# Parameter Estimation in Gaussian Mixture Models with Malicious Noise, without Balanced Mixing Coefficients

Jing Xu, Jakub Marecek

#### **Abstract**

We consider the problem of estimating the means of two Gaussians in a 2-Gaussian mixture, which is not balanced and is corrupted by noise of an arbitrary distribution. We present a robust algorithm to estimate the parameters, together with upper bounds on the numbers of samples required for the estimate to be correct, where the bounds are parametrised by the dimension, ratio of the mixing coefficients, a measure of the separation of the two Gaussians, related to Mahalanobis distance, and a condition number of the covariance matrix. In theory, this is the first sample-complexity result for imbalanced mixtures corrupted by adversarial noise. In practice, our algorithm outperforms the vanilla Expectation-Maximisation (EM) algorithm in terms of estimation error.

### 1 Introduction

Gaussian mixture models are central to both theory and practice of Statistics [1, 2]. As a result of more than a century of study [3], there are algorithms [4] in the noise-free setting with balanced mixing coefficients, which are essentially optimal [5] with respect to their sample complexity and time complexity. Even for the expectation-maximisation algorithm, which is often used in practice [1], there is now some understanding of the performance [6, 7, 8, 9] and its limitations. Within robust statistics [10] and agnostic learning [11, 12], there has been recent progress in estimating parameters of a single Gaussian from a mixture of the Gaussian and noise. Within mixed regression, there has been some progress in estimating parameters of a mixture [13, 14] with balanced coefficients, but very little [15, 16] is known otherwise.

We propose to study the problem that relaxes both the noise-free assumption and the assumption on the balance of the mixing coefficients:

**Definition 1** (Robust Parameter Estimation in Noisy 2-GMM). Given m points in  $\mathbb{R}^n$  that are each, with probability  $w_1 > 0$  from an unknown Gaussian  $N(\mu_1, \Sigma)$ , with probability  $w_2 > 0$  from an unknown Gaussian distribution  $N(\mu_2, \Sigma)$ , and with probability  $w_3 = 1 - w_1 - w_2 > 0$  completely arbitrary, estimate  $\mu_1, \mu_2$  and  $\Sigma$ .

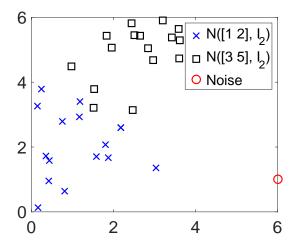
Throughout the paper, we assume:

**Assumption 1.**  $w_1 > w_2 > w_3$ .

Moreover,

- we consider arbitrary, adversarial noise. Our sample complexity is parametrised by the proportion  $w_3$  of noise among the samples.
- We do not make further assumptions on the balance between the Gaussians. Instead, our results are parametrised by the ratios of mixing coefficients  $w_2/w_1$  and  $w_3/w_1$ .

The importance of *not* making further assumptions is hard to overstate. For simplicity, let us illustrate this on a noisy mixture of  $N((1,2),I_2)$  and  $N((3,5),I_2)$  in  $\mathbb{R}^2$ , with  $I_2$  being the  $2 \times 2$  identity matrix. First, we consider the situation, where the mixture is balanced,  $m = 41, w_1 = 20/41, w_2 = 20/41$ , and there is a single sample of noise at  $(6,1) \in \mathbb{R}^2$ . Figure 1 compares the estimate obtained using a standard expectation-maximisation (EM) algorithm with the true mixture and our algorithm. Notice that the one sample of noise



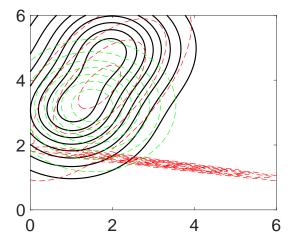


Figure 1: Left: 20 and 20 samples from two Gaussians and one sample of noise at (6,1). Right: A contour plot of the 2-GMM (in black solid lines), an EM estimate from the 41 samples (in red dashed lines), and an estimate by Algorithm 1 using the same 41 samples (in green dashed lines).

causes the EM algorithm to mis-estimate the second component completely. Figure 2 illustrates the impact of varying mixing coefficients. In the three rows of sub-figures, we have used  $w_1 = 25/41,30/41$ , and 35/41, respectively, and the same single element of noise throughout,  $w_3 = 1/41$ . For  $w_1 > 35/41$ , the performance of the EM algorithm deteriorates quickly. We stress that the samples have been obtained with the random seed set to 1 and are not pathological. The EM algorithm is implemented by *fitgmdist* in MathWorks Matlab 2016a, with default parameters. One could conclude that algorithms designed for the noise-free, balanced case should be avoided in applications, where the assumptions may be easily violated.

For robust parameter estimation in noisy 2-GMM, we present an iterative algorithm. The algorithm could be seen as the extension of algorithms for the estimation of the mean and covariance of a single Gaussian in the presence of malicious noise (noisy single gaussian model), as studied by [11], to the noisy 2-GMM and beyond. In our proposed algorithm, each iteration considers one Gaussian, in the decreasing order of their mixing coefficients. In each iteration, parameters of one Gaussian are estimated under the assumption that the remaining samples are either from the Gaussian or the arbitrarily-distributed noise. At the end of each iteration, the samples corresponding to the Gaussian are filtered out. This way, Robust Parameter Estimation in Noisy 2-GMM can be reduced to 2 calls of an algorithm for parameter estimation in Noisy 1-GMM and some additional processing in time  $O(mn^2 + m \log m)$ .

Outside of the meta-algorithm, our contributions are as follows:

- We prove an upper bound on the number of samples required to reach a given precision, considering a spectral method of [11], with the bound parametrised by the dimension n, ratios of the mixing coefficients  $w_2/w_1$  and  $w_3/w_1$ , a condition number of the covariance matrix, and a measure of the separation of the two Gaussians, related to Mahalanobis distance.
- In both theory and computational illustrations, we show that the algorithm is surprisingly robust to the error in the input  $w_1 \in [0, 1]$ .
- In computational tests with Cauchy-distributed noise  $w_3 > 0$ , we demonstrate that the performance of the algorithm employing a spectral method of [11] is superior to the vanilla expectation-maximisation algorithm in terms of estimation error.

# 2 The Algorithm

We present an algorithm that estimates the parameters of Gaussians in the decreasing order of their mixing coefficients. For each Gaussian component in turn, an algorithm for estimating the mean and covariance of one Gaussian from samples corrupted by noise, e.g. [11], is run. Once parameters of one Gaussian are estimated, all samples are ordered by Mahalanobis distance [17] to the Gaussian with the estimated mean and covariance. The closest samples, which are most likely to be samples from this Gaussian with the estimated mean and covariance, are removed from future consideration and the algorithm proceeds with the estimation of the next Gaussian.

#### Algorithm 1 Iterative Parameter Estimation in Noisy 2-GMM

**Require:**  $X = \{x_1, x_2, ..., x_m\}, w_1.$ 

Ensure:  $\hat{\mu}_1, \hat{\mu}_2$ .

1: Let  $\hat{\mu}_1 = AGNOSTICMEAN(X)$ . Let  $\hat{\Sigma} = AGNOSTICCOV(X, 1 - w_1)$ .

2: For i = 1 to m,

$$y_i = (x_i - \hat{\mu}_1)^T \hat{\Sigma}^{-1} (x_i - \hat{\mu}_1).$$

3: Sort

$$y_{(1)} \leq \dots y_{([mw_1])} \leq \dots y_{(m)}$$

4: Let  $X' = \{x_i : y_i >= y_{([mw_1])}\}$ .

5:  $\hat{\mu}_2 = \text{AGNOSTICMEAN}(X')$ .

See Algorithm 1 for the pseudo-code. For completeness, we list the pseudo-code of algorithms AGNOS-TICMEAN and AGNOSTICCOV of [11] in the appendix.

# 3 The Sample Complexity

Our main result is the analysis of sample complexity of Algorithm 1.

**Theorem 3.1** (Sample Complexity of Iterative Parameter Estimation in Noisy 2-GMM). For Problem 1, there exists a  $poly(n, \frac{1}{\varepsilon})$ - time algorithm that takes as input m independent samples  $x_1, x_2, ..., x_m$  and  $w_1$ , then computes the means of two components of noisy 2-GMM model, s.t. If  $\Sigma = \sigma^2 I$ , then

$$\|\mu_2 - \hat{\mu}_2\|_2 = O(\frac{w_3}{w_2} + \varepsilon)\sigma\sqrt{\log n},$$

$$\|\mu_1 - \hat{\mu}_1\|_2 = O(\frac{w_2}{w_1} + \varepsilon)\sigma\sqrt{\log n}.$$

Otherwise, for arbitrary  $\Sigma$ ,

$$\|\mu_2 - \hat{\mu}_2\| = O(\sqrt{\frac{w_3}{w_2}} + \varepsilon) \|\Sigma\|_2^{1/2} \sqrt{\log n},$$

$$\|\mu_1 - \hat{\mu}_1\| = O(\sqrt{\frac{w_2}{w_1}} + \varepsilon) \|\Sigma\|_2^{1/2} \sqrt{\log n},$$

provided that,

- $i) \ \textit{for the spherical case, } m = \Omega\big(\frac{n\frac{1}{w_2}(\log n + \log \frac{1}{\varepsilon})\log n}{\varepsilon^2} + \frac{\frac{1}{w_2}\log(\frac{1}{\varepsilon} + \frac{w_2}{w_3})}{\varepsilon^2 + (\frac{w_2}{w_3})^2} + \frac{\log(\frac{1}{\varepsilon} + \frac{w_2}{w_3})}{w_3}\big).$  For the non-spherical case,  $m = \Omega\big(\frac{n(n + \frac{1}{w_2})(\log n + \log \frac{1}{\varepsilon})\log n}{\varepsilon^2} + \frac{\frac{1}{w_2}\log(\frac{1}{\varepsilon} + \sqrt{\frac{w_2}{w_3}})}{\varepsilon^4 + (\frac{w_3}{w_2})^2} + \frac{\log(\frac{1}{\varepsilon} + \sqrt{\frac{w_2}{w_3}})}{w_3}\big).$
- ii) conditions in Lemma 3.4 are satisfied with  $\eta = O(\varepsilon + \frac{w_3}{w_2})$  in the spherical case  $\Sigma = \sigma^2 I$ , and  $\eta = O(\varepsilon^2 + \frac{w_3}{w_2})$  otherwise where  $\eta$  is a parameter in Lemma 3.4.

This result builds upon the analysis of the separation of the two Gaussians. Condition on recovery of the first mean comes from [11]. Our contribution lies in recovering the second component with mixing coefficient  $w_2$  in a noisy 2-GMM, i.e., the degenerate component of the 2-GMM. In order to derive a robust estimator of  $\mu_2$ , we derive separation conditions on the two Gaussians in (ii) of Theorem 3.1 with  $\eta$  being the auxiliary parameter of accuracy of the estimator  $\hat{\mu}_2$ . Under such conditions, one can filter and subsample according to Mahalanobis-distance criteria, and take subsampled points as an input of the agnostic learning of  $\mu_2$ .

#### 3.1 Main Ideas of the Proof

The complete proof is provided in the appendix. In this section, we show that there exists a polynomial-time algorithm that can estimate mean and covariance matrix of a Gaussian distribution from samples corrupted by malicious noise, i.e., an approximation algorithm for:

**Definition 2** (Robust Parameter Estimation in Noisy 1-GMM). Given points in  $\mathbb{R}^n$  that are each, with probability  $1 - \eta$  from an unknown distribution with mean  $\mu$  and covariance  $\Sigma$ , and with probability  $\eta$  completely arbitrary, estimate  $\mu$  and  $\Sigma$ .

Their most significant results are:

**Lemma 3.2** (Mean recovery in [11]). For Problem 2, there exists a  $poly(n, 1/\varepsilon)$ -time algorithm that takes as input  $m = O(\frac{n(\log n + \log \frac{1}{\varepsilon}) \log n}{\varepsilon^2})$  independent samples  $x_1, x_2, \dots, x_m \sim N_{\eta}(\mu, \Sigma)$  and computes  $\hat{\mu}$  such that the error  $\|\mu - \hat{\mu}\|_2$  is bounded as follows,

$$O(\eta + \varepsilon)\sigma\sqrt{\log n}, \quad \text{if } \Sigma = \sigma^2 I.$$
 $O(\sqrt{\eta} + \varepsilon)\|\Sigma\|_2^{1/2}\sqrt{\log n}, \quad \text{otherwise.}$ 

**Lemma 3.3** (Covariance recovery in [11]). For Problem 2, there exists a poly $(n, 1/\varepsilon)$ -time algorithm that takes as input  $m = \Omega(\frac{n^2(\log n + \log \frac{1}{\varepsilon})\log n}{\varepsilon^2})$  independent samples  $x_1, x_2, \ldots, x_m \sim N_{\eta}(\mu, \Sigma)$  and computes  $\hat{\Sigma}$  such that the error

$$\|\Sigma - \hat{\Sigma}\|_F = O(\eta^{1/2} + C_1(\eta + \varepsilon)^{3/4}) \|\Sigma\|_2 \sqrt{\log n}$$

With the recovery of a Gaussian distribution from malicious noise, we can proceed with the case of imbalanced 2-GMM. The crucial innovation is the condition on the separation between means of the 2-GMM based on the ratio of component weights and a careful analysis of the sample complexity.

**Notation.** Denote the set of the sampled points  $S = \{x_j\}_{j=1}^m$ .  $S = G_1 \cup G_2 \cup N$ , where  $G_k$  denotes the samples from the k<sup>th</sup> component, and N the set of samples belonging to the malicious noise. Denote by

$$l(X) = (X - \hat{\mu}_1)'\hat{\Sigma}^{-1}(X - \hat{\mu}_1),$$

Mahalanobis distance.  $l(X)_{(i)}$  is the  $i^{th}$  smallest random variable among all samples  $\{l(X_j)\}_{j=1}^m$ . Let  $l(X^{G_k})_{(i)}$  denote the  $i^{th}$  smallest term among  $m_k$  samples from the  $k^{th}$  component  $\{l(X_j): X_j \in G_k\}$ . Denote by

$$\lambda = (\mu_2 - \mu_1)^T \Sigma^{-1} (\mu_2 - \mu_1),$$

the 'distance' of two Gaussian distribution in a 2-GMM. The larger  $\lambda$  is, the better-separated the two Gaussian components are. In the spherical case  $\Sigma = \sigma^2 I$ ,

$$\lambda = \frac{\|\mu_2 - \mu_1\|_2^2}{\sigma^2},$$

which can be seen as a signal-to-noise parameter. Let  $tr(\Sigma)$  be the trace of the covariance matrix, and  $tr(\Sigma^2)$  the trace of its squared matrix.

**Proof Sketch of Theorem 3.1.** The identification of  $\mu_1$  is trivial by Lemma 3.2. The challenge lies in efficiently learning the distribution of the second component, which is solved by Step 4 and 5 in Algorithm 1. Formally,

Step 4 (filter) can be translated as follows. Given m i.i.d. samples  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ , we order the  $l(X_i)$  by their magnitude,

$$l(X)_{(1)} \le l(X)_{(2)} \le \ldots \le l(X)_{(mw_1)} \le \ldots \le l(X)_{(m)},$$

and take as input  $\{X_j : l(X_j) \ge l(X)_{(mw_1)}\}$  in Step 5.

Succeeding in identifying samples from the second components leads to Theorem 3.1. In other words, if for any  $\eta > 0$ , given sufficiently large sample size and two sufficiently well-separated Gaussian distributions in an imbalanced case,

$$\Pr(w \in G_1 | l(w) \ge l(X)_{(mw_1)}) < \eta,$$

that is, among samples left over after Step 4 (Filter), those from the 1st component account for a sufficiently small proportion, then the input of Step 5 can be regarded as a Gaussian  $N(\mu_2, \Sigma)$  with malicious noise of weight at most

$$\eta' = \eta + \frac{w_3}{w_2}.$$

Hence, applying the AGNOSTICMEAN algorithm in Step 5 can give a good estimator  $\hat{\mu}_2$  of the second mean given that  $m(w_2 + w_3) = \Omega(\frac{n(\log n + \log \frac{1}{\varepsilon}) \log n}{\varepsilon^2})$ . With the success of step 4 and sufficiently large sample size we have, the estimation error  $\|\hat{\mu}_2 - \mu_2\|$  is bounded by

$$O(\eta + \frac{w_3}{w_2} + \varepsilon)\sigma\sqrt{\log n}, \quad \text{if } \Sigma = \sigma^2 I$$

$$O(\sqrt{\eta + \frac{w_3}{w_2}} + \varepsilon)\|\Sigma\|_2^{1/2}\sqrt{\log n}, \quad \text{otherwise.}$$

Let  $\eta = O(\frac{w_3}{w_2} + \varepsilon)$  in the spherical case  $(\Sigma = \sigma^2 I)$ , or  $\eta = O(\varepsilon^2 + \frac{w_3}{w_2})$  in the non-spherical case, we have the recovery result in Theorem 3.1.

