Distortion Bounds for Transmitting Correlated Sources with Common Part over MAC

Lei Yu, Houqiang Li, *Senior Member, IEEE,* and Chang Wen Chen, *Fellow, IEEE*

***Abstract*—This paper investigates the joint source-channel coding problem of sending two correlated memoryless sources with common part over a memoryless multiple access channel (MAC). An inner bound and two outer bounds on the achievable distortion region are derived. In particular, they respectively recover the existing bounds for several special cases, such as communication without common part, lossless communication, and noiseless communication. When specialized to quadratic Gaussian communication case, the inner bound and outer bound are used to generate two new bounds. Numerical result shows that common part improves the performance of such distributed communication system.**

I. Introduction

The joint source-channel coding (JSCC) problem of transmitting correlated sources (with common part) over multiple access channel was first studied by Cover *et al.* [10] in which a bivariate finite-alphabet source is to be transmitted losslessly over a two-to-one multiple-access channel. As a lossy version of such JSCC problem, Minero *et al.* [7] considered the achievable distortion region of sending memoryless correlated source without common part over memoryless multiple access channel, and they derived an inner bound using a unified framework of hybrid coding (although the result still holds for the communication with common part, however it will become loose especially when the correlated sources are identical). This unified hybrid coding [7] generalizes the JSCC scheme given by Cover *et al.* [10], and can recover the achievability result given in [10]. In addition, specialized to quadratic Gaussian communication case, the inner bound in [7] can also recover the performance of hybrid coding given by Lapidoth *et al.* [12].

As for the converse part, Cover *et al.* [10] gave a tight but uncomputable (multi-letter) outer bound for lossless communication case, and Kang *et al.* [17] single-letterized this outer bound by utilizing a data processing inequality on maximal correlation coefficient. For lossy case, Lapidoth *et al.* [12] gave an outer bound for quadratic Gaussian communication utilizing a similar data processing inequality as well. Recently, Lapidoth *et al.* [18] also derived a new outer bound by the technology of introducing an auxiliary random variable (or remote source). However, the necessary condition of [18] is weaker than the one of [12] due to no data processing inequality applied in the single-letterization processing [18].

Lei Yu and Houqiang Li are with the Department of Electronic Engineering and Information Science, University of Science and Technology of China, Hefei, China (e-mail: yulei@ustc.edu.cn, lihq@ustc.edu.cn). Chang Wen Chen is with the Department of Computer Science and Engineering, State University of New York at Buffalo, Buffalo, USA (e-mail: chencw@buffalo.edu).

As a lossy version of the problem (with common part) studied by Cover *et al.* [10], in this paper, we consider JSCC of transmitting two memoryless correlated sources with common part over multiple access channel, and give an inner bound and two outer bounds on the achievable distortion region. For the inner bound, we propose an extended version of hybrid coding by adding common part into the hybrid coding [7] that is designed for the case with no common part, and hence our inner bound can recover the performance of the hybrid coding [7] by setting the common part to be empty. In addition, the outer bound is derived by introducing auxiliary random variables (or remote sources) as in [1]-[6], and it can recover the existing outer bounds when common part is absent at both encoders. When specialized to Gaussian communication with Gaussian common part, our bounds reduce to a new inner bound and a new outer bound.

The rest of this paper is organized as follows. Section II summarizes basic notations, and formulates the problem. Section III gives the main results for transmitting memoryless sources over memoryless MAC problem. Section IV gives the main results for Gaussian communication case. Finally, Section V gives the concluding remarks.

II. Problem Formulation and Preliminaries

Consider the correlated sources S_1 and S_2 have common part in sense of Gács-Körner-Witsenhausen common information [13], [14].

Definition 1. S_0 is a common part of two correlated sources S_1 and S_2 if there exist two functions $f_k : \mathcal{S}_k \mapsto \mathcal{S}_0, k = 1, 2$ such that $S_0 = f_1(S_1) = f_2(S_2)$ with probability one, where $\mathcal{S}_k$ denotes the alphabet of S_k, $k = 0, 1, 2$. We say that S_1 and S_2 have a common part if there exists a such S_0 as a common part of S_1 and S_2.

Now consider the problem transmitting correlated sources over a multiple access channel as shown in Fig. 1. The sender $k = 1, 2$ first codes discrete memoryless source S_k^n into X_k^n using a source-channel code, then transmits X_k^n to a common receiver through a discrete memoryless multiple access channel (DM-MAC) $p_{Y|X_1,X_2}$, and finally, the receiver produces source reconstructions $\hat{S}_1^n$ and $\hat{S}_2^n$ from the received signal Y^n.

Definition 2. An n-length source-channel code is defined by the two encoding functions $x_k^n : \mathcal{S}_k^n \mapsto \mathcal{X}_k^n, k = 1, 2$ and two decoding functions $\hat{s}_k : \mathcal{Y}^n \mapsto \hat{\mathcal{S}}_k^n, k = 1, 2$, where $\hat{\mathcal{S}}_k, \mathcal{X}_k$ and $\mathcal{Y}$ are the alphabet of source reconstruction $\hat{S}_k$, channel input X_k, and channel output Y.

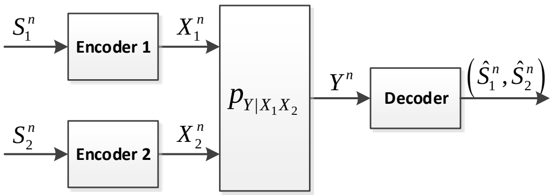

Fig. 1. Communication of memoryless correlated sources over a DM-MAC.

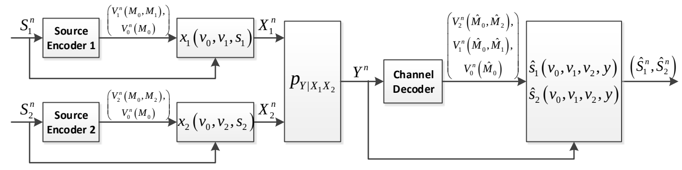

Fig. 2. The hybrid coding used to prove the inner bound in Theorem 1.

For any n-length source-channel code, the induced distortion is defined as

$$\mathbb{E}d_k(S_k^n, \hat{S}_k^n) = \frac{1}{n}\sum_{t=1}^{n}\mathbb{E}d_k(S_{k,t}, \hat{S}_{k,t}), \tag{1}$$

for $k = 1, 2$, where $d_k(s_k, \hat{s}_k) : \mathcal{S}_k \times \hat{\mathcal{S}}_k \mapsto [0, +\infty]$ is a distortion measure function for source S_k.

Definition 3. For transmitting sources (S_1, S_2) over MAC $p_{Y|X_1,X_2}$, we say the distortion tuple (D_1, D_2) is achievable, if there exists a sequence of source-channel codes such that

$$\limsup_{n\to\infty} \mathbb{E}d_k(S_k^n, \hat{S}_k^n) \le D_k. \tag{2}$$

Definition 4. For transmitting source (S_1, S_2) over MAC $p_{Y|X_1,X_2}$, the admissible distortion region is defined as

$$\mathcal{R} \triangleq \{(D_1, D_2) : (D_1, D_2) \text{ is achievable}\}. \tag{3}$$

In symmetric case,

$$\mathcal{R}_{\text{sym}} \triangleq \{D : (D, D) \text{ is achievable}\}. \tag{4}$$

III. General Communication

Now, we bound the distortion region for correlated sources communication over MAC. We first define a distortion region

$$\begin{aligned}
&\mathcal{R}^{(i)} = \Big\{(D_1, D_2) : \text{There exist some pmf } p_{V_0|S_0}p_{V_1|S_1,V_0}p_{V_2|S_2,V_0},\\
&\text{and functions } x_k(v_0, v_k, s_k), \hat{s}_k(v_0, v_1, v_2, y), k = 1, 2 \text{ such that}\\
&\mathbb{E}d_k(S_k, \hat{S}_k) \le D_k, k = 1, 2,\\
&I(V_1; S_1|V_0V_2) < I(V_1; Y|V_0V_2),\\
&I(V_2; S_2|V_0V_1) < I(V_2; Y|V_0V_1),\\
&I(V_1V_2; S_1S_2|V_0) < I(V_1V_2; Y|V_0),\\
&I(V_0V_1V_2; S_1S_2) < I(V_0V_1V_2; Y)\Big\}.
\end{aligned} \tag{5}$$

and another two distortion regions[1]

$$\begin{aligned}
&\mathcal{R}_1^{(o)} =\\
&\Big\{(D_1, D_2) : \text{For any } p_{U_{[1:L]}|S_1,S_2}, \text{ there exist some pmf } p_{\hat{S}_1,\hat{S}_2|S_1,S_2,U} \text{ and}\\
&p_Q\prod p_{S_1,S_2}(s_{1,i}, s_{2,i})p_{U_{[1:L]}|S_1,S_2}(u_{[1:L],i}|s_{1,i}, s_{2,i})p_{X_1|S_1^n,Q}p_{X_2|S_2^n,Q}\\
&\text{such that } \mathbb{E}d_k(S_k, \hat{S}_k) \le D_k, k = 1, 2,\\
&I(S_1S_2; \hat{S}_1\hat{S}_2|U_{\mathcal{A}}) \le I(X_1X_2; Y|U_{\mathcal{A}}^nQ) \text{ for any } \mathcal{A} \subseteq [1:L]\Big\}.
\end{aligned} \tag{6}$$

[1]The L in $\mathcal{R}_1^{(o)}$ is an arbitrary positive integer.

