# One-Shot Variable-Length Secret Key Agreement Approaching Mutual Information 

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#### Abstract

This paper studies an information-theoretic one-shot variable-length secret key agreement problem with public discussion. Let $X$ and $Y$ be jointly distributed random variables, each taking values in some measurable space. Alice and Bob observe $X$ and $Y$ respectively, can communicate interactively through a public noiseless channel, and want to agree on a key length and a key that is approximately uniformly distributed over all bit sequences with the agreed key length. The public discussion is observed by an eavesdropper, Eve. The key should be approximately independent of the public discussion, conditional on the key length. We show that the optimal expected key length is close to the mutual information $I(X ; Y)$ within a logarithmic gap. Moreover, an upper bound and a lower bound on the optimal expected key length can be written down in terms of $I(X ; Y)$ only. This means that the optimal one-shot performance is always within a small gap of the optimal asymptotic performance regardless of the distribution of the pair $(X, Y)$. This one-shot result may find applications in situations where the components of an i.i.d. pair source $\left(X^{n}, Y^{n}\right)$ are observed sequentially and the key is output bit by bit with small delay, or in situations where the random source is not an i.i.d. or ergodic process.


## I. Introduction

The information theoretic secret key agreement problem in the source model has been considered in the asymptotic regime with fixed-length keys by Maurer [1] and Ahlswede and Csiszár [2] (see also [3], [4], [5], [6]). The model is that Alice and Bob observe the components of an i.i.d. pair source $\left(X^{n}, Y^{n}\right)$, where $\left(X_{i}, Y_{i}\right)$ are i.i.d. and $n$ tends to infinity. Alice and Bob want to agree on a secret key with length $n R$ bits using interactive noiseless public discussion, such that an eavesdropper, Eve, observing the public discussion, asymptotically gets zero information about the key. The optimal key rate is given by the mutual information $I(X ; Y)$ [1], [2]. These previous works also consider the case where Eve observes side information $Z^{n}$, but we will not consider side information at Eve in this paper.

An asymptotic model of this sort is not relevant in settings where the number of samples of the pair source available is limited, or its statistics do not follow i.i.d. or other structures that allow asymptotic analysis. This motivates us to consider a general one-shot setting where Alice and Bob observe the components $X$ and $Y$ respectively of a jointly distributed pair of random variables $(X, Y)$, where each component takes values in some measurable space. They wish to agree on the longest secret key possible using noiseless public discussion, which is also observed by Eve. For example, in an Internet of Things (IoT) deployment, one can envision improving the security of communication by generating secret keys in real time using the techniques in this paper. The jointly distributed random variables accessible to the two IoT devices creating such a secret key could be the result of noise associated to prior transmissions in the network or generated deliberately. For a concrete scenario of this kind, if Alice and Bob have earlier received a broadcast message from Charles, they could use the noise from their respective receptions to then create a secret key for private communication with each other. If sufficient resources are available, the broadcast from Charles might even be simply aimed at generating such dependent randomness at Alice and Bob. The important constraint, in contrast to asymptotic approaches, is the real time nature required of secret key generation, since IoT applications are typically delay-constrained.

Previous one-shot secrecy results usually work with a fixed-length setting (e.g. [7], [8], [9]). Here we argue that a variablelength setting is more suitable, for a similar reason as why a variable-length code is more suitable than a fixed-length code for one-shot compression. Each value of $X$ and $Y$ may contain a different amount of information. For example, let the pair $(X, Y)$ be received signals when the same Gaussian signal is sent through an additive Gaussian noise broadcast channel. If $X$ is large, then the amplitude of the signal and the signal-to-noise ratio is likely to be large, allowing Alice and Bob to agree on a longer key. Unlike the asymptotic setting, we cannot invoke the law of large numbers to argue that the amount of usable information is close to the average. If we require the key length to be fixed, we have to reduce the key length to accommodate the worse values of $X$ and $Y$, leading to a waste of information. To make use of all the usable information in $X$ and $Y$, it is natural to consider a variable-length setting where Alice and Bob agree on the length of the secret key and this length can adapt to the values of $X$ and $Y$. We also note that a universal finite blocklength variable-length key agreement problem has been studied in [10]. Nevertheless, the variability of key length in their setting comes from the unknown distribution or type of the source sequence, whereas the variability in this paper comes from the one-shot nature of our setting and the variability of the values of $X, Y$ (or their information density).

In this paper, we show that the optimal expected key length for one-shot variable-length secret key agreement with public discussion is close to $I(X ; Y)$ within a logarithmic gap. An upper bound and a lower bound on the optimal expected key length
can be stated in terms of $I(X ; Y)$ only, meaning that the optimal one-shot expected key length is always within a small gap of the optimal asymptotic rate $I(X ; Y)$, regardless of the distribution of $(X, Y)$. Such a result is impossible for fixed-length keys.

In Section [I, we precisely formulate the one-shot variable-length secret key agreement setting. In Section III we establish upper and lower bounds on the optimal expected key length in terms of $I(X ; Y)$. In Section IV] we show that variable-length keys can be concatenated to form a fixed-length key in the asymptotic regime. In Section $\mathbb{\square}$, we show that a variable-length key can be applied in situations where a task is to be performed multiple times.

## Notation

Throughout this paper, we assume that $\log$ is to base 2 and the entropy $H$ is in bits. The binary entropy function is denoted by $H_{\mathrm{b}}(p)=-p \log p-(1-p) \log (1-p)$, for $p \in[0,1]$. We write $\mathbb{Z}$ for the set of integers, $\mathbb{Z}_{\geq 0}$ for the set of nonnegative integers, and $\mathbb{Z}_{>0}$ for the set of positive integers. We use the notation: $X_{a}^{b}:=\left(X_{a}, \ldots, X_{b}\right), X^{n}:=X_{1}^{n}$ and $[a: b]:=[a, b] \cap \mathbb{Z}$, where $:=$ denotes equality by definition. We also use $\stackrel{\text { def }}{=}$ for equality by definition. We write $\alpha \bmod \beta$ for the remainder of $\alpha$ divided by $\beta>0$; this lies in the range $[0, \beta)$. We write $\{0,1\}^{*}$ for $\bigcup_{i=0}^{\infty}\{0,1\}^{i}$. The length of $w \in\{0,1\}^{*}$ is denoted as $|w|$. The concatenation of $w_{1}, w_{2} \in\{0,1\}^{*}$ is written as $w_{1} \| w_{2}$. For discrete $X$, we write the probability mass function as $p_{X}$. For continuous $X$, we write the probability density function as $f_{X}$. The Bernoulli distribution $p_{X}(0)=1-\alpha, p_{X}(1)=\alpha$ is denoted as $\operatorname{Bern}(\alpha)$. The uniform distribution over a finite set $S$ is denoted as $\operatorname{Unif}(S)$. The total variation distance between probability distributions on the same finite set is denoted as $d_{\mathrm{TV}}\left(p_{X}, q_{X}\right) \in[0,1]$ and equals $\frac{1}{2} \sum_{x}\left|p_{X}(x)-q_{X}(x)\right|$.

## II. Problem Formulation

Suppose Alice and Bob observe $X$ and $Y$ respectively, where $(X, Y)$ is jointly distributed, with each component taking values in some measurable space. Alice sends the finite random variable $W_{1}$ (the output of a stochastic mapping on $X$ ) to Bob. Bob sends the finite random variable $W_{2}$ (the output of a stochastic mapping on $\left(Y, W_{1}\right)$ ) to Alice, and so forth for $W_{3}, \ldots, W_{N}$ until they agree to stop at time $N \in \mathbb{Z}_{\geq 0}$ (which can be random). Here $N=0$ corresponds to the situation with no public discussion. The public discussion $W^{N}$ is also available to Eve. After public discussion, Alice and Bob agree on a key length $L \in \mathbb{Z}_{\geq 0}$ (i.e., $L$ is a deterministic function of $\left(X, W^{N}\right)$, and a deterministic function of $\left(Y, W^{N}\right)$ ). Alice produces $A \in\left[1: 2^{L}\right]$, and Bob produces $B \in\left[1: 2^{L}\right]$. We want $A=B$ with high probability, $A, B$ close to being uniform over $\left[1: 2^{L}\right]$ and close to being independent of Eve's observation $W^{N}$. This is captured by the following condition on the total variation distance

$$
\begin{equation*}
\sup _{l \in \mathbb{Z}_{\geq 0}} d_{\mathrm{TV}}\left(p_{A, B, W^{N} \mid L=l}, \mathrm{U}_{2}\left[1: 2^{l}\right] \times p_{W^{N} \mid L=l}\right) \leq \epsilon \tag{1}
\end{equation*}
$$

where we write $\mathrm{U}_{2}\left[1: 2^{l}\right]$ for the distribution $\operatorname{Unif}\left(\left\{(a, a): a \in\left[1: 2^{l}\right]\right\}\right)$. Here, on the left hand side of (1) the supremum is over $l \in \mathbb{Z}_{\geq 0}$ such that $P(L=l)>0$, and we call the left hand side the distance from the ideal distribution. It measures the distance between the actual distribution $p_{A, B, W^{N} \mid L=l}$ and the ideal distribution where the keys are equal, distributed uniformly over $\left[1: 2^{l}\right]$ and independent of $W^{N}$, and where we require the distance to be small for all $l$, not only averaged over $l$, so we can guarantee the quality of the key for any key length ${ }^{1}$ Define the maximal expected key length at distance $\epsilon$, written as $L_{\epsilon}^{*}(X ; Y)$, as the supremum of $\mathbb{E}[L]$ among all schemes satisfying (1).

To demonstrate the advantage of variable-length keys in one-shot settings, consider $X \sim \operatorname{Unif}\left[1: 2^{m}\right]$ independent of $Q \sim \operatorname{Bern}(7 / 8)$, and $Y=X$ if $Q=0$, otherwise $Y \mid\{X=x, Q=1\} \sim \operatorname{Unif}\left[1: 2^{m}\right]$. To generate a variable-length key, Alice and Bob can send the first $t$ bits of $X$ and $Y$ (containing $m$ bits) respectively through public discussion. If they match, output $L=m-t$ and let $A, B$ be the remaining bits of $X, Y$ respectively. Otherwise, output $L=0$. Then we can achieve $\epsilon=7 \cdot 2^{-t}$, and $\mathbb{E}[L] \geq(m-t) / 8$. On the other hand, for one-shot fixed-length schemes, any $A, B \in\{0,1\}$ generated by Alice and Bob respectively has

$$
\begin{aligned}
& d_{\mathrm{TV}}\left(p_{A, B, W^{N}}, \operatorname{Unif}\{(0,0),(1,1)\} \times p_{W^{N}}\right) \\
& \geq d_{\mathrm{TV}}\left(p_{A, B, W^{N} \mid Q=1}, \operatorname{Unif}\{(0,0),(1,1)\} \times p_{W^{N} \mid Q=1}\right)-2 / 8 \\
& =\sum_{w^{n}} p_{W^{N} \mid Q=1}\left(w^{n}\right) \sum_{a=0}^{1} \max \left\{\frac{1}{2}-p_{A, B \mid W^{N}=w^{n}, Q=1}(a, a), 0\right\}-\frac{1}{4} \\
& \stackrel{(a)}{=} \sum_{w^{n}}\left(p_{W^{N} \mid Q=1}\left(w^{n}\right)\right.
\end{aligned}
$$

[^0]\[

$$
\begin{aligned}
& \left.\cdot \sum_{a=0}^{1} \max \left\{\frac{1}{2}-p_{A \mid W^{N}=w^{n}, Q=1}(a) p_{B \mid W^{N}=w^{n}, Q=1}(a), 0\right\}\right)-\frac{1}{4} \\
& \geq \sum_{w^{n}} p_{W^{N} \mid Q=1}\left(w^{n}\right) \inf _{\alpha, \beta \in[0,1]}\left(\max \left\{\frac{1}{2}-\alpha \beta, 0\right\}\right. \\
& \left.\quad+\max \left\{\frac{1}{2}-(1-\alpha)(1-\beta), 0\right\}\right)-\frac{1}{4} \\
& =\sqrt{2}-1-1 / 4 \approx 0.16 .
\end{aligned}
$$
\]

Here (a) is because conditioned on $Q=1, X, Y$ are independent, and $I\left(X ; Y \mid W^{N}, Q=1\right) \leq I(X ; Y \mid Q=1)=0$ by Lemma 2.2 in [2] since $W^{N}$ is generated by public discussion, and hence $A-W^{N}-B$ forms a Markov chain conditioned on $Q=1$. Further, the last equality can be obtained by direct minimization. This means Alice and Bob cannot even generate 1 bit secret keys that are approximately $\operatorname{Bern}(1 / 2)$, approximately independent of $W^{N}$ and agree with high probability.

