\right) \geq \tau_{n}\right]=0$. On the other hand, if $\lim _{n \rightarrow \infty} \mathbb{P}\left[\mathbb{P}\left(\mathcal{A}_{n} \mid X_{n}\right) \geq \tau_{n}\right]=0$ for some sequence $\tau_{n}$ with $\lim _{n \rightarrow \infty} \tau_{n}=0$, then $\lim _{n \rightarrow \infty} \mathbb{P}\left(\mathcal{A}_{n}\right) \leq$ $\lim _{n \rightarrow \infty}\left\{\mathbb{P}\left[\mathbb{P}\left(\mathcal{A}_{n} \mid X_{n}\right) \geq \tau_{n}\right]+\tau_{n}\right\}=0$.

## Appendix G <br> Proof of Lemma 7

By using a union bound,

$$
\begin{align*}
& \mathbb{P}_{\mathcal{C}}\left(\mathcal{B}^{c}\right) \\
& \leq \mathbb{P}_{\mathcal{C}}\left[\max _{\left(m_{0}, m_{c}, z^{n}\right):\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}} \phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right)>2^{2 m \epsilon}\right] \\
& \quad+\mathbb{P}_{\mathcal{C}}\left[\min _{\left(m_{0}, m_{c}, z^{n}\right):\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}} \phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right)<1\right] . \tag{243}
\end{align*}
$$

In the following, we prove that both the terms of 243) vanish as $m \rightarrow \infty$. Observe

$$
\begin{align*}
& \mathbb{P}_{\mathcal{C}}\left[\max _{\left(m_{0}, m_{c}, z^{n}\right):\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}} \phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right)>2^{2 m \epsilon}\right] \\
& =\mathbb{E}_{\mathcal{C}_{0}} \mathbb{P}_{\mathcal{C}_{1}}\left[\max _{\left(m_{0}, m_{c}, z^{n}\right):\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}}\right. \\
& \left.\phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right)>2^{2 m \epsilon} \mid \mathcal{C}_{0}\right] . \tag{244}
\end{align*}
$$

Hence we only need to show the probability in (244) vanishes for any $\mathcal{C}_{0}$ as $m \rightarrow \infty$. Using union bound, we have

$$
\begin{align*}
& \mathbb{P}_{\mathcal{C}_{1}}\left[\max _{\left(m_{0}, m_{c}, z^{n}\right):\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}} \phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right)>2^{2 m \epsilon} \mid \mathcal{C}_{0}\right] \\
& \leq 2^{m\left(R_{0}+R_{c}\right)}\left|\mathcal{T}_{\delta^{\prime}}^{n}\right| \\
& \max _{\left(m_{0}, m_{c}, z^{n}\right):\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}}  \tag{245}\\
& \mathbb{P}_{\mathcal{C}_{1}}\left[\phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right)>2^{2 m \epsilon} \mid \mathcal{C}_{0}\right]
\end{align*}
$$

Define $\theta_{m_{1}^{\prime}}\left(z^{n}\right) \triangleq 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}^{\prime}, m_{c}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}$, then $\phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right)=\sum_{m_{1}^{\prime}} \theta_{m_{1}^{\prime}}\left(z^{n}\right)$. Given $\left(m_{0}, m_{c}, z^{n}\right)$ and $\mathcal{C}_{0}$ such that $\left(U^{n}\left(m_{0}\right), z^{n}\right) \in \mathcal{T}_{\delta^{\prime}}^{n}$, $\theta_{m_{1}^{\prime}}\left(z^{n}\right), m_{1}^{\prime} \in\left[2^{m R_{1}^{\prime}}\right]$ are i.i.d. random variables, with mean

$$
\begin{align*}
& \mathbb{E}_{\mathcal{C}_{1}} \theta_{m_{1}^{\prime}}\left(z^{n}\right) \\
& =\mathbb{P}\left[\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}^{\prime}, m_{c}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n} \mid U^{n}\left(m_{0}\right)\right] \\
& =\mathbb{P}\left[\left(U^{n}, V^{n}, z^{n}\right) \in \mathcal{T}_{\delta}^{n} \mid U^{n}\right] \\
& \leq 2^{-n\left(I(V ; Z \mid U)-\epsilon_{\delta}\right)}, \tag{246}
\end{align*}
$$

where $\epsilon_{\delta}$ tends to zero as $\delta \rightarrow 0$, and 246 follows from the joint typicality lemma [17]. On the other hand, $\left|\mathcal{T}_{\delta^{\prime}}^{n}\right| \leq$ $2^{n\left(H(Z)+\epsilon_{\delta^{\prime}}\right)}$ for some $\epsilon_{\delta^{\prime}}$ that tends to zero as $\delta^{\prime} \rightarrow 0$. Hence if we can show that the probability in 245 decays doubly exponentially fast with $m$, then the proof will be complete. To that end, we first introduce the following lemma on Chernoff bounds.

Lemma 13. [2], [24] If $X^{l}$ is a sequence of i.i.d. $\operatorname{Bern}(p)$ random variables, then for any $k>0$,

$$
\begin{equation*}
\mathbb{P}\left[\sum_{i=1}^{l} X_{i}>k\right] \leq\left(\frac{e \cdot l \cdot p}{k}\right)^{k} \tag{247}
\end{equation*}
$$

and for any $0 \leq \delta \leq 1$,

$$
\begin{equation*}
\mathbb{P}\left[\sum_{i=1}^{l} X_{i} \leq(1-\delta) l p\right] \leq e^{-\frac{\delta^{2} l_{p}}{2}} \tag{248}
\end{equation*}
$$

By identifying that

$$
\begin{align*}
l & =2^{m R_{1}^{\prime}}  \tag{249}\\
p & \leq 2^{-n\left(I(V ; Z \mid U)-\epsilon_{\delta}\right)}  \tag{250}\\
k & =2^{2 m \epsilon} \tag{251}
\end{align*}
$$

and applying Lemma 13 , we have

$$
\begin{equation*}
\mathbb{P}_{\mathcal{C}_{1}}\left[\phi_{\mathcal{C}}\left(m_{0}, m_{c}, z^{n}\right)>2^{2 m \epsilon} \mid \mathcal{C}_{0}\right] \leq 2^{-m \alpha 2^{m \beta}} \tag{252}
\end{equation*}
$$

where

$$
\begin{align*}
& \alpha=\gamma I(V ; Z \mid U)-\gamma \epsilon_{\delta}-R_{1}^{\prime}+2 \epsilon=\epsilon-\gamma \epsilon_{\delta}  \tag{253}\\
& \beta=2 \epsilon \tag{254}
\end{align*}
$$

For fixed $\epsilon$ and sufficiently small $\delta, \alpha>0$ and $\beta>0$. Hence (252) vanishes doubly exponentially fast. This means that the first term of (243) vanishes as $m \rightarrow \infty$.

In the same way, by utilizing Lemma 13 again, we can prove that for small enough $\delta$, the second term of (243) also vanishes as $m \rightarrow \infty$. This completes the proof of Lemma 7

## Appendix H <br> Proof of Lemma 8

Consider

$$
\begin{align*}
& \mathbb{P}\left(M_{1}=m_{1}, Z^{n}=z^{n}, M_{0}=m_{0}, \mathcal{A}_{3} \mid \mathcal{C}\right) \\
= & 2^{-m R_{t}} \mathbb{P}\left(Z^{n}=z^{n}, \mathcal{A}_{3} \mid U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right)\right) \\
\leq & 2^{-m R_{t}} 2^{-n\left(H(Z \mid V)-\epsilon_{\delta}\right)} \\
& 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\} \tag{255}
\end{align*}
$$

, where 255 follows from the property of typical sequence [17]: for any $\left(v^{n}, z^{n}\right) \in \mathcal{T}_{\delta}^{n}$,

$$
\begin{equation*}
2^{-n\left(H(Z \mid V)+\epsilon_{\delta}\right)} \leq \mathbb{P}\left(Z^{n}=z^{n} \mid V^{n}=v^{n}\right) \leq 2^{-n\left(H(Z \mid V)-\epsilon_{\delta}\right)} \tag{256}
\end{equation*}
$$

with a term $\epsilon_{\delta}$ that vanishes as $m \rightarrow \infty$.
Similarly, we have

$$
\begin{align*}
& \quad \mathbb{P}\left(M_{1}=m_{1}, Z^{n}=z^{n}, M_{0}=m_{0}, \mathcal{A}_{3} \mid \mathcal{C}\right) \\
& \geq 2^{-m R_{t}} 2^{-n\left(H(Z \mid V)+\epsilon_{\delta}\right)} \\
& \quad 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\} . \tag{257}
\end{align*}
$$