Formally, Step 4 (filter) guarantees the following result.

**Lemma 3.4** (Non-Spherical Gaussian Mixture). For any  $0 < \eta < 1$ , given  $m \ge \frac{cw_2 \log \frac{w_2}{w_3}}{w_3 \eta^2}$  i.i.d. samples for some c > 0, we have

$$\Pr(w \in G_1 | l(w) \ge l(X)_{(mw_1)}) < \eta,$$

if the following conditions are satisfied

- $\lambda \geq 2\sqrt{n\log\frac{w_1}{w_2}} + 2\sqrt{n-1} + 2\log\frac{w_1}{w_2}$ .
- Let  $\delta = \eta \frac{w_2}{w_1} + \frac{w_3}{w_1}$ , then

$$\lambda \geq c_1 \sqrt{w_2 \log n} (\|\Sigma\|_2^{\frac{1}{2}} \|\Sigma^{-1}\|_2 + 1) (\operatorname{tr}(\Sigma) + \sqrt{\operatorname{tr}(\Sigma^2) \log \frac{1}{\delta}} + \|\Sigma\| \log \frac{1}{\delta}) + c_2 \sqrt{n \log \frac{1}{\delta}} + \log \frac{1}{\delta}.$$

If  $n = o(\log \frac{1}{\delta})$ , it can be simplified as,

$$\lambda \ge \left(1 + c_1 \sqrt{w_2 \log n}\right) (\|\Sigma\|_2^{\frac{1}{2}} \|\Sigma^{-1}\|_2 + 1) \|\Sigma\|_2 \log \frac{1}{\delta}.$$

• The smallest singular value of  $\Sigma$  is bounded away from 0, i.e.,  $\sigma_{min} = \|\Sigma^{-1}\|_2^{-2} > O(w_2)\|\Sigma\|_2 \log n$ .  $\sigma_{min} \geq O\left(\sqrt{w_2}\|\Sigma\|_2\sqrt{\log n}\left(\frac{2\|\Sigma\|_2^2\sqrt{\log\frac{w_2}{w_3}}}{\sqrt{\operatorname{tr}(\Sigma^2)}}+1\right)\right)$ 

In the spherical case, the filtering guarantee can be simplified.

**Lemma 3.5** (Spherical Gaussian Mixture). For any  $0 < \eta < 1$ , with  $m \ge \frac{cw_2 \log \frac{w_2}{w_3}}{w_3 \eta^2}$  i.i.d. samples for some c > 0,

$$\Pr(w \in G_1 | l(w) \ge l(X)_{(m_1)}) < \eta,$$

if the following conditions are satisfied

• 
$$\lambda \ge 2\sqrt{n\log\frac{w_1}{w_2}} + 2\sqrt{n-1} + 2\log\frac{w_1}{w_2}$$
.

• Let  $\delta = \eta \frac{w_2}{w_1} + \frac{w_3}{w_1}$ , then

$$\lambda \geq c_1 \log \frac{1}{\delta} + c_2 \sqrt{n \log \frac{1}{\delta}} + c_3 \sqrt{\log \frac{1}{\eta} (\log \frac{1}{\delta} + n)},$$

for some  $c_1, c_2, c_3 > 0$ . If  $n = o(\log \frac{1}{\delta})$ , then it is equivalent to,

$$\lambda \geq c_1' \log \frac{1}{\delta},$$

for some  $c'_1$ .

**Remark 1.** The separation of  $\lambda$  depends on three parameters of the 2-GMM models.

- 1. The ratio of two components  $\frac{w_2}{w_1}$ .
- 2. The accuracy of estimation  $\delta = \eta \frac{w_2}{w_1} + \frac{w_3}{w_1}$ .
- 3. The dimension of the problem n.

For  $\frac{w_2}{w_1}$ , it is obvious that the more skewed (imbalanced) the 2-GMM model is, the more difficult it is to learn the means of the two components efficiently. Secondly, notice that  $\delta = \frac{w_2}{w_1}(\eta + \frac{w_3}{w_2})$ , thus the order of  $\delta$  is between the order of the two ratios.  $\frac{w_2}{w_1}$  is the generic upper bound on the accuracy of estimation, while  $\eta$  is an auxiliary parameter for the accuracy of estimation one would like to achieve in estimating the smaller component. Therefore, the more accurate the estimation is for the second mean, i.e., the smaller  $\eta$  is, the more strict separation conditions on  $\lambda$  are. On the other hand, the accuracy term  $\delta$  is bounded below by  $\frac{w_3}{w_1}$  indicating that the underlying bound posed by the malicious noise. Thirdly, separation conditions between the two components depend on the dimension of the problem in an imbalanced case. In particular, it requires that in each one-dimensional direction, the mean of skewed distributed component  $\mu_2$  is roughly  $O(n^{-1/4})$  away from  $\mu_1$  in that direction. This is a stronger separation condition compared to the balanced case ([18]).

**Remark 2.** In the non-spherical case, the third condition in Lemma 3.4 can be translated into  $\frac{\sigma_{\min}}{\sigma_{\max}}$  being bounded away from 0. This contributes to a good approximation of  $\Sigma^{-1}$  using  $\hat{\Sigma}$  which is close to  $\Sigma$  in Frobenius norm.

The following is a proof sketch of the crucial filtering result in the spherical case. The proof for the non-spherical case follows analogously.

**Proof Sketch of Lemma 3.5** Denote by  $m_1 = w_1 m$ . By Bayes rule,

$$\begin{split} &\Pr(w \in G_1 | l(w) \geq l(X)_{(m_1)}) \\ = & \frac{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \Pr(w \in G_1)}{\sum_{J = G_1, G_2, N} \Pr(l(w) \geq l(X)_{(m_1)} | w \in J) \Pr(w \in J)} \\ \leq & \frac{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \Pr(w \in G_1)}{\sum_{J = G_1, G_2} \Pr(l(w) \geq l(X)_{(m_1)} | w \in J) \Pr(w \in J)} \\ = & \frac{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1)}{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1)} . \end{split}$$

Therefore, it suffices to show that,

$$\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_1) 
\le \eta \frac{w_2}{w_1} \Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2).$$
(1)

To show (1), we show an upper bound of LHS and lower bound of RHS and find conditions so that the upper bound of LHS is smaller than the lower bound of RHS. The proof can be decomposed into 2 steps.

(i) An upper bound of  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_1)$ 

$$\begin{split} &\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \\ &\leq \Pr(l(w) \geq l(X^{G_1 \cup G_2})_{(m_1 - m_3)} | w \in G_1) \\ &= \Pr(l(w) \geq l(X^{G_1 \cup G_2})_{(m_1 - m_3)}, l(w) \leq l(X^{G_1})_{(m_1(1 - \beta) - m_3)} | w \in G_1) \\ &+ \Pr(l(w) \geq l(X^{G_1 \cup G_2})_{(m_1 - m_3)}, l(w) \geq l(X^{G_1})_{(m_1(1 - \beta) - m_3)} | w \in G_1) \\ &\leq \sum_{n_1 + n_2 = m_1 - m_3} \sum_{j = n_1}^{m_1(1 - \beta) - m_3} \Pr(l(X^{G_1})_{(j)} = l(w) \geq l(X^{G_2})_{(n_2)} | w \in G_1) + \beta + \frac{m_3}{m_1}. \end{split}$$

Notice that  $\Pr(l(X^{G_1})_{(j)} \ge l(X^{G_2})_{(n_2)})$  is bounded by

$$\begin{split} & \Pr\left(l(X^{G_1})_{(m_1(1-\beta)-m_3)} \geq l(X^{G_2})_{(m_1\beta)}\right) \\ \leq & \Pr\left(l(X^{G_1})_{(m_1(1-\beta)-m_3)} \geq t\right) + \Pr(l(X^{G_2})_{(m_1\beta)} \leq t) \\ \leq & \Pr\left(\frac{S_{m_1}(\underline{p})}{m_1} \leq 1 - \beta - \frac{m_3}{m_1}\right) + \Pr\left(\frac{S_{m_2}(\bar{p})}{m_2} \geq \frac{m_1\beta}{m_2}\right), \end{split}$$

for some  $t, \bar{p}, \underline{p}$  such that  $\underline{p} \leq \Pr(l(X_i^{G_1}) \leq t)$  and  $\bar{p} \geq \Pr(l(X_i^{G_2}) \leq t)$  and  $S_n(p)$  denotes the sum of n i.i.d. Bernoulli trials with parameter p. The reason of introducing t is that  $l(X_i^{G_2})$  is approximately (under  $\hat{\mu}_1, \hat{\Sigma}_1$ ) non-central chi-squared distributed. With a well-chosen t and the condition that the two Gaussians are well-separated, we can approximate the comparison between a central and a non-central chi-squared distribution by finding a cutoff t that well separates the two ellipsoids, instead of working out a joint distribution of the two.

For the first term and  $\underline{p} \leq \Pr(l(X_i^{G_1}) \leq t)$ , we apply a sharp bound on the right tail of central chi-squared distribution [19],

$$\Pr(\chi_n^2 - n \ge 2\sqrt{xn} + 2x) \le \exp(-x).$$

Subsequently, we set *p* accordingly.

For the second term and  $\bar{p} \ge \Pr(l(X_i^{G_2}) \le t)$ , we apply similarly an upper bound on the left tail of non-central chi-squared distribution  $\chi_n^2(\lambda)$  given by [20],

$$\Pr(\chi_n^2(\lambda) \le (n+\lambda) - 2\sqrt{(n+2\lambda)x}) \le \exp(-x).$$

which determines our  $\bar{p}$  accordingly.

In the proof, we show that under the well-separated condition on  $\lambda$ , one could find satisfying threshold t, such that,

$$\Pr\left(\frac{S_{m_1}(\underline{p})}{m_1} \le 1 - \beta - \frac{m_3}{m_1}\right) = O(\exp(-m_1\beta)),$$

$$\Pr\left(\frac{S_{m_2}(\bar{p})}{m_2} \ge \frac{m_1\beta}{m_2}\right) = O(\exp(-m_1\beta)).$$

Hence,

$$\Pr(l(X^{G_1})_{(m_1(1-\beta)-m_3)} \ge l(X^{G_2})_{(m_1\beta)}) = O(\exp(-m_1\beta)).$$

Therefore, under the choice of  $\beta = \frac{m_2}{m_1} \eta$ ,

$$\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_1) = O(\frac{w_2}{w_1} \eta + \frac{w_3}{w_1}).$$

(ii) A lower bound on  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2)$ .

To quantify the right tail of a non-central chi-squared distribution, we apply an alternative characterization of the  $\chi_n^2(\lambda)$  as follows.

<sup>&</sup>lt;sup>1</sup>A non-central chi-squared distribution is denoted as  $\chi_n^2(\lambda)$  with  $\lambda$  being the non-centrality parameter and n the degree of freedom. Let  $(Y_1, Y_2, \ldots, Y_n)$  be n independent normally distributed random variables with mean  $a_i$  and unit variance. Then  $\sum_{j=1}^n Y_j^2$  follows a  $\chi_n^2(\lambda)$  distribution and  $\lambda = \sum_j a_i^2$ .

Lemma 3.6 ([21]).

$$\chi_n^2(\lambda) \stackrel{\mathrm{D}}{=} \chi_{n-1}^2(0) + \chi_1^2(\lambda).$$

In other words, a non-central chi-squared distribution with degree of freedom n and non-centrality parameter  $\lambda$  is, in distribution, equivalent to the sum of a random variable drawn from non-central chi-squared distribution with degree of freedom 1 and non-centrality parameter  $\lambda$  and another random variable drawn from central chi-squared distribution with degree of freedom n-1. Moreover, the two random variables are independent.

Then a lower bound on  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2)$  can be obtained with a proper choice of  $\tilde{m}_2 = O(m_2)$  such that,

$$\begin{split} & \Pr(l(w) \geq l(X)_{(m_1)} | w \in G_2) \\ \geq & \sum_{i+j=m_1} \Pr(l(w) \geq l(X^{G_1})_{(i)}, l(w) \geq l(X^{G_2})_{(j)} | w \in G_2) \\ \geq & \Pr(l(w) > t, l(w) \geq l(X^{G_2})_{(\tilde{m}_2)} | w \in G_2) \Pr(l(X^{G_1})_{(m_1 - \tilde{m}_2)} \leq t) \\ = & \frac{m_2 - \tilde{m}_2}{m_2} \Pr(l(X^{G_2})_{(\tilde{m}_2)} > t) \Pr(l(X^{G_1})_{(m_1 - \tilde{m}_2)} \leq t). \end{split}$$

Similarly we would like to find a ball of 'radius' t that covers at least  $1 - \frac{\tilde{m}_2}{m_1}$  fraction of sampled points from the first component, while overlapping at most a fraction  $\frac{\tilde{m}_2}{m_2}$  of the samples from the second components. Using a tail bound for  $\chi_n^2$  given by [19], one could prove that

$$\Pr(l(X^{G_1})_{(m_1-\tilde{m}_2)} \le t) \ge 1 - c \exp(-\tilde{m}_2).$$

On the other hand, under the well-separated condition, and one could show that with the choice of  $\tilde{m}_2$ ,

$$\Pr(l(X^{G_2})_{(m_2-\tilde{m}_2;m_2)} > t) \ge 1/2.$$

Therefore, for the choice of  $\tilde{m}_2$  to be specified in the appendix,  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2)$  is bounded below by some constant when  $m_2$  is sufficiently large.

One could show that under the separation condition, the upper bound of LHS in (1) is smaller than the lower bound of its RHS, thus completing the proof of Lemma 3.5. The difference between the proofs for the spherical and the non-spherical case is resolved by [22] with the following inequalities. If  $y \sim N(\mu, \Sigma)$ , then

$$\Pr(\|y - \mu\|^2 \ge \operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)x} + 2\|\Sigma\|x) \le \exp(-x).$$

Moreover, if  $y \sim N(\mu_2, \Sigma)$ ,

$$\Pr(\|y - \mu_1\|^2 \ge \operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)x} + 2\|\Sigma\|x + \|\mu_2 - \mu_1\|^2 (1 + \frac{2\|\Sigma\|x}{\sqrt{\operatorname{tr}(\Sigma^2)x}}) \le \exp(-x).$$

Then, applying the same idea of find a ball of 'radius' t for some proper choice of t that separates the two components, we can achieve the desired inequalities.

# 4 Sensitivity

Compared with the widely adopted [1] vanilla expectation-maximisation (EM) algorithm, our algorithm for recovering the mean of 2-GMMs does not requires an initialisation. Moreover, the algorithm significantly outperforms the EM algorithm given the same number of sampled points, as demonstrated in the simulations of the next section. Nevertheless, it is worth noticing that our algorithm does require the mixing coefficient  $w_1$  on the input. In the spherical case, we show that the algorithm is robust to a perturbation in the input  $w_1$ . That is, if instead of  $w_1$ , the input takes an imprecise estimator  $w'_1$ , the output of the estimated means  $\hat{\mu}_1$ ,  $\hat{\mu}_2$  are slightly perturbed.

**Proposition 4.1.** Consider the spherical case and assume that conditions of Theorem 3.1 are satisfied. Then, there exist  $\alpha_k, k = 1, 2, ...$  such that  $\forall 1 > w_1' > 0$  we have  $\|\hat{\mu}_k(w_1') - \hat{\mu}_k(w_1)\|_2$  bounded by:

$$\|\hat{\mu}_k(w_1') - \hat{\mu}_k(w_1)\|_2 \le \alpha_k (\frac{|w_1' - w_1| + w_3}{w_2} + \frac{w_3}{1 - w_1'} + \varepsilon) \sigma \sqrt{\log n}.$$

See Figures 8 and 9 in the next section for a computational illustration.