Moreover, for the case $X=Y$, the expected length of a variable-length key can be within a logarithmic gap from $H(X)$ (which can be observed in the entropy model defined in the next section). This is impossible in general for fixed-length keys due to the nonuniformity of information in $X$.

## III. Main Results

We present our main result, which is a bound on the gap between $L_{\epsilon}^{*}(X ; Y)$ and $I(X ; Y)$, which can be stated in terms of $\epsilon$ and $I(X ; Y)$ only.

Theorem 1. For any $X, Y$ and $0<\epsilon<1$, if $I(X ; Y)<\infty$, we have

$$
\begin{gathered}
L_{\epsilon}^{*}(X ; Y) \geq I(X ; Y)-3 \log (I(X ; Y)+1)-2 \log \frac{1}{\epsilon}-15 \\
L_{\epsilon}^{*}(X ; Y) \leq(1-\epsilon)^{-1}(I(X ; Y)+\log 3+1)
\end{gathered}
$$

If $I(X ; Y)=\infty$ and $\epsilon>0$, then $L_{\epsilon}^{*}(X ; Y)=\infty$.
The following corollary concerns the regimes $\epsilon=(I(X ; Y))^{-\lambda}$ and $\epsilon=2^{-\nu I(X ; Y)}$, when $I(X ; Y)<\infty$.
Corollary 1. For any $X, Y$, we have (write $I=I(X ; Y)$ and assume $I<\infty$ ):

1. If $\lambda \geq 1$ and $I \geq 2$, then

$$
I-(3+2 \lambda) \log (I+1)-15 \leq L_{I^{-\lambda}}^{*}(X ; Y) \leq I+8
$$

2. If $\nu>0$ and $I \geq \nu^{-1}$, then

$$
(1-2 \nu) I-3 \log (I+1)-15 \leq L_{2-\nu I}^{*}(X ; Y) \leq I+\nu^{-1}+6
$$

An implication of this corollary is that the performance of one-shot variable-length key agreement (i.e., $L_{\epsilon}^{*}(X ; Y)$ ), when $\epsilon=(I(X ; Y))^{-\lambda}$ and $\lambda$ is fixed, is always within a logarithmic gap from the performance of asymptotic key agreement (i.e., $I(X ; Y)$ ). For example, if the asymptotic key rate $500 \mathrm{bit} / \mathrm{symbol}$ is achievable, then we know that it is possible to generate a one-shot variable-length key with $\epsilon=1 / 500$ and expected length $\geq 440$ bits.

When applied to i.i.d. $\left(X^{n}, Y^{n}\right)$ for fixed $\epsilon$, the lower bound in Theorem 1 has a $O(\log n)$ gap from $n I(X ; Y)$. This is smaller than the $O(\sqrt{n})$ gap in [7] (for fixed-length keys) due to the inherent advantage of variable-length keys. A similar logarithmic gap also appears in one-shot variable-length channel simulation and source coding results [11], [12], [13], [14].

Also note that the multiplicative gap $(1-\epsilon)^{-1}$ in the upper bound in Theorem 1 is necessary. Consider the erasure source $X \sim \operatorname{Unif}\left[1: 2^{m}\right], Y=X$ with probability $1-\epsilon, Y=e$ with probability $\epsilon$. Then Alice can output $A=X$, and Bob can output $B=Y$ if $Y \neq e$, and output a random $B$ of length $m$ otherwise. The key length is $m$, which has a multiplicative gap from $I(X ; Y)=(1-\epsilon) m$.

Before we prove the main result, we introduce an abstract setting, the entropy model, as an approximation of the variablelength key model. While the entropy model does not have a concrete operational meaning, it is easier to analyze and is an important step in proving the main result.
Definition 1 (Entropy model). Alice and Bob observe $X$ and $Y$ respectively and engage in public discussion $W^{N}$ as in the variable-length key model. After public discussion, Alice and Bob generate $K_{A} \in \mathbb{Z}_{>0}$ and $K_{B} \in \mathbb{Z}_{>0}$ respectively as the "secret key" (instead of $A, B$ ). Here $K_{A}$ is a deterministic function of $\left(X, W^{N}\right)$ and $K_{B}$ is a deterministic function of $\left(Y, W^{N}\right)$. There are no independence requirements between $K_{A}, K_{B}$ and $W^{N}$. Define the maximal coinciding entropy, written as $\kappa(X ; Y)$, as the supremum of

$$
H_{=}\left(K_{A} ; K_{B} \mid W^{N}\right) \stackrel{\text { def }}{=} \mathbb{P}\left\{K_{A}=K_{B}\right\} H\left(K_{A} \mid W^{N}, K_{A}=K_{B}\right)
$$

over all schemes.
We first show that $L_{\epsilon}$ can be upper and lower bounded in terms of $\kappa$.
Lemma 1. For any $X, Y$ and $0<\epsilon<1$, if $\kappa(X ; Y)<\infty$, we have

$$
\begin{gathered}
L_{\epsilon}^{*}(X ; Y) \geq \kappa(X ; Y)-\log (\kappa(X ; Y)+1)-2 \log \frac{1}{\epsilon}-7.082 \\
L_{\epsilon}^{*}(X ; Y) \leq(1-\epsilon)^{-1}(\kappa(X ; Y)+\log 3)
\end{gathered}
$$

If $\kappa(X ; Y)=\infty$ and $\epsilon>0$, then $L_{\epsilon}^{*}(X ; Y)=\infty$.
Proof: We first consider the case $\kappa(X ; Y)<\infty$. We prove the lower bound and upper bound separately. For the lower bound, assume Alice and Bob have created $K_{A}, K_{B} \in \mathcal{K}$ respectively, where $\mathcal{K} \subset \mathbb{Z}_{>0}$ contains the range of $K_{A}$ and $K_{B}$. We may assume without loss of generality that $P\left(K_{A}=K_{B}\right)>0$, since otherwise the lower bound (with $H_{=}\left(K_{A} ; K_{B} \mid W^{N}\right)$ replacing $\kappa(X ; Y)$ ) is trivially true. We show how Alice and Bob can generate secret keys $A, B$ respectively using $K_{A}, K_{B}$ and further public discussion. The main idea is to partition $\mathcal{K}$ into subsets, and have Alice and Bob send which subset $K_{A}$ and $K_{B}$ are in. If the two subsets match, Alice and Bob output the indices of $K_{A}$ and $K_{B}$ within that subset. Otherwise they declare failure (output $L=0$ ). The purpose of the partition is twofold: to group $K_{A}$ 's and $K_{B}$ 's with similar probabilities into the same subset so the final key is close to being uniform conditioned on its length, and to detect errors $\left(K_{A} \neq K_{B}\right)$ by checking whether $K_{A}, K_{B}$ belong to the same subset. Errors are only penalized slightly in the entropy model (we simply do not count the entropy when $K_{A} \neq K_{B}$ ), but are controlled tightly in the key agreement setting to have a probability bounded by $\epsilon$ (though the probability of failure, i.e. $P(L=0$ ), is not bounded by $\epsilon$ ), and hence error detection is necessary. This technique is similar to spectrum slicing [15], [7], but here we perform the slicing or partition on the tentative keys $K_{A}, K_{B}$ at the last stage of the scheme, whereas in [7] the slicing is performed on $X$ at the first stage of the scheme.

For simplicity, we first assume that Alice and Bob have not used any public discussion yet (the general case will be addressed later). Fix $0<\delta \leq 1,0<\epsilon<1$. For $k \in \mathcal{K}$ such that $p_{K_{A} \mid K_{A}=K_{B}}(k)>0$, let $\ell(k):=\left\lfloor-\delta^{-1} \log p_{K_{A} \mid K_{A}=K_{B}}(k)\right\rfloor$. For $t$ in the range of $\ell(\cdot)$, let $\ell^{-1}(t):=\{k: \ell(k)=t\}$. Note that $\left|\ell^{-1}(t)\right| \geq 1$ for all such $t$. Let $\left|\ell^{-1}(t)\right|=\sum_{i=1}^{\beta_{t}} 2^{\alpha_{t, i}}$ be the binary representation of $\left|\ell^{-1}(t)\right|$ for such $t$, where the $\alpha_{t, i}$ 's are sorted in descending order along $i$. We partition $\ell^{-1}(t)$ by first selecting the $2^{\alpha_{t, 1}}$ elements $k \in \ell^{-1}(t)$ with the largest $p_{K_{A} \mid K_{A}=K_{B}}(k)$ and putting them in the first subset, then the next $2^{\alpha_{t, 2}} k$ 's and putting them in the second subset, and so on. Let $\left\{S_{i}\right\}$ be the collection of all these subsets (note that each of them has a size which is a power of 2 ) among the partitions of $\ell^{-1}(t)$ for $t$ in the range of $\ell(\cdot)$. For $k \in \mathcal{K}$ such that $p_{K_{A} \mid K_{A}=K_{B}}(k)>0$, let $c_{S}(k)$ be the index of the $S_{i}$ that contains $k$ (i.e., $k \in S_{c S}(k)$, and write $\ell\left(S_{i}\right)$ for $\ell(k)$ where $k \in S_{i}$ (all $k$ 's in $S_{i}$ have the same $\ell(k)$ ). By the construction of the partition, we have

$$
\begin{align*}
& \mathbb{E}\left[\log \left|S_{c_{S}\left(K_{A}\right)}\right| \mid K_{A}=K_{B}\right] \\
& \geq \mathbb{E}\left[\log \left|\ell^{-1}\left(\ell\left(K_{A}\right)\right)\right| \mid K_{A}=K_{B}\right]-2 \tag{2}
\end{align*}
$$

For each $S_{i}$, let

$$
\begin{gathered}
\rho_{i}=\mathbb{P}\left\{K_{A}=K_{B} \mid K_{A}, K_{B} \in S_{i}\right\}, \\
m_{i}=\max \left\{\left\lfloor\log \left(\epsilon \rho_{i}\left|S_{i}\right|\right)\right\rfloor, 0\right\} .
\end{gathered}
$$