and

$$\begin{aligned}
&\mathcal{R}_2^{(o)} =\\
&\Big\{(D_1, D_2) : \text{For any } p_{U|S_1,S_2} \text{ such that } S_1 \to (S_0, U) \to S_2,\\
&\text{there exist some pmf } p_{\hat{S}_1,\hat{S}_2|S_1,S_2,U} \text{ and}\\
&p_Q\prod p_{S_1,S_2}(s_{1,i}, s_{2,i})\, p_{U|S_1,S_2}(u_i|s_{1,i}, s_{2,i})\, p_{X_1|S_1^n,Q}p_{X_2|S_2^n,Q}\\
&\text{such that } \mathbb{E}d_k(S_k, \hat{S}_k) \le D_k, k = 1, 2,\\
&I(S_1S_2; \hat{S}_1\hat{S}_2) \le I(X_1X_2; Y|Q),\\
&I(S_1S_2; \hat{S}_1\hat{S}_2|S_0) \le I(X_1X_2; Y|S_0^nQ),\\
&I(S_1; \hat{S}_1\hat{S}_2|S_2) \le I(X_1; Y|X_2S_2^nQ),\\
&I(S_2; \hat{S}_1\hat{S}_2|S_1) \le I(X_2; Y|X_1S_1^nQ),\\
&I(S_1S_2; \hat{S}_1\hat{S}_2|S_0U) \le I(X_1X_2; Y|S_0^nU^nQ),\\
&I(S_1; \hat{S}_1\hat{S}_2|S_2U) \le I(X_1; Y|X_2S_2^nU^nQ),\\
&I(S_2; \hat{S}_1\hat{S}_2|S_1U) \le I(X_2; Y|X_1S_1^nU^nQ)\Big\}.
\end{aligned} \tag{7}$$

Note that in $\mathcal{R}_2^{(o)}$, $p_{U|S_1,S_2}$ such that $S_1 \to (S_0, U) \to S_2$ always exists, since $S_1 \to (S_0, S_1) \to S_2$ and $S_1 \to (S_0, S_2) \to S_2$. In addition, it is easy to verify that for such $p_{U|S_1,S_2}$, the random variables $(S_1^n, S_2^n, S_0^n, U^n, X_1, X_2, Q)$ in $\mathcal{R}_2^{(o)}$ satisfy $X_1 \to (S_1^n, Q) \to (S_0^n, U^n, Q) \to (S_2^n, Q) \to X_2$. Now we give the following theorem.

Theorem 1. *For transmitting sources (S_1, S_2) with common part S_0 over MAC $p_{Y|X_1,X_2}$,*

$$\mathcal{R}^{(i)} \subseteq \mathcal{R} \subseteq \mathcal{R}_1^{(o)} \subseteq \mathcal{R}_2^{(o)}. \tag{8}$$

Remark 1. The inner bound in Theorem 1 can be easily extended to Gaussian or any other well-behaved continuous-alphabet source-channel pair by standard discretization method [9, Thm. 3.3], and moreover for this case the outer bound still holds. Theorem 1 can be also extended to the case of source-channel bandwidth mismatch, where m samples of memoryless correlated sources are transmitted through n uses of a DM-MAC. This can be accomplished by replacing the source and channel symbols in Theorem 1 by supersymbols of lengths m and n, respectively. Besides, Theorem 1 can be also extended to the problem with channel input cost (by adding channel input constraint).

Proof: The proof of $\mathcal{R}^{(i)} \subseteq \mathcal{R} \subseteq \mathcal{R}_1^{(o)}$ is given in Appendix A. Now we show that $\mathcal{R}_1^{(o)} \subseteq \mathcal{R}_2^{(o)}$. Actually $\mathcal{R}_2^{(o)}$ is a straightforward consequence of $\mathcal{R}_1^{(o)}$. Choose $L = 4, U_1 = S_1, U_2 = S_2, U_3 = S_0, U_4 = U$ in $\mathcal{R}_1^{(o)}$, where $U|S_1, S_2$ follows $p_{U|S_1,S_2}$. Then setting $\mathcal{A} = \emptyset, \{1\}, \{2\}, \{3\}, \{1,4\}, \{2,4\}, \{3,4\}$ in $\mathcal{R}_1^{(o)}$ respectively gives the inequalities in $\mathcal{R}_2^{(o)}$. Hence $\mathcal{R}_1^{(o)} \subseteq \mathcal{R}_2^{(o)}$. ■

The inner bound $\mathcal{R}^{(i)}$ in Theorem 1 is achieved by a unified hybrid coding scheme depicted in Fig. 2. In this scheme, the codebook has a layered (or superposition) structure, and consists of randomly and independently generated codewords $(V_0^n(m_0), V_1^n(m_0,m_1), V_2^n(m_0,m_2))$, $(m_0,m_1,m_2) \in \prod_{i=0}^{2}[1:2^{nr_i}]$, where (r_0,r_1,r_2) denote the rates of the digital information part and their values are given in Appendix A. At encoder sides, upon source sequence $S_k^n, k=1,2$, the encoder k first generates the common source S_0^n, and produces a common digital message M_0 from S_0^n by joint typicality encoding. Then upon M_0 and S_k^n, the encoder k produces a private digital messages M_k. Finally, the codeword $(V_0^n(M_0), V_k^n(M_0,M_k))$ and the source sequence S_k^n are used to generate channel input X_k^n by symbol-by-symbol mapping $x_k(v_0,v_k,s_k)$. At decoder side, upon received signal Y^n, the decoder reconstruct (M_0,M_1,M_2) (and also $(V_0^n(m_0), V_1^n(m_0,m_1), V_2^n(m_0,m_2))$) losslessly by joint typicality decoding, and then it produces $\hat{S}_k^n, k=1,2$ by symbol-by-symbol mapping $\hat{s}_k(v_0,v_1,v_2,y)$. Such a scheme could achieve any (D_1,D_2) in the inner bound $\mathcal{R}^{(i)}$.

Note that our hybrid coding is a extension of the hybrid coding in [7] to the case of correlated sources with common part. By setting the common part as empty, i.e., $V_0=\emptyset$, our result can recover the performance of the hybrid coding in [7]; see the special cases in the following.

The outer bounds $\mathcal{R}_1^{(o)}$ and $\mathcal{R}_2^{(o)}$ in Theorem 1 are derived by introducing auxiliary random variables (or remote sources) $U_{[1:L]}^n$ or U^n. This bounding technology was originated in the multiple description problem by Ozarow [1], and sequentially used in the distributed source coding problem by Wagner *et al.* [2] and the source broadcast problem [3], [4], [5], [6]. As these works, in our proof one or multiple additional random variables beyond those in the original problem are introduced. Besides, for $\mathcal{R}_2^{(o)}$ the auxiliary random variable U is restricted to following the Markov chain $S_1 \to (S_0,U) \to S_2$. This is inspired the work of Wagner *et al.* [2], where they showed that introducing an auxiliary random variable that follows a Markov chain structure is sufficient to achieve the tight bound for distributed source coding problem. Observe that our problem is a extension of distributed source coding problem to the noisy communication case, hence such a Markov chain structure is also used in our bound. Besides, such a Markov chain structure also makes a sequence of data processing inequalities available, which in turn generates some simpler bounds (see Section IV-C).

A. Special Cases

- Lossy Communication without Common Part

When there is no common part, the inner bound in Theorem 1 reduces to [7, Thm. 1], i.e.,

$$\begin{aligned}\mathcal{R}^{(i)} = \Big\{&(D_1,D_2) : \text{There exist some pmf } p_{V_0}p_{V_1|S_1,V_0}p_{V_2|S_2,V_0},\\ &\text{and functions } x_k(v_0,v_k,s_k), \hat{s}_k(v_0,v_1,v_2,y), k=1,2 \text{ such that}\\ &\mathbb{E}d_k(S_k,\hat{S}_k) \le D_k, k=1,2,\\ &I(V_1;S_1|V_2V_0) < I(V_1;Y|V_2V_0),\\ &I(V_2;S_2|V_1V_0) < I(V_2;Y|V_1V_0),\\ &I(V_1V_2;S_1S_2|V_0) < I(V_1V_2;Y|V_0)\Big\}. \end{aligned} \tag{9}$$

In this case, V_0 is independent of S_1 and S_2, and it becomes a timesharing auxiliary random variable. In addition, the outer bound $\mathcal{R}_2^{(o)}$ reduces to a new outer bound

$$\begin{aligned}\mathcal{R}^{(o)} = \Big\{&(D_1,D_2) : \text{For any } p_{U|S_1,S_2} \text{ such that } S_1 \to U \to S_2,\\ &\text{there exist some pmf } p_{\hat{S}_1,\hat{S}_2|S_1,S_2,U} \text{ and}\\ &p_Q \prod p_{S_1,S_2}(s_{1,i},s_{2,i})\, p_{U|S_1,S_2}(u_i|s_{1,i},s_{2,i})\, p_{X_1|S_1^n,Q}p_{X_2|S_2^n,Q}\\ &\text{such that } \mathbb{E}d_k(S_k,\hat{S}_k) \le D_k, k=1,2,\\ &I(S_1S_2;\hat{S}_1\hat{S}_2) \le I(X_1X_2;Y|Q),\\ &I(S_1;\hat{S}_1\hat{S}_2|S_2) \le I(X_1;Y|X_2S_2^nQ),\\ &I(S_2;\hat{S}_1\hat{S}_2|S_1) \le I(X_2;Y|X_1S_1^nQ),\\ &I(S_1S_2;\hat{S}_1\hat{S}_2|U) \le I(X_1X_2;Y|U^nQ),\\ &I(S_1;\hat{S}_1\hat{S}_2|S_2U) \le I(X_1;Y|X_2S_2^nU^nQ),\\ &I(S_2;\hat{S}_1\hat{S}_2|S_1U) \le I(X_2;Y|X_1S_1^nU^nQ)\Big\}. \end{aligned} \tag{10}$$

- Lossless Communication with Common Part

When specialized to lossless communication of correlated source with common part, by setting $V_0=(S_0,W), V_k=(S_k,X_k), x_k(v_0,v_k,s_k)=x_k, \hat{s}_k(v_0,v_1,v_2,y)=s_k, k=1,2$ where W is a random variable independent of S_1 and S_2, the inner bound in Theorem 1 recovers the inner bound [10, Thm. 1] on the admissible sources region, i.e.,

$$\begin{aligned}\mathcal{R}^{(i)} = \Big\{&p_{S_1,S_2} : \text{There exist some pmf } p_W p_{X_1|S_1,W}p_{X_2|S_2,W} \text{ such that}\\ &H(S_1|S_2) \le I(X_1;Y|X_2S_2W),\\ &H(S_2|S_1) \le I(X_2;Y|X_1S_1W),\\ &H(S_1S_2|S_0) \le I(X_1X_2;Y|S_0W),\\ &H(S_1S_2) \le I(X_1X_2;Y)\Big\}. \end{aligned} \tag{11}$$