Estimation error		n = 10	n = 12	n = 14	n = 16	n = 18	n = 20
$\phantom{aaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa$	Alg. 1	0.54(0.13)	0.51(0.10)	0.53(0.13)	0.56(0.14)	0.55(0.10)	0.55(0.13)
	EM	1.66(2.56)	2.04(3.32)	1.69(1.61)	1.69(1.21)	2.08(2.41)	1.72(0.52)
$\phantom{aaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa$	Alg. 1	1.18(0.19)	1.17(0.13)	1.16(0.19)	1.22(0.17)	1.25(0.16)	1.26(0.16)
	EM	78.71(203.58)	21.07(15.34)	50.21(62.11)	78.04(276.22)	39.97(60.06)	60.27(97.50)
${\ \hat{\mu}_1 - \mu_1\  + \ \hat{\mu}_2 - \mu_2\ }$	Alg. 1	1.72(0.27)	1.68(0.16)	1.70(0.25)	1.78(0.26)	1.80(0.19)	1.81(0.22)
	EM	80.37(203.45)	23.11(14.97)	51.90(62.16)	79.73(276.13)	42.05(59.86)	61.99(97.45)

Table 1: Estimation errors on  $\mu_1$  and  $\mu_2$  using Algorithm 1 and Vanilla EM

Estimation error		n = 10	n = 20	n = 40	n = 60	n = 80
$\phantom{aaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa$	Alg. 1	0.54(0.13)	0.55(0.13)	0.77(0.26)	0.74(0.32)	0.77(0.33)
	EM	1.66(2.56)	1.72(0.52)	2.99(2.46)	11.04(43.41)	7.95(11.42)
$\phantom{aaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa$	Alg. 1	1.18(0.19)	1.26(0.16)	1.59(0.20)	3.54(0.81)	1.88(0.19)
	EM	78.71(203.58)	60.27(97.50)	156.80(491.43)	221.60(040.94)	98.24(161.98)
${\ \hat{\mu}_1 - \mu_1\  + \ \hat{\mu}_2 - \mu_2\ }$	Alg. 1	1.72(0.27)	1.81(0.22)	2.37(0.35)	2.47(0.31)	2.64(0.44)
	EM	80.37(203.45)	61.99(97.45)	159.78(491.24)	232.64(1043.11)	106.20(161.84)

Table 2: Estimation errors on  $\mu_1$  and  $\mu_2$  using Algorithm 1 and Vanilla EM on Higher Dimensions

# 5 Computational Illustrations

We performed 50 simulations on 10,000 samples from a noisy 2-GMM each, with mixing coefficients  $w_1 = 0.8$ ,  $w_2 = 0.16$ ,  $w_3 = 0.04$  throughout. The noise is Cauchy distributed and dimensions n = 10, 12, 14, 16, 18, 20, 40, 60, 80. Table 1 compares the estimation error on  $\mu_1$  and  $\mu_2$  achieved by Algorithm 1 to the estimation error achieved by the vanilla EM algorithm, implemented as fitgmdist in Mathworks Matlab 2016a. Following the conventional rules, we measure the estimation error as

$$\min_{\pi \in P_2} \|\hat{\mu}_{\pi(1)} - \mu_1\|_2 + \|\hat{\mu}_{\pi(2)} - \mu_1\|_2.$$

The true sampling error in the graph denotes the labeled mean  $\|mean(X^{G_1}) - \mu_1\|$ ,  $\|mean(X^{G_2}) - \mu_2\|$ . We start with the 2-GMM with malicious noise in n = 10, 12, 14, 16, 18, 20 dimensional cases.

Figures 4,5,3 show that the estimation errors by vanilla EM are much larger than that by Algorithm 1, especially in the second component. Based on Table 1, the variance of the predictions by Algorithm 1 is also smaller than that of EM.

From Figures 6,7, one could notice that as dimension goes up, vanilla EM algorithm witnesses much larger error and variance, compared with the more robust Algorithm 1, which does not requires any initialization.

In Figures 5 and 3, while the average  $\|\hat{\mu}_2 - \mu_2\|$  obtained by Algorithm 1 is below 5, the averaged estimation error for the second component is above 20 for vanilla EM. From Table 1 we can further notice that the outputs of the vanilla EM algorithm also suffer from large variation both compared with Algorithm 1 as well as across different dimensions, which is due to its sensitivity to the initialization points.

Next, we conduct a more detailed experiment on the sensitivity to the input of  $w_1$  in Algorithm 1. Figures 8 and 9 show the estimation error on  $\mu_1, \mu_2$  when varying

$$\alpha = \frac{1 - w_1'}{1 - w_1}.$$

Clearly, we overestimate the mixing coefficient of the larger component, whenever  $\alpha < 1$ . For  $\alpha > 1$ , the proportion of the smaller component and malicious noise, i.e.  $w_1$ , is overestimated. Figures 8 and 9 outline the results, suggesting the performance of Algorithm 1 when the input  $w'_1$  is distorted.

# 6 Discussion

We have presented a meta-algorithm for Robust Parameter Estimation in Noisy 2-GMM, which works best when the differences in mixing coefficients are large, i.e.,  $w_1 \gg w_2 \gg w_3$ , and the coefficients are known

	n = 10	n = 12	n = 14	n = 16	n = 18	n = 20
$\alpha = 0.50$	3.05(0.40)	3.09(0.33)	3.22(0.44)	3.18(0.37)	3.28(0.40)	3.39(0.44)
$\alpha = 0.67$	2.65(0.40)	2.72(0.28)	2.82(0.41)	2.89(0.48)	2.90(0.34)	2.97(0.44)
$\alpha = 0.83$	2.24(0.36)	2.25(0.27)	2.31(0.32)	2.39(0.37)	2.41(0.29)	2.53(0.33)
$\alpha = 1.00$	1.72(0.27)	1.68(0.16)	1.70(0.25)	1.78(0.26)	1.80(0.18)	1.81(0.22)
$\alpha = 1.17$	1.83(0.29)	1.78(0.20)	1.84(0.26)	1.95(0.29)	1.98(0.18)	2.05(0.24)
$\alpha = 1.33$	2.05(0.32)	2.05(0.23)	2.16(0.30)	2.30(0.30)	2.31(0.24)	2.38(0.26)
$\alpha = 1.50$	2.43(0.36)	2.53(0.29)	2.69(0.30)	2.82(0.35)	2.87(0.30)	3.00(0.32)

Table 3: Estimation error of  $\|\hat{\mu}_1 - \mu_1\| + \|\hat{\mu}_2 - \mu_2\|$  as  $\alpha = \frac{1-w_1'}{1-w_1}$  varies for higher dimensions

exactly. There, it outperforms the widely-used expectation-maximisation (EM) algorithm considerably, as documented in Section 5. The algorithm does not require any initialisation and it is rather stable with respect to the error in the estimate of the mixing coefficients, as detailed in Section 4. Our main result is an analysis of the sample complexity of the algorithm, utilising spectral methods of [11].

To continue in this direction, one could parametrise our analysis by the performance of the algorithm for Problem 2, and analyse whether EM algorithms for Problem 2 would improve the performance. Using information-theoretic arguments [5], one could also study the tightness of the bound on the sample complexity.

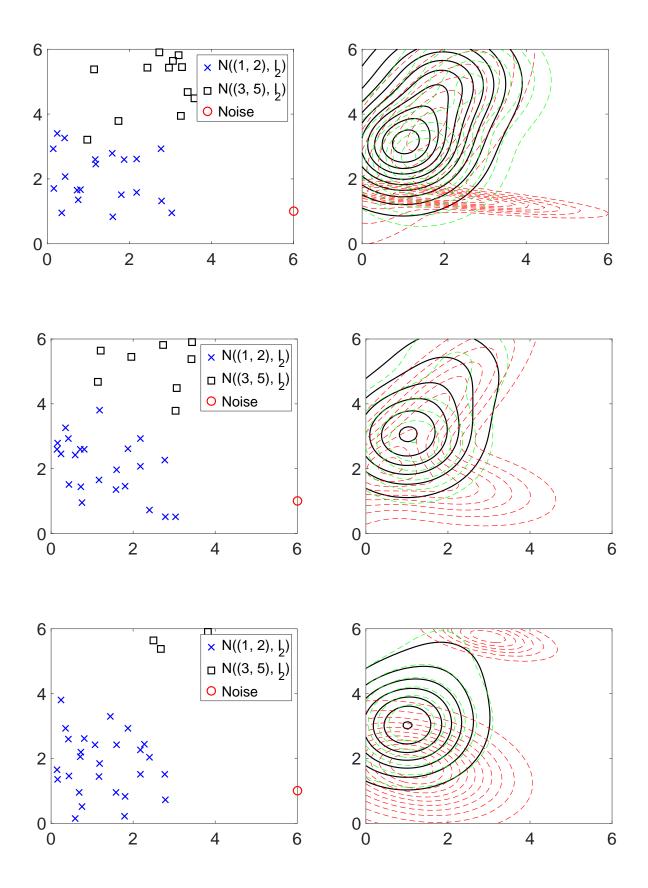


Figure 2: Left: 25 (top), 30 (middle), and 35 (bottom) samples from one Gaussian and 15, 10, and 5 samples from the other Gaussian, with one sample of noise at (6,1). Right: A contour plot of the 2-GMM (in black solid lines), an EM estimate from the 41 samples (in red dashed lines), and an estimate of Algorithm 1 using the same 41 samples (in green dashed lines).

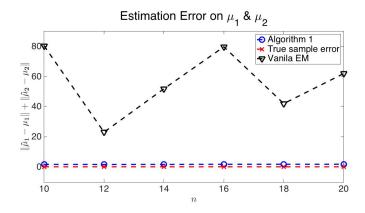


Figure 3: A comparison of vanilla EM with Algorithm 1 in terms of estimation error  $\|\hat{\mu}_1 - \mu_1\|_2 + \|\hat{\mu}_2 - \mu_2\|_2$ 

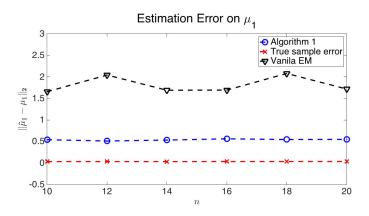


Figure 4: A comparison of vanilla EM with Algorithm 1 in terms of estimation error  $\|\hat{\mu}_1 - \mu_1\|_2$ 

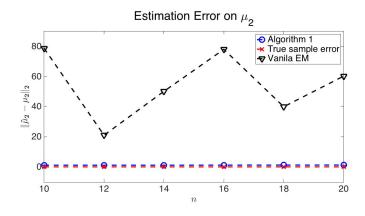


Figure 5: A comparison of vanilla EM with Algorithm 1 in terms of estimation error  $\|\hat{\mu}_2 - \mu_2\|_2$ 

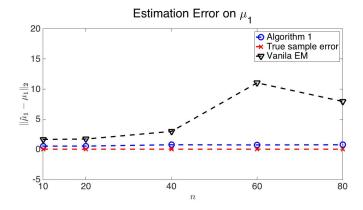


Figure 6: A comparison of vanilla EM with Algorithm 1 in terms of estimation error  $\|\hat{\mu}_1 - \mu_1\|_2$ 

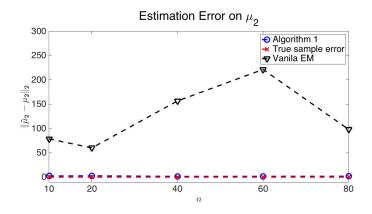


Figure 7: A comparison of vanilla EM with Algorithm 1 in terms of estimation error  $\|\hat{\mu}_2 - \mu_2\|_2$ 

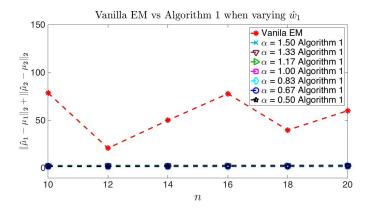


Figure 8: A comparison of vanilla EM with Algorithm 1 in terms of estimation error  $\|\hat{\mu}_1 - \mu_1\|_2 + \|\hat{\mu}_2 - \mu_2\|_2$ 

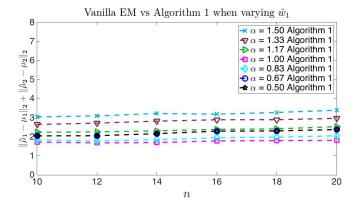


Figure 9: A comparison of vanilla EM with Algorithm 1 in terms of estimation error  $\|\hat{\mu}_1 - \mu_1\|_2 + \|\hat{\mu}_2 - \mu_2\|_2$ 

#### References

- [1] D. Titterington, A. Smith, and U. Makov, *Statistical analysis of finite mixture distributions*. Wiley series in probability and mathematical statistics: Applied probability and statistics, Wiley, 1985.
- [2] G. McLachlan and D. Peel., *Finite Mixture Models*. Wiley series in probability and mathematical statistics, Wiley, 2000.
- [3] K. Pearson, "Contributions to the mathematical theory of evolution," *Philosophical Transactions of the Royal Society of London. A*, vol. 185, pp. 71–110, 1894.
- [4] M. Hardt and E. Price, "Tight bounds for learning a mixture of two gaussians," in *Proceedings of the Forty-seventh Annual ACM Symposium on Theory of Computing*, STOC '15, (New York, NY, USA), pp. 753–760, ACM, 2015.
- [5] Y. Yang and A. Barron, "Information-theoretic determination of minimax rates of convergence," *The Annals of Statistics*, vol. 27, pp. 1564–1599, 10 1999.
- [6] S. Faria and G. Soromenho, "Fitting mixtures of linear regressions," *Journal of Statistical Computation and Simulation*, vol. 80, no. 2, pp. 201–225, 2010.
- [7] X. Yi, C. Caramanis, and S. Sanghavi, "Alternating minimization for mixed linear regression," in *Proceedings of the 31st International Conference on Machine Learning* (E. P. Xing and T. Jebara, eds.), vol. 32 of *Proceedings of Machine Learning Research*, (Bejing, China), pp. 613–621, PMLR, 22–24 Jun 2014.
- [8] S. Balakrishnan, M. J. Wainwright, and B. Yu, "Statistical guarantees for the em algorithm: From population to sample-based analysis," *The Annals of Statistics*, vol. 45, pp. 77–120, 02 2017.
- [9] T. T. Cai, J. Ma, and L. Zhang, "Chime: Clustering of high-dimensional gaussian mixtures with em algorithm and its optimality," in *submitted*, 2017.
- [10] P. J. Huber, "Robust estimation of a location parameter," *The Annals of Mathematical Statistics*, vol. 35, pp. 73–101, 03 1964.
- [11] K. A. Lai, A. B. Rao, and S. Vempala, "Agnostic estimation of mean and covariance," in *Foundations of Computer Science (FOCS)*, 2016 IEEE 57th Annual Symposium on, pp. 665–674, IEEE, 2016.
- [12] I. Diakonikolas, G. Kamath, D. M. Kane, J. Li, A. Moitra, and A. Stewart, "Robustly learning a gaussian: Getting optimal error, efficiently," in *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms*, 2018. arXiv preprint arXiv:1704.03866.
- [13] Y. Chen, X. Yi, and C. Caramanis, "A convex formulation for mixed regression with two components: Minimax optimal rates," in *Proceedings of The 27th Conference on Learning Theory* (M. F. Balcan, V. Feldman, and C. Szepesvári, eds.), vol. 35 of *Proceedings of Machine Learning Research*, (Barcelona, Spain), pp. 560–604, PMLR, 13–15 Jun 2014.
- [14] P. Hand and B. Joshi, "A convex program for mixed linear regression with a recovery guarantee for well-separated data," *arXiv* preprint arXiv:1612.06067, 2016.
- [15] A. Galimzianova, F. Pernuš, B. Likar, and Ž. Špiclin, "Robust estimation of unbalanced mixture models on samples with outliers," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 37, pp. 2273–2285, Nov 2015.
- [16] I. Naim and D. Gildea, "Convergence of the em algorithm for gaussian mixtures with unbalanced mixing coefficients," in *Proceedings of the 29th International Conference on Machine Learning (ICML-12)*, pp. 1655–1662, 2012.
- [17] P. C. Mahalanobis, "On the generalised distance in statistics," *Proceedings of the National Institute of Sciences of India, 1936*, pp. 49–55, 1936.

- [18] S. Balakrishnan, M. J. Wainwright, B. Yu, *et al.*, "Statistical guarantees for the em algorithm: From population to sample-based analysis," *The Annals of Statistics*, vol. 45, no. 1, pp. 77–120, 2017.
- [19] B. Laurent and P. Massart, "Adaptive estimation of a quadratic functional by model selection," *Annals of Statistics*, pp. 1302–1338, 2000.
- [20] L. Birgé, "An alternative point of view on lepski's method," *Lecture Notes-Monograph Series*, pp. 113–133, 2001.
- [21] T. Cacoullos and M. Koutras, "Quadratic forms in spherical random variables: Generalized noncentral x2 distribution," *Naval Research Logistics (NRL)*, vol. 31, no. 3, pp. 447–461, 1984.
- [22] D. Hsu, S. Kakade, T. Zhang, *et al.*, "A tail inequality for quadratic forms of subgaussian random vectors," *Electronic Communications in Probability*, vol. 17, 2012.