Therefore,

$$
\begin{align*}
& \mathbb{P}\left(M_{1}=m_{1}, \mathcal{A}_{3} \mid \mathcal{C} Z^{n} M_{0}\right) \\
= & \frac{\mathbb{P}\left(M_{1}=m_{1}, Z^{n}=z^{n}, M_{0}=m_{0}, \mathcal{A}_{3} \mid \mathcal{C}\right)}{\sum_{m_{1} \in\left[2^{m R_{1}}\right]} \mathbb{P}\left(M_{1}=m_{1}, Z^{n}=z^{n}, M_{0}=m_{0} \mid \mathcal{C}\right)} \\
\leq & \frac{\mathbb{P}\left(M_{1}=m_{1}, Z^{n}=z^{n}, M_{0}=m_{0}, \mathcal{A}_{3} \mid \mathcal{C}\right)}{\sum_{m_{1} \in\left[2^{m R_{1}}\right]} \mathbb{P}\left(M_{1}=m_{1}, Z^{n}=z^{n}, M_{0}=m_{0}, \mathcal{A}_{3} \mid \mathcal{C}\right)} \\
\leq & \frac{2^{-m\left(R_{0}+R_{1}\right)} 2^{-n\left(H(Z \mid V)-\epsilon_{\delta}\right)}}{2^{-m\left(R_{0}+R_{1}\right)} 2^{-n\left(H(Z \mid V)+\epsilon_{\delta}\right)}} \\
& \times \frac{1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}}{\sum_{m_{1} \in\left[2^{m R_{1}}\right]} 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}} \\
\leq & \frac{2^{2 n \epsilon_{\delta}} 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}}{\sum_{m_{1} \in\left[2^{m R_{1}}\right]} 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}} \\
\leq & 2^{-m\left(R_{c}-2 \gamma \epsilon_{\delta}\right)} 1\left\{\left(U^{n}\left(m_{0}\right), V^{n}\left(m_{0}, m_{1}\right), z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\}, \tag{258}
\end{align*}
$$

where (258) follows from (199).

## Appendix I

## Proof of Converse of Theorem 5

From source-channel coding theorem for Gaussian communication, we have $\frac{\gamma}{2} \log \left(1+\frac{P}{N_{B}}\right) \geq \frac{1}{2} \log ^{+} \frac{N_{S}}{D_{B}}$. Besides, similar to the discrete memoryless case, if $R_{L}>\frac{1}{2} \log ^{+} \frac{N_{S}}{D_{E}}$, then no matter what scheme Nodes A and B use, the henchman and the wiretapper can ignore $Z^{n}$ altogether and simply use a point-to-point rate-distortion code to describe $S^{m}$ within distortion $D_{E}$ (with probability 1). Hence we only need prove

$$
\begin{equation*}
R_{L} \leq R_{K}+\frac{\gamma}{2} \log ^{+}\left(\frac{1+P / N_{B}}{1+P / N_{E}}\right)+\frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}} \tag{259}
\end{equation*}
$$

To that end, we follow similar steps to the proof of the outer bound of Theorem 3 Since the admissible region of ( $R_{K}, R_{L}, D_{B}, D_{E}$ ) only depends on the margin distribution of the wiretap channel, it is sufficient to consider the channel to be physically degraded. Furthermore, observe that the derivation up to 128 holds verbatim. Hence to complete the proof, we only need to prove

$$
\begin{align*}
\bar{I}(\mathbf{Y} ; \mathbf{X} \mid \mathbf{Z}) & \leq \frac{1}{2} \log ^{+}\left(\frac{1+P / N_{B}}{1+P / N_{E}}\right)  \tag{260}\\
\underline{R_{\mathbf{S} \mid \hat{\mathbf{S}}}}\left(D_{E}\right) & \leq \frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}} \tag{261}
\end{align*}
$$

Actually, for physically degraded Gaussian wiretap channel (as considered here), (260) was proven by Tan [18. Thm. 5]. Next we will show 261 also holds. From Theorem 1 it is equivalent to showing that upon the two-sided information $\hat{S}^{m}$ (within the distortion $D_{B}$ ), there exists a source code with rate $\frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}}$ achieving the distortion $D_{E}$ with positive probability. The corresponding part of the proof of Theorem 3 requires that the alphabet of the source or its reproduction is finite, hence it cannot be applied to the case of continuous alphabets, such as the Gaussian case. From the assumption, $S^{m}$ is in the balls of center $\hat{S}^{m}$ and radius $\sqrt{m D_{B}}$ with high probability. Hence, we use a sphere covering lemma to prove (261).

Lemma 14. [22], [23] Let $R>1$ and let $\nu_{R, l}$ be the minimal number of (closed) balls of radius 1 which can cover a (closed) ball of radius $R$ in $\mathbb{R}^{l}$. If $l \geq 9$, then we have
$1<\nu_{R, l} \leq \frac{4 e R^{l} l \sqrt{l}}{\ln l-2}(l \ln l+l \ln (\ln l)+l \ln R+12 \ln (144 l))$
for all $1<R<\frac{l}{2 \ln l}$.
Observe that for any fixed $R, \frac{4 e R^{l} l \sqrt{l}}{\ln l-2}(l \ln l+l \ln (\ln l)+$ $l \ln R+12 \ln (144 l))=2^{l(\log R+o(1))}$. Hence from Lemma 14 , we can easily get that for large enough $m$, it suffices to cover a ball of radius $\sqrt{m D_{B}}$ using $2^{m\left(\frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}}+o(1)\right)}$ balls of radius $\sqrt{m D_{E}}$. This implies upon $\hat{S}^{m}$, there exists a source code with rate $\frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}}$ achieving the distortion $D_{E}$ (with high probability). Hence $R_{\mathbf{S} \mid \hat{\mathbf{S}}}\left(D_{E}\right) \leq \frac{1}{2} \log ^{+} \frac{D_{B}}{D_{E}}$ holds. This completes the proof of the converse part.

## APPENDIX J <br> Proof of Achievability of Theorem 5

The proof of $\mathcal{R}_{\text {sep }}^{(i)}$ of Theorem 3 requires that the alphabets of the channel input and output, and the alphabets of the source and its reproduction, are all finite, hence it cannot be applied to the Gaussian case directly. Now, we prove the achievability part for the Gaussian case by exploiting the techniques of d-tilted information, weak typicality, and discretization.

## A. Weak typicality and d -tilted information

Before proving the achievability part of Theorem 55 we need introduce some preliminaries. To extend $\mathcal{R}_{\text {sep }}^{(i)}$ of Theorem 3 to the Gaussian case, we need to replace the strong typicality (200) with weak typicality. The $\delta$-typical set and the $\delta$ -jointly-typical set are defined according to the notion of weak typicality ${ }^{11}$, see [21]:

$$
\begin{aligned}
& \mathcal{T}_{\delta}^{n}(X) \triangleq \\
& \left\{x^{n} \in \mathcal{X}^{n}:\left|-\frac{1}{n} \log \prod_{i=1}^{n} f_{X}\left(x_{i}\right)-h(X)\right| \leq \frac{\delta}{2} \log e\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathcal{T}_{\delta}^{n}(X, Z) \triangleq \\
& \left\{\left(x^{n}, z^{n}\right) \in \mathcal{X}^{n} \times \mathcal{Z}^{n}:\right. \\
& \left|-\frac{1}{n} \log \prod_{i=1}^{n} f_{X}\left(x_{i}\right)-h(X)\right| \leq \frac{\delta}{2} \log e \\
& \left|-\frac{1}{n} \log \prod_{i=1}^{n} f_{Z}\left(z_{i}\right)-h(Z)\right| \leq \frac{\delta}{2} \log e \\
& \left.\left|-\frac{1}{n} \log \prod_{i=1}^{n} f_{X Z}\left(x_{i}, z_{i}\right)-h(X Z)\right| \leq \frac{\delta}{2} \log e\right\}
\end{aligned}
$$

For jointly Gaussian variables $X$ and $Z$, where $Z=X+U$ and $U$ is independent of $X$, the $\delta$-typical set and the $\delta$-jointlytypical set become

$$
\begin{equation*}
\mathcal{T}_{\delta}^{n}(X) \triangleq\left\{x^{n} \in \mathbb{R}^{n}:\left|\frac{\left\|x^{n}\right\|^{2}}{n N_{X}}-1\right| \leq \delta\right\} \tag{263}
\end{equation*}
$$

and

$$
\begin{align*}
\mathcal{T}_{\delta}^{n}(X, Z) \triangleq & \left\{\left(x^{n}, z^{n}\right) \in \mathbb{R}^{2 n}:\left|\frac{\left\|x^{n}\right\|^{2}}{n N_{X}}-1\right| \leq \delta\right. \\
& \left|\frac{\left\|z^{n}\right\|^{2}}{n N_{Z}}-1\right| \leq \delta \\
& \left.\left|\frac{\left\|x^{n}\right\|^{2}}{n N_{X}}+\frac{\left\|z^{n}-x^{n}\right\|^{2}}{n N_{U}}-2\right| \leq \delta\right\} \tag{264}
\end{align*}
$$

respectively, where $\left\|x^{n}\right\|=\sqrt{\sum_{i=1}^{n} x_{i}^{2}}$ denotes Euclidean norm of $x^{n}$, and $N_{Z}, N_{X}$ and $N_{U}$ denote the variances of $Z, X$ and $U$.