In addition, the outer bound in Theorem 1 reduces to a new outer bound

$$\begin{aligned}\mathcal{R}^{(o)} = \Big\{&p_{S_1,S_2} : \text{For any } p_{U|S_1,S_2} \text{ such that } S_1 \to (S_0,U) \to S_2,\\ &\text{there exist some pmf}\\ &p_Q \prod p_{S_1,S_2}(s_{1,i},s_{2,i})\, p_{U|S_1,S_2}(u_i|s_{1,i},s_{2,i})\, p_{X_1|S_1^n,Q}p_{X_2|S_2^n,Q} \text{ such that}\\ &H(S_1S_2) \le I(X_1X_2;Y|Q),\\ &H(S_1|S_2) \le I(X_1;Y|X_2S_2^nQ),\\ &H(S_2|S_1) \le I(X_2;Y|X_1S_1^nQ),\\ &H(S_1S_2|S_0) \le I(X_1X_2;Y|S_0^nQ),\\ &H(S_1S_2|S_0U) \le I(X_1X_2;Y|S_0^nU^nQ),\\ &H(S_1|S_2U) \le I(X_1;Y|X_2S_2^nU^nQ),\\ &H(S_2|S_1U) \le I(X_2;Y|X_1S_1^nU^nQ)\Big\}. \end{aligned} \tag{12}$$

Furthermore, if (S_1,S_2) satisfy $S_1 \to S_0 \to S_2$, then

$$\begin{aligned}\mathcal{R} = \mathcal{R}^{(i)} = \mathcal{R}^{(o)} = \Big\{&p_{S_1,S_2} : \text{There exist some pmf}\\ &p_W p_{X_1|S_1,W}p_{X_2|S_2,W} \text{ such that}\\ &H(S_1|S_2) \le I(X_1;Y|X_2S_2W),\\ &H(S_2|S_1) \le I(X_2;Y|X_1S_1W),\\ &H(S_1S_2|S_0) \le I(X_1X_2;Y|S_0W),\\ &H(S_1S_2) \le I(X_1X_2;Y)\Big\}. \end{aligned} \tag{13}$$

This implies the admissible sources region for transmitting the correlated sources that are conditionally independent given the common part has been characterized completely. As a counterpart, the admissible sources region for broadcasting conditionally independent sources has been given in [6].

When S_0, S_1, S_2 are independent, the distortion region can be used to derive the capacity region of Multiple Access Channel with Common Message.

- Multiple Access Channel with Common Message

Consider lossless communication of independent sources S_0, S_1, S_2 with $H(S_k) = R_k, k = 0, 1, 2$, then the problem becomes Multiple Access Channel with Common Message [11]. Specialized to this case, (13) reduces to the capacity region [11, Eqn. 11], i.e.,

$$\begin{aligned}\mathcal{R} = \mathcal{R}^{(i)} = \mathcal{R}^{(o)} = \Big\{ & (R_0, R_1, R_2) : \text{There exist some pmf} \\ & p_W p_{X_1|W} p_{X_2|W} \text{ such that} \\ & R_1 \le I(X_1; Y|X_2 W), \\ & R_2 \le I(X_2; Y|X_1 W), \\ & R_1 + R_2 \le I(X_1 X_2; Y|W), \\ & R_0 + R_1 + R_2 \le I(X_1 X_2; Y) \Big\}. \end{aligned} \tag{14}$$

- Distributed Source Coding with Common Part

Consider the MAC $p_{Y|X_1,X_2}$ is noiseless, i.e., $Y = (X_1, X_2)$, and constrain $H(X_k) \le R_k, k = 1, 2$, then the problem becomes Distributed Source Coding with Common Part [8]. Set $V_k = (V_k, X_k), x_k(v_0, v_k, s_k) = x_k, \hat{s}_k(v_0, v_1, v_2, y) = \hat{s}_k(v_0, v_1, v_2), k = 1, 2$, where X_1 and X_2 are two random variables independent of each other and other variables. Then the inner bound of Theorem 1 recovers the inner bound [8, Thm.1] on the achievable distortion region, i.e.,

$$\begin{aligned}\mathcal{R}^{(i)} = \Big\{ & (D_1, D_2) : \text{There exist some pmf } p_{V_0|S_0} p_{V_1|S_1,V_0} p_{V_2|S_2,V_0}, \\ & \text{and functions } \hat{s}_k(v_0, v_1, v_2), k = 1, 2 \text{ such that} \\ & \mathbb{E} d_k(S_k, \hat{S}_k) \le D_k, k = 1, 2, \\ & I(V_1; S_1|V_0 V_2) < R_1, \\ & I(V_2; S_2|V_0 V_1) < R_2, \\ & I(V_0 V_1 V_2; S_1 S_2) < R_1 + R_2 \Big\}. \end{aligned} \tag{15}$$

IV. Quadratic Gaussian Communication

In this section, we apply the result for general communication to the quadratic Gaussian communication case. Consider sending jointly Gaussian sources $S_k = (S_0, S_k'), k = 1, 2$ with $(S_0, S_1', S_2') \sim \mathcal{N}\left(\mathbf{0}, \Sigma_{(S_0, S_1', S_2')}\right)$ and[2]

$$\Sigma_{(S_0, S_1', S_2')} = \begin{pmatrix} 1 & \rho_{01} & \rho_{02} \\ \rho_{01} & 1 & \rho_{12} \\ \rho_{02} & \rho_{12} & 1 \end{pmatrix} \tag{16}$$

over a power-constrained Gaussian MAC $Y = X_1 + X_2 + Z$ with $\mathbb{E}\left(X_k^2\right) \le P_k, k = 1, 2$ and $Z \sim \mathcal{N}(0, 1)$[3]. We also assume distortion is measured by quadratic distortion function on $S_k', k = 1, 2$, i.e., $d_k(s_k, \hat{s}_k) = d(s_k', \hat{s}_k) \triangleq (s_k' - \hat{s}_k)^2, k = 1, 2$, and source bandwidth and channel bandwidth are matched.

Without loss of generality, (S_0, S_1', S_2') can be expressed as

$$S_1' = \rho_{01} S_0 + \sqrt{1 - \rho_{01}^2} U_1, \tag{17}$$

$$S_2' = \rho_{02} S_0 + \sqrt{1 - \rho_{02}^2} U_2, \tag{18}$$

with

$$U_1 = \beta_1 U + \sqrt{1 - \beta_1^2} B_1, \tag{19}$$

$$U_2 = \beta_2 U + \sqrt{1 - \beta_2^2} B_2. \tag{20}$$

where $U \sim \mathcal{N}(0, 1)$ and $B_k \sim \mathcal{N}(0, 1), k = 1, 2$ are mutually independent Gaussian variables and also independent of S_0, and

$$\beta_1 \beta_2 = \frac{\rho_{12} - \rho_{01}\rho_{02}}{\sqrt{(1 - \rho_{01}^2)(1 - \rho_{02}^2)}}. \tag{21}$$

Obviously $S_1 \to (S_0, U) \to S_2$ holds.

A. Hybrid Coding Scheme

In the following, we obtain the performance of hybrid coding scheme by specializing the inner bound of Theorem 1. Let

$$V_0 = S_0 + W_0 \tag{22}$$

$$V_k = F_k(S_0, S_k, V_0)^T + W_k, k = 1, 2, \tag{23}$$

and set $x_k(v_0, v_k, s_k), k = 1, 2$ to the linear functions

$$X_k = G_k(S_0, S_k, V_0, V_k)^T, k = 1, 2, \tag{24}$$

where $W_k \sim \mathcal{N}(0, \omega_k), k = 0, 1, 2$ are mutually independent and also independent of $S_k, k = 0, 1, 2$, and $F_k = (f_{k,1}, f_{k,2}, f_{k,3})$ and $G_k = (g_{k,1}, g_{k,2}, g_{k,3}, g_{k,4})$ are two row vectors of coefficients.

This induces the relationship

$$(S_0, S_1, S_2, V_0, V_1, V_2, Y)^T = A(S_0, S_1, S_2, W_0, W_1, W_2, Z)^T,$$

where A is given in (25) with

$$a_{71} = g_{1,1} + g_{2,1} + g_{1,3} + g_{2,3} + g_{1,4}(f_{1,1} + f_{1,3}) + g_{2,4}(f_{2,1} + f_{2,3}).$$

Hence the covariance of $(S_0, S_1, S_2, V_0, V_1, V_2, Y)$ is given by

$$\Sigma_{(S_0, S_1, S_2, V_0, V_1, V_2, Y)} = A \Sigma_{(S_0, S_1, S_2, W_0, W_1, W_2, Z)} A^T. \tag{26}$$

Set $\hat{s}_k(v_0, v_1, v_2, y), k = 1, 2$ to the linear functions

$$\hat{S}_k = \Sigma_{S_k, (V_0, V_1, V_2, Y)} \Sigma^{-1}_{(V_0, V_1, V_2, Y)} (V_0, V_1, V_2, Y)^T, \tag{27}$$

then the covariance of error $E_k \triangleq S_k - \hat{S}_k, k = 1, 2$ is given by

$$\Sigma_{E_k} = \Sigma_{S_k} - \Sigma_{S_k, (V_0, V_1, V_2, Y)} \Sigma^{-1}_{(V_0, V_1, V_2, Y)} \Sigma^T_{S_k, (V_0, V_1, V_2, Y)}.$$

In addition, owing to power constraint,

$$\Sigma_{X_k} \le P_k, \tag{28}$$

where

$$\Sigma_{X_k} = G_k \Sigma_{(S_0, S_k, V_0, V_k)} G_k^T. \tag{29}$$

Substitute these random variables and functions into $\mathcal{R}^{(i)}$ in Theorem 1, then we get the performance of the hybrid coding.

[2] Throughout this paper, we use $\Sigma_{(X,Y)}$ to denote the covariance of (X, Y) and $\Sigma_{X,Y}$ to denote the cross-covariance of X and Y.