Since $p_{K_{A} \mid K_{A}=K_{B}}(k)>0$ for each $k \in S_{i}$, we have $\rho_{i}>0$. Further partition $S_{i}$ into $\tilde{S}_{i, 1}, \ldots, \tilde{S}_{i, 2^{-m_{i}}\left|S_{i}\right|}$ each with size $2^{m_{i}}$. If we select the partition uniformly at random,

$$
\begin{aligned}
& \mathbb{P}\left\{K_{A} \neq K_{B}, \exists j: K_{A}, K_{B} \in \tilde{S}_{i, j} \mid K_{A}, K_{B} \in S_{i}\right\} \\
& =\sum_{k_{A} \neq k_{B} \in S_{i}} p_{K_{A} K_{B} \mid K_{A}, K_{B} \in S_{i}}\left(k_{A}, k_{B}\right) \mathbb{P}\left\{\exists j: k_{A}, k_{B} \in \tilde{S}_{i, j}\right\} \\
& =\sum_{k_{A} \neq k_{B} \in S_{i}} p_{K_{A} K_{B} \mid K_{A}, K_{B} \in S_{i}}\left(k_{A}, k_{B}\right) \cdot \frac{2^{m_{i}}-1}{\left|S_{i}\right|-1} \\
& \leq \epsilon \rho_{i}
\end{aligned}
$$

where the last line can be obtained by considering the 2 cases of $m_{i}$. Hence there exists a fixed partition $\tilde{S}_{i, 1}, \ldots, \tilde{S}_{i, 2^{m_{i}}}$ satisfying

$$
\begin{aligned}
& \mathbb{P}\left\{K_{A} \neq K_{B} \mid \exists j: K_{A}, K_{B} \in \tilde{S}_{i, j}\right\} \\
& =\mathbb{P}\left\{K_{A} \neq K_{B} \wedge \exists j: K_{A}, K_{B} \in \tilde{S}_{i, j} \mid K_{A}, K_{B} \in S_{i}\right\} \\
& \\
& \quad / \mathbb{P}\left\{\exists j: K_{A}, K_{B} \in \tilde{S}_{i, j} \mid K_{A}, K_{B} \in S_{i}\right\}
\end{aligned}
$$

$$
\begin{equation*}
\leq \epsilon \rho_{i} / \rho_{i}=\epsilon \tag{3}
\end{equation*}
$$

Let $c_{\tilde{S}}(k)$ be the index $j$ of $\tilde{S}_{i, j}$ containing $k$ (i.e., $k \in \tilde{S}_{c S}(k), c_{\tilde{S}}(k)$. Like $c_{S}(k), c_{\tilde{S}}(k)$ is defined for $k \in \mathcal{K}$ such that $p_{K_{A} \mid K_{A}=K_{B}}(k)>0$.

If $c_{S}\left(K_{A}\right)$ and $c_{\tilde{S}}\left(K_{A}\right)$ are well defined, Alice sends $W_{1}=\left(c_{S}\left(K_{A}\right), c_{\tilde{S}}\left(K_{A}\right)\right)$ through public discussion; otherwise Alice sends a failure symbol. If $c_{S}\left(K_{B}\right)$ and $c_{\tilde{S}}\left(K_{B}\right)$ are well defined, Bob sends $W_{2}=\left(c_{S}\left(K_{B}\right), c_{\tilde{S}}\left(K_{B}\right)\right)$; otherwise Bob sends a failure symbol. Declare failure (output $L=0$ ) if either party sends a failure symbol. If $W_{1}=W_{2}$, Alice outputs the index $A$ of $K_{A}$ in $\tilde{S}_{c_{S}\left(K_{A}\right), c_{\tilde{S}}\left(K_{A}\right)}$ (containing $L=\log \left|\tilde{S}_{c_{S}\left(K_{A}\right), c_{\tilde{S}}\left(K_{A}\right)}\right|$ bits), and Bob outputs the index $B$ of $K_{B}$ in $\tilde{S}_{c_{S}\left(K_{B}\right), c_{\tilde{S}}\left(K_{B}\right)}$. Declare failure if $W_{1} \neq W_{2}$ (output $L=0$ ). We have

$$
\mathbb{E}[L] \geq \mathbb{E}\left[L \mathbf{1}\left\{K_{A}=K_{B}\right\}\right]=\mathbb{P}\left\{K_{A}=K_{B}\right\} \mathbb{E}\left[L \mid K_{A}=K_{B}\right]
$$

where

$$
\begin{align*}
& \mathbb{E}\left[L \mid K_{A}=K_{B}\right] \\
& =\mathbb{E}\left[m_{c_{S}\left(K_{A}\right)} \mid K_{A}=K_{B}\right] \\
& \geq \mathbb{E}\left[\left.\log \left|S_{c_{S}\left(K_{A}\right)}\right|-\log \frac{1}{\epsilon \rho_{c_{S}\left(K_{A}\right)}}-1 \right\rvert\, K_{A}=K_{B}\right] \\
& \geq \mathbb{E}\left[\log \left|\ell^{-1}\left(\ell\left(K_{A}\right)\right)\right| \mid K_{A}=K_{B}\right] \\
& \quad-\mathbb{E}\left[\left.\log \frac{1}{\rho_{c_{S}\left(K_{A}\right)}} \right\rvert\, K_{A}=K_{B}\right]-\log \frac{1}{\epsilon}-3 \tag{4}
\end{align*}
$$

where the last inequality is by (2). For the first term,

$$
\begin{aligned}
& \mathbb{E}\left[\log \left|\ell^{-1}\left(\ell\left(K_{A}\right)\right)\right| \mid K_{A}=K_{B}\right] \\
& =\sum_{t=0}^{\infty} \mathbb{P}\left\{\ell\left(K_{A}\right)=t \mid K_{A}=K_{B}\right\} \log \left|\ell^{-1}(t)\right| \\
& \stackrel{(a)}{\geq} \sum_{t=0}^{\infty} \mathbb{P}\left\{\ell\left(K_{A}\right)=t \mid K_{A}=K_{B}\right\} \log \frac{\mathbb{P}\left\{\ell\left(K_{A}\right)=t \mid K_{A}=K_{B}\right\}}{2^{-\delta t}} \\
& =\delta \mathbb{E}\left[\ell\left(K_{A}\right) \mid K_{A}=K_{B}\right]-H\left(\ell\left(K_{A}\right) \mid K_{A}=K_{B}\right) \\
& \stackrel{(b)}{\geq} \delta \mathbb{E}\left[\ell\left(K_{A}\right) \mid K_{A}=K_{B}\right]-\log e \\
& \quad-\log \left(\mathbb{E}\left[\ell\left(K_{A}\right) \mid K_{A}=K_{B}\right]+1\right) \\
& (c) \\
& \geq \delta \mathbb{E}\left[-\delta^{-1} \log p_{K_{A}}\left|K_{A}=K_{B}\left(K_{A}\right)-1\right| K_{A}=K_{B}\right]-\log e \\
& \quad-\log \left(\mathbb{E}\left[-\delta^{-1} \log p_{K_{A} \mid K_{A}=K_{B}}\left(K_{A}\right) \mid K_{A}=K_{B}\right]+1\right) \\
& =H\left(K_{A} \mid K_{A}=K_{B}\right)-\log \left(\delta^{-1} H\left(K_{A} \mid K_{A}=K_{B}\right)+1\right)-\delta-\log e \\
& (d) \\
& \geq H\left(K_{A} \mid K_{A}=K_{B}\right)-\log \left(H\left(K_{A} \mid K_{A}=K_{B}\right)+1\right) \\
& \quad-\log \frac{1}{\delta}-\delta-\log e,
\end{aligned}
$$

where (a) and (c) are because $2^{-\delta(i+1)} \leq p_{K_{A} \mid K_{A}=K_{B}}(k) \leq 2^{-\delta i}$ for all $k \in \ell^{-1}(i)$, (b) is due to $H(J) \leq \log (\mathbb{E}[J]+1)+$ $\mathbb{E}[J] \log (1+1 / \mathbb{E}[J]) \leq \log (\mathbb{E}[J]+1)+\log e$ for any random variable $J \in \mathbb{Z}_{\geq 0}$ since the geometric distribution maximizes the entropy for a given mean, and (d) is because $\delta \leq 1$. For the second term,

$$
\begin{aligned}
& \mathbb{E}\left[\left.\log \frac{1}{\rho_{c_{S}\left(K_{A}\right)}} \right\rvert\, K_{A}=K_{B}\right] \\
& =\sum_{i} \mathbb{P}\left\{c_{S}\left(K_{A}\right)=c_{S}\left(K_{B}\right)=i \mid K_{A}=K_{B}\right\} \log \frac{1}{\rho_{i}} \\
& =\sum_{i} \frac{\mathbb{P}\left\{c_{S}\left(K_{A}\right)=c_{S}\left(K_{B}\right)=i\right\}}{\mathbb{P}\left\{K_{A}=K_{B}\right\}} \rho_{i} \log \frac{1}{\rho_{i}} \\
& \leq \sum_{i} \frac{\mathbb{P}\left\{c_{S}\left(K_{A}\right)=c_{S}\left(K_{B}\right)=i\right\}}{\mathbb{P}\left\{K_{A}=K_{B}\right\}}\left(e^{-1} \log e\right) \\
& \leq \frac{e^{-1} \log e}{\mathbb{P}\left\{K_{A}=K_{B}\right\}}
\end{aligned}
$$

Substituting back to (4),

$$
\begin{align*}
& \mathbb{E}[L] \\
& \geq \mathbb{P}\left\{K_{A}=K_{B}\right\} \\
& \quad \cdot\left(H\left(K_{A} \mid K_{A}=K_{B}\right)-\log \left(H\left(K_{A} \mid K_{A}=K_{B}\right)+1\right)\right) \\
& \quad-\log \frac{1}{\delta}-\delta-\log \frac{1}{\epsilon}-e^{-1} \log e-\log e-3 \\
& \geq \mathbb{P}\left\{K_{A}=K_{B}\right\} H\left(K_{A} \mid K_{A}=K_{B}\right) \\
& \quad-\log \left(\mathbb{P}\left\{K_{A}=K_{B}\right\} H\left(K_{A} \mid K_{A}=K_{B}\right)+1\right) \\
& \quad-\log \frac{1}{\delta}-\delta-\log \frac{1}{\epsilon}-4.974 \tag{5}
\end{align*}
$$

Note that the length of the key is $L=1\left\{W_{1}=W_{2}\right\} m_{c_{S}\left(K_{A}\right)}$. Next we analyze the distribution of the key. Fix $i, j$. For any $a \in\left[1: 2^{m_{i}}\right]$,

$$
\begin{aligned}
& \mathbb{P}\left\{A=B=a \mid K_{A}, K_{B} \in \tilde{S}_{i, j}\right\} \\
& =\mathbb{P}\left\{K_{A}=K_{B} \mid K_{A}, K_{B} \in \tilde{S}_{i, j}\right\} \mathbb{P}\left\{A=a \mid K_{A}=K_{B} \in \tilde{S}_{i, j}\right\} \\
& \geq \mathbb{P}\left\{K_{A}=K_{B} \mid K_{A}, K_{B} \in \tilde{S}_{i, j}\right\} \frac{2^{-\delta\left(\ell\left(S_{i}\right)+1\right)}}{2^{-\delta \ell\left(S_{i}\right)} 2^{m_{i}}} \\
& =\mathbb{P}\left\{K_{A}=K_{B} \mid K_{A}, K_{B} \in \tilde{S}_{i, j}\right\} 2^{-m_{i}-\delta}
\end{aligned}
$$

since $2^{-\delta\left(\ell\left(S_{i}\right)+1\right)} \leq p_{K_{A} \mid K_{A}=K_{B}}(k) \leq 2^{-\delta \ell\left(S_{i}\right)}$ for all $k \in S_{i}$. Write $\mathrm{U}_{2}\left(\left[1: 2^{l}\right]\right)=\operatorname{Unif}\left(\left\{(a, a): a \in\left[1: 2^{l}\right]\right\}\right)$.