Similar to 26, the rate-distortion function for continuous source is defined as

$$
\begin{equation*}
R_{S}(D)=\min _{f_{\tilde{S} \mid S}: \mathbb{E} d(S, \check{S}) \leq D} I(S ; \check{S}) . \tag{265}
\end{equation*}
$$

We impose the following basic restrictions on pdf $f_{S}$ and the distortion measure $d$ :

- Restriction 1: $R_{S}(D)$ is finite for some $D$, i.e. $D_{\min }<$ $\infty$, where

$$
\begin{equation*}
D_{\min }=\inf \left\{D: \quad R_{S}(D)<\infty\right\} \tag{266}
\end{equation*}
$$

[^8]- Restriction 2: The minimum in 265) is achieved by a pdf $f_{\tilde{S}^{\star} \mid S}$, which is unique up to $f_{\tilde{S}^{\star} S^{S}}$-null sets, that is, if $g_{\tilde{S}^{\star} \mid S}$ is another pdf achieving the minimum in 265, then $f_{\check{S}^{\star} \mid S}=g_{\check{S}^{\star} \mid S}, f_{\check{S}^{\star} S}$-almost everywhere.
Definition 9 (d-tilted information [27]). For $D>D_{\text {min }}$, the d -tilted information in $s$ is defined as

$$
\begin{equation*}
\jmath_{S}(s, D) \triangleq \log \frac{1}{\mathbb{E}\left[\exp \left(\lambda^{\star} D-\lambda^{\star} d\left(s, \check{S}^{\star}\right)\right)\right]} \tag{267}
\end{equation*}
$$

where the expectation is with respect to $f_{\check{S}_{\star}}$, i.e. the unconditional distribution of the reproduction random variable that achieves the minimum in 265, and

$$
\begin{equation*}
\lambda^{\star}=-R_{S}^{\prime}(D) \tag{268}
\end{equation*}
$$

The following properties of d -tilted information, proven in [28], are used in the sequel.

$$
\begin{align*}
& \jmath_{S}(s, D)=\imath_{S ; \check{S}^{\star}}(s ; \check{s})+\lambda^{\star} d(s, \check{s})-\lambda^{\star} D  \tag{269}\\
& \mathbb{E}\left[\jmath_{S}(s, D)\right]=R_{S}(D)  \tag{270}\\
& \mathbb{E}\left[\exp \left(\lambda^{\star} D-\lambda^{\star} d(S, \check{s})+\jmath_{S}(S, D)\right)\right] \leq 1 \tag{271}
\end{align*}
$$

where (269) holds for $f_{\check{S}^{\star}}$-almost every $\check{s}$, while 271 holds for all $\check{s} \in \mathcal{S}$, and

$$
\begin{equation*}
{ }^{\imath}{ }_{S ; \check{S}}(s ; \check{s})=\log \frac{f_{\check{S} \mid S}(\check{s} \mid s)}{f_{\check{S}}(\check{s})} \tag{272}
\end{equation*}
$$

denotes the information density of the joint distribution $f_{S \check{S}}$ at $(s, \check{s})$ (c.f. (7).

Similarly, we can define conditional d-tilted information and prove the corresponding properties. Similar to 30, the conditional rate-distortion function for continuous source is defined as

$$
\begin{equation*}
R_{S \mid \hat{S}}(D)=\min _{f_{\tilde{S} \mid S \hat{S}}: \mathbb{E} d(S, \check{S}) \leq D} I(S ; \check{S} \mid \hat{S}) \tag{273}
\end{equation*}
$$

Now we can establish the following lemma, the proof of which is given in Appendix K

Lemma 15. The minimization in (273) can be divided into two optimization subproblems:

$$
\begin{equation*}
R_{S \mid \hat{S}}(D)=\min _{b(\hat{s}): \mathbb{E}_{\hat{S}}[b(\hat{S})] \leq D} \mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}(b(\hat{S})) \tag{274}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{S \mid \hat{S}=\hat{s}}(\beta)=\min _{f_{\tilde{S} \mid S, \hat{S}=\hat{s}}: \mathbb{E}[d(S, \check{S}) \mid \hat{S}=\hat{s}] \leq \beta} I(S ; \check{S} \mid \hat{S}=\hat{s}) \tag{275}
\end{equation*}
$$

denotes the rate-distortion function of source $S$ under condition that $\hat{S}=\hat{s}$. Moreover, assume $b^{\star}(\hat{s})$ achieves the minimum in 274, and $f_{\check{S} \star \mid S, \hat{S}=\hat{s}}$ achieves the minimum in (275) with $\beta=b^{\star}(\hat{s})$, then $f_{\tilde{S}^{\star} \mid S, S_{S}}$ minimizes (273) as well, i.e., $f_{\tilde{S}^{\star} \mid S, \hat{S}}$ is a solution to the minimization in 273.

Similar to Restrictions 1 and 2 , for conditional ratedistortion function, we impose the following basic restrictions on $f_{S \hat{S}}$ and the distortion measure:

- Restriction 3: For all $\hat{s} \in \hat{\mathcal{S}}, R_{S \mid \hat{S}=\hat{s}}(\beta)$ is finite for some $\beta$, i.e. $\beta_{\text {min }}(\hat{s})<\infty$, where

$$
\begin{equation*}
\beta_{\min }(\hat{s})=\inf \left\{\beta: \quad R_{S \mid \hat{S}=\hat{s}}(\beta)<\infty\right\} \tag{276}
\end{equation*}
$$

- Restriction 4: For all $\hat{s} \in \hat{\mathcal{S}}$, the minimum in 275 is achieved by a pdf $f_{\tilde{S}^{\star} \mid S, \hat{S}=\hat{s}}$ which is unique up to $f_{\tilde{S}^{\star} S| | \hat{S}=\hat{s}}$-null sets, that is, if $g_{\tilde{S}^{\star} \mid S, \hat{S}=\hat{s}}$ is another pdf achieving the minimum in (26), then $f_{\check{S}^{\star} \mid S, \hat{S}=\hat{s}}=$ $g_{\check{S} \star \mid S, \hat{S}=\hat{s}}, f_{\check{S} \star S \mid \hat{S}=\hat{s}^{2}}$-almost everywhere;
- Restriction 5: $R_{S \mid \hat{S}}(D)$ is strictly decreasing in $\left(D_{\min }, D_{\max }\right)$ where

$$
\begin{equation*}
D_{\min }=\inf \left\{D: \quad R_{S \mid \hat{S}}(D)<\infty\right\} \tag{277}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{\max }=\inf \left\{D: \quad R_{S \mid \hat{S}}(D)=0\right\} \tag{278}
\end{equation*}
$$

Assume $b^{\star}(\hat{s})$ achieves the minimum in 274, and $f_{\tilde{S}^{\star} \mid S, \hat{S}=\hat{s}}$ achieves the minimum in 275) with $\beta=b^{\star}(\hat{s})$, as assumed in Lemma 15, then the following lemma holds. The proof is given in Appendix $\square$

## Lemma 16.

$$
\begin{align*}
& R_{S \mid \hat{S}}(D)=\mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}\left(b^{\star}(\hat{s})\right)  \tag{279}\\
& \mathbb{E}_{\hat{S}}\left[b^{\star}(\hat{S})\right]=D  \tag{280}\\
& \mathbb{E}\left[d\left(S, \check{S}^{\star}\right) \mid \hat{S}=\hat{s}\right]=b^{\star}(\hat{s})  \tag{281}\\
& R_{S \mid \hat{S}=\hat{s}}^{\prime}\left(b^{\star}(\hat{s})\right)=-\lambda, \forall \hat{s} \in \hat{\mathcal{S}} \tag{282}
\end{align*}
$$

for some constant $\lambda \geq 0$.
Definition 10 (Conditional $d$-tilted information). For $b^{\star}(\hat{s})>\beta_{\text {min }}(\hat{s})$, the conditional d -tilted information in $s$ under condition $\hat{S}=\hat{s}$ is defined as

$$
\begin{align*}
& \jmath_{S \mid \hat{S}=\hat{s}}\left(s, b^{\star}(\hat{s})\right) \\
\triangleq & \log \frac{1}{\mathbb{E}_{\check{S}^{\star} \mid \hat{S}=\hat{s}}\left[\exp \left(\lambda^{\star}(\hat{s}) b^{\star}(\hat{s})-\lambda^{\star}(\hat{s}) d\left(s, \check{S}^{\star}\right)\right)\right]} \tag{283}
\end{align*}
$$

where the expectation is with respect to $f_{\tilde{S}^{\star} \mid \hat{S}=\hat{s}}$, and

$$
\begin{equation*}
\lambda^{\star}(\hat{s})=-R_{S \mid \hat{S}=\hat{s}}^{\prime}\left(b^{\star}(\hat{s})\right) \tag{284}
\end{equation*}
$$