[3] For simplicity, we assume source variances are unit and so is the channel noise power, which can cover general cases by scaling P_k and D_k.

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 0 & 0 & 0\\ f_{1,1}+f_{1,3} & f_{1,2} & 0 & f_{1,3} & 1 & 0 & 0\\ f_{2,1}+f_{2,3} & 0 & f_{2,2} & f_{2,3} & 0 & 1 & 0\\ a_{71} & g_{1,2}+g_{1,4}f_{1,2} & g_{2,2}+g_{2,4}f_{2,2} & g_{1,3}+g_{1,4}f_{1,3}+g_{2,3}+g_{2,4}f_{2,3} & g_{1,4} & g_{2,4} & 1 \end{pmatrix} \tag{25}$$

Theorem 2. *For transmitting Gaussian source with common part over Gaussian MAC,*

$$\begin{aligned}\mathcal{R}\supseteq\mathcal{R}_h^{(i)}\triangleq\Big\{&(D_1,D_2): \textit{There exist } F_k,G_k,\omega_0,\omega_1,\omega_2,k=1,2\\ &\textit{such that } \Sigma_{E_k}\le D_k,\Sigma_{X_k}\le P_k,k=1,2,\\ &\frac{\left|\Sigma_{(V_0,V_2,S_0,S_1)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,S_0,S_1)}\right|}<\frac{\left|\Sigma_{(V_0,V_2,Y)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,Y)}\right|},\\ &\frac{\left|\Sigma_{(V_0,V_1,S_0,S_2)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,S_0,S_2)}\right|}<\frac{\left|\Sigma_{(V_0,V_1,Y)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,Y)}\right|},\\ &\frac{\left|\Sigma_{(V_0,S_0,S_1,S_2)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,S_0,S_1,S_2)}\right|}<\frac{\left|\Sigma_{(V_0,Y)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,Y)}\right|},\\ &\frac{\left|\Sigma_{(S_0,S_1,S_2)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,S_0,S_1,S_2)}\right|}<\frac{\left|\Sigma_{Y}\right|}{\left|\Sigma_{(V_0,V_2,V_1,Y)}\right|}\Big\}.\end{aligned}\tag{30}$$

Proof: Substitute the random variables and functions set above into $\mathcal{R}^{(i)}$ of Theorem 1, then

$$I\left(V_1;S_1|V_0V_2\right)=\frac{1}{2}\log\frac{\left|\Sigma_{(V_0,V_2,S_1)}\right|\left|\Sigma_{(V_0,V_2,V_1)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,S_1)}\right|\left|\Sigma_{(V_0,V_2)}\right|}$$

and

$$I\left(V_1;Y|V_0V_2\right)=\frac{1}{2}\log\frac{\left|\Sigma_{(V_0,V_2,Y)}\right|\left|\Sigma_{(V_0,V_2,V_1)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,Y)}\right|\left|\Sigma_{(V_0,V_2)}\right|}.$$

Hence the inequality $I\left(V_1;S_1|V_0V_2\right)<I\left(V_1;Y|V_0V_2\right)$ in $\mathcal{R}^{(i)}$ is equivalent to $\frac{\left|\Sigma_{(V_0,V_2,S_0,S_1)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,S_0,S_1)}\right|}<\frac{\left|\Sigma_{(V_0,V_2,Y)}\right|}{\left|\Sigma_{(V_0,V_2,V_1,Y)}\right|}$. Similarly, the last three inequalities in $\mathcal{R}^{(i)}$ are equivalent to the last three inequalities in $\mathcal{R}_h^{(i)}$. ∎

B. Uncoded Scheme

Now we consider an uncoded scheme which adopts linear symbol-by-symbol encoders

$$X_1=g_{10}S_0+g_{11}U_1 \tag{31}$$

$$X_2=g_{20}S_0+g_{22}U_2 \tag{32}$$

and MMSE (minimum mean square error) decoders (which is optimal given the encoder (31) and (32))

$$\hat{S}_1=\mathbb{E}\left(S_1'|Y\right)=\frac{\left(\rho_{01}(g_{10}+g_{20})+g_{11}\sqrt{1-\rho_{01}^2}+g_{22}\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{1-\rho_{02}^2}}\right)Y}{(g_{10}+g_{20})^2+g_{11}^2+g_{22}^2+2g_{11}g_{22}\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{(1-\rho_{01}^2)(1-\rho_{02}^2)}}+1},$$

$$\hat{S}_2=\mathbb{E}\left(S_2'|Y\right)=\frac{\left(\rho_{02}(g_{10}+g_{20})+g_{22}\sqrt{1-\rho_{02}^2}+g_{11}\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{1-\rho_{01}^2}}\right)Y}{(g_{10}+g_{20})^2+g_{11}^2+g_{22}^2+2g_{11}g_{22}\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{(1-\rho_{01}^2)(1-\rho_{02}^2)}}+1}$$

where $\left(g_{k0},g_{kk}\right),k=1,2$ satisfy power constraint $g_{k0}^2+g_{kk}^2\le P_k,k=1,2$. Note that such uncoded scheme is a special case of hybrid coding above.

Theorem 3. *The distortion pairs (D_1^u,D_2^u) resulting from the described uncoded scheme are given by*

$$D_1^u=1-\frac{\left(\rho_{01}(g_{10}+g_{20})+g_{11}\sqrt{1-\rho_{01}^2}+g_{22}\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{1-\rho_{02}^2}}\right)^2}{(g_{10}+g_{20})^2+g_{11}^2+g_{22}^2+2g_{11}g_{22}\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{(1-\rho_{01}^2)(1-\rho_{02}^2)}}+1},$$

$$D_2^u=1-\frac{\left(\rho_{02}(g_{10}+g_{20})+g_{22}\sqrt{1-\rho_{02}^2}+g_{11}\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{1-\rho_{01}^2}}\right)^2}{(g_{10}+g_{20})^2+g_{11}^2+g_{22}^2+2g_{11}g_{22}\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{(1-\rho_{01}^2)(1-\rho_{02}^2)}}+1}.$$

Hence

$$\begin{aligned}\mathcal{R}\supseteq\mathcal{R}_u^{(i)}\triangleq\Big\{&(D_1,D_2): \textit{There exist } \left(g_{k0},g_{kk}\right),k=1,2 \textit{ such that}\\ &g_{k0}^2+g_{kk}^2\le P_k,k=1,2,D_1\ge D_1^u,D_2\ge D_2^u\Big\}.\end{aligned}$$

C. Outer Bound

Substitute the random variable U such that (19) and (20) into the outer bound $\mathcal{R}_2^{(o)}$ of Theorem 1, then the following outer bound on Gaussian communication is recovered.

Theorem 4. *For transmitting Gaussian source with common part over Gaussian MAC,*

$$\begin{aligned}\mathcal{R}\subseteq\mathcal{R}^{(o)}\triangleq\Big\{&(D_1,D_2): \textit{There exist some values } 0\le\hat{\rho}\le1,\\ &0\le\hat{\rho}_0\le\rho_{12|0}\triangleq\frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{\left(1-\rho_{01}^2\right)\left(1-\rho_{02}^2\right)}} \textit{ such that for any } \rho_{12|0}\le\beta_1\le1,\\ &R_{S_1S_2}(D_1,D_2)\le\frac{1}{2}\log\left(1+P_1+P_2+2\hat{\rho}\sqrt{P_1P_2}\right),\\ &R_{S_1S_2|S_0}(D_1,D_2)\\ &\quad\le\frac{1}{2}\log\left(1+\left[\frac{1-\hat{\rho}^2}{1-\rho_{12|0}^2},1\right]^-\left(P_1+P_2+2\hat{\rho}_0\sqrt{P_1P_2}\right)\right),\\ &\frac{\left(1-\rho_{01}^2\right)\left(1-\rho_{12|0}^2\right)}{D_1}\le1+\left[1-\hat{\rho}^2,1-\hat{\rho}_0^2\right]^-P_1,\\ &\frac{\left(1-\rho_{02}^2\right)\left(1-\rho_{12|0}^2\right)}{D_2}\le1+\left[1-\hat{\rho}^2,1-\hat{\rho}_0^2\right]^-P_2,\\ &\left[\frac{\left(1-\rho_{01}^2\right)\left(1-\beta_1^2\right)}{D_1},1\right]^+\left[\frac{\left(1-\rho_{02}^2\right)\left(1-\beta_2^2\right)}{D_2},1\right]^+\le1+\\ &\qquad\left[(1-\theta_1^2)P_1+(1-\theta_2^2)P_2,\left(1-\frac{\hat{\rho}_0^2}{\beta_2^2}\right)P_1+\left(1-\frac{\hat{\rho}_0^2}{\beta_1^2}\right)P_2\right]^-,\\ &\frac{\left(1-\rho_{01}^2\right)\left(1-\beta_1^2\right)}{D_1}\le1+\left[1-\theta_1^2,1-\frac{\hat{\rho}_0^2}{\beta_2^2}\right]^-P_1,\\ &\frac{\left(1-\rho_{02}^2\right)\left(1-\beta_2^2\right)}{D_2}\le1+\left[1-\theta_2^2,1-\frac{\hat{\rho}_0^2}{\beta_1^2}\right]^-P_2,\\ &\textit{for some } \theta_1,\theta_2 \textit{ such that } 0\le\theta_1,\theta_2\le1,\hat{\rho}\le\theta_1\theta_2\Big\},\end{aligned}\tag{33}$$

where $[x,y]^+ \triangleq \max\{x,y\}$, $[x,y]^- \triangleq \min\{x,y\}$, $\beta_2 = \frac{\rho_{12}-\rho_{01}\rho_{02}}{\beta_1\sqrt{(1-\rho_{01}^2)(1-\rho_{02}^2)}}$,