$$
\begin{aligned}
& d_{\mathrm{TV}}\left(p_{A, B \mid K_{A}, K_{B} \in \tilde{S}_{i, j}}, \mathrm{U}_{2}\left(\left[1: 2^{m_{i}}\right]\right)\right) \\
& =\sum_{a=1}^{2^{m_{i}}} \max \left\{2^{-m_{i}}-\mathbb{P}\left\{A=B=a \mid K_{A}, K_{B} \in \tilde{S}_{i, j}\right\}, 0\right\} \\
& \leq \sum_{a=1}^{2^{m_{i}}} \max \left\{2^{-m_{i}}-\mathbb{P}\left\{K_{A}=K_{B} \mid K_{A}, K_{B} \in \tilde{S}_{i, j}\right\} 2^{-m_{i}-\delta}, 0\right\} \\
& =1-2^{-\delta} \mathbb{P}\left\{K_{A}=K_{B} \mid K_{A}, K_{B} \in \tilde{S}_{i, j}\right\}
\end{aligned}
$$

Hence

$$
\begin{aligned}
& d_{\mathrm{TV}}\left(p_{A, B, W^{2} \mid W_{1}=W_{2}, c_{S}\left(K_{A}\right)=i},\right. \\
& \left.=\mathrm{U}_{2}\left(\left[1: 2^{m_{i}}\right]\right) \times p_{W^{2} \mid W_{1}=W_{2}, c_{S}\left(K_{A}\right)=i}\right) \\
& \quad \sum_{j=1}^{2^{-m_{i}}\left|S_{i}\right|}\left(p_{W^{2} \mid W_{1}=W_{2}, c_{S}\left(K_{A}\right)=i}((i, j),(i, j))\right. \\
& \left.\quad \cdot d_{\mathrm{TV}}\left(p_{A, B \mid W_{1}=W_{2}=(i, j)}, \mathrm{U}_{2}\left(\left[1: 2^{m_{i}}\right]\right)\right)\right) \\
& =\sum_{j=1}^{2^{-m_{i}}\left|S_{i}\right|}\left(\mathbb{P}\left\{K_{A} \in \tilde{S}_{i, j} \mid \exists j^{\prime}: K_{A}, K_{B} \in \tilde{S}_{i, j^{\prime}}\right\}\right. \\
& \leq \sum_{j=1}^{2^{-m_{i}}\left|S_{i}\right|}\left(\mathbb{P}\left\{K_{A} \in \tilde{S}_{i, j} \mid \exists j^{\prime}: K_{A}, K_{B} \in \tilde{S}_{i, j^{\prime}}\right\}\right. \\
& \left.\leq \cdot\left(1-2^{-\delta} \mathbb{P}\left\{K_{A}=K_{B} \mid K_{A}, K_{B} \in \tilde{S}_{i, j}\right\}\right)\right) \\
& =1-2^{-\delta} \sum_{j=1}^{2^{-m_{i}}\left|S_{i}\right|} \mathbb{P}\left\{K_{A}=K_{B} \in \tilde{S}_{i, j} \mid \exists j^{\prime}: K_{A}, K_{B} \in \tilde{S}_{i, j^{\prime}}\right\} \\
& \leq 1-2^{-\delta}(1-\epsilon),
\end{aligned}
$$

where the last inequality is by (3). For $l \geq 1$,

$$
\begin{aligned}
& d_{\mathrm{TV}}\left(p_{A, B, W^{2} \mid L=l}, \mathrm{U}_{2}\left(\left[1: 2^{l}\right]\right) \times p_{W^{2} \mid L=l}\right) \\
& =\sum_{i}\left(\mathbb{P}\left\{c_{S}\left(K_{A}\right)=i \mid W_{1}=W_{2}, m_{c_{S}\left(K_{A}\right)}=l\right\}\right. \\
& \quad \cdot d_{\mathrm{TV}}\left(p_{A, B, W^{2} \mid W_{1}=W_{2}, c_{S}\left(K_{A}\right)=i}\right. \\
& \left.\left.\quad \mathrm{U}_{2}\left(\left[1: 2^{m_{i}}\right]\right) \times p_{W^{2} \mid W_{1}=W_{2}, c_{S}\left(K_{A}\right)=i}\right)\right) \\
& \leq 1-2^{-\delta}(1-\epsilon) \\
& \leq 1-(1-\delta / \log e)(1-\epsilon) \\
& \leq \epsilon+\delta / \log e
\end{aligned}
$$

For any $0<\epsilon^{\prime}<1$, let $\epsilon=(3 / 5) \epsilon^{\prime}, \delta=(2 / 5) \epsilon^{\prime} \log e$, then $d_{\mathrm{TV}}\left(p_{A, B, W^{2} \mid L=l}, \mathrm{U}_{2}\left(\left[1: 2^{l}\right]\right) \times p_{W^{2} \mid L=l}\right) \leq \epsilon^{\prime}$, and by (5]),

$$
\begin{align*}
\mathbb{E}[L] \geq & \mathbb{P}\left\{K_{A}=K_{B}\right\} H\left(K_{A} \mid K_{A}=K_{B}\right) \\
& -\log \left(\mathbb{P}\left\{K_{A}=K_{B}\right\} H\left(K_{A} \mid K_{A}=K_{B}\right)+1\right) \\
& -2 \log \frac{1}{\epsilon^{\prime}}-7.082 \tag{6}
\end{align*}
$$

The case $\kappa(X ; Y)=\infty$ can be handled by considering a sequence of schemes with $H_{=}\left(K_{A} ; K_{B} \mid W^{N}\right)$ finite and tending to infinity.

For the case where Alice and Bob have already used some public discussion $W^{N}$ to generate $K_{A}, K_{B}$, we apply the same arguments for $p_{K_{A}, K_{B} \mid W^{N}=w^{n}}$ for each $w^{n}$. The additional public discussion is appended at the end of $W^{N}$ so the overall public discussion is $W^{N+2}$. We still have $d_{\mathrm{TV}}\left(p_{A, B, W^{N+2} \mid L=l}, \mathrm{U}_{2}\left(\left[1: 2^{l}\right]\right) \times p_{W^{N+2} \mid L=l}\right) \leq \epsilon^{\prime}$ by convexity of $d_{\mathrm{TV}}$. Further, we have

$$
\begin{aligned}
& \mathbb{E}_{w^{n} \sim p_{W^{N}}}\left[\mathbb{P}\left\{K_{A}=K_{B} \mid W^{N}=w^{n}\right\} H\left(K_{A} \mid W^{N}=w^{n}, K_{A}=K_{B}\right)\right] \\
& =\mathbb{P}\left\{K_{A}=K_{B}\right\} H\left(K_{A} \mid W^{N}, K_{A}=K_{B}\right)
\end{aligned}
$$

Therefore (6) still holds after replacing $H\left(K_{A} \mid K_{A}=K_{B}\right)$ with $H\left(K_{A} \mid W^{N}, K_{A}=K_{B}\right)$.

For the upper bound, assume Alice and Bob have $A$ and $B$ respectively. Let $C$ satisfy $C \mid\left\{L=l, W^{N}=w^{n}\right\} \sim \operatorname{Unif}\left[1: 2^{l}\right]$ for any $l, w^{n}$. By the coupling characterization of total variation distance, we have

$$
\begin{aligned}
& \mathbb{P}\{A=B=C \mid L=l\} \\
& =1-d_{\mathrm{TV}}\left(p_{A, B, W^{N} \mid L=l}, \mathrm{U}_{2}\left(\left[1: 2^{l}\right]\right) \times p_{W^{N} \mid L=l}\right) \geq 1-\epsilon
\end{aligned}
$$

We have

$$
\begin{aligned}
& \kappa(X ; Y) \\
& \geq \mathbb{P}\{A=B\} H\left(A \mid W^{N}, A=B\right) \\
& =H\left(A \mathbf{1}\{A=B\} \mid W^{N}, \mathbf{1}\{A=B\}\right) \\
& \geq H\left(A \mathbf{1}\{A=B\}, \mathbf{1}\{A=B=C\} \mid W^{N}\right) \\
& \quad-H(\mathbf{1}\{A=B=C\}, \mathbf{1}\{A=B\}) \\
& \geq H\left(C \mathbf{1}\{A=B=C\} \mid W^{N}\right)-\log 3 \\
& \geq \sum_{l=0}^{\infty} \mathbb{P}\{L=l\} H\left(C \mathbf{1}\{A=B=C\} \mid W^{N}, L=l\right)-\log 3 \\
& \stackrel{(a)}{\geq} \sum_{l=0}^{\infty} \mathbb{P}\{L=l\} \cdot l \cdot \mathbb{P}\{A=B=C \mid L=l\}-\log 3 \\
& \geq \\
& \geq(1-\epsilon) \mathbb{E}[L]-\log 3
\end{aligned}
$$

where (a) is because $C \mid\left\{L=l, W^{N}=w^{n}\right\} \sim \operatorname{Unif}\left[1: 2^{l}\right]$ and

$$
\begin{aligned}
& H\left(C \mathbf{1}\{A=B=C\} \mid W^{N}=w^{n}, L=l\right) \\
& \geq-\sum_{c \in\left[1: 2^{l}\right]}\left(\mathbb{P}\left\{C \mathbf{1}\{A=B=C\}=c \mid W^{N}=w^{n}, L=l\right\}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.\cdot \log \mathbb{P}\left\{C \mathbf{1}\{A=B=C\}=c \mid W^{N}=w^{n}, L=l\right\}\right) \\
\geq & -\sum_{c \in\left[1: 2^{l}\right]} \mathbb{P}\left(\left\{C \mathbf{1}\{A=B=C\}=c \mid W^{N}=w^{n}, L=l\right\}\right. \\
& \left.\cdot \log \mathbb{P}\left\{C=c \mid W^{N}=w^{n}, L=l\right\}\right) \\
= & \sum_{c \in\left[1: 2^{l}\right]} \mathbb{P}\left\{C \mathbf{1}\{A=B=C\}=c \mid W^{N}=w^{n}, L=l\right\} \cdot l \\
= & l \cdot \mathbb{P}\left\{A=B=C \mid W^{N}=w^{n}, \hat{L}=l\right\} .
\end{aligned}
$$

Since $L_{\epsilon}^{*}$ can be upper and lower bounded by $\kappa$, in order to prove Theorem 1 we can bound $\kappa$ instead. The following lemma bounds $\kappa$ in terms of $I(X ; Y)$.

Lemma 2. For any $X, Y$, if $I(X ; Y)<\infty$, we have

$$
\begin{gathered}
\kappa(X ; Y) \geq I(X ; Y)-2 \log (I(X ; Y)+1)-7.034 \\
\kappa(X ; Y) \leq I(X ; Y)+1
\end{gathered}
$$

If $I(X ; Y)=\infty$, then $\kappa(X ; Y)=\infty$.
Proof: Assume that $I(X ; Y)<\infty$. We first prove the upper bound.

$$
\begin{aligned}
I(X ; Y)= & I(X ; Y)-I\left(X ; Y \mid W^{N}\right)+I\left(X ; Y \mid W^{N}, K_{B}\right) \\
& +I\left(X ; K_{B} \mid W^{N}\right)-I\left(X ; K_{B} \mid W^{N}, Y\right) \\
& \stackrel{(a)}{\geq} I\left(X ; K_{B} \mid W^{N}\right) \\
& \stackrel{(b)}{\geq} I\left(K_{A} ; K_{B} \mid W^{N}\right) \\
\geq & I\left(K_{A} ; K_{B} \mid W^{N}, \mathbf{1}\left\{K_{A}=K_{B}\right\}\right)-1 \\
\geq & \mathbb{P}\left\{K_{A}=K_{B}\right\} H\left(K_{A} \mid W^{N}, K_{A}=K_{B}\right)-1
\end{aligned}
$$

where (a) is due to $I\left(X ; Y \mid W^{N}\right) \leq I(X ; Y)$ by Lemma 2.2 in [2] since $W^{N}$ is generated by interactive communication, and the Markov chain $X-\left(Y, W^{N}\right)-K_{B}$, and (b) is due to the Markov chain $K_{A}-\left(X, W^{N}\right)-K_{B}$.