Combining (282) and 284, we have for all $\hat{s}$,

$$
\begin{equation*}
\lambda^{\star}(\hat{s})=\lambda \tag{285}
\end{equation*}
$$

Obviously, the conditional d-tilted information for the distribution $f_{S \mid \hat{S}=\hat{s}}$ and distortion $b^{\star}(\hat{s})$ can be considered as a special unconditional d-tilted information for $f_{S^{\prime}}$ and $D^{\prime}$ such that $f_{S^{\prime}}=f_{S \mid \hat{S}=\hat{s}}$ and $D^{\prime}=b^{\star}(\hat{s})$. Hence 269)-271, still hold for $\int_{S \mid \hat{S}=\hat{s}}\left(s, b^{\star}(\hat{s})\right)$, i.e.,

$$
\begin{align*}
& \jmath_{S \mid \hat{S}=\hat{s}}\left(s, b^{\star}(\hat{s})\right) \\
& =\imath_{S ; \breve{S}^{\star} \mid \hat{S}=\hat{s}}(s ; \check{s})+\lambda^{\star}(\hat{s}) d(s, \check{s})-\lambda^{\star}(\hat{s}) b^{\star}(\hat{s})  \tag{286}\\
& \mathbb{E}_{S \mid \hat{S}=\hat{s}}\left[\jmath_{S \mid \hat{S}=\hat{s}}\left(S, b^{\star}(\hat{s})\right)\right]=R_{S \mid \hat{S}=\hat{s}}\left(b^{\star}(\hat{s})\right)  \tag{287}\\
& \mathbb{E}_{S \mid \hat{S}=\hat{s}}\left[\exp \left(\lambda^{\star}(\hat{s}) b^{\star}(\hat{s})-\lambda^{\star}(\hat{s}) d(S, \check{s})+\jmath_{S \mid \hat{S}=\hat{s}}\left(S, b^{\star}(\hat{s})\right)\right)\right] \\
& \leq 1 \tag{288}
\end{align*}
$$

where 286 holds for $f_{\check{S} \star \mid \hat{S}=\hat{s}^{2}}$-almost every $\check{s}$, while 288) holds for all $\check{s} \in \mathcal{S}$, and

$$
\begin{equation*}
{ }^{2} S_{S ; \tilde{S}^{\star} \mid \hat{S}=\hat{s}}(s ; \check{s})=\log \frac{f_{\check{S} \mid S, \hat{S}=\hat{s}}(\check{s} \mid s)}{f_{\check{S} \mid \hat{S}=\hat{s}}(\check{s})} \tag{289}
\end{equation*}
$$

denotes the conditional information density of the joint distribution $f_{S \check{S} \mid \hat{S}=\hat{s}}$ at $(s, \check{s})$.

## B. Subproblem: lossy compression of a codeword drawn uniformly from a random codebook with side information

Next we return to proving the achievability part of Theorem 5 We follow similar steps to that of the discrete memoryless case. Consider the subproblem described in Appendix E-A for the Gaussian source-channel case. Then we can prove that Theorem 77 still holds. To show the achievability for the Gaussian case, it is sufficient to consider the case of $U=\emptyset, V=X$.

Theorem 8. Theorem 7 with $U=\emptyset, V=X$ holds for Gaussian communication case.

Proof of Theorem 8. Since $d_{B}(x, y)=d_{E}(x, y)=$ $(x-y)^{2}$, in the following, we use $d(x, y)$ to denote both of them. Furthermore, for the memoryless Gaussian source $S$, set $\hat{S}$ to be a jointly Gaussian variable with $S$ such that $S=\hat{S}+W, \hat{S}$ and $W$ are independent and $\mathbb{E}\left[W^{2}\right]=D_{B}$. It is easy to verify that $f_{S \hat{S}}$ and $d(s, \check{s})$ satisfy Restrictions 1-5.

Next we follow similar steps to the proof of $\mathcal{R}_{\text {sep }}^{(i)}$ of Theorem 7. except for some modifications. First, we need to replace the strong typicality 200 with the weak typicality, since strong typicality only works for the variables with finite alphabets. Second, we need re-define $\mathcal{A}_{1}$ as

$$
\begin{align*}
& \mathcal{A}_{1} \triangleq\left\{S^{m} \in \mathcal{T}_{\delta}^{m}, \frac{1}{m} \sum_{i=1}^{m} \jmath_{S}\left(S_{i}, D_{E}\right) \geq R_{S}\left(D_{E}\right)-\delta\right. \\
& \frac{1}{m} \sum_{i=1}^{m} \jmath_{S \mid \hat{S}=\hat{S}_{i}}\left(S_{i}, b^{\star}\left(\hat{S}_{i}\right)\right) \geq R_{S \mid \hat{S}}\left(D_{E}\right)-\delta \\
&\left.\frac{1}{m} \sum_{i=1}^{m} b^{\star}\left(\hat{S}_{i}\right) \geq D_{E}-\delta\right\} \tag{290}
\end{align*}
$$

for $\delta>0 . \mathcal{A}_{2}, \mathcal{A}_{3}, \mathcal{A}_{123}, \mathcal{A}_{23}, \mathcal{B}$ remain the same but restricted to the case with $U=\emptyset, V=X$. Then we have following lemmas, the proofs of which are given in Appendices $M$ and N , respectively.
Lemma 17. For Gaussian wiretap channel, $\lim _{m \rightarrow \infty} \mathbb{P}(\mathcal{B})=$ 1.

Lemma 18. For any stationary memoryless source $f_{S \hat{S}}$ with general distortion function $d(s, \check{s})$ satisfying Restrictions 1-5 (not restricted to the quadratic Gaussian case), $\lim _{m \rightarrow \infty} \mathbb{P}\left(\mathcal{A}_{123}\right)=1$.

Re-define $\eta_{m, k, \breve{s}^{m}}$ as

$$
\begin{aligned}
& \eta_{m, k, \check{s}^{m}}\left(D_{E}\right) \triangleq \sum_{s^{m} \in \mathcal{S}^{m}} \prod_{i=1}^{m} P_{S \mid \hat{S}}\left(s_{i} \mid \hat{S}_{i}(j(m, k))\right) \\
& \times 1\left\{d\left(s^{m}, \check{s}^{m}\right) \leq D_{E}, s^{m} \in \mathcal{T}_{\delta}^{m}\right. \\
& \quad \frac{1}{m} \sum_{i=1}^{m} \jmath_{S}\left(s_{i}, D_{E}\right) \geq R_{S}\left(D_{E}\right)-\delta \\
& \frac{1}{m} \sum_{i=1}^{m} \jmath_{S \mid \hat{S}=\hat{s}_{i}}\left(s_{i}, b^{\star}\left(\hat{s}_{i}\right)\right) \geq R_{S \mid \hat{S}}\left(D_{E}\right)-\delta
\end{aligned}
$$

$$
\begin{equation*}
\left.\frac{1}{m} \sum_{i=1}^{m} b^{\star}\left(\hat{s}_{i}\right) \geq D_{E}-\delta\right\} \tag{291}
\end{equation*}
$$

Then the derivation up to 214 still holds, i.e.,

$$
\begin{align*}
& \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{R_{L} \text { codes }} \mathbb{P}\left[d\left(S^{m}, \check{S}^{m}\right) \leq D_{E}\right]>\tau\right] \\
& \leq \mathbb{P}_{\mathcal{C} Z^{n} M_{0}[ }^{\max } \sum_{\check{s}^{m} \in \mathbb{R}^{m}, m_{1}^{\prime} \in\left[2^{m R_{1}^{\prime}}\right]} \sum_{k=1}^{2^{m R_{K}}} \sum_{m_{c}=1}^{2^{m R_{c}}} \\
& \left.\eta_{m, k, \check{s}^{m}}\left(D_{E}\right)>\tau^{\prime} 2^{-m \lambda^{\prime}}\right]+\epsilon_{m}^{\prime} \tag{292}
\end{align*}
$$

where $\eta_{m, k,,^{m}}\left(D_{E}\right)$ is defined in 216. Now to take the maximizing operation out of the probability in (292), we use a discretization technology that is also used in proof of Lemma 17. Quantize $\check{s}^{m}$ as $[\check{s}]^{m} \in \mathcal{N}^{m}$ by the following manner, where $\mathcal{N}$ is defined in 322).