$$R_{S_1S_2}(D_1,D_2) = \inf_{p_{\hat{S}_1\hat{S}_2|S_1S_2}:\mathbb{E}(S'_k-\hat{S}_k)^2\le D_k, k=1,2} I(S_1S_2;\hat{S}_1\hat{S}_2) = \begin{cases} \frac{1}{2}\log^+\frac{1}{D_1}, & \text{if } \rho_{12}^2 \ge \frac{1-D_2}{1-D_1}; \\ \frac{1}{2}\log^+\frac{1-\rho_{12}^2}{D_1D_2}, & \text{if } \rho_{12}^2 \le (1-D_1)(1-D_2); \\ \frac{1}{2}\log^+\frac{1-\rho_{12}^2}{D_1D_2-\left(|\rho_{12}|-\sqrt{(1-D_1)(1-D_2)}\right)^2}, & \text{otherwise} \end{cases}$$

with $\log^+ x \triangleq \max\{\log x, 0\}$, *under the assumption that* $D_1 \le D_2$*, denotes the minimum sum rate needed to achieve both* D_1 *and* D_2 *at the receiver when the encoders cooperate to encode their observations [12, Thm. III.1], and*

$$R_{S_1S_2|S_0}(D_1,D_2) = \inf_{p_{\hat{S}_1\hat{S}_2|S_0S_1S_2}:\mathbb{E}(S'_k-\hat{S}_k)^2\le D_k, k=1,2} I(S_1S_2;\hat{S}_1\hat{S}_2|S_0) = \inf_{p_{\hat{U}_1\hat{U}_2|U_1U_2}:\mathbb{E}(U_k-\hat{U}_k)^2\le D_k, k=1,2} I(U_1U_2;\hat{U}_1\hat{U}_2) = \begin{cases} \frac{1}{2}\log^+\frac{1}{D'_1}, & \text{if } \rho_{12|0}^2 \ge \frac{1-D'_2}{1-D'_1}; \\ \frac{1}{2}\log^+\frac{1-\rho_{12|0}^2}{D'_1D'_2}, & \text{if } \rho_{12|0}^2 \le (1-D'_1)(1-D'_2); \\ \frac{1}{2}\log^+\frac{1-\rho_{12|0}^2}{D'_1D'_2-\left(|\rho_{12|0}|-\sqrt{(1-D'_1)(1-D'_2)}\right)^2}, & \text{otherwise} \end{cases}$$

with

$$D'_1 = \frac{D_1}{1-\rho_{01}^2}, \qquad D'_2 = \frac{D_2}{1-\rho_{02}^2},$$

under the assumption that $D'_1 \le D'_2$*, denotes the minimum sum rate needed to achieve both* D_1 *and* D_2 *at the receiver when the side information* S_0 *is available at both the encoders and the decoder and the encoders cooperate to encode their observations with help of* S_0 *.*

The proof of Theorem 4 is given in Appendix B. The Maximal Correlation Theory (Hirschfeld–Gebelein–Rényi maximal correlation) is exploited in the proof. When $\rho_{01} = \rho_{02} = 0$, Theorem 4 can recover the outer bound without common part [12, Thm. IV.1]. Besides, Theorem 4 can be extended to any other source-channel pair by following similar steps to the proof.

When specialized to the symmetric case, Theorem 4 reduces to the following result.

Corollary 1. *In the symmetric case,*

$$\begin{aligned}
&\mathcal{R}_{sym} \subseteq \mathcal{R}_{sym}^{(o)} \triangleq \Big\{ D : \text{There exist some values } 0 \le \hat{\rho} \le 1, \\
&0 \le \hat{\rho}_0 \le \rho_{12|0} \triangleq \frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{(1-\rho_{01}^2)(1-\rho_{02}^2)}} \text{ such that for any } \rho_{12|0} \le \beta_1 \le 1, \\
&R_{S_1S_2}(D,D) \le \frac{1}{2}\log\left(1+2(1+\hat{\rho})P\right), \\
&R_{S_1S_2|S_0}(D,D) \le \frac{1}{2}\log\left(1+\left[\frac{1-\hat{\rho}^2}{1-\rho_{12|0}^2},1\right]^- \cdot 2(1+\hat{\rho}_0)P\right), \\
&\frac{(1-\rho_{01}^2)(1-\rho_{12|0}^2)}{D} \le 1+[1-\hat{\rho}^2, 1-\hat{\rho}_0^2]^- P, \\
&\frac{(1-\rho_{02}^2)(1-\rho_{12|0}^2)}{D} \le 1+[1-\hat{\rho}^2, 1-\hat{\rho}_0^2]^- P, \\
&\left[\frac{(1-\rho_{01}^2)(1-\beta_1^2)}{D},1\right]^+\left[\frac{(1-\rho_{02}^2)(1-\beta_2^2)}{D},1\right]^+ \\
&\quad \le 1+\left[2-\theta_1^2-\theta_2^2, 2-\frac{\hat{\rho}_0^2}{\beta_2^2}-\frac{\hat{\rho}_0^2}{\beta_1^2}\right]^- P, \\
&\frac{(1-\rho_{01}^2)(1-\beta_1^2)}{D} \le 1+\left[1-\theta_1^2, 1-\frac{\hat{\rho}_0^2}{\beta_2^2}\right]^- P, \\
&\frac{(1-\rho_{02}^2)(1-\beta_2^2)}{D} \le 1+\left[1-\theta_2^2, 1-\frac{\hat{\rho}_0^2}{\beta_1^2}\right]^- P, \\
&\text{for some } \theta_1,\theta_2 \text{ such that } 0 \le \theta_1,\theta_2 \le 1, \hat{\rho} \le \theta_1\theta_2 \Big\},
\end{aligned} \tag{34}$$

where

$$R_{S_1S_2}(D,D) = \begin{cases} \frac{1}{2}\log^+\frac{1-\rho_{12}^2}{D^2}, & \text{if } |\rho_{12}| \le 1-D; \\ \frac{1}{2}\log^+\frac{1+|\rho_{12}|}{2D-(1-|\rho_{12}|)}, & \text{otherwise} \end{cases}$$

and

$$R_{S_1S_2|S_0}(D,D) = \begin{cases} \frac{1}{2}\log^+\frac{1-\rho_{12|0}^2}{D'^2}, & \text{if } |\rho_{12|0}| \le 1-D'; \\ \frac{1}{2}\log^+\frac{1+|\rho_{12|0}|}{2D'-(1-|\rho_{12|0}|)}, & \text{otherwise} \end{cases}$$

with

$$D' = \frac{D}{1-\rho_{01}^2}.$$

Fig. 3 illustrates the various bounds on the achievable distortion.

V. Concluding Remarks

In this paper, we focused on the joint source-channel coding problem of sending memoryless correlated sources with common part over memoryless multiple access channel, and developed an inner bound and two outer bounds for this problem. The inner bound is achieved by a unified hybrid coding scheme with common part, and as special cases, it can recover the performance of existing hybrid coding without common part. Similarly, our outer bound can also recover several outer bounds in the literature. When specialized to transmitting Gaussian sources over Gaussian MAC, the inner bound and outer bound are used to generate a new inner bound and a new outer bound, which can recover the best known inner bound and outer bound without common part in the literature.

It is worth noting that in our results, two kinds of common informations are involved. They are respectively in sense of Gács-Körner-Witsenhausen common information [13], [14],

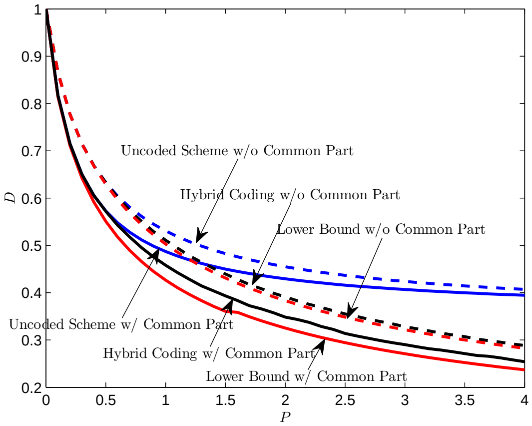

Fig. 3. Distortion bounds in the symmetric case for sending Gaussian sources over Gaussian MAC. Uncoded Scheme w/ Common Part, Hybrid Coding w/ Common Part, and Lower Bound w/ Common Part are the bounds for the case with common part where $\rho_{12} = 0.3, \rho_{01} = \rho_{02} = 0.8$. They respectively correspond to the inner bound in Theorem 3, the inner bound in Theorem 2, the outer bound in Corollary 1. Uncoded Scheme w/o Common Part, Hybrid Coding w/o Common Part, and Lower Bound w/o Common Part are the bounds for the case with no common part where $\rho_{12} = 0.3, \rho_{01} = \rho_{02} = 0$.

and in sense of Wyner's common information [15]. In our problem, the Gács-Körner-Witsenhausen common information, i.e., the common part, has been exploited to improve the performance of communication system, and the Wyner's common information has been exploited to obtain the outer bounds. Besides, correlation ratio and maximal correlation coefficient are also utilized to derive the outer bound for Gaussian communication case. These concepts and tools are expected to be exploited to derive achievability and converse results for other problems in network information theory.

Appendix A
Proof of Theorem 1

A. Inner Bound

We use the hybrid coding shown in Fig. 2 to prove the inner bound.

Codebook Generation: Fix conditional pmf $p_{V_0|S_0} p_{V_1|S_1,V_0} p_{V_2|S_2,V_0}$, encoding functions $x_k(v_0, v_k, s_k), k = 1,2$ and decoding functions $\hat{s}_k(v_0, v_1, v_2, y), k = 1,2$ that satisfy all the inequalities in the inner bound (5). Randomly and independently generate a set of sequences $v_0^n(m_0), m_0 \in [1 : 2^{nr_0}]$, with each distributed according to $\prod_{i=1}^n p_{V_0}(v_{0,i})$. For $k = 1,2$ and for each $m_0 \in [1 : 2^{nr_0}]$, randomly and independently generate a set of sequences $v_k^n(m_0, m_k), m_k \in [1 : 2^{nr_k}]$, with each distributed according to $\prod_{i=1}^n p_{V_k|V_0}(v_{k,i}|v_{0,i}(m_0))$. The codebook

$$\mathcal{C} = \Big\{ (v_0^n(m_0), v_1^n(m_0, m_1), v_2^n(m_0, m_2)) : (m_0, m_1, m_2) \in [1 : 2^{nr_0}] \times [1 : 2^{nr_1}] \times [1 : 2^{nr_2}] \Big\}.$$

is revealed to both the encoders and the decoder.