We now prove the lower bound. The main idea is to transmit $X$ from Alice to Bob, who has side information $Y$, using interactive communication, and then use the part of $X$ not leaked by the interactive communication as the key. While this is similar to Slepian-Wolf coding [16] studied in a one-shot interactive setting in [17], [18], here we are concerned with the leakage of information by the interactive communication, not the amount of communication. Note that if we use the results in [17], [18], we obtain a gap on the order of $\sqrt{H(X \mid Y)}$ instead of $\log I(X ; Y)$, which is undesirable since $H(X \mid Y)$ can be much larger than $I(X ; Y)$, or even infinite.

We design a scheme for the entropy model as follows. First consider the case where $X \sim \operatorname{Unif}[0,1]$ and $Y$ is discrete and finite. The general case will be addressed later. Fix $m \in \mathbb{Z}_{>0}, 0<\epsilon<1 / 2$. Alice generates $\tilde{X}_{2}, \ldots, \tilde{X}_{2^{m}} \stackrel{i i d}{\sim}$ Unif $[0,1]$. Let $S_{1}:=\left\{X, \tilde{X}_{2}, \ldots, \tilde{X}_{2^{m}}\right\}$. At time $i$, Alice sends $S_{i}$ (as an unordered set, or equivalently a sorted sequence, of size $2^{m-i+1}$ ) through public discussion, then Bob finds $\hat{X}_{i}=\arg \max _{x \in S_{i}} f_{X \mid Y}(x \mid Y)$. If $f_{X \mid Y}\left(\hat{X}_{i} \mid Y\right) / \sum_{x \in S_{i}} f_{X \mid Y}(x \mid Y) \geq 1-\epsilon$, then Bob declares through public discussion to stop, and Alice produces $K_{A} \in\left[1: 2^{m-i+1}\right]$ as the rank of $X$ in $S_{i}$, Bob produces $K_{B}$ as the rank of $\hat{X}_{i}$ in $S_{i}$. Otherwise, Bob declares through public discussion to continue, Alice selects $S_{i+1} \subseteq S_{i}$ uniformly among all subsets with size $2^{m-(i+1)+1}$ that contain $X$, and continues to time $i+1$. The scheme will continue up to at most time $m+1$ (at which only $S_{m+1}=\{X\}$ is left). While in this scheme the variable $S_{1}$, which is part of the public discussion, is not finite (the other $S_{i}$ 's can be transmitted as indices with reference to $S_{1}$ and are therefore finite), we will later see that it can also be reduced to a finite discrete random variable.

We now analyze the scheme. Let the time at which Bob declares to stop be $T$. Note that the posterior probability of $\{X=x\}$ (where $x \in S_{i}$ ) at Bob at time $i$ is $f_{X \mid Y}(x \mid Y) / \sum_{x^{\prime} \in S_{i}} f_{X \mid Y}\left(x^{\prime} \mid Y\right)$. The posterior error probability is always less than or equal to $\epsilon$ when Bob declares to stop. Hence $\mathbb{P}\left\{K_{A} \neq K_{B} \mid Y=y, T=t, S^{t}=s^{t}\right\} \leq \epsilon$ for any $y, t, s^{t}$, implying $\mathbb{P}\left\{K_{A} \neq K_{B}\right\} \leq \epsilon$ and $\mathbb{P}\left\{K_{A} \neq K_{B} \mid Y\right\} \leq \epsilon$ almost surely. Let $Q \sim$ Unif $[0,1]$, independent of all random variables defined before. Define the event

$$
E=\left\{K_{A} \neq K_{B} \text { or } Q \leq 1-\frac{1-\epsilon}{1-\mathbb{P}\left\{K_{A} \neq K_{B} \mid Y\right\}}\right\}
$$

then $\mathbb{P}\{E \mid Y\}=\epsilon$ almost surely, and

$$
\begin{align*}
p_{\mathrm{c}}(X, Y) & \stackrel{\text { def }}{=} \mathbb{P}\left\{E^{\mathrm{c}} \mid X, Y\right\}  \tag{7}\\
& =\mathbb{P}\left\{K_{A}=K_{B} \mid X, Y\right\} \cdot \frac{1-\epsilon}{1-\mathbb{P}\left\{K_{A} \neq K_{B} \mid Y\right\}}
\end{align*}
$$

Condition on the event $\{X=x, Y=y\}$ from now on until otherwise stated. Let $\gamma=f_{X \mid Y}(x \mid y)$. Assume Alice continues to generate $S_{i}$ 's after time $T$. Let $S_{m+1}=\left\{\bar{X}_{1}\right\}, S_{m}=\left\{\bar{X}_{1}, \bar{X}_{2}\right\}, S_{m-1}=\left\{\bar{X}_{1}, \ldots, \bar{X}_{4}\right\}, \ldots, S_{1}=\left\{\bar{X}_{1}, \ldots, \bar{X}_{2^{m}}\right\}$. Then $\bar{X}_{1}=x$ and $\bar{X}_{2}, \ldots, \bar{X}_{2^{m}} \stackrel{i i d}{\sim}$ Unif $[0,1]$. We also define $\bar{X}_{2^{m}+1}, \bar{X}_{2^{m}+2}, \ldots$ so that $\bar{X}_{2}, \bar{X}_{3}, \ldots \stackrel{i i d}{\sim}$ Unif $[0,1]$. Assume $\bar{X}_{2}^{\infty}$ is independent of $Q$. Let $V_{i}=f_{X \mid Y}\left(\bar{X}_{i} \mid y\right)$. Note that $V_{i}$ has expectation 1 (conditioned on $\{X=x, Y=y\}$ ). Let

$$
R=\min \left\{i: \sum_{j=2}^{i+1} V_{j}>\frac{\gamma \epsilon}{1-\epsilon}\right\}
$$

then $T \leq m+1-\min \{\lfloor\log R\rfloor, m\}$ (since by that time we have $\left.f_{X \mid Y}(x \mid y) / \sum_{x^{\prime} \in S_{i}} f_{X \mid Y}\left(x^{\prime} \mid y\right) \geq 1-\epsilon\right)$. For $\alpha<1$, by Markov inequality,

$$
\mathbb{P}\left\{R \leq \frac{\alpha \gamma \epsilon}{1-\epsilon}\right\}=\mathbb{P}\left\{\sum_{j=2}^{\left\lfloor\frac{\alpha \gamma \epsilon}{1-\epsilon}\right\rfloor+1} V_{j}>\frac{\gamma \epsilon}{1-\epsilon}\right\} \leq \alpha
$$

Hence,

$$
\begin{aligned}
& \mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right] \\
& \geq \mathbb{E}\left[\min \{\lfloor\log R\rfloor, m\} \mid E^{\mathrm{c}}\right] \\
& =\sum_{i=1}^{m} \mathbb{P}\left\{\lfloor\log R\rfloor \geq i \mid E^{\mathrm{c}}\right\} \\
& \geq \sum_{i=1}^{m} \max \left\{1-\frac{\mathbb{P}\left\{R<^{i}\right\}}{\mathbb{P}\left(E^{\mathrm{c}}\right)}, 0\right\} \\
& \geq \sum_{i=1}^{m} \max \left\{1-2^{i} \frac{1-\epsilon}{\gamma \in \mathbb{P}\left(E^{\mathrm{c}}\right)}, 0\right\} \\
& \geq \\
& \geq \int_{1}^{m+1} \max \left\{1-2^{t} \frac{1-\epsilon}{\gamma \epsilon \mathbb{P}\left(E^{\mathrm{c}}\right)}, 0\right\} d t \\
& \geq \mathbf{1}\left\{\frac{1-\epsilon}{\gamma \epsilon \mathbb{P}\left(E^{\mathrm{c}}\right)} \leq \frac{1}{2}\right\}\left(2 \frac{1-\epsilon}{\gamma \epsilon \mathbb{P}\left(E^{\mathrm{c}}\right)} \log e-\log \frac{1-\epsilon}{\gamma \in \mathbb{P}\left(E^{\mathrm{c}}\right)}-1-\log e\right) \\
& \\
& \quad-\max \left\{-\log \left(\frac{1-\epsilon}{\gamma \in \mathbb{P}\left(E^{\mathrm{c}}\right)}\right)-(m+1), 0\right\} \\
& \geq \\
& \geq \\
& \\
& \quad-\log \frac{1-\epsilon}{\gamma \in \mathbb{P}\left(E^{\mathrm{c}}\right)}-1-\log e \\
& \max \left\{-\log \left(\frac{1-\epsilon}{\gamma \in \mathbb{P}\left(E^{\mathrm{c}}\right)}\right)-(m+1), 0\right\} .
\end{aligned}
$$

Also note that $K_{A} \neq K_{B}$ if and only if there exists $t \in[1: m+1], i \in\left[2: 2^{m+1-t}\right]$ such that $V_{i} \geq(1-\epsilon) \sum_{j=1}^{2^{m+1-t}} V_{j}$, which is equivalent to

$$
V_{i} \geq \frac{1-\epsilon}{\epsilon}\left(\gamma+\sum_{j \in\left[2: 2^{m+1-t}\right] \backslash\{i\}} V_{j}\right)
$$

Hence $\mathbb{P}\left\{K_{A}=K_{B} \mid X=x, Y=y\right\}$ only depends on $y$ and $\gamma=f_{X \mid Y}(x \mid y)$, and is nondecreasing in $\gamma$ for fixed $y$. By (7), $p_{\mathrm{c}}(x, y)=\mathbb{P}\left\{E^{\mathrm{c}} \mid X=x, Y=y\right\}$ is nondecreasing in $\mathbb{P}\left\{K_{A}=K_{B} \mid X=x, Y=y\right\}$ for fixed $y$, and therefore is nondecreasing in $\gamma$ for fixed $y$.