# **Algorithms**

#### **Algorithm 2** OUTLIERDAMPING(S)

**Require:**  $S = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\} \subset \mathbb{R}^n$ .

**Ensure:**  $S \subset \mathbb{R}^n$ ,  $\omega = (w_1, \dots, w_m)$ .

- 1: **If** n = 1:
  - **Return** (S, -1)
- 2: Let **a** be the coordinate-wise media of X. Let  $s^2 = C \text{Tr}(\Sigma)$ . Estimate  $\text{tr}(\Sigma)$  by estimating 1d variance along n orthogonal directions and adding up.<sup>2</sup>
- 3: Set  $w_i = \exp(-\frac{\|x_i \mathbf{a}\|_2^2}{s^2})$  for every  $\mathbf{x}_i \in X$ .

# **Algorithm 3** OUTLIERTRUNCATION( $S, \eta, \varepsilon$ )

**Require:**  $S = \{x_1, x_2, ..., x_m\} \subset \mathbb{R}^n, \, \eta \in [0, 1].$ 

**Ensure:**  $S \subset \mathbb{R}^n$ ,  $\mathbf{w} = 1$ .

1: **If** n = 1:

Let [a,b] be the smallest interval containing  $(1-\eta-\varepsilon)(1-\eta)$  fraction of the points,

 $\tilde{S} \leftarrow S \cap [a,b]$ . Return  $(\tilde{S},1)$ 

- 2: For each d = 1, ..., m,
  - (i) Let  $[a_d, b_d]$  be the smallest interval containing  $(1 \eta \varepsilon)(1 \eta)$  fraction of the points  $\{x_d : \mathbf{x} \in S\}$ ,
  - (ii) let  $a_i \leftarrow \text{MEAN}(\{x_d : \mathbf{x} \in S\} \cap [a_d, b_d])$ .  $\mathbf{a} \leftarrow (a_1, \dots, a_n)$ .
- 3: Set  $B(r, \mathbf{a}) \leftarrow \text{ball of minium radius } r \text{ centered at } \mathbf{a} \text{ that contains } (1 \eta \varepsilon)(1 \eta) \text{ fraction of } S.$
- 4:  $\tilde{S} \leftarrow S \cap B(r, \mathbf{a})$ . Return  $(\tilde{S}, \mathbf{1})$ .

#### **Algorithm 4** AGNOSTICMEAN(⋅)

**Require:**  $S = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\} \subset \mathbb{R}^n$ .

Ensure:  $\hat{\mu}$ .

- 1: Let  $(\tilde{S}, \mathbf{w}) = \text{OUTLIERDAMPING}(\mathbf{S}) / \text{Gaussian Case}$ Let  $(\tilde{S}, \omega) = \text{OUTLIERREMOVAL}(S, 1 - w_1) / \text{Non-Gaussian Case}$
- 2: **If** n = 1:
  - (a) if  $\omega = -1$ , Return median( $\tilde{S}$ ). // Gaussian Case
  - (b) **else Return** mean( $\tilde{S}$ ). // Non-Gaussian Case

$$\Sigma_{\tilde{S},\mathbf{w}} \leftarrow \frac{1}{m} \sum_{i} w_{x_i} (\mathbf{x}_i - \mu_{\tilde{S},\mathbf{w}}) (\mathbf{x}_i - \mu_{\tilde{S},\mathbf{w}})^T.$$

3:  $\mu_{\tilde{S},\mathbf{w}} \leftarrow \frac{1}{m} \sum_{i \in S} w_{x_i} \mathbf{x}_i$ .  $\Sigma_{\tilde{S},\mathbf{w}} \leftarrow \frac{1}{m} \sum_{i} w_{x_i} (\mathbf{x}_i - \mu_{\tilde{S},\mathbf{w}}) (\mathbf{x}_i - \mu_{\tilde{S},\mathbf{w}})^T$ . Let V be the span of the top n/2 principal components of  $\Sigma_{\tilde{S},\mathbf{w}}$ , and W be its complment.

- 4: Set  $S_1 \leftarrow \mathbf{P}_V(S)$  where  $\mathbf{P}_V$  is the projection operation on to V.
- 5: Set  $\hat{\mu}_V \leftarrow \text{AGNOSTICMEAN}(S_1)$  and  $\hat{m}u_W \leftarrow \text{MEAN}(\mathbf{P}_W \tilde{S})$
- 6: Let  $\hat{\mu} \in \mathbb{R}^n$  be such that  $\mathbf{P}_V \hat{\mu} = \hat{\mu}_V$ , and  $\mathbf{P}_W \hat{\mu} = \hat{\mu}_W$ .
- 7: **Return**  $\hat{\mu}$ .

<sup>&</sup>lt;sup>2</sup> Suppose  $D = N(\mu, \sigma^2)$ . Denote by  $D_n$  the 1-GMM with malicious noise of fraction  $\eta$ . There are several ways to estimate  $\sigma$ . One is, let  $x_{med} = \text{MEDIAN}\{x_i\}$ . Let  $\Phi(x)$  be the c.d.f. of N(0,1). Note that  $c_1 = \Phi(1) \approx 0.851$ . Let  $C_{\sigma}$  be the  $c_1 \times 100\%$ th quantile of S. Let the estimator  $\hat{\sigma} = C - \hat{\mu}$  where  $\hat{\mu}$  is the estimated mean. One can prove that  $|\hat{\sigma}^2 - \sigma^2| = O(\eta \sigma^2)$ .

#### Algorithm 5 AGNOSTICCOV $(S, w_2 + w_3)$

**Require:**  $S = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\} \subset \mathbb{R}^n$ .

Ensure:  $\hat{\mu}$ .

- 1:  $x_i' = \frac{\dot{x}_i x_{i+m/2}}{\sqrt{2}}$ , for  $i \in \{1, ..., m/2\}$ . (Notice that  $\mathbb{E}[x'(x')^T] = \Sigma$ .)
- 2: Let  $S^{(2)} \leftarrow \{x_i'(x_i')^T | i = 1, 2, ..., m/2\}$
- 3: Run the mean estimation algorithm on  $S^{(2)}$ , where elements of  $S^{(2)}$  are viewed as vectors in  $\mathbb{R}^{n^2}$ . Let the output be  $\hat{\Sigma}$ .
- 4: **Return**  $\hat{\Sigma}$ .

## B Proof of Lemma 3.4

Let 
$$l(x) = (x - \hat{\mu}_1)'\hat{\Sigma}^{-1}(x - \hat{\mu}_1)$$
, and

$$l^*(x) = (x - \mu_1)' \Sigma^{-1}(x - \mu_1),$$

be the true value of the Mahalanobis distance. Then,

$$\begin{split} l(x) = & (x - \mu_1 + \mu_1 - \hat{\mu}_1)' \hat{\Sigma}_1^{-1} (x - \mu_1 + \mu_1 - \hat{\mu}_1) \\ = & (x - \mu)' \hat{\Sigma}^{-1} (x - \mu_1) + 2(\mu_1 - \hat{\mu}_1)' \hat{\Sigma}^{-1} (x - \mu_1) + (\mu_1 - \hat{\mu}_1)' \hat{\Sigma}^{-1} (\mu_1 - \hat{\mu}_1) \\ = & (x - \mu_1)' \Sigma^{-1} (x - \mu_1) + (x - \mu_1)' \hat{\Sigma}^{-1} (\Sigma_1 - \hat{\Sigma}) \Sigma^{-1} (x - \mu_1) \\ & + 2(\mu_1 - \hat{\mu}_1)' \hat{\Sigma}^{-1} (x - \mu_1) + (\mu_1 - \hat{\mu}_1)' \hat{\Sigma}^{-1} (\mu_1 - \hat{\mu}_1). \end{split}$$

And

$$|(x - \mu_1)' \hat{\Sigma}^{-1} (\Sigma_1 - \hat{\Sigma}) \Sigma^{-1} (x - \mu_1)|$$

$$\leq ||\Sigma - \hat{\Sigma}||_F ||\hat{\Sigma}_1^{-1}||_2 ||\Sigma^{-1}||_2 ||x - \mu_1||_2^2$$

$$\leq 2||\Sigma - \hat{\Sigma}||_F ||\Sigma^{-1}||_2^2 ||x - \mu_1||_2^2.$$

the last inequality is due to the fact that  $||X||_2 \le ||X||_F$ , and  $||(A+E)^{-1}||_2 \le \frac{||A^{-1}||_2}{1-||A^{-1}E||_2}$  as long as A is non-singular and  $||A^{-1}E||_2 < 1$ , such that

$$\begin{split} \|\hat{\Sigma}^{-1}\|_{2} &= \|(\Sigma + \hat{\Sigma} - \Sigma)^{-1}\| \\ &\leq \frac{\|\Sigma^{-1}\|_{2}}{1 - \|\Sigma^{-1}(\Sigma - \hat{\Sigma})\|_{2}} \\ &\leq \frac{\|\Sigma^{-1}\|_{2}}{1 - \|\Sigma^{-1}\|_{2}\|\Sigma - \hat{\Sigma}\|_{F}} \\ &\leq 2\|\Sigma^{-1}\|_{2}. \end{split}$$

as long as  $\|\Sigma - \hat{\Sigma}\|_F \le \frac{1}{2} \|\Sigma^{-1}\|_2^{-1}$ . By Theorem 1.1, 1.3 from [11],

$$\|\mu_1 - \hat{\mu}_1\|_2 = O(\eta^{1/2} + \varepsilon) \|\Sigma\|_2^{1/2} \log^{1/2} n$$
  
 $\|\Sigma - \hat{\Sigma}\|_F = O(\eta^{1/2} + C_{4,2}^{1/4} (\eta + \varepsilon)^{3/4}) C_4^{1/2} \|\Sigma\|_2 \log^{1/2} n,$ 

with  $C_4 = 3$ ,  $C_{4,2} = 7!!/4$ . Therefore,

$$|l(x) - l^*(x)| \le 2\|\Sigma^{-1}\|_2^2 \|x - \mu_1\|_2^2 \|\Sigma - \hat{\Sigma}\|_F + 4\|\Sigma^{-1}\|_2 \|x - \mu_1\|_2 \|\mu_1 - \hat{\mu}_1\|_2 + \|\Sigma^{-1}\|_2 \|\mu_1 - \hat{\mu}_1\|_2^2.$$

Therefore, we can approximate l(x) using  $l^*(x)$  which follows (non-)central chi-squared distribution. Next, notice that

$$\begin{split} &\Pr(w \in G_{1} | l(w) \geq l(X)_{(m_{1})}) \\ = & \frac{\Pr(l(w) \geq l(X)_{(m_{1})} | w \in G_{1}) \Pr(w \in G_{1})}{\sum_{k=1}^{2} \Pr(l(w) \geq l(X)_{(m_{1})} | w \in G_{k}) \Pr(w \in G_{k}) + \Pr(l(w) \geq l(X)_{(m_{1})} | w \in N) \Pr(w \in N)} \\ \leq & \frac{\Pr(l(w) \geq l(X)_{(m_{1})} | w \in G_{1}) \Pr(w \in G_{1})}{\Pr(l(w) \geq l(X)_{(m_{1})} | w \in G_{1}) \Pr(w \in G_{1}) + \Pr(l(w) \geq l(X)_{(m_{1})} | w \in G_{2}) \Pr(w \in G_{2})}. \end{split}$$

To show that the above is bounded, we need to find (i) an upper bound on  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_1)$ ,(ii) a lower bound for  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2)$  in (2)..

(i) First need to prove an upper bound on  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_1)$ .

$$\begin{split} &\Pr(l(w) \geq l(X)_{(m_1)}|w \in G_1) \\ \leq &\Pr(l(w) \geq l(X^{G_1 \cup G_2})_{(m_1 - m_3)}|w \in G_1) \\ &= \Pr(l(w) \geq l(X^{G_1 \cup G_2})_{(m_1 - m_3)}, l(w) \geq l(X^{G_1})_{(m_1(1 - \beta) - m_3)}|w \in G_1) \\ &+ \Pr(l(w) \geq l(X^{G_1 \cup G_2})_{(m_1 - m_3)}, l(w) \leq l(X^{G_1})_{(m_1(1 - \beta) - m_3)}|w \in G_1) \\ \leq &\beta + \frac{m_3}{m_1} + \sum_{n_1 + n_2 = m_1 - m_3} \Pr(l(X^{G_1})_{(m_1(1 - \beta) - m_3)} \geq l(w) \geq l(X^{G_1})_{(n_1)}, l(w) \geq l(X^{G_2})_{(n_2)}|w \in G_1) \\ \leq &\beta + \frac{m_3}{m_1} + \sum_{n_1 + n_2 = m_1 - m_3} \sum_{j = n_1}^{m_1(1 - \beta) - m_3} \Pr(l(X^{G_1})_{(j)} = l(w) \geq l(X^{G_2})_{(n_2)}|w \in G_1) \\ \leq &\beta + \frac{m_3}{m_1} + \sum_{n_1 + n_2 = m_1 - m_3} \sum_{j = n_1}^{m_1(1 - \beta) - m_3} \frac{1}{m_1} \Pr(l(X^{G_1})_{(m_1(1 - \beta) - m_3)} \geq l(X^{G_2})_{(m_1\beta)}|w \in G_1) \\ \leq &\beta + \frac{m_3}{m_1} + \frac{(m_2 - m_1\beta)^2}{2m_1} \Pr(l(X^{G_1})_{(m_1(1 - \beta) - m_3)} \geq l(X^{G_2})_{(m_1\beta)}). \end{split}$$

The last inequality above can be upper bounded as follows,

$$\Pr(l(X^{G_{1}})_{(m_{1}(1-\beta)-m_{3})} \geq l(X^{G_{2}})_{(m_{1}\beta)}) \\
\leq \Pr(l(X^{G_{1}})_{(m_{1}(1-\beta)-m_{3})} \geq l(X^{G_{2}})_{m_{1}\beta} | l(X^{G_{2}})_{(m_{1}\beta)} \geq t) \Pr(l(X^{G_{2}})_{m_{1}\beta} \geq t) \\
+ \Pr(l(X^{G_{2}})_{(m_{1}\beta)} \leq t) \\
\leq \Pr(l(X^{G_{1}})_{(m_{1}(1-\beta)-m_{3})} \geq t) \Pr(l(X^{G_{2}})_{(m_{1}\beta)} \geq t) + \Pr(l(X^{G_{2}})_{(m_{1}\beta)} \leq t) \\
\leq \Pr(\frac{\sum_{i} \mathbb{1}(l(X_{i}^{G_{1}}) \leq t)}{m_{1}} \leq 1 - \beta - \frac{m_{3}}{m_{1}}) + \Pr(\frac{\sum_{i} \mathbb{1}(l(X_{i}^{G_{2}}) \leq t)}{m_{2}} \geq \frac{m_{1}\beta}{m_{2}}) \\
\leq \Pr(\frac{S_{m_{1}}(\underline{p})}{m_{1}} \leq 1 - \beta - \frac{m_{3}}{m_{1}}) + \Pr(\frac{S_{m_{2}}(\bar{p})}{m_{2}} \geq \frac{m_{1}\beta}{m_{2}}).$$
(3)

Here  $\Pr(l(X_i^{G_1}) \le t) \ge \underline{p}$  and  $\Pr(l(X_i^{G_2}) \le t) \le \overline{p}$  and  $S_n(p)$  is a sum of n i.i.d. Bernoulli trials with p being the probability of success. The last inequality is due to the following lemma. The proof for the lemma is standard calculus and thus omitted.

**Lemma B.1.** Denote  $S_m(p)$  the sum of m Bernoulli random variables with parameter p. Then  $\Pr(S_m(p) \le t)$  is non-decreasing in p.

For 
$$\underline{p}$$
, let  $\underline{p} = 1 - \exp(-\frac{t' - \sqrt{(2t' - n)n}}{2}) - \exp(-x')$ , where
$$t' = t - 2\|\Sigma - \hat{\Sigma}\|_F \|\Sigma^{-1}\|_2^2 (\operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)x'} + 2\|\Sigma\|x')$$

$$- 4\|\mu_1 - \hat{\mu}_1\|_2 \|\Sigma^{-1}\|_2 \sqrt{\operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)x'}} + 2\|\Sigma\|x'$$

$$- 2\|\mu_1 - \hat{\mu}_1\|_2^2 \|\Sigma^{-1}\|_2^2,$$
(4)

we have

$$\begin{split} \Pr(l(X_i^{G_1}) \leq t) \geq & \Pr(l^*(X_i^{G_1}) \leq t') \\ \geq & 1 - \Pr(\chi_n^2 \geq t' | \|X_i^{G_1} - \mu_1\|_2 \leq \operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)x'} + 2\|\Sigma\|x') - \exp(-x') \\ \geq & 1 - \exp(-\frac{t' - \sqrt{(2t' - n)n}}{2}) - \exp(-x'). \end{split}$$

To get the second inequality above, we apply Proposition 1 in [22] that for  $Y \sim N(\mu, \Sigma)$ ,

$$\Pr(\|Y - \mu\|_2 \ge \operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)y} + 2\|\Sigma\|y) \le \exp(-x).$$

The last inequality above is due to [19] on the concentration inequality of central chi-squared distribution.