Encoding: We use joint typicality encoding. Let $\epsilon > \epsilon_0$. Given s_0^n, both encoders 1 and 2 find the smallest index m_0 such that $(s_0^n, v_0^n(m_0)) \in \mathcal{T}_{\epsilon_0}^{(n)}$. If there is no such index, let $m_0 = 1$. For $k = 1,2$, given s_k^n and $v_0^n(m_0)$, encoder k finds the smallest index m_k such that $(s_0^n, s_k^n, v_0^n(m_0), v_k^n(m_0, m_k)) \in \mathcal{T}_\epsilon^{(n)}$. If there is no such index, let $m_k = 1$. Then the encoder k transmits the signal

$$x_{k,i} = x_k(v_{0,i}(m_0), v_{k,i}(m_0, m_k), s_{k,i}), 1 \le i \le n. \quad (35)$$

Decoding: We use joint typicality decoding. Let $\epsilon' > \epsilon$. Upon receiving signal y^n, the decoder of the receiver finds the smallest index vector $(\hat{m}_0, \hat{m}_1, \hat{m}_2)$ such that

$$(v_0^n(\hat{m}_0), v_1^n(\hat{m}_0, \hat{m}_1), v_2^n(\hat{m}_0, \hat{m}_2), y^n) \in \mathcal{T}_{\epsilon'}^{(n)}. \quad (36)$$

If there is no such index vector, let $(\hat{m}_0, \hat{m}_1, \hat{m}_2) = (1,1,1)$. The decoder reconstructs the sources as for $k = 1,2$,

$$\hat{s}_{k,i} = \hat{s}_k(v_{0,i}(\hat{m}_0), v_{1,i}(\hat{m}_0, \hat{m}_1), v_{2,i}(\hat{m}_0, \hat{m}_2), y_i), 1 \le i \le n. \quad (37)$$

Analysis of Expected Distortion: We bound the distortion averaged over (S_1^n, S_2^n), and the random choice of the codebook $\mathcal{C}$. Define the "error" event

$$\mathcal{E} = \Big\{ \Big(S_0^n, S_1^n, S_2^n, V_0^n(\hat{M}_0), V_1^n(\hat{M}_0, \hat{M}_1), V_2^n(\hat{M}_0, \hat{M}_2), Y^n \Big) \notin \mathcal{T}_{\epsilon'}^{(n)} \Big\}.$$

Then we have

$$\mathcal{E} \subseteq \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \cup \mathcal{E}_5, \quad (38)$$

where

$$\mathcal{E}_0 = \Big\{ (S_0^n, V_0^n(m_0)) \notin \mathcal{T}_{\epsilon_0}^{(n)} \text{ for all } m_0 \Big\},$$
$$\mathcal{E}_1 = \Big\{ (S_0^n, S_1^n, V_0^n(M_0), V_1^n(M_0, m_1)) \notin \mathcal{T}_{\epsilon}^{(n)} \text{ for all } m_1 \Big\},$$
$$\mathcal{E}_2 = \Big\{ (S_0^n, S_2^n, V_0^n(M_0), V_2^n(M_0, m_2)) \notin \mathcal{T}_{\epsilon}^{(n)} \text{ for all } m_2 \Big\},$$
$$\mathcal{E}_3 = \Big\{ (S_0^n, S_1^n, S_2^n, V_0^n(M_0), V_1^n(M_0, M_1), V_2^n(M_0, M_2)) \notin \mathcal{T}_{\epsilon_1}^{(n)} \Big\},$$
$$\mathcal{E}_4 = \Big\{ (S_0^n, S_1^n, S_2^n, V_0^n(M_0), V_1^n(M_0, M_1), V_2^n(M_0, M_2), Y^n) \notin \mathcal{T}_{\epsilon'}^{(n)} \Big\},$$
$$\mathcal{E}_5 = \Big\{ \Big(V_0^n(m_0'), V_1^n(m_0', m_1'), V_2^n(m_0', m_2'), Y^n \Big) \in \mathcal{T}_{\epsilon'}^{(n)} \text{ for some } (m_0', m_1', m_2') \ne (M_0, M_1, M_2) \Big\},$$

for some ϵ_1 such that $\epsilon_0 < \epsilon < \epsilon_1 < \epsilon'$. Using union bound, we have

$$\mathbb{P}(\mathcal{E}) \le \mathbb{P}(\mathcal{E}_0) + \mathbb{P}(\mathcal{E}_0^c \cap \mathcal{E}_1) + \mathbb{P}(\mathcal{E}_0^c \cap \mathcal{E}_2) + \mathbb{P}(\mathcal{E}_0^c \cap \mathcal{E}_1^c \cap \mathcal{E}_2^c \cap \mathcal{E}_3) + \mathbb{P}(\mathcal{E}_3^c \cap \mathcal{E}_4) + \mathbb{P}(\mathcal{E}_5). \quad (39)$$

Now we claim that if all the inequalities in the inner bound (5) hold, then $\mathbb{P}(\mathcal{E})$ tends to zero as $n \to \infty$. Before proving it, we show that this claim implies the distortions in the inner bound (5) are achievable. The expected distortions are bounded by

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E} d_k(S_k^n, \hat{S}_k^n) \\
&= \limsup_{n\to\infty} \Big(\mathbb{P}\left(\mathcal{E}_{4,k}\right) \mathbb{E}\left[d_k(S_k^n, \hat{S}_k^n) | \mathcal{E}_{4,k} \right] \\
&\qquad\qquad + \mathbb{P}\left(\mathcal{E}_{4,k}^c\right) \mathbb{E}\left[d_k(S_k^n, \hat{S}_k^n) | \mathcal{E}_{4,k}^c \right] \Big) \qquad (40) \\
&= \limsup_{n\to\infty} \mathbb{E}\left[d_k(S_k^n, \hat{S}_k^n) | \mathcal{E}_{4,k}^c \right] \qquad (41) \\
&\le (1+\epsilon') \, \mathbb{E} d_k(S_k, \hat{S}_k) \qquad (42) \\
&\le (1+\epsilon') \, D_k, \qquad (43)
\end{aligned}$$

for $k = 1, 2$, where (42) follows from typical average lemma [9]. Therefore, the desired distortions are achieved for sufficiently small ϵ'.

Next we turn back to prove the claim above. Following from covering lemma [9, Sec. 3.7], the first three terms of (39), $\mathbb{P}\left(\mathcal{E}_0\right) + \mathbb{P}\left(\mathcal{E}_0^c \cap \mathcal{E}_1\right) + \mathbb{P}\left(\mathcal{E}_0^c \cap \mathcal{E}_2\right)$, vanishes as $n \to \infty$ if

$$r_0 > I\left(V_0; S_0\right), \qquad (44)$$

and according to Markov lemma [9, Sec. 12.1.1], the fourth item tends to zero as $n \to \infty$ if

$$r_k > I\left(V_k; S_k | V_0\right), k = 1, 2. \qquad (45)$$

Then by conditional typicality lemma [9, Sec. 2.5], the fifth item tends to zero as $n \to \infty$.

Now we focus on the last term of (39). $\mathcal{E}_5$ can be writen as

$$\mathcal{E}_5 = \mathcal{E}_{51} \cup \mathcal{E}_{52} \cup \mathcal{E}_{53} \cup \mathcal{E}_{54}, \qquad (46)$$

where

$$\begin{aligned}
\mathcal{E}_{51} &= \Big\{ \left(V_0^n(m_0'), V_1^n(m_0', m_1'), V_2^n(m_0', m_2'), Y^n \right) \in \mathcal{T}_{\epsilon'}^{(n)} \\
&\qquad \text{for some } m_0' \neq M_0, m_1' \neq M_1, m_2' \neq M_2 \Big\}, \\
\mathcal{E}_{52} &= \Big\{ \left(V_0^n(M_0), V_1^n(M_0, m_1'), V_2^n(M_0, m_2'), Y^n \right) \in \mathcal{T}_{\epsilon'}^{(n)} \\
&\qquad \text{for some } m_1' \neq M_1, m_2' \neq M_2 \Big\}, \\
\mathcal{E}_{53} &= \Big\{ \left(V_0^n(M_0), V_1^n(M_0, M_1), V_2^n(M_0, m_2'), Y^n \right) \in \mathcal{T}_{\epsilon'}^{(n)} \\
&\qquad \text{for some } m_2' \neq M_2 \Big\}, \\
\mathcal{E}_{54} &= \Big\{ \left(V_0^n(M_0), V_1^n(M_0, m_1'), V_2^n(M_0, M_2), Y^n \right) \in \mathcal{T}_{\epsilon'}^{(n)} \\
&\qquad \text{for some } m_1' \neq M_1 \Big\},
\end{aligned}$$

Using union bound we have

$$\mathbb{P}\left(\mathcal{E}_5\right) \le \mathbb{P}\left(\mathcal{E}_{51}\right) + \mathbb{P}\left(\mathcal{E}_{52}\right) + \mathbb{P}\left(\mathcal{E}_{53}\right) + \mathbb{P}\left(\mathcal{E}_{54}\right). \qquad (47)$$

Following similar steps to the proof of [7, Thm. 1], one can prove $\mathbb{P}\left(\mathcal{E}_{51}\right)$ vanishes as $n \to \infty$ if

$$r_0 + r_1 + r_2 < I\left(V_0 V_1 V_2; Y\right) + I\left(V_1; V_2 | V_0\right), \qquad (48)$$

$\mathbb{P}\left(\mathcal{E}_{52}\right)$ vanishes as $n \to \infty$ if

$$r_1 + r_2 < I\left(V_1 V_2; Y | V_0\right) + I\left(V_1; V_2 | V_0\right), \qquad (49)$$

$\mathbb{P}\left(\mathcal{E}_{53}\right)$ vanishes as $n \to \infty$ if

$$r_2 < I\left(V_2; Y | V_0 V_1\right), \qquad (50)$$

and $\mathbb{P}\left(\mathcal{E}_{54}\right)$ vanishes as $n \to \infty$ if

$$r_1 < I\left(V_1; Y | V_0 V_2\right). \qquad (51)$$

Combining (44), (45), and (48)-(51) leads to the sufficient condition, which completes the proof of the inner bound.