We now remove the conditioning on $\{X=x, Y=y\}$.

$$
\begin{aligned}
& \mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right] \\
& \geq \mathbb{E}\left[\left.-\log \frac{1-\epsilon}{f_{X \mid Y}(X \mid Y) \epsilon p_{\mathrm{c}}(X, Y)} \right\rvert\, E^{\mathrm{c}}\right]-1-\log e \\
& \quad-\mathbb{E}\left[\left.\max \left\{-\log \frac{1-\epsilon}{f_{X \mid Y}(X \mid Y) \epsilon p_{\mathrm{c}}(X, Y)}-m-1,0\right\} \right\rvert\, E^{\mathrm{c}}\right]
\end{aligned}
$$

$$
\begin{equation*}
\geq \mathbb{E}\left[\left.\log \frac{f_{X \mid Y}(X \mid Y) p_{\mathrm{c}}(X, Y)}{1-\epsilon} \right\rvert\, E^{\mathrm{c}}\right]+\log \epsilon-\frac{\delta_{\epsilon, m}}{1-\epsilon}-1-\log e \tag{8}
\end{equation*}
$$

where

$$
\begin{aligned}
& \delta_{\epsilon, m} \stackrel{\text { def }}{=} \mathbb{E}\left[\max \left\{-\log \frac{1-\epsilon}{f_{X \mid Y}(X \mid Y) \epsilon}-(m+1), 0\right\}\right] \\
&= I(X ; Y)+\log \frac{\epsilon}{1-\epsilon} \\
&-\mathbb{E}\left[\min \left\{-\log \frac{1-\epsilon}{f_{X \mid Y}(X \mid Y) \epsilon}, m+1\right\}\right]
\end{aligned}
$$

which tends to 0 as $m \rightarrow \infty$ by Fatou's lemma. For the other term, since $\mathbb{P}\left\{E^{c} \mid Y\right\}=1-\epsilon, E^{c}$ is independent of $Y$, and

$$
\begin{aligned}
& \mathbb{E}\left[\left.\log \frac{f_{X \mid Y}(X \mid Y) p_{\mathrm{c}}(X, Y)}{1-\epsilon} \right\rvert\, E^{\mathrm{c}}\right] \\
& =\iint_{0}^{1} \frac{f_{X \mid Y}(x \mid y) p_{\mathrm{c}}(x, y)}{1-\epsilon} \log \frac{f_{X \mid Y}(x \mid y) p_{\mathrm{c}}(x, y)}{1-\epsilon} d x d P_{Y}(y)
\end{aligned}
$$

Fix $y$. Let $G_{y}=\left\{x \in[0,1]: p_{c}(x, y) \leq 1-\epsilon\right\}, G_{y}^{c}=[0,1] \backslash G_{y}$. Since $p_{c}(x, y)$ is nondecreasing in $f_{X \mid Y}(x \mid y)$, we have $f_{X \mid Y}\left(x_{1} \mid y\right) \leq f_{X \mid Y}\left(x_{2} \mid y\right)$ for any $x_{1} \in G_{y}, x_{2} \in G_{y}^{\mathrm{c}}$. Let $\ell(t)=t \log t$, then $\ell^{\prime}(t)=\log t+\log e$ is increasing,

$$
\begin{aligned}
& \int_{0}^{1} \ell\left(\frac{f_{X \mid Y}(x \mid y) p_{\mathrm{c}}(x, y)}{1-\epsilon}\right) d x-\int_{0}^{1} \ell\left(f_{X \mid Y}(x \mid y)\right) d x \\
& \stackrel{(a)}{=}-\int_{G_{y}} \int_{f_{X \mid Y}(x \mid y) p_{c}(x, y) /(1-\epsilon)}^{f_{X \mid Y}(x \mid y)} \ell^{\prime}(t) d t d x \\
& \quad+\int_{G_{y}^{\mathrm{c}}} \int_{f_{X \mid Y}(x \mid y)}^{f_{X \mid Y}(x \mid y) p_{\mathrm{c}}(x, y) /(1-\epsilon)} \ell^{\prime}(t) d t d x \\
& \geq 0
\end{aligned}
$$

since $\ell^{\prime}(t)$ is increasing, all the $t$ 's in the negative integral in (a) is not greater than the $t$ 's in the positive integral in (a), and

$$
\begin{aligned}
& \int_{G_{y}^{\mathrm{c}}} \int_{f_{X \mid Y}(x \mid y)}^{f_{X \mid Y}(x \mid y) p_{\mathrm{c}}(x, y) /(1-\epsilon)} d t d x-\int_{G_{y}} \int_{f_{X \mid Y}(x \mid y) p_{\mathrm{c}}(x, y) /(1-\epsilon)}^{f_{X \mid Y}(x \mid y)} d t d x \\
& =\int_{0}^{1}\left(\frac{f_{X \mid Y}(x \mid y) p_{\mathrm{c}}(x, y)}{1-\epsilon}-f_{X \mid Y}(x \mid y)\right) d x \\
& =\frac{\mathbb{P}\left\{E^{\mathrm{c}} \mid Y=y\right\}}{1-\epsilon}-1=0 .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \iint_{0}^{1} \ell\left(\frac{f_{X \mid Y}(x \mid y) p_{\mathrm{c}}(x, y)}{1-\epsilon}\right) d x d P_{Y}(y) \\
& \geq \iint_{0}^{1} \ell\left(f_{X \mid Y}(x \mid y)\right) d x d P_{Y}(y)=I(X ; Y)
\end{aligned}
$$

Substituting back to (8),

$$
\begin{align*}
& \mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right] \\
& \geq I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m}-1-\log e . \tag{9}
\end{align*}
$$

Assume Alice selects $S_{i}$ in the following way: Alice observes $X$, generates $\tilde{X}_{2}, \ldots, \tilde{X}_{2^{m}} \stackrel{i i d}{\sim} \operatorname{Unif}[0,1]$, and $S_{1}=\left\{X, \tilde{X}_{2}, \ldots, \tilde{X}_{2^{m}}\right\}$ (let $\tilde{X}_{1}=X$ ). Alice generates a permutation $\Phi$ over $\left[1: 2^{m}\right]$ uniformly at random. At time $i$, Alice selects $S_{i}=\left\{\tilde{X}_{j}: \Phi(j) \equiv\right.$ $\left.\Phi(1)\left(\bmod 2^{i-1}\right)\right\}$. It is straightforward to check that $S_{i+1}$ is distributed uniformly among all subsets of $S_{i}$ with size $2^{m-(i+1)+1}$ that contains $X$. Hence we can assume $S_{i}$ 's are generated this way.

$$
\begin{aligned}
& H\left(K_{A} \mid T, S^{T}, \Phi, E^{\mathrm{c}}\right) \\
& =H\left(X \mid T, S^{T}, \Phi, E^{\mathrm{c}}\right) \\
& =\mathbb{E}_{t \sim p_{T \mid E^{\mathrm{c}}}}\left[H\left(X \mid S_{1}, \Phi, E^{\mathrm{c}}, T=t\right)\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.-\sum_{i=2}^{t} I\left(X ; S_{i} \mid S^{i-1}, \Phi, E^{\mathrm{c}}, T=t\right)\right] \\
= & H\left(X \mid S_{1}, \Phi, T, E^{\mathrm{c}}\right) \\
& -\sum_{i=2}^{\infty} \mathbb{P}\left\{T \geq i \mid E^{\mathrm{c}}\right\} I\left(X ; S_{i} \mid S^{i-1}, \Phi, T, E^{\mathrm{c}}, T \geq i\right) \\
\geq & H\left(X \mid S_{1}, \Phi, E^{\mathrm{c}}\right)-H\left(T \mid E^{\mathrm{c}}\right)-\sum_{i=2}^{\infty} \mathbb{P}\left\{T \geq i \mid E^{\mathrm{c}}\right\} \tag{10}
\end{align*}
$$

where the last inequality is because $S_{i}$ only has two possibilities given $S_{i-1}$ and $\Phi$ (depending on whether $\Phi(1) \bmod 2^{i-1}=$ $\Phi(1) \bmod 2^{i-2}$ or $\left.\left(\Phi(1) \bmod 2^{i-2}\right)+2^{i-2}\right)$. For the first term,

$$
\begin{aligned}
& H\left(X \mid S_{1}, \Phi, E^{\mathrm{c}}\right) \\
& =\mathbb{E}\left[\left.\sum_{x \in S_{1}} \mathbb{P}\left\{X=x \mid S_{1}, \Phi, E^{\mathrm{c}}\right\} \log \frac{1}{\mathbb{P}\left\{X=x \mid S_{1}, \Phi, E^{\mathrm{c}}\right\}} \right\rvert\, E^{\mathrm{c}}\right] \\
& \geq \mathbb{E}\left[\left.\sum_{x \in S_{1}} \mathbb{P}\left\{X=x \mid S_{1}, \Phi, E^{\mathrm{c}}\right\} \log \frac{\mathbb{P}\left\{E^{\mathrm{c}} \mid S_{1}, \Phi\right\}}{\mathbb{P}\left\{X=x \mid S_{1}, \Phi\right\}} \right\rvert\, E^{\mathrm{c}}\right] \\
& =\mathbb{E}\left[\left.\sum_{x \in S_{1}} \mathbb{P}\left\{X=x \mid S_{1}, \Phi, E^{\mathrm{c}}\right\} \log \frac{\mathbb{P}\left\{E^{\mathrm{c}} \mid S_{1}, \Phi\right\}}{2^{-m}} \right\rvert\, E^{\mathrm{c}}\right] \\
& =m+\mathbb{E}\left[\log \mathbb{P}\left\{E^{\mathrm{c}} \mid S_{1}, \Phi\right\} \mid E^{\mathrm{c}}\right] \\
& =m+(1-\epsilon)^{-1} \mathbb{E}\left[\mathbb{P}\left\{E^{\mathrm{c}} \mid S_{1}, \Phi\right\} \log \mathbb{P}\left\{E^{\mathrm{c}} \mid S_{1}, \Phi\right\}\right] \\
& \geq m-(1-\epsilon)^{-1} e^{-1} \log e \\
& \geq m-2 e^{-1} \log e,
\end{aligned}
$$

where the last inequality is by $\epsilon<1 / 2$. For the second and third term in 10,

$$
\begin{align*}
& -H\left(T \mid E^{\mathrm{c}}\right)-\sum_{i=2}^{\infty} \mathbb{P}\left\{T \geq i \mid E^{\mathrm{c}}\right\} \\
& =-m-H\left(T \mid E^{\mathrm{c}}\right)+\mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right] \\
& \stackrel{(a)}{\geq}-m-\left(\mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right]+1\right) H_{\mathrm{b}}\left(\frac{1}{\mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right]+1}\right) \\
& \quad+\mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right] \\
& \geq-m+\mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right]-\log \left(\mathbb{E}\left[m-T+1 \mid E^{\mathrm{c}}\right]+1\right)-\log e \\
& \stackrel{(b)}{\geq}-m+\max \left\{I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m}-1-\log e, 0\right\} \\
& \quad-\log \left(\max \left\{I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m}-1-\log e, 0\right\}+1\right) \\
& \quad-\log e-0.0861 \\
& \geq-m+I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m} \\
& \quad-\log \left(\max \left\{I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m}, 0\right\}+1\right)-3.9715 . \tag{11}
\end{align*}
$$

where (a) is because $H\left(T \mid E^{c}\right)=H\left(m-T+1 \mid E^{c}\right)$ and the geometric distribution maximizes the entropy of a nonnegative integer-valued random variable with fixed mean, and (b) is by (9) and that $t \mapsto t-\log (t+1)$ decreases by at most 0.0861 . Substituting back in (10),

$$
\begin{aligned}
& H\left(K_{A} \mid T, S^{T}, \Phi, E^{\mathrm{c}}\right) \\
& \geq I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m} \\
& \quad-\log \left(\max \left\{I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m}, 0\right\}+1\right)-5.033
\end{aligned}
$$

Recall that $E^{\mathrm{c}} \subseteq\left\{K_{A}=K_{B}\right\}$. Hence

$$
\begin{aligned}
& H_{=}\left(K_{A} ; K_{B} \mid T, S^{T}, \Phi\right) \\
& =\mathbb{P}\left\{K_{A}=K_{B}\right\} H\left(K_{A} \mid T, S^{T}, \Phi, K_{A}=K_{B}\right) \\
& =H\left(K_{A} \mathbf{1}\left\{K_{A}=K_{B}\right\} \mid T, S^{T}, \Phi, \mathbf{1}\left\{K_{A}=K_{B}\right\}\right)
\end{aligned}
$$

$$
\begin{align*}
& \geq H\left(K_{A} \mathbf{1}\left\{K_{A}=K_{B}\right\}, \mathbf{1}\left\{E^{c}\right\} \mid T, S^{T}, \Phi, \mathbf{1}\left\{K_{A}=K_{B}\right\}, \mathbf{1}\left\{E^{c}\right\}\right)-1 \\
& \geq H\left(K_{A} \mathbf{1}\left\{E^{c}\right\} \mid T, S^{T}, \Phi, \mathbf{1}\left\{K_{A}=K_{B}\right\}, \mathbf{1}\left\{E^{\mathrm{c}}\right\}\right)-1 \\
& =H\left(K_{A} \mathbf{1}\left\{E^{c}\right\} \mid T, S^{T}, \Phi, \mathbf{1}\left\{E^{\mathrm{c}}\right\}\right)-1 \\
& =(1-\epsilon) H\left(K_{A} \mid T, S^{T}, \Phi, E^{\mathrm{c}}\right)-1 \\
& \geq(1-\epsilon)\left(I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m}\right. \\
& \left.-\log \left(\max \left\{I(X ; Y)+\log \epsilon-(1-\epsilon)^{-1} \delta_{\epsilon, m}, 0\right\}+1\right)\right)-6.033 . \tag{12}
\end{align*}
$$