**Lemma B.2** ([19]). Let  $X \sim \chi_n^2$ , then for any  $x \ge 0$ ,

$$\Pr(X - n \ge 2\sqrt{xn} + 2x) \le \exp(-x),$$

$$\Pr(X - n \le -2\sqrt{xn}) \le \exp(-x).$$

By simple calculation, we can deduce that  $\Pr(\chi_n^2 \ge t) \le \exp(-\frac{(\sqrt{2t-n}-\sqrt{n})^2}{4})$ . Therefore, by the Chernoff-like bound in Lemma B.3, we have

$$\Pr(\frac{S_{m_1}(\underline{p})}{m_1} \le 1 - \beta - \frac{m_3}{m_1}) \le \exp(-m_1 \frac{[\beta + \frac{m_3}{m_1} - \exp(-\frac{t' - \sqrt{(2t' - n)n}}{2}) - \exp(-x')]^2}{2(1 - \beta - \frac{m_3}{m_1})(\beta + \frac{m_3}{m_1})}),$$

when

$$\beta + \frac{m_3}{m_1} \ge 2\exp(-\frac{t' - \sqrt{(2t' - n)n}}{2}) + 2\exp(-x'). \tag{5}$$

**Lemma B.3** (Relative Entropy Chernoff Bound). Assume that  $X \in [0,1]$ , and  $\mathbb{E}[X] = \mu$ . Fix  $\varepsilon$ , Define

$$MaxVar[a,b] = \max_{p \in [a,b]} Var_p,$$

where  $Var_p$  is the variance of a random variable which is 1 with probability p and 0 with probability 1-p. Then we have the following inequalities.

$$\Pr(\bar{X}_n \ge \mu + \varepsilon) \le \exp(-n\frac{\varepsilon^2}{2MaxVar[\mu, \mu + \varepsilon]}),$$

$$\Pr(\bar{X}_n \leq \mu - \varepsilon) \leq \exp(-n \frac{\varepsilon^2}{2MaxVar[\mu - \varepsilon, \mu]}).$$

The proof for the Relative Entropy Chernoff Bound is standard and is omitted here. Let  $x' = \log \frac{\beta + \frac{m_3}{m_1}}{4}$ , to show (5) it suffices to show that

$$\frac{t' - \sqrt{(2t' - n)n}}{2} \ge \log(\beta/4 + \frac{m_3}{4m_1}),$$

which is equivalent to

$$t' \ge \frac{\left(\sqrt{n} + 2\sqrt{\log(\frac{\beta}{4} + \frac{m_3}{4m_1})}\right)^2 + n}{2}.$$
 (6)

For the second term of (3), firstly, we show that  $l^*(w)$  for any  $w \in G_2$  follows a non-central chi-squared distribution.

Suppose  $\Sigma = Q^T DQ$  with Q being real orthogonal matrix. Suppose that  $Y \sim N(\mu_2, \Sigma)$ ,

$$(Y - \mu_1)^T \Sigma^{-1} (Y - \mu_1) = (Y - \mu_1)^T Q^T D^{-1} Q (Y - \mu_1)$$
$$= [D^{-1/2} Q (Y - \mu_1)]^T [D^{-1/2} Q (Y - \mu_1)].$$

Notice that  $D^{-1/2}Q(Y-\mu_2)$  follows some Gaussian distribution with  $\mathbb{E}[D^{-1/2}Q(Y-\mu_1)] = D^{-1/2}Q(\mu_2-\mu_1)$ ,

$$Var[D^{-1/2}Q(Y - \mu_1)] = D^{-1/2}Q(Q^TDQ)(D^{-1/2}Q)^T = I.$$

Hence,  $D^{-1/2}Q(Y - \mu_2) \sim N(D^{-1/2}Q^T(\mu_2 - \mu_1), I)$ . Therefore,

$$l^*(w) = (w - \mu_2)^T \Sigma^{-1}(x - \mu_2) \sim \chi_n^2(\lambda),$$

where  $\lambda = \|D^{-1/2}Q^T(\mu_2 - \mu_1)\|_2^2 = (\mu_2 - \mu_1)^T \Sigma^{-1}(\mu_2 - \mu_1)$ . Similarly we have, for some t'' to be specified later,

$$\Pr(l(X_i^{G_2}) \le t) \le \Pr(\chi_n^2(\lambda) \le t'') + \exp(-x'')$$

$$\le \exp(-(\frac{n+\lambda-t''}{2\sqrt{n+2\lambda}})^2) + \exp(-x'')$$

$$= \bar{p},$$

where

$$\begin{split} t'' = & t + 2\|\Sigma - \hat{\Sigma}\|_F \|\Sigma^{-1}\|_2^2 (\operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)x''} + 2\|\Sigma\|x'' + \|\mu_2 - \mu_1\|^2 (\frac{2\|\Sigma\|x''}{\sqrt{\operatorname{tr}(\Sigma^2)x''}} + 1) \\ & + 4\|\mu_1 - \hat{\mu}_1\|_2 \|\Sigma^{-1}\|_2 \sqrt{\operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)x''}} + 2\|\Sigma\|x + \|\mu_2 - \mu_1\|^2 (\frac{2\|\Sigma\|x''}{\sqrt{\operatorname{tr}(\Sigma^2)x''}} + 1) \\ & + 2\|\mu_1 - \hat{\mu}_1\|_2^2 \|\Sigma^{-1}\|_2^2. \end{split}$$

The result is due to the following lemma.

**Lemma B.4** ([20]). Let  $X \sim \chi_n^2(\lambda)$ , then for all x > 0,

$$\Pr(X \ge (n+\lambda) + 2\sqrt{(n+2\lambda)x} + 2x) \le \exp(-x),$$

$$\Pr(X \le (n+\lambda) - 2\sqrt{(n+2\lambda)x}) \le \exp(-x).$$
By Lemma B.3,  $\Pr(\frac{S_{m_2}(\bar{p})}{m_2} \ge \frac{m_1\beta}{m_2}) \le \exp(-m_2 \frac{[\beta \frac{m_1}{m_2} - \exp(-(\frac{n+\lambda-t''}{2\sqrt{n+2\lambda}})^2) - \exp(-x'')]^2}{\beta \frac{m_1}{m_2}(1-\beta \frac{m_1}{m_2})})$  when
$$\beta \frac{m_1}{m_2} \ge 2 \exp(-(\frac{n+\lambda-t''}{2\sqrt{n+2\lambda}})^2) + 2 \exp(-x''). \tag{7}$$

where  $x'' = -\log \frac{\eta}{4}$ . Then it suffices to show  $\beta \frac{m_1}{m_2} \ge 4 \exp(-(\frac{n+\lambda-t''}{2\sqrt{n+2\lambda}})^2)$ , which is equivalent to,

$$t'' \le n + \lambda - 2\sqrt{(n+2\lambda)\log\frac{m_2}{m_1\beta}}. (8)$$

Then we have

$$\Pr(l(X^{G_1})_{m_1(1-\beta)-m_3} \ge l(X^{G_2})_{m_1\beta}) \sim O(\exp(-m_1\beta)).$$

Therefore, let  $\beta = \frac{m_2}{m_1} \eta$ , we have

$$\beta + \frac{m_3}{m_1} + \frac{(m_2 - m_1 \beta)^2}{2m_1} \Pr(l(X^{G_1})_{m_1(1-\beta) - m_3} \ge l(X^{G_2})_{m_1 \beta})$$

$$= \frac{m_2}{m_1} \eta + \frac{m_3}{m_1} + c \frac{m_2}{m_1} \exp(-m_2 \eta + \log m_2).$$

as long as  $\eta > \frac{\log m_2}{m_2}$ , we have

$$\frac{m_2}{m_1}\eta + \frac{m_3}{m_1} + c\frac{m_2}{m_1}\exp(-m_2\eta + \log m_2) = o(\frac{m_2}{m_1}).$$

The only thing left is to find t', t'' that satisfy (19) and (21). By (19)

$$t \geq 2\|\Sigma - \hat{\Sigma}\|_{F} \|\Sigma^{-1}\|_{2}^{2} (\operatorname{tr}(\Sigma) + 2\sqrt{-\operatorname{tr}(\Sigma^{2})\log(\frac{\beta}{4} + \frac{m_{3}}{4m_{1}})} - 2\|\Sigma\|\log(\frac{\beta}{4} + \frac{m_{3}}{4m_{1}}))$$

$$+ 4\|\mu_{1} - \hat{\mu}_{1}\|_{2} \|\Sigma^{-1}\|_{2} \sqrt{-\operatorname{tr}(\Sigma) - 2\sqrt{\operatorname{tr}(\Sigma^{2})\log(\frac{\beta}{4} + \frac{m_{3}}{4m_{1}})} - 2\|\Sigma\|\log(\frac{\beta}{4} + \frac{m_{3}}{4m_{1}})}$$

$$+ 2\|\mu_{1} - \hat{\mu}_{1}\|_{2}^{2} \|\Sigma^{-1}\|_{2}^{2}$$

$$+ \frac{(\sqrt{n} + \sqrt{-2\log(\frac{\beta}{4} + \frac{m_{3}}{4m_{1}})})^{2} + n}{2}.$$
(9)

For (21), we have

$$\begin{split} &n + \lambda - 2\sqrt{(n+2\lambda)\log\frac{m_2}{m_1\beta}} \\ &\geq t + 2\|\Sigma_1 - \hat{\Sigma}_1\|_F \|\Sigma_1^{-1}\|_2^2 (\operatorname{tr}(\Sigma_2) + 2\sqrt{\operatorname{tr}(\Sigma_2^2)\log\frac{4}{\eta}} + 2\|\Sigma_2\|\log\frac{4}{\eta} + \|\mu_2 - \mu_1\|^2 (\frac{2\|\Sigma_2\|\sqrt{\log\frac{4}{\eta}}}{\sqrt{\operatorname{tr}(\Sigma_2^2)}} + 1) \\ &+ 4\|\mu_1 - \hat{\mu}_1\|_2 \|\Sigma_1^{-1}\|_2 \sqrt{\operatorname{tr}(\Sigma_2) + 2\sqrt{\operatorname{tr}(\Sigma_2^2)\log\frac{4}{\eta}}} + 2\|\Sigma_2\|\log\frac{4}{\eta} + \|\mu_2 - \mu_1\|^2 (\frac{2\|\Sigma_2\|\sqrt{\log\frac{4}{\eta}}}{\sqrt{\operatorname{tr}(\Sigma_2^2)}} + 1) \\ &+ 2\|\mu_1 - \hat{\mu}_1\|_2^2 \|\Sigma_1^{-1}\|_2^2, \end{split}$$

combined with the lower bound on t in (9), we have

$$\begin{split} &n + \lambda - 2\sqrt{(n + 2\lambda)\log\frac{1}{\eta}} \\ &\geq 2\|\Sigma - \hat{\Sigma}\|_{F} \|\Sigma^{-1}\|_{2}^{2} \times \left[2\mathrm{tr}(\Sigma) + 2\sqrt{\mathrm{tr}(\Sigma^{2})}(\sqrt{\log\frac{4}{\eta}} + \sqrt{-\log(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}})}) \right. \\ &+ 2\|\Sigma\|(\log\frac{4}{\eta} - \log(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}})) + \|\mu_{2} - \mu_{1}\|^{2}(\frac{2\|\Sigma\|\sqrt{\log\frac{4}{\eta}}}{\sqrt{\mathrm{tr}(\Sigma^{2})}} + 1)] \\ &+ 4\|\mu_{1} - \hat{\mu}_{1}\|_{2}\|\Sigma^{-1}\|_{2}\sqrt{\mathrm{tr}(\Sigma) + 2\sqrt{\mathrm{tr}(\Sigma^{2})\log\frac{4}{\eta}} + 2\|\Sigma\|\log\frac{4}{\eta} + \|\mu_{2} - \mu_{1}\|^{2}(\frac{2\|\Sigma\|\sqrt{\log\frac{4}{\eta}}}{\sqrt{\mathrm{tr}(\Sigma^{2})}} + 1)} \\ &+ 4\|\mu_{1} - \hat{\mu}_{1}\|_{2}\|\Sigma^{-1}\|_{2}\sqrt{\mathrm{tr}(\Sigma) + 2\sqrt{-\mathrm{tr}(\Sigma^{2})\log(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}})} - 2\|\Sigma\|\log(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}})} \\ &+ 4\|\mu_{1} - \hat{\mu}_{1}\|_{2}^{2}\|\Sigma^{-1}\|_{2}^{2} + \frac{(\sqrt{n} + \sqrt{2\log(\frac{\beta}{4} + \frac{m_{3}}{4m_{1}})})^{2} + n}{2}. \end{split}$$

Notice that

$$\log \frac{4}{\eta} = O(-\log(\frac{\eta}{4}\frac{m_2}{m_1} + \frac{m_3}{4m_1})),$$

and

$$\lambda = (\mu_2 - \mu_1)' \Sigma^{-1} (\mu_2 - \mu_1) \geq \|\Sigma\|_2^{-1} \|\mu_2 - \mu_1\|_2^2.$$

Hence, to show (22) it suffices to show that

$$n + \lambda - 2\sqrt{(n+2\lambda)\log\frac{1}{\eta}} - 2\|\Sigma - \hat{\Sigma}\|_{F}\|\Sigma^{-1}\|_{2}^{2}\lambda\left(\frac{2\|\Sigma\|^{2}\sqrt{\log\frac{4}{\eta}}}{\sqrt{\operatorname{tr}(\Sigma^{2})}} + 1\right)$$

$$-8\|\mu_{1} - \hat{\mu}_{1}\|_{2}\|\Sigma^{-1}\|_{2}\sqrt{\lambda\left(\frac{2\|\Sigma\|^{2}\sqrt{\log\frac{4}{\eta}}}{\sqrt{\operatorname{tr}(\Sigma^{2})}} + 1\right)}$$

$$\geq 2\|\Sigma - \hat{\Sigma}\|_{F}\|\Sigma^{-1}\|_{2}^{2} \times \left[2\operatorname{tr}(\Sigma) + 4\sqrt{\operatorname{tr}(\Sigma^{2})}\sqrt{\log\left(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}}\right)^{-1}}\right) + 4\|\Sigma\|\log\left(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}}\right)^{-1}\right]$$

$$+8\|\mu_{1} - \hat{\mu}_{1}\|_{2}\|\Sigma^{-1}\|_{2}\sqrt{2\operatorname{tr}(\Sigma) + 4\sqrt{\operatorname{tr}(\Sigma^{2})\log\left(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}}\right)^{-1}} + 4\|\Sigma\|\log\left(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}}\right)^{-1}}$$

$$+4\|\mu_{1} - \hat{\mu}_{1}\|_{2}^{2}\|\Sigma^{-1}\|_{2}^{2} + \frac{\left(\sqrt{n} + \sqrt{2\log\left(\frac{\eta}{4}\frac{m_{2}}{m_{1}} + \frac{m_{3}}{4m_{1}}\right)^{-1}}\right)^{2} + n}{2}.$$

$$(11)$$

Subsequently, (11) can be reduced to the following lower bound on  $\lambda$ .

$$\begin{split} \sqrt{\lambda} \geq & a + \sqrt{a^2 + 2b} = O(\sqrt{a^2 + 2b}), \\ a = & 8\|\mu_1 - \hat{\mu}_1\|_2 \|\Sigma^{-1}\|_2 \sqrt{\frac{2\|\Sigma\|^2 \sqrt{\log \frac{4}{\eta}}}{\sqrt{\operatorname{tr}(\Sigma^2)}}} + 1, \\ b = & 2\|\Sigma - \hat{\Sigma}\|_F \|\Sigma^{-1}\|_2^2 \times \left[2\operatorname{tr}(\Sigma) + 4\sqrt{\operatorname{tr}(\Sigma^2)}\sqrt{\log(\frac{\eta}{4}\frac{m_2}{m_1} + \frac{m_3}{4m_1})^{-1}}) + 4\|\Sigma\|\log(\frac{\eta}{4}\frac{m_2}{m_1} + \frac{m_3}{4m_1})^{-1}\right] \\ & + 8\|\mu_1 - \hat{\mu}_1\|_2 \|\Sigma^{-1}\|_2 \sqrt{2\operatorname{tr}(\Sigma) + 4\sqrt{\operatorname{tr}(\Sigma^2)\log(\frac{\eta}{4}\frac{m_2}{m_1} + \frac{m_3}{4m_1})^{-1}} + 4\|\Sigma\|\log(\frac{\eta}{4}\frac{m_2}{m_1} + \frac{m_3}{4m_1})^{-1}} \\ & + 4\|\mu_1 - \hat{\mu}_1\|_2^2 \|\Sigma^{-1}\|_2^2 + \sqrt{2n\log(\frac{\eta}{4}\frac{m_2}{m_1} + \frac{m_3}{4m_1})^{-1}} + \log(\frac{\eta}{4}\frac{m_2}{m_1} + \frac{m_3}{4m_1})^{-1}. \end{split}$$