B. Outer Bound

For fixed $p_{U_{[1:L]}|S_1,S_2}$, we introduce a set of auxiliary random variables $U_{[1:L]}^n$ that follow $\prod_{i=1}^n p_{U_{[1:L]}|S_1,S_2}\left(u_{[1:L],i} | s_{1,i}, s_{2,i}\right)$. Then the Markov chain $U_{[1:L]}^n \to (S_1^n, S_2^n) \to (X_1^n, X_2^n) \to Y^n \to (\hat{S}_1^n, \hat{S}_2^n)$ holds. Assume $\mathcal{A} \subseteq [1:L]$. Next, we derive a lower bound for $I\left(S_1^n S_2^n; Y^n | U_{\mathcal{A}}^n\right)$.

$$\begin{aligned}
&I\left(S_1^n S_2^n; Y^n | U_{\mathcal{A}}^n\right) \\
&= \sum_{t=1}^n I\left(S_{1,t} S_{2,t}; Y^n | U_{\mathcal{A}}^n S_1^{t-1} S_2^{t-1}\right) \qquad (52) \\
&= \sum_{t=1}^n H\left(S_{1,t} S_{2,t} | U_{\mathcal{A}}^n S_1^{t-1} S_2^{t-1}\right) \\
&\qquad - H\left(S_{1,t} S_{2,t} | Y^n U_{\mathcal{A}}^n S_1^{t-1} S_2^{t-1}\right) \qquad (53) \\
&= \sum_{t=1}^n H\left(S_{1,t} S_{2,t} | U_{\mathcal{A},t}\right) - H\left(S_{1,t} S_{2,t} | Y^n U_{\mathcal{A}}^n S_1^{t-1} S_2^{t-1}\right) \qquad (54) \\
&= \sum_{t=1}^n I\left(S_{1,t} S_{2,t}; Y^n U_{\mathcal{A}}^n S_1^{t-1} S_2^{t-1} | U_{\mathcal{A},t}\right) \qquad (55) \\
&\ge \sum_{t=1}^n I(S_{1,t} S_{2,t}; \hat{S}_{1,t} \hat{S}_{2,t} | U_{\mathcal{A},t}) \qquad (56) \\
&= nI(S_{1,Q} S_{2,Q}; \hat{S}_{1,Q} \hat{S}_{2,Q} | U_{\mathcal{A},Q} Q) \qquad (57) \\
&= nI(S_{1,Q} S_{2,Q}; \hat{S}_{1,Q} \hat{S}_{2,Q} Q | U_{\mathcal{A},Q}) \qquad (58) \\
&\ge nI(S_{1,Q} S_{2,Q}; \hat{S}_{1,Q} \hat{S}_{2,Q} | U_{\mathcal{A},Q}) \qquad (59) \\
&= nI(S_1 S_2; \hat{S}_1 \hat{S}_2 | U_{\mathcal{A}}), \qquad (60)
\end{aligned}$$

where Q is a time-sharing random variable uniformly distributed $[1:n]$ and independent of all other random variables, and in (60), $S_k \triangleq S_{k,Q}, \hat{S}_k \triangleq \hat{S}_{k,Q}, U_l \triangleq U_{l,Q}, k = 1, 2, 1 \le l \le L$.

Now, we turn to upper-bounding $I\left(S_1^n S_2^n; Y^n | U_{\mathcal{A}}^n\right)$.

$$\begin{aligned}
&I\left(S_1^n S_2^n; Y^n | U_{\mathcal{A}}^n\right) \\
&\le I\left(X_1^n X_2^n; Y^n | U_{\mathcal{A}}^n\right) \qquad (61) \\
&= \sum_{t=1}^n I\left(Y_t; X_1^n X_2^n | U_{\mathcal{A}}^n Y^{t-1}\right) \qquad (62) \\
&\le \sum_{t=1}^n I\left(Y_t; X_1^n X_2^n Y^{t-1} | U_{\mathcal{A}}^n\right) \qquad (63) \\
&= \sum_{t=1}^n I\left(Y_t; X_{1,t} X_{2,t} | U_{\mathcal{A}}^n\right) \qquad (64) \\
&= nI\left(Y_Q; X_{1,Q} X_{2,Q} | U_{\mathcal{A}}^n Q\right), \qquad (65) \\
&= nI\left(Y; X_1 X_2 | U_{\mathcal{A}}^n Q\right), \qquad (66)
\end{aligned}$$

where (64) follows from $\left(X_1^n, X_2^n, Y^{t-1}\right) \to (X_{1,t}, X_{2,t}) \to Y_t$, Q is the time-sharing random variable defined above, and $Y \triangleq Y_Q, X_k \triangleq X_{k,Q}, k = 1, 2$.

Combine (60) and (66), then we have

$$I(S_1 S_2; \hat{S}_1 \hat{S}_2 | U_{\mathcal{A}}) \le I\left(X_1 X_2; Y | U_{\mathcal{A}}^n Q\right) \text{ for any } \mathcal{A} \subseteq [1:L]. \qquad (67)$$

In addition, $(Q, S_1^n, S_2^n, X_1, X_2, Y)$ follows the distribution $p_Q \prod p_{S_1,S_2}(s_{1,i}, s_{2,i}) p_{U_{[1:L]}|S_1,S_2}(u_{[1:L],i}|s_{1,i}, s_{2,i})$ $p_{X_1|S_1^n,Q} p_{X_2|S_2^n,Q} p_{Y|X_1,X_2}$. This completes the proof of the outer bound $\mathcal{R}_1^{(o)}$.

Appendix B
Proof of Theorem 4

Before proving Theorem 4, we need introduce several correlations and their properties, including correlation coefficient, correlation ratio, maximal correlation coefficient, as well as the corresponding conditional correlations.

Definition 5. For any random variables W_1 and W_2 with alphabets $\mathcal{W}_1 \subseteq \mathbb{R}$ and $\mathcal{W}_2 \subseteq \mathbb{R}$, the (Pearson) correlation coefficient of W_1 and W_2 is defined by

$$\rho(W_1, W_2) = \frac{\text{cov}(W_1, W_2)}{\sqrt{\text{var}(W_1)}\sqrt{\text{var}(W_2)}}.$$

Similarly, the conditional correlation coefficient of W_1 and W_2 given another random variable W_0 is defined by

$$\rho(W_1, W_2|W_0) = \frac{\mathbb{E}[\text{cov}(W_1, W_2|W_0)]}{\sqrt{\mathbb{E}[\text{var}(W_1|W_0)]}\sqrt{\mathbb{E}[\text{var}(W_2|W_0)]}}.$$

Definition 6. For any random variables W_1 and W_2 with alphabets $\mathcal{W}_1 \subseteq \mathbb{R}$ and $\mathcal{W}_2$, the correlation ratio of W_1 on W_2 is defined by

$$\theta(W_1, W_2) = \sup_f \rho(W_1, f(W_2)),$$

where the supremum is taken over all the functions $f : \mathcal{W}_2 \mapsto \mathbb{R}$ satisfying

$$0 < \mathbb{E}\left[f^2(W_2)\right] < \infty. \tag{68}$$

Similarly, the conditional correlation ratio of W_1 on W_2 given another random variable W_0 is defined by

$$\theta(W_1, W_2|W_0) = \sup_f \rho(W_1, f(W_2, W_0)|W_0),$$

where the supremum is taken over all the functions $f : \mathcal{W}_2 \times \mathcal{W}_0 \mapsto \mathbb{R}$ satisfying

$$0 < \mathbb{E}\left[f^2(W_2, W_0)\right] < \infty. \tag{69}$$

Definition 7. For any random variables W_1 and W_2 with alphabets $\mathcal{W}_1$ and $\mathcal{W}_2$, the maximal correlation coefficient of W_1 and W_2 is defined by

$$\rho_m(W_1, W_2) = \sup_{f_1, f_2} \rho(f_1(W_1), f_2(W_2)),$$

where the supremum is taken over all the functions $f_k : \mathcal{W}_k \mapsto \mathbb{R}$ for $k = 1, 2$, satisfying

$$0 < \mathbb{E}\left[f_k^2(W_k)\right] < \infty, \tag{70}$$

Moreover, the conditional maximal correlation coefficient of W_1 and W_2 given another random variable W_0 is defined by

$$\rho_m(W_1, W_2|W_0) = \sup_{f_1, f_2} \rho(f_1(W_1, W_0), f_2(W_2, W_0)|W_0),$$

where the supremum is taken over all the functions $f_k : \mathcal{W}_k \times \mathcal{W}_0 \mapsto \mathbb{R}$ for $k = 1, 2$, satisfying

$$0 < \mathbb{E}\left[f_k^2(W_k, W_0)\right] < \infty. \tag{71}$$

Lemma 1. *[19] For any random variables W_0, W_1 and W_2, (conditional) correlation coefficient, (conditional) correlation ratio, and (conditional) maximal correlation coefficient have the following properties:*

$$0 \le |\rho(W_1, W_2)| \le \theta(W_1, W_2) \le \rho_m(W_1, W_2) \le 1; \tag{72}$$

$$\begin{aligned} 0 \le |\rho(W_1, W_2|W_0)| &\le \theta(W_1, W_2|W_0) \\ &\le \rho_m(W_1, W_2|W_0) \le 1; \end{aligned} \tag{73}$$

$$\theta(W_1, W_2W_0) \ge \theta(W_1, W_0); \tag{74}$$

$$\rho_m(W_1, W_2W_0) \ge \rho_m(W_1, W_0); \tag{75}$$

$$\begin{aligned} \theta(W_1, W_2) &= \sqrt{\frac{var(\mathbb{E}[W_1|W_2])}{var(W_1)}} \\ &= \sqrt{1 - \frac{\mathbb{E}[var(W_1|W_2)]}{var(W_1)}}; \end{aligned} \tag{76}$$

$$\begin{aligned} \theta(W_1, W_2|W_0) &= \sqrt{\frac{\mathbb{E}[var(\mathbb{E}[W_1|W_2W_0]|W_0)]}{\mathbb{E}[var(W_1|W_0)]}} \\ &= \sqrt{1 - \frac{\mathbb{E}[var(W_1|W_2W_0)]}{\mathbb{E}[var(W_1|W_0)]}}; \end{aligned} \tag{77}$$