Since $\delta_{\epsilon, m} \rightarrow 0$ as $m \rightarrow \infty$, for $m$ large enough, we have (write $I=I(X ; Y)$ )

$$
\kappa \geq(1-\epsilon)(I+\log \epsilon-\log (\max \{I+\log \epsilon, 0\}+1))-6.034 .
$$

If $I>2$, substitute $\epsilon=I^{-1}$, we have

$$
\begin{aligned}
\kappa \geq & I-\log I-\log (I-\log I+1) \\
& +\frac{\log I+\log (I-\log I+1)}{I}-7.034 \\
& \geq I-2 \log (I+1)-7.034
\end{aligned}
$$

It can also be checked that the lemma is true when $I \leq 2$, since the right hand side is negative.
Next we consider the case where $X, Y \in \mathbb{Z}_{>0}$ are discrete are finite. Let $\hat{X} \mid\{X=x\} \sim \operatorname{Unif}\left[F_{X}(x-1), F_{X}(x)\right]$. Then $\hat{X} \sim \operatorname{Unif}[0,1]$ and $I(\hat{X} ; Y)=I(X ; Y)$. We apply the above scheme over $(\hat{X}, Y)$. Since the scheme makes no distinction between values of $\hat{x}$ in the same interval $\left(F_{X}(x-1), F_{X}(x)\right]$ mapped to the same $x$ (they have the same $f_{\hat{X} \mid Y}(\hat{x} \mid y)$ for all $y$ ), to transmit $S_{1}$ we only need to transmit the sizes $\mid S_{1} \cap\left(F_{X}(x-1), F_{X}(x)\right]$, which are finite.

For the general case where each component of the pair $(X, Y)$ lies in a general measurable space, we apply the above scheme over $\left(g_{1}(X), g_{2}(Y)\right)$, where $g_{1}(X)$ and $g_{2}(Y)$ are discretized version of $X$ and $Y$ lying in finite sets. Since (see [19])

$$
I(X ; Y)=\sup _{g_{1}, g_{2}: g_{1}(\mathcal{X}), g_{2}(\mathcal{Y}) \text { finite }} I\left(g_{1}(X) ; g_{2}(Y)\right),
$$

the proof is completed by considering a sequence of discretizations approaching the mutual information. This approach also handles the case where $I(X ; Y)=\infty$.

We now complete the proof of Theorem 1
Proof of Theorem $\square$. The upper bound follows from Lemma $\square$ and 2 For the lower bound, by Lemma 1 and 2

$$
\begin{aligned}
& L_{\epsilon}^{*}(X ; Y) \\
& \geq \kappa(X ; Y)-\log (\kappa(X ; Y)+1)-2 \log \frac{1}{\epsilon}-7.082 \\
& \stackrel{(a)}{\geq} I(X ; Y)-2 \log (I(X ; Y)+1)-7.034 \\
& \quad-\log (\max \{I(X ; Y)-2 \log (I(X ; Y)+1)-6.238,0\}+1) \\
& \quad-2 \log \frac{1}{\epsilon}-0.0861-7.082 \\
& \geq I(X ; Y)-3 \log (I(X ; Y)+1)-2 \log \frac{1}{\epsilon}-14.2021
\end{aligned}
$$

where (a) is because $t \mapsto t-\log (t+1)$ decreases by at most 0.0861 .

## IV. Concatenating Variable-Length Keys

Consider the situation where Alice and Bob observe the respective coordinates of a random process $\left\{\left(X_{i}, Y_{i}\right)\right\}_{i \in \mathbb{Z}_{>0}}$ sequentially, where we assume that the pairs $\left(X_{i}, Y_{i}\right)$ are independent over $i$. Instead of grouping the source symbols into large blocks to allow the generation of fixed-length keys, they may reduce the delay of key generation by generating a variable-length key upon observing the respective coordinates of each source symbol pair. These variable-length keys can be concatenated to form a stream of secret key bits that can be used as soon as they become available.

Suppose we have two independent variable-length keys with expected lengths $\mathbb{E}\left[L_{1}\right], \mathbb{E}\left[L_{2}\right]$ and distances from ideal distributions $\epsilon_{1}, \epsilon_{2}$ respectively. Then we can concatenate them to form a variable-length key with expected length $\mathbb{E}\left[L_{3}\right]=$ $\mathbb{E}\left[L_{1}\right]+\mathbb{E}\left[L_{2}\right]$ and distance from ideal distribution $\epsilon_{3} \leq \epsilon_{1}+\epsilon_{2}$. The distance from ideal distribution grows linearly with the number of variable-length keys concatenated, which prevents us from concatenating too many keys. Instead of considering the distance from ideal distribution, we may consider the entropy and bit error probability instead, as shown below.

Proposition 1. Let $(A, B)$ be a variable-length key with expected length $\mathbb{E}[L]$ and distance from ideal distribution $\epsilon$. Then, for all $l \in \mathbb{Z}_{\geq 0}$ and $i \in[1: l]$, we have

$$
\begin{equation*}
\mathbb{P}\{A[i] \neq B[i] \mid L=l\} \leq \epsilon \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
H\left(A \mid W^{N}, L=l\right), H\left(B \mid W^{N}, L=l\right) \geq l(1-2 \epsilon) \tag{14}
\end{equation*}
$$

where we write $A[i]$ for the $i$-th bit of $A$, and $W^{N}$ denotes the public discussion, stopping at the random time $N$. As a result, if we concatenate two independent keys $\left(A_{1}, B_{1}\right),\left(A_{2}, B_{2}\right)$ with lengths $L_{1}, L_{2}$ and public discussions $W_{1}^{N_{1}}, W_{2}^{N_{2}}$ respectively, both with distances from ideal distributions bounded by $\epsilon$, i.e., $A=A_{1}\left\|A_{2}, B=B_{1}\right\| B_{2}, L=L_{1}+L_{2}$, then the same guarantees are preserved, i.e., $\mathbb{P}\{A[i] \neq B[i] \mid L=l\} \leq \epsilon$, which is (13) for the concatenated key, and

$$
\begin{equation*}
H\left(A \mid W_{1}^{N_{1}}, W_{2}^{N_{2}}, L=l\right), H\left(B \mid W_{1}^{N_{1}}, W_{2}^{N_{2}}, L=l\right) \geq l(1-2 \epsilon) \tag{15}
\end{equation*}
$$

Proof: It is straightforward to prove (13). We first prove (14). Let $g(t)=-t \log t$ for $t \in[0,1]$. Then, by the concavity of $g$, for any $\gamma \in[0,1]$,

$$
g(t) \geq g(\gamma)\left(1-\frac{\max \{\gamma-t, 0\}}{\gamma}-\frac{\max \{t-\gamma, 0\}}{1-\gamma}\right)
$$

For $l \geq 1$,

$$
\begin{aligned}
& H\left(A \mid W^{N}=w^{n}, L=l\right) \\
& =\sum_{a=1}^{2^{l}} g\left(p_{A \mid W^{N}=w^{n}, L=l}(a)\right) \\
& \geq \sum_{a=1}^{2^{l}} g\left(2^{-l}\right)\left(1-\frac{\max \left\{2^{-l}-p_{A \mid W^{N}=w^{n}, L=l}(a), 0\right\}}{2^{-l}}\right. \\
& \left.\quad-\frac{\max \left\{p_{A \mid W^{N}=w^{n}, L=l}(a)-2^{-l}, 0\right\}}{1-2^{-l}}\right) \\
& \geq l 2^{-l}\left(2^{l}-\frac{d_{\mathrm{TV}}\left(p_{A \mid W^{N}=w^{n}, L=l}, \operatorname{Unif}\left[1: 2^{l}\right]\right)}{2^{-l}}\right. \\
& \left.\quad-\frac{d_{\mathrm{TV}}\left(p_{A \mid W^{N}=w^{n}, L=l}, \operatorname{Unif}\left[1: 2^{l}\right]\right)}{1-2^{-l}}\right) \\
& \geq l\left(1-2 d_{\mathrm{TV}}\left(p_{A \mid W^{N}=w^{n}, L=l}, \operatorname{Unif}\left[1: 2^{l}\right]\right)\right) .
\end{aligned}
$$

Since $d_{\mathrm{TV}}\left(p_{A, W^{N} \mid L=l}, \operatorname{Unif}\left[1: 2^{l}\right] \times p_{W^{N} \mid L=l}\right) \leq \epsilon$, we have

$$
H\left(A \mid W^{N}, L=l\right) \geq l(1-2 \epsilon)
$$

Suppose now that we concatenate two independent keys $\left(A_{1}, B_{1}\right),\left(A_{2}, B_{2}\right)$ with lengths $L_{1}, L_{2}$ and public discussions $W_{1}^{N_{1}}, W_{2}^{N_{2}}$ respectively, both with distances from ideal distributions bounded by $\epsilon$, i.e., $A=A_{1}\left\|A_{2}, B=B_{1}\right\| B_{2}, L=L_{1}+L_{2}$. It is straightforward to prove (13) for the concatenated key. To prove (15), note that

$$
\begin{aligned}
& H\left(A \mid W_{1}^{N_{1}}, W_{2}^{N_{2}}, L=l\right) \\
& \geq H\left(A \mid W_{1}^{N_{1}}, W_{2}^{N_{2}}, L_{1}, L=l\right) \\
& =\sum_{t=0}^{l} \mathbb{P}\left\{L_{1}=t \mid L=l\right\}\left(H\left(A_{1} \mid W_{1}^{N_{1}}, L_{1}=t\right)\right. \\
& \left.\quad+H\left(A_{2} \mid W_{2}^{N_{2}}, L_{2}=l-t\right)\right) \\
& \geq \sum_{t=0}^{l} \mathbb{P}\left\{L_{1}=t \mid L=l\right\}(t(1-2 \epsilon)+(l-t)(1-2 \epsilon)) \\
& =l(1-2 \epsilon)
\end{aligned}
$$

Then we show that it is possible to construct a fixed-length key in the asymptotic regime using i.i.d. variable-length keys and a simple outer code. The following proposition shows that the asymptotic fixed-length result is implied by the one-shot variable-length result (by applying Theorem 1 on $X^{t}, Y^{t}, \epsilon=t^{-2}$, and taking $t \rightarrow \infty$ ).

Proposition 2. Fix $R<\mu\left(1-H_{\mathrm{b}}(2 \epsilon)\right)$. For $i \in \mathbb{Z}_{>0}$, let $\left(A_{i}, B_{i}\right)$ be i.i.d. variable-length keys with respective public discussion $W_{i}$ (we let $W_{i}:=W_{i}^{N_{i}}$ and omit the superscript), expected length $\mu$ and distance from ideal distribution $\epsilon$. Then we can construct a sequence of fixed-length keys $\left\{\left(K_{A, n}, K_{B, n}\right)\right\}_{n=1}^{\infty}$, where $K_{A, n}, K_{B, n} \in\left[1: 2^{\lfloor n R\rfloor}\right]$ is generated using $A^{n}, B^{n}$, and possibly using additional public discussion, where

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \mathbb{P}\left\{K_{A, n} \neq K_{B, n}\right\}=0 \\
\liminf _{n \rightarrow \infty} \frac{1}{n} H\left(K_{A, n}\right) \geq R-2 \epsilon \mu
\end{gathered}
$$

and

$$
\limsup _{n \rightarrow \infty} \frac{1}{n} I\left(K_{A, n} ; \tilde{W}_{n}\right) \leq 2 \epsilon \mu
$$

where $\tilde{W}_{n}$ denotes all the public discussion used to generate $K_{A, n}, K_{B, n}$ (including the $W_{i}$ 's and the additional public discussion). Similar conditions hold for $K_{B, n}$.