To reduce this complicated inequality, it can be shown that,

$$a \le O((w_2 \log n)^{\frac{1}{4}}).$$

Combined with

$$\|\Sigma - \hat{\Sigma}\|_F = O(\sqrt{w_2}) \|\Sigma\| \sqrt{\log n}, \quad \|\mu_1 - \hat{\mu}_1\|_F = O(\sqrt{w_2}) \|\Sigma\|^{1/2} \sqrt{\log n},$$

we have

$$b = c_1 \sqrt{w_2 \log n} (\|\Sigma\|^{\frac{1}{2}} \|\Sigma^{-1}\| + 1) (\operatorname{tr}(\Sigma) + \sqrt{\operatorname{tr}(\Sigma^2) \log \frac{1}{\delta}} + \|\Sigma\| \log \frac{1}{\delta}) + \frac{1}{\sqrt{2}} \sqrt{n \log \frac{1}{\delta}} + \log \frac{1}{\delta},$$

where  $\eta \frac{w_2}{w_1} + \frac{w_2}{w_3} = \delta$ , and  $c_1 > 0$  being some constant. Hence, by (12),

$$\lambda \geq c_1 \sqrt{w_2 \log n} (\|\Sigma\|^{\frac{1}{2}} \|\Sigma^{-1}\| + 1) (\operatorname{tr}(\Sigma) + \sqrt{\operatorname{tr}(\Sigma^2) \log \frac{1}{\delta}} + \|\Sigma\| \log \frac{1}{\delta}) + \frac{1}{\sqrt{2}} \sqrt{n \log \frac{1}{\delta}} + \log \frac{1}{\delta}.$$

If  $n = o(\log \frac{1}{\varepsilon})$ , then we have

$$\lambda \ge \left(1 + c\sqrt{w_2 \log n} (\|\Sigma\|^{\frac{1}{2}} \|\Sigma^{-1}\| + 1) \|\Sigma\|\right) \log \frac{1}{\varepsilon}. \tag{13}$$

With this lower bound on  $\lambda$ , we can choose t', t'' to complete the upper bound of (3).

(ii) Next, we show a lower bound for  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2)$  in (2).

$$\begin{split} &\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_2) \\ &= \sum_{i+j+k=m_1} \Pr(l(w) \geq l(X^{G_1})_{(i)}, l(w) \geq l(X^{G_2})_{(j)}, l(w) \geq l(X^N)_{(k)} | w \in G_2) \\ &\geq \sum_{i+j=m_1} \Pr(l(w) \geq l(X^{G_1})_{(i)}, l(w) \geq l(X^{G_2})_{(j)} | w \in G_2) \\ &\geq \Pr(l(w) > l(X^{G_1})_{(m_1-\tilde{m}_2)}, l(w) \geq l(X^{G_2})_{(\tilde{m}_2)} | w \in G_2, l(X^{G_1})_{(m_1-\tilde{m}_2)} \leq t) \\ &\times \Pr(l(X^{G_1})_{(m_1-\tilde{m}_2)} \leq t) \\ &\geq \Pr(l(w) > t, l(w) \geq l(X^{G_2})_{(\tilde{m}_2)} | w \in G_2) \times \Pr(l(X^{G_1})_{(m_1-\tilde{m}_2)} \leq t) \\ &= \frac{m_2 - \tilde{m}_2}{m_2} \Pr(l(X^{G_2})_{(\tilde{m}_2)} > t) \times \Pr(l(X^{G_1})_{(m_1-\tilde{m}_2)} \leq t). \end{split}$$

for some  $t, \tilde{m}_2$  to be specified later.

**Claim 1.**  $\Pr(l(X^{G_1})_{(m_1-\tilde{m}_2)} \ge t) \le O(\exp[-\tilde{m}_2(1-\frac{\bar{p}}{\tilde{m}_2/m_1})^2]).$  *Proof.* 

$$\begin{split} \Pr(l(X^{G_1})_{(m_1 - \tilde{m}_2)} \geq t) &= \Pr(\sum_{i=1}^{m_1} \mathbb{1}(l(X_i^{G_1}) \geq t) \geq \tilde{m}_2) \\ &= \Pr(\frac{\sum_{i=1}^{m_1} \mathbb{1}(l(X_i^{G_1}) \geq t)}{m_1} \geq \frac{\tilde{m}_2}{m_1}) \\ &= \Pr(\frac{S_{m_1}(p)}{m_1} \geq \frac{\tilde{m}_2}{m_1}) \\ &\leq \Pr(\frac{S_{m_1}(\bar{p})}{m_1} \geq \frac{\tilde{m}_2}{m_1}) \\ &\leq \exp[-m_1 \frac{(\frac{\tilde{m}_2}{m_1} - \bar{p})^2}{2\text{MaxVar}[\bar{p}, \frac{\tilde{m}_2}{m_1}]}]. \end{split}$$

By Lemma B.2,  $\Pr(\chi_n^2 \ge t) \le \exp(-\frac{(\sqrt{2t-n}-\sqrt{n})^2}{4})$ . Let  $\bar{p} = \exp(-\frac{(\sqrt{2t-n}-\sqrt{n})^2}{4}) = \exp(-\frac{t-\sqrt{(2t-n)n}}{2})$ , we have

$$\begin{split} \Pr(l(X^{G_1})_{(m_1 - \tilde{m}_2)} \geq t) &\leq \exp[-m_1 \frac{(\frac{\tilde{m}_2}{m_1} - \bar{p})^2}{2 \text{MaxVar}[\bar{p}, \frac{\tilde{m}_2}{m_1}]}] \\ &= \exp[-m_1 \frac{(\frac{\tilde{m}_2}{m_1} - \bar{p})^2}{2\frac{\tilde{m}_2}{m_1}(1 - \frac{\tilde{m}_2}{m_1})]}] \\ &= O(\exp[-\tilde{m}_2(1 - \frac{\bar{p}}{\tilde{m}_2/m_1})^2]). \end{split}$$

under

$$\frac{\tilde{m}_2}{m_1} > \bar{p} = \exp(-\frac{t - \sqrt{(2t - n)n}}{2}).$$
 (14)

**Claim 2.**  $\Pr(l(X^{G_2})_{(\tilde{m}_2)} > t) \ge 1/2$ . *Proof.* 

$$\begin{aligned} \Pr(l(X^{G_2})_{(\tilde{m}_2)}) &\leq t) = \Pr(\sum_{i=1}^{m_2} \mathbb{1}(l(X_i^{G_2}) \leq t) \geq \tilde{m}_2) \\ &= \Pr(\frac{S_{m_2}(q)}{m_2} \geq \frac{\tilde{m}_2}{m_2}) \\ &\leq \Pr(\frac{S_{m_2}(\bar{q})}{m_2} \geq \frac{\tilde{m}_2}{m_2}). \end{aligned}$$

Need to find  $\tilde{m}_2, \bar{q}$  such that

$$\frac{\tilde{m}_2}{m_2} \ge \bar{q} \ge \Pr(l(X^{G_2}) \le t)),$$

which then leads to  $\Pr(\frac{S_{m_2}(\bar{q})}{m_2} \ge \frac{\tilde{m}_2}{m_2}) \le \frac{1}{2}$ . According to [20],  $\forall t < n + \lambda$ ,

$$\Pr(\chi_n^2(\lambda) \le t) \le \exp(-(\frac{n+\lambda-t}{2\sqrt{n+2\lambda}})^2) = \bar{q}.$$

Therefore, combined with (14) in Claim 1, we have

$$\frac{\tilde{m}_2}{m_2} \ge \max\{\exp(-(\frac{n+\lambda-t}{2\sqrt{n+2\lambda}})^2), \exp(-\frac{t-\sqrt{(2t-n)n}}{2} + \log\frac{m_1}{2m_2})\},$$

which is equivalent to,

$$n + \lambda - 2\sqrt{-\log\frac{\tilde{m}_2}{m_2}}\sqrt{n + 2\lambda} \ge t \ge \frac{(2\sqrt{\log\frac{m_2}{\tilde{m}_2} + \log\frac{m_1}{2m_2}} + \sqrt{n})^2 + n}{2}.$$

Hence, if

$$n+\lambda-2\sqrt{-\log\frac{\tilde{m}_2}{m_2}}\sqrt{n+2\lambda}\geq \frac{(2\sqrt{\log\frac{m_2}{\tilde{m}_2}+\log\frac{m_1}{2m_2}}+\sqrt{n})^2+n}{2},$$

i.e.,

$$\lambda \ge 2(\sqrt{\log \frac{m_2}{\tilde{m}_2} + \log \frac{m_1}{2m_2}} + \sqrt{\log \frac{m_2}{\tilde{m}_2}})^2 + 2(\sqrt{\log \frac{m_2}{\tilde{m}_2} + \log \frac{m_1}{2m_2}} + \sqrt{\log \frac{m_2}{\tilde{m}_2}})\sqrt{n}. \tag{15}$$

Let  $\tilde{m}_2 = \frac{1}{2}m_2$  we have,

$$\lambda \ge 3\log\frac{m_1}{2m_2} + 3\sqrt{n\log\frac{m_1}{2m_2}},\tag{16}$$

which suffices to prove Claim 2.

The following presents an alternative way to prove Claim 2. In particular, it relies on the following characterization of a non-central chi-squared distribution.

#### Lemma B.5.

$$\chi_n^2(\lambda) \stackrel{\mathbf{D}}{=} \chi_{n-1}^2(0) + \chi_1^2(\lambda).$$

The CDF of non-central chi-square with one degree of freedom is quite explicit.

$$\Pr(\chi_1^2(\lambda) \le t) = \Phi(\sqrt{t} - \sqrt{\lambda}) - \Phi(-\sqrt{t} - \sqrt{\lambda}).$$

Hence,

$$\Pr(l(X^{G_2})_{(\tilde{m}_2)}) \le t) = \Pr(\sum_{i=1}^{m_2} \mathbb{1}(l(X_i^{G_2}) \ge t) \le m_2 - \tilde{m}_2) \\
= \Pr(\frac{S_{m_2}(p)}{m_2} \le \frac{m_2 - \tilde{m}_2}{m_2}) \\
\le \Pr(\frac{S_{m_2}(\underline{p})}{m_2} \le \frac{m_2 - \tilde{m}_2}{m_2}),$$

where  $p = \Pr(\chi_n^2(\lambda) \ge \underline{p})$ . Need to find a lower bound  $\underline{p}$  and  $\tilde{m}_2$ , such that  $\underline{p} \ge \frac{m_2 - \tilde{m}_2}{m_2}$  and thus

$$\Pr(l(X^{G_2})_{(\tilde{m}_2)}) \le t) \le \frac{1}{2},$$

which then completes the proof of Claim 2.

Denote by  $\bar{\Phi}(t) = 1 - \Phi(t)$ . By Lemma 3.6 and the explicit CDF of non-central chi-square with one degree of freedom,

$$\begin{split} & \Pr(\chi_{1}^{2}(\lambda) + \chi_{n-1}^{2}(0) \geq t) \\ \geq & \Pr(\chi_{1}^{2}(\lambda) + \chi_{n-1}^{2}(0) \geq t | \chi_{n-1}^{2}(0) \geq n - 1 - 2\sqrt{(n-1)x}) \Pr(\chi_{n-1}^{2}(0) \geq n - 1 - 2\sqrt{(n-1)x}) \\ \geq & \Pr(\chi_{1}^{2}(\lambda) \geq t - (n-1) + 2\sqrt{(n-1)x}) \Pr(\chi_{n-1}^{2}(0) \geq n - 1 - 2\sqrt{(n-1)x}) \\ \geq & \bar{\Phi}(\sqrt{t - (n-1) + 2\sqrt{(n-1)x}} - \sqrt{\lambda}) (1 - \exp(-x)) \\ = & \underline{p}. \end{split}$$

Combined with (14) in Claim 1, it suffices to find  $t, \tilde{m}_2$  satisfying

$$\frac{\tilde{m}_2}{m_2} \ge 1 - \underline{p}, 
\frac{\tilde{m}_2}{m_2} \ge \exp\left(-\frac{t - \sqrt{(2t - n)n}}{2} + \log\frac{m_1}{2m_2}\right).$$
(17)

Let 
$$t - (n-1) + 2\sqrt{(n-1)x} = \lambda + 1$$
,  $x = 1$ , and

$$\frac{3}{4} \ge \frac{\tilde{m}_2}{m_2} = 1 - \bar{\Phi}(1)(1 - e^{-1}) \ge \frac{1}{2},$$

which yields  $\Pr(\chi_1^2(\lambda) + \chi_{n-1}^2(0) \ge t) \ge \underline{p} = \overline{\Phi}(1)(1 - e^{-1})$ . Moreover, we need to show that

$$\frac{t-\sqrt{(2t-n)n}}{2}-\log\frac{m_1}{2m_2}\geq -\log\frac{\tilde{m}_2}{m_2}=\log(1-\underline{p})^{-1}.$$

by plugging in  $t - (n-1) + 2\sqrt{(n-1)} = \lambda + 1$ ,  $\lambda \ge \frac{(\sqrt{n} + 2\sqrt{\log \frac{m_1}{m_2}})^2 + n}{2} - n + 2\sqrt{n-1}$ , which finally yields,

$$\lambda \geq 2\sqrt{n\log\frac{m_1}{m_2}} + 2\sqrt{n-1} + 2\log\frac{m_1}{m_2}.$$

Notice that this is actually quite close to (16). Hence, all of the above combined guarantees that  $\tilde{m}_2$  exists, and thus completes the alternative proof of Claim 2.

Therefore,

$$\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2) \ge \frac{m_2 - \tilde{m}_2}{m_2} \Pr(l(X^{G_2})_{(m_2 + 1 - \tilde{m}_2 : m_2)} > t) \times \Pr(l(X^{G_1})_{(m_1 - \tilde{m}_2)} \le t) \\
\ge \frac{1}{20} (1 - \exp(-cm_2)),$$

for some c > 0.