$$
\begin{equation*}
\left[\check{s}_{i}\right]=\Delta \cdot \text { Round }\left(\frac{\check{s}_{i}}{\Delta}\right) \tag{293}
\end{equation*}
$$

Then similar to (328), we can prove

$$
\begin{align*}
\frac{\left\|[\check{s}]^{m}-s^{m}\right\|^{2}}{m}-\epsilon_{\Delta} & \leq \frac{\left\|\check{s}^{m}-s^{m}\right\|^{2}}{m} \\
& \leq \frac{\left\|[\check{s}]^{m}-s^{m}\right\|^{2}}{m}+\epsilon_{\Delta} \tag{294}
\end{align*}
$$

for some $\epsilon_{\Delta}$ that vanishes as $\Delta \rightarrow 0$. Hence

$$
\begin{equation*}
\eta_{m, k, \check{s}^{m}}\left(D_{E}\right) \leq \eta_{m, k,[\check{s}]^{m}}\left(D_{E}+\epsilon_{\Delta}\right) \tag{295}
\end{equation*}
$$

Define $\mathcal{F}^{m} \triangleq\left\{[\check{s}]^{m} \in \mathcal{N}^{m}:\left\|[\check{s}]^{m}\right\|^{2} \leq m N_{S}(1+\delta)\right\}$, then we have

$$
\begin{align*}
& \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{R_{L} \text { codes }} \mathbb{P}\left[d\left(S^{m}, \check{S}^{m}\right) \leq D_{E}\right]>\tau\right] \\
& \leq \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{[\check{s}]^{m} \in \mathcal{N}^{m}, m_{1}^{\prime} \in\left[2^{\left.m R_{1}^{\prime}\right]}\right.} \sum_{k=1}^{2^{m R_{K}}} \sum_{2_{c}=1}^{2^{m R_{c}}}\right. \\
& \left.\eta_{m, k,[\check{s}]^{m}}\left(D_{E}+\epsilon_{\Delta}\right)>\tau^{\prime} 2^{-m \lambda^{\prime}}\right]+\epsilon_{m}^{\prime}  \tag{296}\\
& =\mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\max _{[\check{s}]^{m} \in \mathcal{F}^{m}, m_{1}^{\prime} \in\left[2^{\left.m R_{1}^{\prime}\right]}\right.}^{2^{m R_{K}}} \sum_{k=1}^{2^{m R_{c}}} \sum_{m_{c}=1}^{\left.\eta_{m, k,[\check{s}]^{m}}\left(D_{E}+\epsilon_{\Delta}\right)>\tau^{\prime} 2^{-m \lambda^{\prime}}\right]+\epsilon_{m}^{\prime}}\right. \\
& \leq 2^{m R_{1}^{\prime}}\left|\mathcal{F}^{m}\right|  \tag{297}\\
& \max _{[\check{s}]^{m} \in \mathcal{F}^{m}, m_{1}^{\prime} \in\left[2^{m R_{1}^{\prime}}\right]} \\
& \mathbb{P}_{\mathcal{C} Z^{n} M_{0}}\left[\sum_{k=1}^{2^{m R_{K}}} \sum_{2_{c}^{m R_{c}}}^{\sum_{m}} \eta_{m, k,[\check{s}]^{m}}\left(D_{E}+\epsilon_{\Delta}\right)>\tau^{\prime} 2^{-m \lambda^{\prime}}\right]+\epsilon_{m}^{\prime} \tag{298}
\end{align*}
$$

where (296) follows from 292 and 295, and 297) follows from that $s^{m}$ belongs in the $m$-ball with radius $m N_{S}(1+\delta)$ (since $s^{m} \in \mathcal{T}_{\delta}^{m}$ ), hence it suffices to use the points in the ball as reconstructions.

Furthermore, similar to (336, we have

$$
\begin{equation*}
\left|\mathcal{F}^{m}\right| \leq 2^{m\left(\frac{1}{2} \log m+o(\log m)\right)} \tag{299}
\end{equation*}
$$

Hence if we can show that the probability in 298 decays doubly exponentially fast with $m$, then the proof will be
complete. To that end, we first introduce the following lemmas which are related to Lemmas 9 and 10. The proof of Lemma 20 is given in Appendix 0

Lemma 19. [25] Thm. 4] If $f_{S}$ and $d(s, \check{s})$ satisfy Restrictions 1 and 2, and $S^{m}$ is i.i.d. according to $f_{S}$, then for any $\hat{s}^{m} \in$ $\hat{\mathcal{S}}^{m}$,

$$
\begin{align*}
& \mathbb{P}\left[d\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \frac{1}{m} \sum_{i=1}^{m} \jmath_{S}\left(S_{i}, D_{E}\right) \geq R_{S}\left(D_{E}\right)-\delta\right] \\
& \leq 2^{-m\left(R_{S}\left(D_{E}\right)-\epsilon_{m, \delta}\right)} \tag{300}
\end{align*}
$$

where $\epsilon_{m, \delta}$ is a term that vanishes as $\delta \rightarrow 0$ and $m \rightarrow \infty$.
Lemma 20. Fix $f_{S \hat{S}}$ and $\hat{s}^{m} \in \hat{\mathcal{S}}^{m}$. If $f_{S \hat{S}}$ and $d(s, \check{s})$ satisfy Restrictions 1-5, and $S^{m}$ is distributed according to $\prod_{i=1}^{m} f_{S \mid \hat{S}=\hat{s} i}$, then for any $\check{s}^{m}$,

$$
\begin{align*}
& \mathbb{P}\left[d\left(S^{m}, \check{s}^{m}\right) \leq D_{E}\right. \\
& \quad \frac{1}{m} \sum_{i=1}^{m} \jmath_{S \mid \hat{S}=\hat{s}_{i}}\left(S_{i}, b^{\star}\left(\hat{s}_{i}\right)\right) \geq R_{S \mid \hat{S}}\left(D_{E}\right)-\delta \\
& \left.\left.\quad \frac{1}{m} \sum_{i=1}^{m} b^{\star}\left(\hat{s}_{i}\right) \geq D_{E}-\delta \right\rvert\, \hat{S}^{m}=\hat{s}^{m}\right] \\
& \leq 2^{-m\left(R_{S \mid \hat{S}}\left(D_{E}\right)-\epsilon_{m, \delta}\right)} \tag{301}
\end{align*}
$$

where $\epsilon_{m, \delta}$ is a term that vanishes as $\delta \rightarrow 0$ and $m \rightarrow \infty$.
Note that Lemmas 19 and 20 hold for any stationary memoryless source with general distortion measure (not restricted to the Gaussian source with quadratic distortion), and hence they can be used to replace Lemmas 9 and 10 in the proof of Theorem 7

Apply Lemmas 11, 19 and 20, then we have the probability in (298) decays doubly exponentially fast with $m$. This completes the proof of Theorem 8 .

## C. Likelihood encoder

All the derivations in Appendix E-B still hold for the source and channel with continuous alphabets. Hence the inner bound $\mathcal{R}_{\text {sep }}^{(i)}$ (with $U=\emptyset, V=X$ ) still holds for the Gaussian communication. It is easy to verify $\mathcal{R}_{\text {sep }}^{(i)}$ is just the region of Theorem 55, hence the achievability of Theorem 5 holds.

## Appendix K <br> Proof of Lemma 15

Proof of " $\geq$ ". Assume $f_{\tilde{S}^{\prime} \mid S \hat{S}}$ achieves the minimum in 273) and define $b^{\prime}(\hat{s})=\mathbb{E}\left[d\left(S, \check{S}^{\prime}\right) \mid \hat{S}=\hat{s}\right]$, then

$$
\begin{align*}
& \mathbb{E}_{\hat{S}^{\prime}}\left[b^{\prime}\left(\hat{S}^{\prime}\right)\right] \leq D, \text { and } \\
& R_{S \mid \hat{S}}(D)= I\left(S ; \check{S}^{\prime} \mid \hat{S}\right) \\
&= \int d \hat{s} f(\hat{s}) I\left(S ; \check{S}^{\prime} \mid \hat{S}=\hat{s}\right) \\
& \geq \int d \hat{s} f(\hat{s}) \\
& \times \min _{f_{\tilde{S} \mid S, \hat{s}=\hat{s}}: \mathbb{E}[d(S, \check{S}) \mid \hat{S}=\hat{s}] \leq b^{\prime}(\hat{s})} I(S ; \check{S} \mid \hat{S}=\hat{s}) \\
&= \mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}\left(b^{\prime}(\hat{s})\right) \\
& \geq \min _{b(\hat{s}): \mathbb{E}_{\hat{S}}[b(\hat{S})] \leq D} \mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}(b(\hat{s})) . \tag{302}
\end{align*}
$$