$$\begin{aligned} \rho_m(W_1, W_2) &= \sup_f \sqrt{\frac{var(\mathbb{E}[f(W_1)|W_2])}{var(f(W_1))}} \\ &= \sup_f \sqrt{1 - \frac{\mathbb{E}[var(f(W_1)|W_2)]}{var(f(W_1))}}; \end{aligned} \tag{78}$$

$$\begin{aligned} \rho_m(W_1, W_2|W_0) &= \sup_f \sqrt{\frac{\mathbb{E}[var(\mathbb{E}[f(W_1, W_0)|W_2W_0]|W_0)]}{\mathbb{E}[var(f(W_1, W_0)|W_0)]}} \\ &= \sup_f \sqrt{1 - \frac{\mathbb{E}[var(f(W_1, W_0)|W_2W_0)]}{\mathbb{E}[var(f(W_1, W_0)|W_0)]}}; \end{aligned} \tag{79}$$

$$\begin{aligned} &1 - \theta^2(W_1, W_2W_0) \\ &\quad = \left(1 - \theta^2(W_1, W_0)\right)\left(1 - \theta^2(W_1, W_2|W_0)\right); \end{aligned} \tag{80}$$

and

$$\begin{aligned} &1 - \theta^2(W_1, W_2W_0|Z) \\ &\quad = \left(1 - \theta^2(W_1, W_0|Z)\right)\left(1 - \theta^2(W_1, W_2|W_0Z)\right). \end{aligned} \tag{81}$$

Besides, some other remarkable properties are also needed in proving Theorem 4.

Lemma 2. *[14, Thm. 1] For a sequence of pairs of independent random variables $(W_{1,i}, W_{2,i})_{i=1}^n$, we have*

$$\rho_m(W_1^n, W_2^n) \le \sup_{1 \le i \le n} \rho_m(W_{1,i}, W_{2,i}), \tag{82}$$

where $W_k^n = (W_{k,1}, W_{k,2}, \cdots, W_{k,n})$ for $k = 1, 2$.

Lemma 3. *[16, Sec. IV, Lem. 10.2] For jointly Gaussian random variables W_0, W_1 and W_2, we have*

$$\rho_m(W_1, W_2) = |\rho(W_1, W_2)|, \tag{83}$$

$$\rho_m(W_1, W_2|W_0) = |\rho(W_1, W_2|W_0)|. \tag{84}$$

Lemma 4 (Data Processing Inequality). *[19] If random variable W and non-degenerate random variables X,Y,Z form a Markov chain $X \to (Y,W) \to Z$, then*

$$\rho(X,Z|W) \le \theta(X,Y|W)\,\theta(Z,Y|W), \tag{85}$$
$$\theta(X,Z|W) \le \theta(X,Y|W)\,\rho_m(Z,Y|W), \tag{86}$$
$$\rho_m(X,Z|W) \le \rho_m(X,Y|W)\,\rho_m(Z,Y|W). \tag{87}$$

Moreover, the equalities hold in (85)-(87), if (X,Y,W) and (Z,Y,W) have the same distribution. In particular, if W is degenerate, then

$$\rho(X,Z) \le \theta(X,Y)\,\theta(Z,Y), \tag{88}$$
$$\theta(X,Z) \le \theta(X,Y)\,\rho_m(Z,Y), \tag{89}$$
$$\rho_m(X,Z) \le \rho_m(X,Y)\,\rho_m(Z,Y). \tag{90}$$

Now we use $\mathcal{R}_2^{(o)}$ to prove Theorem 4. For $\mathcal{R}_2^{(o)}$, denote $\hat\rho$ as the correlation coefficient between X_1 and X_2, i.e., $\hat\rho \triangleq \rho(X_1,X_2)$, and θ_k as correlation ratio of X_k on (S_0^n,U^n,Q) , i.e.,

$$\theta_k \triangleq \theta(X_k, S_0^nU^nQ), k=1,2. \tag{91}$$

It should hold that $0 \le \hat\rho,\theta_1,\theta_2 \le 1$. Observe that in $\mathcal{R}_2^{(o)}$, $X_1 \to (S_1^n,Q) \to (S_0^n,U^n,Q) \to (S_2^n,Q) \to X_2$ holds. Hence from Lemma 4, we have

$$\hat\rho \le \theta_1\theta_2. \tag{92}$$

From Property (76) of Lemma 1, we have

$$\mathbb{E}\left[\mathrm{var}(X_k|S_0^nU^nQ)\right] = \left(1-\theta_k^2\right)\mathbb{E}\left[\mathrm{var}(X_k)\right], k=1,2. \tag{93}$$

In addition, denote $\hat\rho_0 = \rho(X_1,X_2|S_0^nQ)$, $\rho_{12|0} = \rho(S_1,S_2|S_0) = \frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{(1-\rho_{01}^2)(1-\rho_{02}^2)}}$ and $\theta_k' = \theta(X_k,U^n|S_0^nQ), k=1,2.$ Then utilizing Lemmas 2, 3 and 4, we have

$$\hat\rho_0 \le \rho_m(S_1^n,S_2^n|S_0^nQ) = \rho_m(S_1,S_2|S_0) = \rho_{12|0}, \tag{94}$$

$$\hat\rho_0 \le \theta(X_1,S_2^n|S_0^nQ) \tag{95}$$
$$\le \theta(X_1,U^n|S_0^nQ)\,\rho_m(S_2^n,U^n|S_0^nQ) \tag{96}$$
$$= \theta_1'\beta_2, \tag{97}$$

and

$$\hat\rho_0 \le \theta_2'\beta_1. \tag{98}$$

Now based on the inequalities above and utilizing the outer bound $\mathcal{R}_2^{(o)}$ of Theorem 1, we can obtain a sequence of desired results. Specifically, Combining the inequality $I(S_1S_2;\hat S_1\hat S_2) \le I(X_1X_2;Y|Q)$ in $\mathcal{R}_2^{(o)}$ with

$$I(S_1S_2;\hat S_1\hat S_2) \ge R_{S_1S_2}(D_1,D_2) \tag{99}$$

and

$$I(X_1X_2;Y|Q) = h(Y|Q) - h(Y|X_1X_2) \tag{100}$$
$$\le h(Y) - h(Y|X_1X_2) \tag{101}$$
$$\le \frac{1}{2}\log(1+\mathrm{var}(X_1+X_2)) \tag{102}$$
$$= \frac{1}{2}\log\Big(1+\mathrm{var}(X_1)+\mathrm{var}(X_2) + 2\rho(X_1,X_2)\sqrt{\mathrm{var}(X_1)\mathrm{var}(X_2)}\Big) \tag{103}$$
$$\le \frac{1}{2}\log\left(1+P_1+P_2+2\hat\rho\sqrt{P_1P_2}\right), \tag{104}$$

gives

$$R_{S_1S_2}(D_1,D_2) \le \frac{1}{2}\log\left(1+P_1+P_2+2\hat\rho\sqrt{P_1P_2}\right). \tag{105}$$

In addition, from Property (80) of Lemma 1, we have

$$1-\theta^2(X_1,S_0^nQ) = \frac{1-\theta^2(X_1,X_2S_0^nQ)}{1-\theta^2(X_1,X_2|S_0^nQ)} \tag{106}$$
$$\le \min\left(\frac{1-\hat\rho^2}{1-\rho_{12|0}^2},1\right), \tag{107}$$

where the inequality (107) follows from

$$\theta(X_1,X_2S_0^nQ) \ge \theta(X_1,X_2) \ge \rho(X_1,X_2), \tag{108}$$

and

$$\theta(X_1,X_2|S_0^nQ) \le \rho_m(X_1,X_2|S_0^nQ) \le \rho_{12|0}. \tag{109}$$

Then combining the inequality $I(S_1S_2;\hat S_1\hat S_2|S_0) \le I(X_1X_2;Y|S_0^nQ)$ in $\mathcal{R}_2^{(o)}$ with

$$I(S_1S_2;\hat S_1\hat S_2|S_0) \ge R_{S_1S_2|S_0}(D_1,D_2) \tag{110}$$

and

$$\begin{aligned} &I(X_1X_2;Y|S_0^nQ) \\ &= h(Y|S_0^nQ) - h(Y|X_1X_2) \\ &\le \frac{1}{2}\log(1+\mathbb{E}\mathrm{var}(X_1+X_2|S_0^nQ)) \\ &= \frac{1}{2}\log\Big(1+\mathbb{E}\mathrm{var}(X_1|S_0^nQ)+\mathbb{E}\mathrm{var}(X_2|S_0^nQ) \\ &\quad + 2\rho(X_1,X_2|S_0^nQ)\sqrt{\mathbb{E}\mathrm{var}(X_1|S_0^nQ)\mathbb{E}\mathrm{var}(X_2|S_0^nQ)}\Big) \\ &\le \frac{1}{2}\log\Big(1+\left(1-\theta^2(X_1,S_0^nQ)\right)P_1+\left(1-\theta^2(X_2,S_0^nQ)\right)P_2 \\ &\quad + 2\hat\rho_0\sqrt{\left(1-\theta^2(X_1,S_0^nQ)\right)\left(1-\theta^2(X_2,S_0^nQ)\right)P_1P_2}\Big) \\ &\le \frac{1}{2}\log\Big(1+\min\Big(\frac{1-\hat\rho^2}{1-\rho_{12|0}^2},1\Big)\left(P_1+P_2+2\hat\rho_0\sqrt{P_1P_2}\right)\Big) \end{aligned} \tag{111}$$

gives

$$\begin{aligned} &R_{S_1S_2|S_0}(D_1,D_2) \\ &\le \frac{1}{2}\log\Big(1+\min\Big(\frac{1-\hat\rho^2}{1-\rho_{12|0}^2},1\Big)\left(P_1+P_2+2\hat\rho_0\sqrt{P_1P_2}\right)\Big). \end{aligned} \tag{112}$$

Similarly, the last five inequalities in (33) can be obtained as well. This completes the proof.

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