Moreover, given enough sampled points, the fraction of sampled points belonging to each component is approximately the true weight  $w_i$ , that is, given  $m \ge \frac{2\log\frac{1}{\eta}}{w_2\eta^2}$ , we have  $\exp(-\frac{mw_2\eta^2}{2}) \le \eta$ , which yields the concentration inequality of Bernoulli trials,

$$\Pr(|\frac{m_2}{m} - w_2| \ge w_2 \eta) \le \exp(-mw_2 \eta^2) \le \eta.$$

Similarly, with  $m \ge \frac{2\log\frac{1}{\eta}}{w_3}$ , we can bound the probability of Bernoulli trials,

$$\Pr(\left|\frac{m_3}{m} - w_3\right| \ge w_3) \le 2\exp(-2mw_3).$$

Hence, by union bounds,

$$\begin{split} &\Pr(w \in G_1 | l(w) \geq l(X)_{(m_1)}) \\ &= \frac{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \Pr(w \in G_1)}{\sum_{k=1}^{2} \Pr(l(w) \geq l(X)_{(m_1)} | w \in G_k) \Pr(w \in G_k) + \Pr(l(w) \geq l(X)_{(m_1)} | w \in N) \Pr(w \in N)} \\ &\leq \frac{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \Pr(w \in G_1)}{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \Pr(w \in G_1) + \Pr(l(w) \geq l(X)_{(m_1)} | w \in G_2) \Pr(w \in G_2)} \\ &\leq \frac{2\eta w_2 w_1}{2\eta w_2 w_1 + w_2 \frac{1}{20}} + 2 \exp(-\frac{m(w_2 \eta)^2}{2w_2}) + 2 \exp(-\frac{mw_3}{2}) \\ &\leq 32\eta. \end{split}$$

To conclude, under the condition of Lemma 3.4 and Lemma 3.2, 3.3, in order to complete the proof of Theorem 3.1, *m* must satisfy,

$$\begin{split} m &= \Omega(\frac{n(\frac{1}{w_2} + n)(\log n + \log \frac{1}{\varepsilon})\log n}{\varepsilon^2}). \\ m &\geq \frac{2\log \frac{1}{\varepsilon^2 + \frac{w_3}{w_2}}}{w_2(\varepsilon^2 + \frac{w_3}{w_2})^2}. \\ m &\geq \frac{2\log \frac{1}{\varepsilon^2 + \frac{w_2}{w_3}}}{w_3}. \end{split}$$

Hence,

$$m = \Omega\left(\frac{n(n + \frac{1}{w_2})(\log n + \log \frac{1}{\varepsilon})\log n}{\varepsilon^2} + \frac{\frac{1}{w_2}\log(\frac{1}{\varepsilon} + \sqrt{\frac{w_2}{w_3}})}{\varepsilon^4 + (\frac{w_3}{w_2})^2} + \frac{\log(\frac{1}{\varepsilon} + \sqrt{\frac{w_2}{w_3}})}{w_3}\right).$$

#### C Proof of Lemma 3.5

Let  $l^*(x) = \frac{\|x - \mu_1\|_2^2}{\sigma_1^2}$  be the true value of Mahalanobis distance, and  $l(x) = \frac{\|x - \hat{\mu}_1\|_2^2}{\hat{\sigma}^2}$  the computed estimator for l(x) in the algorithm. Then

$$l(x) = \frac{\|x - \mu_1\|_2^2}{\hat{\sigma}_1^2} + 2\frac{(\mu_1 - \hat{\mu}_1)'(x - \mu_1)}{\hat{\sigma}_1^2} + \frac{\|\mu_1 - \hat{\mu}_1\|_2^2}{\hat{\sigma}_1^2}.$$

 $\hat{\sigma}_1^2 \leq 2\sigma_1^2$  By Theorem 1.1, 1.3 from [11],

$$\|\mu_1 - \hat{\mu}_1\|_2 = O(\eta + \varepsilon)\sigma_1\sqrt{\log n}.$$
  
 $|\sigma_1^2 - \hat{\sigma}_1^2| = O(\eta + \varepsilon)\sigma_1^2.$ 

Similarly by Bayes Rule,

$$\begin{split} & \Pr(w \in G_1 | l(w) \geq l(X)_{(m_1)}) \\ = & \frac{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \Pr(w \in G_1)}{\sum_{k=1}^2 \Pr(l(w) \geq l(X)_{(m_1)} | w \in G_k) \Pr(w \in G_k) + \Pr(l(w) \geq l(X)_{(m_1)} | w \in N) \Pr(w \in N)} \\ \leq & \frac{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \Pr(w \in G_1)}{\Pr(l(w) \geq l(X)_{(m_1)} | w \in G_1) \Pr(w \in G_1) + \Pr(l(w) \geq l(X)_{(m_1)} | w \in G_2) \Pr(w \in G_2)}. \end{split}$$

Need to prove an upper bound on  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_1)$  and a lower bound on  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2)$ . Similar to the non-spherical case,

$$\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_1) \le \beta + \frac{m_3}{m_1} + \frac{(m_2 - m_1 \beta)^2}{2m_1} \Pr(l(X^{G_1})_{(m_1(1 - \beta) - m_3)} \ge l(X^{G_2})_{(m_1 \beta)}),$$

and we have

$$\begin{split} & \Pr(l(X^{G_1})_{(m_1(1-\beta)-m_3)} \geq l(X^{G_2})_{(m_1\beta)}) \\ \leq & \Pr(l(X^{G_1})_{m_1(1-\beta)-m_3:m_1} \geq l(X^{G_2})_{m_1\beta:m_2} | l(X^{G_2})_{m_1\beta:m_2} \geq t) \Pr(l(X^{G_2})_{m_1\beta:m_2} \geq t) \\ & + \Pr(l(X^{G_2})_{m_1\beta:m_2} \leq t) \\ \leq & \Pr(\frac{S_{m_1}(\underline{p})}{m_1} \leq 1 - \beta - \frac{m_3}{m_1}) + \Pr(\frac{S_{m_2}(\bar{p})}{m_2} \geq \frac{m_1\beta}{m_2}). \end{split}$$

Here  $\Pr(l(X_i^{G_1}) \le t) \ge p$  and  $\Pr(l(X_i^{G_2}) \le t) \le \bar{p}$ .

Notice  $l(X_i^{G_2})$  is approximately  $||y - \mu_1||_2^2$  with  $y \sim N(\mu_2, \sigma^2 I)$ . By Theorem 1 in Hsu, Kakade & Zhang(2011), for  $y \sim N(\mu_2, \sigma^2 I)$ ,

$$\Pr\left(\|y - \mu_1\|_2^2 \ge n\sigma^2 + 2\sqrt{nt}\sigma^2 + 2\sigma^2t + \|\mu_2 - \mu_1\|_2^2(1 + 2\sqrt{\frac{t}{n}})\right) \le \exp(-t),$$

while for the first term we can still apply  $\Pr(\chi_n^2 \ge t) \le \exp(-\frac{t - \sqrt{n(2t - n)}}{2})$ . Therefore,  $\Pr(l(X_i^{G_1}) \le t)$  can be reduced to

$$\begin{split} \Pr(l(X_i^{G_1}) \leq t) = & \Pr(\frac{\|x - \mu_1\|_2^2}{\hat{\sigma}_1^2} + 2\frac{(\mu_1 - \hat{\mu}_1)'(x - \mu_1)}{\hat{\sigma}_1^2} + \frac{\|\mu_1 - \hat{\mu}_1\|_2^2}{\hat{\sigma}_1^2} \leq t) \\ \geq & \Pr(\frac{\|x - \mu_1\|_2^2}{\hat{\sigma}_1^2} + 2\frac{\|\mu_1 - \hat{\mu}_1\|\|x - \mu_1\|}{\hat{\sigma}_1^2} + \frac{\|\mu_1 - \hat{\mu}_1\|_2^2}{\hat{\sigma}_1^2} \leq t) \\ = & \Pr(\frac{\|x - \mu_1\|_2}{\sigma_1} \leq -\frac{\|\mu_1 - \hat{\mu}_1\|_2}{\sigma_1} + \frac{\hat{\sigma}_1}{\sigma_1}\sqrt{t}) \\ \geq & 1 - \exp(-\frac{(-\frac{\|\mu_1 - \hat{\mu}_1\|_2}{\sigma_1} + \frac{\hat{\sigma}_1}{\sigma_1}\sqrt{t})^2 - \sqrt{n(2[-\frac{\|\mu_1 - \hat{\mu}_1\|_2}{\sigma_1} + \frac{\hat{\sigma}_1}{\sigma_1}\sqrt{t}]^2 - n)}}{2}). \end{split}$$

Therefore, let  $\underline{p} = 1 - \exp(-\frac{(-\frac{\|\mu_1 - \hat{\mu}_1\|_2}{\sigma} + \frac{\hat{\sigma}_1}{\sigma_1}\sqrt{t})^2 - \sqrt{n(2[-\frac{\|\mu_1 - \hat{\mu}_1\|_2}{\sigma} + \frac{\hat{\sigma}_1}{\sigma_1}\sqrt{t}]^2 - n)}}{2})$ , by the Relative Entropy Chernoff Bound,

$$\Pr(\frac{S_{m_1}(\underline{p})}{m_1} \le 1 - \beta - \frac{m_3}{m_1}) \le \exp(-m_1 \frac{\left[\beta + \frac{m_3}{m_1} - \exp(-\frac{(\frac{\|\mu_1 - \mu_1\|_2}{\sigma} + \frac{\sigma_1}{\sigma_1}\sqrt{t})^2 - \sqrt{n(2[-\frac{\|\mu_1 - \mu_1\|_2}{\sigma} + \frac{\sigma_1}{\sigma_1}\sqrt{t}]^2 - n)}}{2(1 - \beta - \frac{m_3}{m_1})(\beta + \frac{m_3}{m_1})}\right),$$

and suppose that

$$\beta + \frac{m_3}{m_1} \ge 2 \exp\left(-\frac{\left(-\frac{\|\mu_1 - \hat{\mu}_1\|_2}{\sigma} + \frac{\hat{\sigma}_1}{\sigma_1}\sqrt{t}\right)^2 - \sqrt{n(2\left[-\frac{\|\mu_1 - \hat{\mu}_1\|_2}{\sigma} + \frac{\hat{\sigma}_1}{\sigma_1}\sqrt{t}\right]^2 - n)}}{2}\right). \tag{18}$$

To prove (18), it suffices to show the following,

$$\left(-\frac{\|\mu_{1} - \hat{\mu}_{1}\|_{2}}{\sigma} + \frac{\hat{\sigma}_{1}}{\sigma^{1}}\sqrt{t}\right)^{2} \ge \frac{\left(\sqrt{n} + 2\sqrt{\log(\frac{\beta}{4} + \frac{m_{3}}{4m_{1}})^{-1}}\right)^{2} + n}{2}.$$
 (19)

For  $\Pr(l(X_i^{G_2}) \le t)$ , similarly, we can get

$$\begin{aligned} \Pr(l(X_{i}^{G_{2}}) \leq t) &= \Pr(\frac{\|x - \mu_{1}\|_{2}^{2}}{\hat{\sigma}_{1}^{2}} + 2\frac{(\mu_{1} - \hat{\mu}_{1})'(x - \mu_{1})}{\hat{\sigma}_{1}^{2}} + \frac{\|\mu_{1} - \hat{\mu}_{1}\|_{2}^{2}}{\hat{\sigma}_{1}^{2}} \leq t) \\ &\leq \Pr(\frac{\|x - \mu_{1}\|_{2}^{2}}{\hat{\sigma}_{1}^{2}} - 2\frac{\|\mu_{1} - \hat{\mu}_{1}\|\|x - \mu_{1}\|}{\hat{\sigma}_{1}^{2}} + \frac{\|\mu_{1} - \hat{\mu}_{1}\|_{2}^{2}}{\hat{\sigma}_{1}^{2}} \leq t) \\ &= \Pr(\|x - \mu_{1}\|_{2} \leq \hat{\sigma}_{1}\sqrt{t} + \|\mu_{1} - \hat{\mu}_{1}\|) \\ &= \Pr(\chi_{n}^{2}(\lambda) \leq \frac{(\hat{\sigma}_{1}\sqrt{t} + \|\mu_{1} - \hat{\mu}_{1}\|)^{2}}{\sigma_{1}^{2}}) \\ &\leq \bar{n} \end{aligned}$$

thus we can choose  $\bar{p}$  such that  $\beta \frac{m_1}{m_2} > \bar{p}$ . By the Chernoff-like bound,

$$\Pr(\frac{S_{m_2}(\bar{p})}{m_2} \ge \frac{m_1\beta}{m_2}) \le \exp(-m_2 \frac{[\beta \frac{m_1}{m_2} - \exp(-(\frac{n+\lambda-x}{2\sqrt{n+2\lambda}})^2]^2}{\beta \frac{m_1}{m_2}(1-\beta \frac{m_1}{m_2})}),$$

when

$$\beta \frac{m_1}{m_2} \ge 2 \exp\left(-\left(\frac{n+\lambda-x}{2\sqrt{n+2\lambda}}\right)^2.$$
 (20)

where  $x = (\frac{\hat{\sigma}_1 \sqrt{t} + \|\mu_1 - \hat{\mu}_1\|_2}{\sigma_1})^2$ . Then it suffices to show that

$$\left(\frac{\hat{\sigma}_{1}\sqrt{t} + \|\mu_{1} - \hat{\mu}_{1}\|_{2}}{\sigma_{1}}\right)^{2} \le n + \lambda - 2\sqrt{(n + 2\lambda)\log\frac{m_{2}}{2m_{1}\beta}}.$$
(21)

If it holds, we have

$$\Pr(l(X^{G_1})_{m_1(1-\beta)-m_3:m_1} \ge l(X^{G_2})_{m_1\beta:m_2}) \sim O(\exp(-m_1\beta))$$

Therefore, let  $\beta = \frac{m_2}{m_1} \eta$ , then

$$\beta + \frac{m_3}{m_1} + \frac{(m_2 - m_1 \beta)^2}{2m_1} \Pr(l(X^{G_1})_{m_1(1-\beta) - m_3: m_1} \ge l(X^{G_2})_{m_1 \beta: m_2})$$

$$= \frac{m_2}{m_1} \eta + \frac{m_3}{m_1} + c \frac{m_2}{m_1} \exp(-m_2 \eta + \log m_2),$$

and as long as  $\eta > \frac{\log m_2}{m_2}$ , we have

$$\frac{m_2}{m_1}\eta + \frac{m_3}{m_1} + c\frac{m_2}{m_1}\exp(-m_2\eta + \log m_2) = o(\frac{m_2}{m_1}).$$

This complete the proof for the upper bound of  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_1)$ . To show that such choices of  $\bar{p}, p$  exist, combining (19) (21), we have

$$n + \lambda - 2\sqrt{(n+2\lambda)\log\frac{m_2}{2m_1\beta}} \ge (\sqrt{b} + 2\sqrt{c})^2,$$

$$b = \frac{\left(\sqrt{n} + 2\sqrt{\log(\frac{\beta}{4} + \frac{m_3}{4m_1})^{-1}}\right)^2 + n}{2},$$

$$c = \frac{\|\mu_1 - \hat{\mu}_1\|_2^2}{\sigma_1^2},$$
(22)

which yields,

$$\lambda \ge 4\log\frac{2}{\eta} + 2\log\frac{1}{\delta} + 2\sqrt{n\log\frac{1}{\delta}} + 2\sqrt{\log\frac{1}{\eta}}\sqrt{4\log\frac{1}{\eta} + 8\log\frac{1}{\delta}} + 4\sqrt{n\log\frac{1}{\delta}} + n + 2\Delta + \Delta,$$

$$\Delta = 4c + \sqrt{c(2n + 4\sqrt{n\log\frac{1}{\delta}} + 4\log\frac{1}{\delta})} = c_0'w_2\sqrt{(n + \log\frac{1}{\delta})\log n},$$

$$\delta = \frac{\beta}{4} + \frac{m_3}{4m_1}.$$

One could prove that for  $\eta > w_3$ ,  $\delta = O(\eta)$ . After some simplification, we get

$$\lambda \ge c_1' \log \frac{1}{\delta} + c_2' \sqrt{n \log \frac{1}{\delta}} + c_3' \sqrt{\log \frac{1}{\eta} (\log \frac{1}{\delta} + n)},$$

$$\delta = \frac{\beta}{4} + \frac{m_3}{4m_1},$$
(23)

where  $c_1' = 6, c_2' = 2, c_3' = 8 + 4c_0'$ . If  $n = o(\log \frac{1}{\delta})$ , then

$$\lambda \ge (c_1' + c_2' + c_3') \log \frac{1}{\delta}. \tag{24}$$

Similarly we can find a lower bound for  $\Pr(l(w) \ge l(X)_{(m_1)} | w \in G_2) \ge 1/2$ .