Proof of " $\leq$ ". Observe that

$$
\begin{align*}
& \quad \min _{b(\hat{s}): \mathbb{E}_{\hat{S}}[b(\hat{S})] \leq D} \mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}(b(\hat{s})) \\
& =\min _{b(\hat{s}): \mathbb{E}_{\hat{S}}[b(\hat{S})] \leq D} \int d \hat{s} f(\hat{s}) \\
& \quad \times \min _{f_{\tilde{S} \mid S, \hat{s}=\hat{s}}: \mathbb{E}[d(S, \tilde{S}) \mid \hat{S}=\hat{s}] \leq b(\hat{s})} I(S ; \check{S} \mid \hat{S}=\hat{s}) . \tag{303}
\end{align*}
$$

Assume $b^{\prime \prime}(\hat{s})$ and $f_{\breve{S}^{\prime \prime} \mid S, \hat{S}=\hat{s}}$ achieves the minimum in (303). Then we have

$$
\begin{equation*}
\mathbb{E} d\left(S, \check{S}^{\prime \prime}\right)=\mathbb{E}_{\hat{S}} \mathbb{E}\left[d\left(S, \check{S}^{\prime \prime}\right) \mid \hat{S}=\hat{s}\right] \leq \mathbb{E}_{\hat{S}} \hat{b}^{\prime \prime}(\hat{S}) \leq D \tag{304}
\end{equation*}
$$

and

$$
\begin{align*}
& \min _{b(\hat{s}): \mathbb{E}_{\hat{S}}[b(\hat{S})] \leq D} \mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}(b(\hat{s})) \\
& =\int d \hat{s} f(\hat{s}) I\left(S ; \check{S}^{\prime \prime} \mid \hat{S}=\hat{s}\right)  \tag{305}\\
& =I\left(S ; \check{S}^{\prime \prime} \mid \hat{S}\right)  \tag{306}\\
& \geq \min _{f_{\tilde{S} \mid S \hat{S}}: \mathbb{E} d(S, \check{S}) \leq D} I(S ; \check{S} \mid \hat{S})  \tag{307}\\
& =R_{S \mid \hat{S}}(D) \tag{308}
\end{align*}
$$

where (307) follows from (304).
Combining (302) and (308) gives us

$$
\begin{equation*}
R_{S \mid \hat{S}}(D)=\min _{b(\hat{s}): \mathbb{E}_{\hat{S}}[b(\hat{S})] \leq D} \mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}(b(\hat{s})) \tag{309}
\end{equation*}
$$

Furthermore, it is easy to verify that $f_{\tilde{S}^{\prime \prime \prime} \mid S, \hat{S}=\hat{s}}$ is also a solution to the minimization in 273. Hence the lemma holds.

## Appendix L

Proof of Lemma 16
Equation (279) follows straightforwardly by the assumption that $b^{\star}(\hat{s})$ achieves the minimum in (274).

Furthermore, from the assumptions, we have

$$
\begin{equation*}
\mathbb{E}_{\hat{S}}\left[b^{\star}(\hat{S})\right] \leq D \tag{310}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{E}\left[d\left(S, \check{S}^{\star}\right) \mid \hat{S}=\hat{s}\right] \leq b^{\star}(\hat{s}) \tag{311}
\end{equation*}
$$

Then we have

$$
\begin{align*}
\mathbb{E}\left[d\left(S, \check{S}^{\star}\right)\right] & =\mathbb{E}_{\hat{S}} \mathbb{E}\left[d\left(S, \check{S}^{\star}\right) \mid \hat{S}=\hat{s}\right]  \tag{312}\\
& \leq \mathbb{E}_{\hat{S}} b^{\star}(\hat{s})  \tag{313}\\
& \leq D \tag{314}
\end{align*}
$$

where (313) follows from 311, and (314) follows from 310.
Since $R_{S \mid \hat{S}}(D)$ is strictly decreasing (Restriction 5), $R_{S \mid \hat{S}}(D)$ is achieved by some $f_{\check{S} \mid S, \hat{S}}$ only if $\mathbb{E}[d(S, \breve{S})]=D$. On the other hand, Lemma 15 tells us $f_{\check{S} \star \mid S, \hat{S}}$ achieves $R_{S \mid \hat{S}}(D)$. Hence

$$
\begin{equation*}
\mathbb{E}\left[d\left(S, \check{S}^{\star}\right)\right]=D \tag{315}
\end{equation*}
$$

Combining (314) and 315) gives us

$$
\begin{align*}
& \mathbb{E}_{\hat{S}}\left[b^{\star}(\hat{S})\right]=D  \tag{316}\\
& \mathbb{E}\left[d\left(S, \check{S}^{\star}\right) \mid \hat{S}=\hat{s}\right]=b^{\star}(\hat{s}) \tag{317}
\end{align*}
$$

Furthermore, since $R_{S \mid \hat{S}=\hat{s}}(\cdot)$ is convex, minimizing 274 is equivalent to minimizing its Lagrangian function, i.e.,

$$
\begin{equation*}
\min _{b(\hat{s})}\left\{\mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}(b(\hat{s}))+\lambda \mathbb{E}_{\hat{S}}[b(\hat{S})]\right\} \tag{318}
\end{equation*}
$$

where $\lambda \geq 0$ is a Lagrangian multiplier. By calculus of variations, the solution to (318) or 274) satisfies the necessary condition

$$
\begin{equation*}
f(\hat{s}) R_{S \mid \hat{S}=\hat{s}}^{\prime}(b(\hat{s}))+\lambda f(\hat{s})=0 . \tag{319}
\end{equation*}
$$

That is

$$
\begin{equation*}
\lambda=-R_{S \mid \hat{S}=\hat{s}}^{\prime}(b(\hat{s})) . \tag{320}
\end{equation*}
$$

On the other hand, $b^{\star}(\hat{s})$ is assumed to be the solution to (274), hence

$$
\begin{equation*}
\lambda=-R_{S \mid \hat{S}=\hat{s}}^{\prime}\left(b^{\star}(\hat{s})\right) \tag{321}
\end{equation*}
$$

## Appendix M

Proof of Lemma 17
Before proving Lemma 17 , we need introduce the discretization of Euclidean space, and corresponding properties. Let

$$
\begin{equation*}
[Z] \in \mathcal{N} \triangleq\{\cdots,-2 \Delta,-\Delta, 0, \Delta, 2 \Delta, \cdots\} \tag{322}
\end{equation*}
$$

be a quantized version of $Z$, obtained by mapping $Z$ to the closest quantization point, i.e.,

$$
\begin{equation*}
\left[z_{i}\right]=\Delta \cdot \operatorname{Round}\left(\frac{z_{i}}{\Delta}\right) \tag{323}
\end{equation*}
$$

Then for any $z^{n} \in \mathbb{R}^{n}$,

$$
\begin{equation*}
0 \leq \frac{\left\|z^{n}-[z]^{n}\right\|^{2}}{n} \leq \frac{\Delta^{2}}{4} \tag{324}
\end{equation*}
$$

Furthermore, using triangle inequality we have

$$
\begin{equation*}
-\frac{\left\|z^{n}-[z]^{n}\right\|^{2}}{n} \leq \frac{\left\|z^{n}\right\|^{2}}{n}-\frac{\left\|[z]^{n}\right\|^{2}}{n} \leq \frac{\left\|z^{n}-[z]^{n}\right\|^{2}}{n} \tag{325}
\end{equation*}
$$

Hence

$$
\begin{equation*}
-\frac{\Delta^{2}}{4} \leq \frac{\left\|z^{n}\right\|^{2}}{n}-\frac{\left\|[z]^{n}\right\|^{2}}{n} \leq \frac{\Delta^{2}}{4} \tag{326}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
\frac{\left\|[z]^{n}\right\|^{2}}{n N_{Z}}-\epsilon_{\Delta} \leq \frac{\left\|z^{n}\right\|^{2}}{n N_{Z}} \leq \frac{\left\|[z]^{n}\right\|^{2}}{n N_{Z}}+\epsilon_{\Delta} \tag{327}
\end{equation*}
$$

for some $\epsilon_{\Delta}$ that vanishes as $\Delta \rightarrow 0$. Similarly, it holds that

$$
\begin{equation*}
\frac{\left\|[z]^{n}-x^{n}\right\|^{2}}{n N_{E}}-\epsilon_{\Delta} \leq \frac{\left\|z^{n}-x^{n}\right\|^{2}}{n N_{E}} \leq \frac{\left\|[z]^{n}-x^{n}\right\|^{2}}{n N_{E}}+\epsilon_{\Delta} \tag{328}
\end{equation*}
$$