Proof: If $\epsilon>0$ is small enough, then for all $\xi>0$ small enough we have $R<(\mu+\xi)\left(1-H_{\mathrm{b}}(2(\epsilon+\xi))\right)$. Fix such a $\xi>0$ and fix $n$. Let $\tilde{K}_{A} \in \mathbb{F}_{2}^{n(\mu+\xi)}$ be the first $\lceil n(\mu+\xi)\rceil$ bits of $A_{1}\left\|A_{2}\right\| \cdots \| A_{n}$ (append zeroes if there are not enough bits), and similarly define $\tilde{K}_{B}$. Let $P \in \mathbb{F}_{2}^{(\lceil n(\mu+\xi)\rceil-\lfloor n R\rfloor) \times\lceil n(\mu+\xi)\rceil}$ be the parity check matrix of a linear code with minimum distance at least $2(\epsilon+\xi) n(\mu+\xi)$. This is possible by the Gilbert-Varshamov bound [20], [21] since $\lfloor n R\rfloor<n(\mu+\xi)\left(1-H_{\mathrm{b}}(2(\epsilon+\xi))\right)$. Alice sends $P \tilde{K}_{A}$ through public discussion, and Bob finds $\hat{K}$ with the smallest Hamming distance from $\tilde{K}_{B}$ satisfying $P \hat{K}=P \tilde{K}_{A}$. By Proposition 1 and law of large numbers, $\mathbb{P}\left\{\left|\left\{i: \tilde{K}_{A}[i] \neq \tilde{K}_{B}[i]\right\}\right| \leq \epsilon n(\mu+\xi)\right\} \rightarrow 1$, and hence the code can correct the error and $\hat{K}=\tilde{K}_{A}$ with probability tending to 1 . Alice outputs $K_{A} \in \mathbb{F}_{2}^{\lfloor n R\rfloor}$, the coordinates of $\tilde{K}_{A}$ in the affine subspace $\left\{v: P v=P \tilde{K}_{A}\right\}$. Bob outputs $K_{B}$, the coordinates of $\hat{K}$ in the affine subspace $\left\{v: P v=P \tilde{K}_{A}\right\}$ (Alice and Bob agree beforehand on the same basis of the subspace). Note that the public discussion is $\left(W_{1}, \ldots, W_{n}, P \tilde{K}_{A}\right)=\left(W^{n}, P \tilde{K}_{A}\right)$. We have

$$
\begin{aligned}
& H\left(K_{A} \mid W^{n}, P \tilde{K}_{A}\right) \\
& \geq H\left(K_{A} \mid W^{n}, L^{n}, P \tilde{K}_{A}\right) \\
& \geq \mathbb{P}\left\{\left|\frac{1}{n} \sum_{i=1}^{n} L_{i}-\mu\right| \leq \xi\right\} \\
& \quad \cdot H\left(K_{A}\left|W^{n}, L^{n}, P \tilde{K}_{A},\left|\frac{1}{n} \sum_{i=1}^{n} L_{i}-\mu\right| \leq \xi\right)\right.
\end{aligned}
$$

By law of large numbers, $\mathbb{P}\left\{\left|(1 / n) \sum_{i} L_{i}-\mu\right| \leq \xi\right\} \rightarrow 1$. We have

$$
\begin{aligned}
& H\left(K_{A}\left|W^{n}, L^{n}, P \tilde{K}_{A},\left|\frac{1}{n} \sum_{i} L_{i}-\mu\right| \leq \xi\right)\right. \\
& \stackrel{(a)}{=} H\left(A^{n}\left|W^{n}, L^{n}, P \tilde{K}_{A},\left|\frac{1}{n} \sum L_{i}-\mu\right| \leq \xi\right)\right. \\
& \geq H\left(A^{n}\left|W^{n}, L^{n},\left|\frac{1}{n} \sum L_{i}-\mu\right| \leq \xi\right)-n(\mu-\xi)+\lfloor n R\rfloor\right. \\
& \stackrel{(b)}{\geq} n(\mu-\xi)(1-2 \epsilon)-n(\mu-\xi)+\lfloor n R\rfloor \\
& =\lfloor n R\rfloor-2 \epsilon n(\mu-\xi)
\end{aligned}
$$

where (a) is because $A_{1}\left\|A_{2}\right\| \cdots \| A_{n}$ has length at most $n(\mu+\xi)$ if $\left|(1 / n) \sum_{i} L_{i}-\mu\right| \leq \xi$, and $\tilde{K}_{A}$ is a function of $P \tilde{K}_{A}$ and $K_{A}$; and (b) is by Proposition 1 Hence for sufficiently large $n$,

$$
\frac{1}{n} H\left(K_{A} \mid W^{n}, P \tilde{K}_{A}\right) \geq R-2 \epsilon(\mu-\xi)-\xi
$$

Since $I\left(K_{A} ; W^{n}, P \tilde{K}_{A}\right) \leq\lfloor n R\rfloor-H\left(K_{A} \mid W^{n}, P \tilde{K}_{A}\right)$, for sufficiently large $n$,

$$
\frac{1}{n} I\left(K_{A} ; W^{n}, P \tilde{K}_{A}\right) \leq 2 \epsilon(\mu-\xi)+\xi
$$

The proof is completed by letting $\xi \rightarrow 0$.

## V. Splitting a Variable-Length Key

Another way to obtain fixed-length keys from a variable-length secret key is by splitting the key. Suppose Alice and Bob share a variable-length key $A, B$ with length $L$. They want to perform a task multiple times (e.g. communicating an encrypted message), each time requiring a fixed-length key with length $t$. Alice and Bob can perform the task $M=\lfloor L / t\rfloor$ times using different segments of $A$ and $B$ (treated as bit sequences) as the keys. Let the segments be $\tilde{A}^{\lfloor L / t\rfloor}=\tilde{A}_{1}, \ldots, \tilde{A}_{\lfloor L / t\rfloor}$ and $\tilde{B}^{\lfloor L / t\rfloor}$, defined similarly. By the definition of variable-length keys, we have the following secrecy guarantee for any $m$ :

$$
d_{\mathrm{TV}}\left(p_{\tilde{A}^{m}, \tilde{B}^{m}, W^{N} \mid M=m}, \mathrm{U}_{2}\left(\left[1: 2^{t}\right]^{\otimes m}\right) \times p_{W^{N} \mid M=m}\right) \leq \epsilon,
$$

where $\mathrm{U}_{2}\left(\left[1: 2^{t}\right]^{\otimes m}\right)$ denotes Unif $\left\{\left(a^{m}, a^{m}\right): a_{1}, \ldots, a_{m} \in\left[1: 2^{t}\right]\right\}$. This means the total variation distance between the actual distribution and the ideal one (where $\tilde{A}^{m}=\tilde{B}^{m}$, i.i.d. uniform over $\left[1: 2^{t}\right]$ independent of $W^{N}$ ) is bounded by $\epsilon$. Any event on $\tilde{A}^{m}, \tilde{B}^{m}, W^{N}$ (e.g. an error event, Eve correctly guessing some functions of $\tilde{A}, \tilde{B}$, etc.) has a probability within $\epsilon$ from the probability of that event measured in the ideal distribution (the probability of error, the advantage of Eve, etc. are bounded by $\epsilon$ ). Therefore Alice and Bob can perform the task an expected $\mathbb{E}[M] \geq \mathbb{E}[L] / t-1$ times while guaranteeing the advantage of Eve is bounded by $\epsilon$.
Consider the payoff function $g(\tilde{a}, \tilde{b}, v) \in\left[g_{\min }, g_{\max }\right]$ (which can be negative), where $\tilde{a}, \tilde{b} \in\left[1: 2^{t}\right]$ are the keys, and $v$ is Eve's action (e.g. Eve's guess of the message). The total payoff is $\sum_{i=1}^{M} g\left(\tilde{A}_{i}, \tilde{B}_{i}, V_{i}\right)$. To make the secrecy guarantee stronger, we allow for the hypothetical possibility that Eve observes $A_{i}, B_{i}$ strictly causally, i.e., $V_{i}$ can depend on $W^{N}, \tilde{A}^{i-1}, \tilde{B}^{i-1}, V^{i-1}$. This rules out the possibility of simply reusing the same key for each $i$ and provides a stronger guarantee without actually implying that Eve has access to the previous keys (which would result in compromising previous communications). Let $g^{*}=\inf _{v} \mathbb{E}[g(C, C, v)]$ be the worst-case expected payoff in the ideal distribution where $C \sim \operatorname{Unif}\left[1: 2^{t}\right]$ (since Eve's observation $W^{N}$ is independent of $C$ in the ideal distribution she can only fix her output at some $v$ ). Assume $g^{*}>0$ (otherwise we cannot have a positive payoff even if we have a perfect secret key). The expected payoff

$$
\begin{align*}
& \mathbb{E}\left[\sum_{i=1}^{M} g\left(\tilde{A}_{i}, \tilde{B}_{i}, V_{i}\right)\right] \\
& \stackrel{(a)}{\geq} \mathbb{E}\left[\mathbb{E}\left[\sum g\left(C_{i}, C_{i}, V_{i}\right) \mid M\right]\right. \\
& \left.\quad-d_{\mathrm{TV}}\left(p_{\tilde{A}^{M}, \tilde{B}^{M}, W^{N} \mid M}, p_{C^{M}, C^{M}, W^{N} \mid M}\right) M\left(g_{\max }-g_{\min }\right)\right] \\
& \stackrel{(b)}{\geq} \mathbb{E}\left[M\left(g^{*}-\epsilon\left(g_{\max }-g_{\min }\right)\right)\right] \\
& \geq \\
& \geq\left(\frac{\mathbb{E}[L]+1}{t}-1\right)\left(g^{*}-\epsilon\left(g_{\max }-g_{\min }\right)\right), \tag{16}
\end{align*}
$$

where in (a), $C_{i}$ are i.i.d. uniform over [1:2 $\left.2^{t}\right]$ independent of $W^{N}$, and we assume $V_{i} \mid\left\{W^{N}=w^{n}, C^{i-1}=c^{i-1}, V^{i-1}=\right.$ $\left.v^{i-1}\right\} \sim p_{V_{i} \mid W^{N}, \tilde{A}^{i-1}, \tilde{B}^{i-1}, V^{i-1}}\left(\cdot \mid w^{n}, c^{i-1}, c^{i-1}, v^{i-1}\right)$, and $(\mathrm{b})$ is because $C_{i}$ is independent of $W^{N}, C^{i-1}, V^{i-1}$ and therefore $C_{i}$ is independent of $V_{i}$. We can see that this is close to the ideal payoff $\mathbb{E}[L] g^{*} / t$ when $\epsilon$ is small.

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[^0]:    ${ }^{1}$ In [10], a variable-length key with a constraint on the average total variation distance over key lengths is studied. Considering the average distance (instead of the maximum distance in our paper) is undesirable because it is possible for Alice and Bob to declare an extremely long key with low probability, which has an arbitrarily small impact on the average distance, but can increase the expected key length arbitrarily.