Hence, when  $m \ge \frac{2\log\frac{1}{\eta}}{w_0\eta^2}$ , we have  $\exp(-\frac{m(w_2\eta)^2}{2w_2}) \le \eta$ , which bounds the probability of Bernoulli trials,

$$\Pr(|\frac{m_2}{m} - w_2| \ge w_2 \eta) \le 2 \exp(-\frac{mw_2 \eta^2}{2}),$$

Similarly, with  $m \ge \frac{2\log\frac{1}{\eta}}{w_3}$ , we can bound the probability of Bernoulli trials,

$$\Pr(\left|\frac{m_3}{m} - w_3\right| \ge w_3) \le 2\exp(-\frac{mw_3}{2}).$$

Hence, by union bounds,  $\Pr(w \in G_1 | l(w) \ge l(X)_{(m_1)}) \le 32\eta$ . Also,

$$\begin{split} m &= \Omega(\frac{\frac{n}{w_2}(\log n + \log \frac{1}{\varepsilon})\log n}{\varepsilon^2}), \\ m &\geq \frac{2\log \frac{1}{\varepsilon + \frac{w_3}{w_2}}}{w_2(\varepsilon + \frac{w_3}{w_2})^2}, \\ m &\geq \frac{2\log \frac{1}{\varepsilon + \frac{w_2}{w_3}}}{w_2}. \end{split}$$

Hence,

$$m = \Omega\left(\frac{n\frac{1}{w_2}(\log n + \log\frac{1}{\varepsilon})\log n}{\varepsilon^2} + \frac{\frac{1}{w_2}\log\left(\frac{1}{\varepsilon} + \frac{w_2}{w_3}\right)}{\varepsilon^2 + \left(\frac{w_3}{w_3}\right)^2} + \frac{\log\left(\frac{1}{\varepsilon} + \frac{w_2}{w_3}\right)}{w_3}\right).$$

# D Proof of Proposition 4.1

Assume that  $w'_1 > w_1$ . We are interested in the following sensitivity of the filter step to misestimated  $w_i$  on the input.

$$\begin{split} &|\Pr(w \in G_1|l(w) \geq l(X)_{(mw'_1)}) - \Pr(w \in G_1|l(w) \geq l(X)_{(mw_1)})| \\ &= |\frac{\Pr(l(w) \geq l(X)_{(mw'_1)}|w \in G_1) \Pr(w \in G_1)}{\sum_{k=1}^2 \Pr(l(w) \geq l(X)_{(mw'_1)}|w \in G_k) \Pr(w \in G_k) + \Pr(l(w) \geq l(X)_{(mw'_1)}|w \in N) \Pr(w \in N)} \\ &- \frac{\Pr(l(w) \geq l(X)_{(mw_1)}|w \in G_k) \Pr(w \in G_k) + \Pr(l(w) \geq l(X)_{(mw_1)}|w \in N) \Pr(w \in N)}{\sum_{k=1}^2 \Pr(l(w) \geq l(X)_{(mw_1)}|w \in G_k) \Pr(l(w) \geq l(X)_{(mw_1)}|w \in N) \Pr(w \in N)}| \\ &\leq |f(\Pr(l(w) \geq l(X)_{(mw_1)}|w \in G_1), \Pr(l(w) \geq l(X)_{(mw_1)}|w \in G_2)) \\ &- f(\Pr(l(w) \geq l(X)_{(mw'_1)}|w \in G_1), \Pr(l(w) \geq l(X)_{(mw'_1)}|w \in G_2))| \\ &+ |g(\Pr(l(w) \geq l(X)_{(mw'_1)}|w \in G_1), \Pr(l(w) \geq l(X)_{(mw'_1)}|w \in G_2), \Pr(l(w) \geq l(X)_{(mw_1)}|w \in N)) \\ &- g(\Pr(l(w) \geq l(X)_{(mw'_1)}|w \in G_1), \Pr(l(w) \geq l(X)_{(mw'_1)}|w \in G_2), \Pr(l(w) \geq l(X)_{(mw'_1)}|w \in N))| \\ &\leq \max \left|\frac{\partial f}{\partial x}(x,y)||\Pr(l(w) > l(X)_{(mw'_1)}|w \in G_1) - \Pr(l(w) > l(X)_{(mw_1)}|w \in G_1)| \\ &+ \max \left|\frac{\partial f}{\partial y}(x,y,z)||\Pr(l(w) > l(X)_{(mw'_1)}|w \in G_2) - \Pr(l(w) > l(X)_{(mw_1)}|w \in G_2)| \\ &+ \max \left|\frac{\partial g}{\partial x}(x,y,z)||\Pr(l(w) > l(X)_{(mw'_1)}|w \in G_2) - \Pr(l(w) > l(X)_{(mw_1)}|w \in G_2)| \\ &+ 2\max \left|\frac{\partial g}{\partial z}(x,y,z)||\Pr(l(w) > l(X)_{(mw'_1)}|w \in G_2) - \Pr(l(w) > l(X)_{(mw_1)}|w \in G_2)| \\ &+ 2\max \left|\frac{\partial g}{\partial z}(x,y,z)| \\ &\leq \frac{2w_1}{w_2}|\Pr(l(w) > l(X)_{(mw'_1)}|w \in G_1) - \Pr(l(w) > l(X)_{(mw_1)}|w \in G_2)| \\ &+ \frac{4w_1w_3}{w_2}|\Pr(l(w) > l(X)_{(mw'_1)}|w \in G_1) - \Pr(l(w) > l(X)_{(mw_1)}|w \in G_1)| \\ &+ \frac{16w_1w_3\eta}{w_2}|\Pr(l(w) > l(X)_{(mw'_1)}|w \in G_2) - \Pr(l(w) > l(X)_{(mw_1)}|w \in G_2)| \\ &+ \frac{8w_1w_3\eta}{w_2}, \end{aligned}$$

where

$$f(x,y) = \frac{w_1 x}{w_1 x + w_2 y}.$$

$$g(x,y,z) = \frac{w_1 w_3 xz}{(w_1 x + w_2 y)(w_1 x + w_2 y + w_3 z)}.$$

The third inequality is due to the intermediate value theorem.

Now the problem reduces to comparing

(a) 
$$\Pr(l(w) > l(X)_{(mw_1)} | w \in G_1)$$
 and  $\Pr(l(w) > l(X)_{(mw_1)} | w \in G_1)$ .

(b) 
$$\Pr(l(w) > l(X)_{(mw')} | w \in G_2)$$
 and  $\Pr(l(w) > l(X)_{(mw_1)} | w \in G_2)$ .

For (a),

$$\begin{split} &\Pr(l(X)_{(mw_1')} > l(w) > l(X)_{(mw_1)} | w \in G_1) \\ &\leq \Pr(l(X)_{(mw_1')} > l(w) > l(X)_{(mw_1)}, l(w) \leq l(X^{G_1})_{(m_1(1-\tau))} | w \in G_1) + \tau \\ &\leq \Pr(l(w) > l(X^{G_1 \cup G_2})_{(mw_1 - m_3)}, l(w) \leq l(X^{G_1})_{(m_1(1-\tau))} | w \in G_1) \\ &- \Pr(l(w) > l(X^{G_1 \cup G_2})_{(mw_1')}, l(w) \leq l(X^{G_1})_{(m_1(1-\tau))} | w \in G_1) \\ &\leq \sum_{n_1 + n_2 = mw_1 - m_3} \Pr(l(w) > l(X^{G_1})_{(n_1)}, l(w) \leq l(X^{G_2})_{(n_2)}, l(w) \leq l(X^{G_1})_{(m_1(1-\tau))} | w \in G_1) \\ &- \sum_{n'_1 + n'_2 = mw'_1} \Pr(l(w) > l(X^{G_1})_{(n'_1)}, l(w) \leq l(X^{G_1})_{(m_1(1-\tau))}, l(w) > l(X^{G_2})_{(n'_2)} | w \in G_1) + 2\tau \\ &\leq \sum_{n_1 + n_2 = mw_1 - m_3} \sum_{j = n_1}^{m_1(1-\tau)} \frac{1}{m_1} \Pr(l(X^{G_1})_{(j)} > l(X^{G_2})_{(n_2)}) - \sum_{n'_1 + n'_2 = mw'_1} \sum_{q = n'_1}^{m_1(1-\tau)} \frac{1}{m_1} \Pr(l(X^{G_1})_{(q)} > l(X^{G_2})_{(m'_1 - n'_1)}) + 2\tau \\ &\leq \sum_{n_1 = mw_1 - m_3} \sum_{j = n_1}^{m_1(1-\tau)} \frac{1}{m_1} \Pr(l(X^{G_1})_{(j)} > l(X^{G_2})_{(mw_1 - m_3 - n_1)}) - \sum_{n'_1 = mw'_1 - m_2}^{mw_1} \sum_{j = n'_1}^{m_1(1-\tau)} \frac{1}{m_1} \Pr(l(X^{G_1})_{(j)} > l(X^{G_2})_{(mw'_1 - n'_1)}) + 2\tau. \end{split}$$

Let  $n'_1 = l_1 + m(w'_1 - w_1) + m_3$ , then it is equivalent to

$$\begin{split} &\sum_{n_1=mw_1-m_3-m_2}^{(mw_1-m_3)} \sum_{j=n_1}^{m_1(1-\tau)} \frac{1}{m_1} \Pr(l(X^{G_1})_{(j)} > l(X^{G_2})_{(mw_1-m_3-n_1)}) \\ &- \sum_{l_1=mw_1-m_2-m_3}^{mw_1-m(w_1'-w_1)-m_3} \sum_{j=l_1+m(w_1'-w_1)+m_3}^{m_1(1-\tau)} \frac{1}{m_1} \Pr(l(X^{G_1})_{(j)} > l(X^{G_2})_{(mw_1-m_3-l_1)}) + 2\tau \\ &= \sum_{i=mw_1-m_3-m_1(1-\tau)}^{m(w_1'-w_1)} \sum_{j=mw_1-m_3-i}^{m_1(1-\tau)} \frac{1}{m_1} \Pr(l(X^{G_1})_{(j)} > l(X^{G_2})_{(i)}) + \sum_{i=mw_1'-m_1(1-\tau)}^{m_2} \sum_{j=mw_1-m_3-i}^{mw_1'-i} \frac{1}{m_1} \Pr(l(X^{G_1})_{(j)} > l(X^{G_2})_{(i)}) + 2\tau \\ &\leq \frac{[m_1(1-\tau)-mw_1+m_3+m(w_1'-w_1)]^2}{2m_1} \Pr(l(X^{G_1})_{(m_1(1-\tau))} > l(X^{G_2})_{mw_1-m_3-m_1(1-\tau)}) \\ &+ \frac{[m(w_1'-w_1)+m_3][m_2+m_1(1-\tau)-mw_1']}{m_1} \Pr(l(X^{G_1})_{(m_1(1-\tau))} > l(X^{G_2})_{(mw_1'-m_1(1-\tau))}) + 2\tau \\ &\leq c_5 \frac{[m(w_1'-w_1)+m_3-m_1\tau]^2}{2m_1} \Pr(l(X^{G_1})_{(m_1(1-\tau))} > l(X^{G_2})_{(m_1\tau-m_3)}) \\ &+ \frac{[m(w_1'-w_1)+m_3][m_2-m_1\tau-m(w_1'-w_1)]}{m_1} \Pr(l(X^{G_1})_{(m_1(1-\tau))} > l(X^{G_2})_{(m(w_1'-w_1)+m_1\tau)}) + 2\tau \\ &= \frac{[m(w_1'-w_1)+m_3-m_1\tau]^2}{2m_1} c_5 \exp(-m_1\tau) + \frac{[m(w_1'-w_1)+m_3][m_2-m_1\tau-m(w_1'-w_1)]}{m_1} c_5 \exp(-m_1\tau) + 2\tau, \end{split}$$

under Lemma 3.5. Let  $\tau = \min\{\frac{w_3}{w_1} + \frac{w_1' - w_1}{w_1}, \frac{w_2}{w_1} - \frac{w_1' - w_1}{w_1}\}$ , we have

$$\begin{split} &\Pr(l(X)_{(mw'_1)} > l(w) > l(X)_{(mw_1)} | w \in G_1) \\ &= \Pr(l(w) > l(X)_{(mw_1)} | w \in G_1) - \Pr(l(w) > l(X)_{(mw'_1)} | w \in G_1) \\ &\leq c_5 m_1 \tau \exp(-m_1 \tau) (\frac{w_2}{w_1} - \frac{w'_1 - w_1}{w_1} - \tau) + c_5 m_1 \exp(-m_1 \tau) (\frac{w_3}{w_1} + \frac{w'_1 - w_1}{w_1} - \tau)^2 + 2\tau \\ &\leq c_5 m_1 \exp(-m_1 \tau) \frac{w_2 + w_3}{w_1} (\frac{w_3}{w_1} + \frac{w'_1 - w_1}{w_1}) + 2 (\frac{w_3}{w_1} + \frac{w'_1 - w_1}{w_1}) \\ &\leq c_6 (\frac{w_3}{w_1} + \frac{w'_1 - w_1}{w_1}), \end{split}$$

for some  $c_6 > 0$ .

Similarly, the following argues for the case when  $w \in G_2$ .

$$\begin{split} &\Pr(I(X)_{(mn'_1)} > I(w) > I(X)_{(mnn)}|w \in G_2) \\ &= \Pr(I(X)_{mn'_1} > I(w), |w \in G_2) - \Pr(I(X)_{(mnn)}|w \in G_2) \\ &\leq \Pr(I(X)_{(mn'_1)} > I(w), I(w) \geq I(X^{G_2})_{(mn'_1 - m_1(1 - \tau))}|w \in G_2) \\ &= \Pr(I(X)_{(mn'_1)} > I(w), I(w) \geq I(X^{G_2})_{(mn'_1 - m_1(1 - \tau))}|w \in G_2) \\ &+ \frac{mn'_1 - m_1}{m_2} = \frac{m_1 + m_1 + m_1 - m_1}{m_2} \\ &\leq \sum_{\substack{n_1 + n'_1 = m_1, n_2 = n_1 \neq n_1 = m_1, n_2 \geq n_1 \neq n_1 = n_1}} \Pr(I(w) < I(X^{G_2})_{(m_1 - n'_1)}, I(w) < I(X^{G_2})_{(m_2 - n'_2)}, I(w) > I(X^{G_2})_{(mn'_1 - m_1(1 - \tau))}|w \in G_2) \\ &- \sum_{\substack{n_1 + n_2 = m_1, n_2 \geq n_1 \neq n_1 = n_1, n_2 \geq n_1 \neq n_1 = n_1}} \Pr(I(w) < I(X^{G_1})_{(m_1 - n'_1)}, I(w) < I(X^{G_2})_{(m_2 - n'_2)}, I(w) \geq I(X^{G_2})_{(m_1 + mn'_1 - m_1, 1 - \tau)}|w \in G_2) \\ &+ \frac{mn'_1 - m_1(1 - \tau)}{m_2} + \frac{mnu_1 - m_1(1 - \tau) - m_1}{m_2} \\ &= \sum_{\substack{n'_1 = n_1 \neq n_1 = n_1, n_2 \neq n_1 = n_2, n_2 \neq n_2, n_2$$

Therefore,

$$\begin{split} &|\operatorname{Pr}(w \in G_1|l(w) \geq l(X)_{(mw'_1)}) - \operatorname{Pr}(w \in G_1|l(w) \geq l(X)_{(mw_1)})| \\ \leq & \frac{2w_1}{w_2} |\operatorname{Pr}(l(w) > l(X)_{(mw'_1)}|w \in G_1) - \operatorname{Pr}(l(w) > l(X)_{(mw_1)}|w \in G_1)| \\ &+ \frac{2w_1\eta}{w_2} |\operatorname{Pr}(l(w) > l(X)_{(mw'_1)}|w \in G_2) - \operatorname{Pr}(l(w) > l(X)_{(mw_1)}|w \in G_2)| \\ &+ \frac{4w_1w_3}{w_2^2} |\operatorname{Pr}(l(w) > l(X)_{(mw'_1)}|w \in G_1) - \operatorname{Pr}(l(w) > l(X)_{(mw_1)}|w \in G_1)| \\ &+ \frac{16w_1w_3\eta}{w_2} |\operatorname{Pr}(l(w) > l(X)_{(mw'_1)}|w \in G_2) - \operatorname{Pr}(l(w) > l(X)_{(mw_1)}|w \in G_2)| \\ &+ \frac{8w_1w_3\eta}{w_2} \\ \leq &(\frac{2w_1}{w_2} + \frac{4w_1w_3}{w_2^2}) |\operatorname{Pr}(l(w) > l(X)_{(mw'_1)}|w \in G_1) - \operatorname{Pr}(l(w) > l(X)_{(mw_1)}|w \in G_1)| \\ &+ (\frac{2w_1\eta}{w_2} + \frac{16w_1w_3\eta}{w_2}) |\operatorname{Pr}(l(w) > l(X)_{(mw'_1)}|w \in G_2) - \operatorname{Pr}(l(w) > l(X)_{(mw_1)}|w \in G_2)| + \frac{8w_1w_3\eta}{w_2} \\ \leq &c_9 \frac{w'_1 - w_1 + w_3}{w_2}, \end{split}$$

for some  $c_9 > 0$ .

Hence, when reporting  $w'_1$ , the malicious noise of weight accounts for at most,

$$\eta' = \eta + c_9 \frac{w_1' - w_1 + w_3}{w_2},$$

and after the agnostic recovery of the second mean,  $\|\hat{\mu}_2(w_1') - \mu_2\|$  is bounded by

$$O(\eta + c_9 \frac{w_1' - w_1 + w_3}{w_2} + \frac{w_3}{1 - w_1'} + \varepsilon) \sigma \sqrt{\log n}),$$

given the same condition of sampling complexity regarding  $\eta$ . Here  $\frac{w_3}{1-w_1'}$  is the upper bound for the weight of malicious noise in the new input in the second run. In the spherical case,  $\|\hat{\mu}_2(w_1) - \mu_2\|$  is bounded by  $O(\eta + \frac{w_3}{w_2} + \varepsilon)\sigma\sqrt{\log n}$ . Hence,  $\|\hat{\mu}_2(w_1') - \hat{\mu}_2(w_1)\|$  is bounded by

$$O(\frac{w_1' - w_1 + w_3}{w_2} + \frac{w_3}{1 - w_1'} + \varepsilon)\sigma\sqrt{\log n}).$$

Similarly, one could prove the same for the case  $w'_1 < w_1$ . Therefore,

$$\|\hat{\mu}_2(w_1') - \hat{\mu}_2(w_1)\| = O(\frac{|w_1' - w_1| + w_3}{w_2} + \frac{w_3}{1 - w_1'} + \varepsilon)\sigma\sqrt{\log n}).$$