Therefore, combining (327) and (328), and according to definition of weak typicality, we have if $\left(x^{n},[z]^{n}\right) \in \mathcal{T}_{\delta-\epsilon_{\Delta}}^{n}$, then $\left(x^{n}, z^{n}\right) \in \mathcal{T}_{\delta}^{n}$; and in turn if $\left(x^{n}, z^{n}\right) \in \mathcal{T}_{\delta}^{n}$, then $\left(x^{n},[z]^{n}\right) \in \mathcal{T}_{\delta+\epsilon_{\Delta}}^{n}$, where $\mathcal{T}_{\delta}^{n}, \mathcal{T}_{\delta-\epsilon_{\Delta}}^{n}$ and $\mathcal{T}_{\delta+\epsilon_{\Delta}}^{n}$ correspond to jointly typical sets of $(X, Z)$. This implies

$$
\begin{align*}
1\left\{\left(x^{n},[z]^{n}\right) \in \mathcal{T}_{\delta-\epsilon_{\Delta}}^{n}\right\} & \leq 1\left\{\left(x^{n}, z^{n}\right) \in \mathcal{T}_{\delta}^{n}\right\} \\
& \leq 1\left\{\left(x^{n},[z]^{n}\right) \in \mathcal{T}_{\delta+\epsilon_{\Delta}}^{n}\right\} \tag{329}
\end{align*}
$$

Now we start to prove Lemma 17 From (329, we have (330) (given at the top of next page), where $\mathcal{T}_{\delta^{\prime}+\epsilon_{\Delta}}^{n}$ corresponds to a typical set of $Z$, and $\mathcal{F}^{n} \triangleq \mathcal{N}^{n} \cap \mathcal{T}_{\delta^{\prime}+\epsilon_{\Delta}}^{n}$.

Using union bound we have

$$
\begin{align*}
& \mathbb{P}_{\mathcal{C}}\left[\max _{m_{c} \in\left[2^{\left.m R_{c}\right],[z]^{n} \in \mathcal{F}^{n}}\right.}\right. \\
& \left.\quad \sum_{m_{1}^{\prime}} 1\left\{\left(X^{n}\left(m_{c}, m_{1}^{\prime}\right),[z]^{n}\right) \in \mathcal{T}_{\delta+\epsilon_{\Delta}}^{n}\right\}>2^{2 m \epsilon}\right] \\
& \leq 2^{m R_{c}}\left|\mathcal{F}^{n}\right|_{m_{c} \in\left[2^{m R_{c}}\right],[z]^{n} \in \mathcal{F}^{n}} \\
& \quad \mathbb{P}_{\mathcal{C}}\left(\sum_{m_{1}^{\prime}} 1\left\{\left(X^{n}\left(m_{c}, m_{1}^{\prime}\right),[z]^{n}\right) \in \mathcal{T}_{\delta+\epsilon_{\Delta}}^{n}\right\}>2^{2 m \epsilon}\right) . \tag{331}
\end{align*}
$$

Furthermore, $\left|\mathcal{F}^{n}\right|$ is upper-bounded by $2^{n\left(\frac{1}{2} \log n+o(\log n)\right)}$ as shown in (336), and it is easy to verify that (246)-254) still hold for the Gaussian case. Hence the probability in (331) decays doubly exponentially fast with $n$, which further means (331) vanishes as $m \rightarrow \infty$.

Similarly, we can also prove

$$
\begin{align*}
\mathbb{P}_{\mathcal{C}} & \min _{m_{c} \in\left[2^{\left.m R_{c}\right]},[z]^{n} \in \mathcal{F}^{n}\right.} \\
& \left.\sum_{m_{1}^{\prime}} 1\left\{\left(X^{n}\left(m_{c}, m_{1}^{\prime}\right),[z]^{n}\right) \in \mathcal{T}_{\delta-\epsilon_{\Delta}}^{n}\right\}<1\right] \rightarrow 0 \tag{337}
\end{align*}
$$

as $m \rightarrow \infty$. Hence Lemma 17 holds.

## Appendix N <br> PROOF OF LEMMA 18

Similar to 187, for continuous random variables the following holds.

$$
\begin{equation*}
\mathbb{E}_{\mathcal{C}}\left[Q_{S^{m} \hat{S}^{m}}\left(s^{m}, \hat{s}^{m}\right)\right]=\prod_{i=1}^{m} f_{S \hat{S}}\left(s_{i}, \hat{s}_{i}\right) \tag{338}
\end{equation*}
$$

Hence $\left(S^{m}, \hat{S}^{m}\right)$ is i.i.d. Since (weakly) typical set has total probability close to one [21], we have

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{P}\left(S^{m} \in \mathcal{T}_{\delta}^{m}\right)=1 \tag{339}
\end{equation*}
$$

$$
\begin{align*}
\mathbb{P}_{\mathcal{C}}\left(\mathcal{B}^{c}\right) \leq & \mathbb{P}_{\mathcal{C}}\left[\min _{m_{c} \in\left[2^{m R_{c}}\right],[z]^{n} \in \mathcal{T}_{\delta^{\prime}+\epsilon_{\Delta}}^{n}} \sum_{m_{1}^{\prime}} 1\left\{\left(X^{n}\left(m_{c}, m_{1}^{\prime}\right),[z]^{n}\right) \in \mathcal{T}_{\delta-\epsilon_{\Delta}}^{n}\right\}<1\right. \\
& \text { or } \left.\max _{m_{c} \in\left[2^{m R_{C}}\right],[z]^{n} \in \mathcal{T}_{\delta^{\prime}+\epsilon_{\Delta}}^{n}} \sum_{m_{1}^{\prime}} 1\left\{\left(X^{n}\left(m_{c}, m_{1}^{\prime}\right),[z]^{n}\right) \in \mathcal{T}_{\delta+\epsilon_{\Delta}}^{n}\right\}>2^{2 m \epsilon}\right] \\
= & \mathbb{P}_{\mathcal{C}}\left[\min _{m_{c} \in\left[2^{\left.m R_{c}\right],[z]^{n} \in \mathcal{F}^{n}}\right.} \sum_{m_{1}^{\prime}} 1\left\{\left(X^{n}\left(m_{c}, m_{1}^{\prime}\right),[z]^{n}\right) \in \mathcal{T}_{\delta-\epsilon_{\Delta}}^{n}\right\}<1\right] \\
& +\mathbb{P}_{\mathcal{C}}\left[\max _{m_{c} \in\left[2^{m R_{c}}\right],[z]^{n} \in \mathcal{F}^{n}} \sum_{m_{1}^{\prime}} 1\left\{\left(X^{n}\left(m_{c}, m_{1}^{\prime}\right),[z]^{n}\right) \in \mathcal{T}_{\delta+\epsilon_{\Delta}}^{n}\right\}>2^{2 m \epsilon}\right] \tag{330}
\end{align*}
$$

$$
\begin{align*}
\left|\mathcal{F}^{n}\right| & \leq \frac{\text { Volume of } n \text {-ball with radius } \sqrt{n N_{Z}\left(1+\delta^{\prime}+\epsilon_{\Delta}\right)}+\sqrt{n \Delta^{2}}}{\Delta^{n}}  \tag{332}\\
& =\frac{\pi^{n / 2}\left(\sqrt{n N_{Z}\left(1+\delta^{\prime}+\epsilon_{\Delta}\right)}+\sqrt{n \Delta^{2}}\right)^{n}}{\Delta^{n} \Gamma\left(\frac{n}{2}+1\right)}  \tag{333}\\
& \leq \frac{\pi^{n / 2}\left(\sqrt{n N_{Z}\left(1+\delta^{\prime}+\epsilon_{\Delta}\right)}+\sqrt{n \Delta^{2}}\right)^{n}}{\Delta^{n}}  \tag{334}\\
& =2^{n\left(\frac{1}{2} \log \pi-\log \Delta+\log \left(\sqrt{n N_{Z}\left(1+\delta^{\prime}+\epsilon_{\Delta}\right)}+\sqrt{n \Delta^{2}}\right)\right)}  \tag{335}\\
& \leq 2^{n\left(\frac{1}{2} \log n+o(\log n)\right)} \tag{336}
\end{align*}
$$

By the law of large numbers, we also have for any $\delta>0$,

$$
\begin{gather*}
\lim _{m \rightarrow \infty} \mathbb{P}\left(\frac{1}{m} \sum_{i=1}^{m} \jmath_{S}\left(S_{i}, D_{E}\right)>\mathbb{E}_{\jmath_{S}}\left(S, D_{E}\right)-\delta\right)=1  \tag{340}\\
\lim _{m \rightarrow \infty} \mathbb{P}\left(\frac{1}{m} \sum_{i=1}^{m} b^{\star}\left(\hat{S}_{i}\right) \geq \mathbb{E}_{\hat{S}}\left[b^{\star}(\hat{S})\right]-\delta\right)=1 \tag{341}
\end{gather*}
$$

and

$$
\begin{align*}
& \lim _{m \rightarrow \infty} \mathbb{P}\left(\frac{1}{m} \sum_{i=1}^{m} \jmath_{S \mid \hat{S}=\hat{S}_{i}}\left(S_{i}, b^{\star}\left(\hat{S}_{i}\right)\right)>\right. \\
&\left.\mathbb{E}_{\hat{S}} \mathbb{E}_{S}\left[\jmath_{S \mid \hat{S}=\hat{s}}\left(S, b^{\star}(\hat{s})\right)\right]-\delta\right)=1 \tag{342}
\end{align*}
$$

Combining (340) with the property (270), we have

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{P}\left(\frac{1}{m} \sum_{i=1}^{m} \jmath_{S}\left(S_{i}, D_{E}\right)>R_{S}\left(D_{E}\right)-\delta\right)=1 \tag{343}
\end{equation*}
$$

Combining (341) with 280 gives us

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{P}\left(\frac{1}{m} \sum_{i=1}^{m} b^{\star}\left(\hat{S}_{i}\right) \geq D_{E}-\delta\right)=1 \tag{344}
\end{equation*}
$$

Furthermore,

$$
\begin{align*}
\mathbb{E}_{\hat{S}} \mathbb{E}_{S}\left[\jmath_{S \mid \hat{S}=\hat{s}}\left(S, b^{\star}(\hat{s})\right)\right] & =\mathbb{E}_{\hat{S}} R_{S \mid \hat{S}=\hat{s}}\left(b^{\star}(\hat{s})\right)  \tag{345}\\
& =R_{S \mid \hat{S}}\left(D_{E}\right), \tag{346}
\end{align*}
$$

where (345) follows from (287), and (346) follows from 279).
Combining (342) with 346 gives
$\left.\lim _{m \rightarrow \infty} \mathbb{P}\left(\frac{1}{m} \sum_{i=1}^{m} \jmath_{S \mid \hat{S}=\hat{S}_{i}}\left(S_{i}, b^{\star}\left(\hat{S}_{i}\right)\right)\right)>R_{S \mid \hat{S}}\left(D_{E}\right)-\delta\right)=1$.
(339), (343), (344) and (347) imply the lemma holds.

## Appendix O <br> Proof of Lemma 20

Lemma 20 is proven in (348)-(352), where $\lambda \geq 0$ is given in (282) (or 285), and (352) follows from 285) and the property 288).

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$$
\begin{align*}
& \mathbb{P}\left[d\left(S^{m}, \check{s}^{m}\right) \leq D_{E}, \frac{1}{m} \sum_{i=1}^{m} \jmath_{S \mid \hat{S}=\hat{s}_{i}}\left(S_{i}, b^{\star}\left(\hat{s}_{i}\right)\right) \geq R_{S \mid \hat{S}}\left(D_{E}\right)-\delta, \left.\frac{1}{m} \sum_{i=1}^{m} b^{\star}\left(\hat{s}_{i}\right) \geq D_{E}-\delta \right\rvert\, \hat{S}^{m}=\hat{s}^{m}\right] \tag{348}
\end{align*}
$$

$$
\begin{align*}
& \begin{aligned}
& \sum_{s^{m}: d\left(s^{m}, \check{s}^{m}\right) \leq D_{E}, \frac{1}{m} \sum_{i=1}^{m} J_{S \mid \hat{S}=\hat{s}_{i}}\left(s_{i}, b^{\star}\left(\hat{s}_{i}\right)\right) \geq R_{S \mid \hat{s}}\left(D_{E}\right)-\delta} P\left(s^{m} \mid \hat{s}^{m}\right) 2^{m \lambda\left(D_{E}-d\left(s^{m}, \check{s}^{m}\right)\right)} \\
& \times 2^{\sum_{i=1}^{m} J_{S \mid \hat{s}=\hat{s}_{i}}\left(s_{i}, b^{\star}\left(\hat{s}_{i}\right)\right)-m\left(R_{S \mid \hat{S}}\left(D_{E}\right)-\delta\right)} 1\left\{\frac{1}{m} \sum_{i=1}^{m} b^{\star}\left(\hat{s}_{i}\right) \geq D_{E}-\delta\right\}
\end{aligned}  \tag{349}\\
& \left.\leq 2^{-m\left(R_{S \mid \hat{S}}\left(D_{E}\right)-\delta\right.}\right)_{\mathbb{E}}\left[2^{m \lambda\left(D_{E}-d\left(S^{m}, \check{s}^{m}\right)\right)+\sum_{i=1}^{m} \jmath_{S \mid \hat{S}=\hat{s}_{i}}\left(S_{i}, b^{\star}\left(\hat{s}_{i}\right)\right)} \mid \hat{S}^{m}=\hat{s}^{m}\right] 1\left\{\frac{1}{m} \sum_{i=1}^{m} b^{\star}\left(\hat{s}_{i}\right) \geq D_{E}-\delta\right\}  \tag{350}\\
& \left.\leq 2^{-m\left(R_{S \mid \hat{S}}\left(D_{E}\right)-\delta-\lambda \delta\right.}\right) \mathbb{E}\left[2^{\left.\lambda\left(\sum_{i=1}^{m} b^{\star}\left(\hat{s}_{i}\right)-\sum_{i=1}^{m} d\left(S_{i}, \hat{s}_{i}\right)\right)\right)+\sum_{i=1}^{m} \jmath_{S \mid \hat{S}=\hat{s}_{i}}\left(S_{i}, b^{\star}\left(\hat{s}_{i}\right)\right)} \mid \hat{S}^{m}=\hat{s}^{m}\right]  \tag{351}\\
& \leq 2^{-m\left(R_{S \mid \hat{S}}\left(D_{E}\right)-\delta-\lambda \delta\right)} \text {, } \tag{352}
\end{align*}
$$

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[^1]:    ${ }^{1}$ Throughout this paper, we use the boldface to denote a sequence of random variables, e.g., $\mathbf{B}=\left\{B_{n}\right\}_{n \in \mathbb{N}}$. Note that the alphabets of these random variables are not restricted to be Cartesian product, and the information spectrum quantities defined in this subsection will be applied for sequences of random variables of different length in the latter sections.
    ${ }^{2}$ For convenience, all log's and exp's in this paper are with respect to base 2.
    ${ }^{3}$ In this paper, the set $\{1, \ldots, l\}$ is sometimes denoted by $[l]$.

[^2]:    ${ }^{4}$ Similar to [8], it can be shown that one cannot benefit from replacing the decoder with a stochastic one.
    ${ }^{5}$ Here $\lfloor x\rfloor$ denotes the maximum integer not larger than $x$

[^3]:    ${ }^{6}$ The admissible region $\mathcal{R}$ does not change if we replace the constraint $\lim \sup _{m \rightarrow \infty} \frac{1}{m} \log \left|\mathcal{L}_{m}\right| \leq R_{L}$ of Definition 2 with $\frac{1}{m} \log \left|\mathcal{L}_{m}\right| \leq R_{L}$ for all $m$. This is because it only affects the achievability of the points on the boundary of $\mathcal{R}$, and however, $\mathcal{R}$ is defined as a closed set (all the boundary points are incorporated into it). To keep consistent with the standard formulation of rate constraint in [11], here we write $\lim \sup _{m \rightarrow \infty} \frac{1}{m} \log \left|\mathcal{L}_{m}\right| \leq$ $R_{L}$, instead of the other one.

[^4]:    ${ }^{7}$ Furthermore, by setting $d_{E}\left(s^{m}, \breve{s}^{m}\right)=1\left\{s^{m} \neq \check{s}^{m}\right\}$ and $D_{E}=0$, the admissible region for losslessly reconstructing the source at the wiretapper can be defined as well. By checking our proof, it can be verified that the admissible region for this case is the same as that obtained by specializing our result to the case of $d_{E}\left(s^{m}, \check{s}^{m}\right)=\frac{1}{m} \sum_{i=1}^{m} 1\left\{s_{i} \neq \check{s}_{i}\right\}$ and $D_{E}=0$.

[^5]:    ${ }^{8}$ Both the distributions $P_{X}$ and $P_{\hat{S} \mid S}$ are unique; see [21 Thm. 2.7.4].

[^6]:    ${ }^{9}$ Assume $R_{K} \leq R_{t}-R_{c}$. If $R_{K}>R_{t}-R_{c}$, then only $R_{t}-R_{c}$ rate of key is used in our scheme.

[^7]:    ${ }^{10}$ The subscripts means "key", "public", and "channel", respectively.

[^8]:    ${ }^{11}$ For ease in writing, we replace $\delta$ with $\frac{\delta}{2} \log e$ in the definitions of $\delta$ typical set and the $\delta$-jointly-typical set.

