# Three Formulations of the Kuramoto Model as a System of Polynomial Equations 

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#### Abstract

We compare three formulations of stationary equations of the Kuramoto model as systems of polynomial equations. In the comparison, we present bounds on the numbers of real equilibria based on the work of Bernstein, Kushnirenko, and Khovanskii, and performance of methods for the optimisation over the set of equilibria based on the work of Lasserre, both of which could be of independent interest.


Index Terms-Kuramoto model, Dynamic systems, Algebraic approaches, Polynomial methods, Multivariable polynomials

## I. Introduction

The Kuramoto model is a prototypical model for studying many phenomena including the synchronization of power systems, neural networks, chemical oscillators, particle coordinations, rhythmic applause, and so on. See [1]-[4]. The general Kuramoto model is given by the differential equation:

$$
\begin{equation*}
\frac{d \theta_{i}}{d t}=\omega_{i}-\frac{1}{N} \sum_{j=1}^{N} K_{i, j} \sin \left(\theta_{i}-\theta_{j}\right), \text { for } i=1, \ldots, N \tag{1}
\end{equation*}
$$

where $N$ is the number of oscillators, $K_{i, j}$ is the coupling strength between the $i$-th and $j$-th oscillators. The matrix $K=\left[K_{i, j}\right]$ may also be viewed as the adjacency matrix for the underlying weighted graph. $\Omega=\left(\omega_{1}, \ldots, \omega_{N}\right)$ contains the natural frequencies of the $N$ oscillators. Of crucial importance in studying the phase space of this system of ordinary differential equations are the equilibria which are values of $\theta_{1}, \ldots, \theta_{N}$ for which $\frac{d \theta_{i}}{d t}=0$ for all $i=1, \ldots, N$. The stability analysis of the equilibria can reveal the behavior of the dynamical system near the equilibria. One can also study synchronization phenomenon of the Kuramoto model with the knowledge of the equilibria of the model [5]. Interpreted as a special case [6]-[8] of the power flow equations in alternating-current power systems with harmonic currents, the equilibria of the Kuramoto model provide crucial information for planning and designing power grids.

Equilibria of the Kuramoto model can be found by solving the equilibrium conditions $\frac{d \theta_{i}}{d t}=0$ for all $i$. Since the

[^0]equilibrium conditions are invariant under transformations of the form $\theta_{i} \rightarrow \theta_{i}+\alpha$ for all $\theta_{i}$ with any fixed $\alpha \in(-\pi, \pi]$, they possess infinitely many solutions. To remove this degree of freedom, we fix $\theta_{N}=0$ and remove the $N$-th equation $\frac{d \theta_{N}}{d t}=0$. Thus, we are left with $N-1$ nonlinear equations in $N-1$ angles. For example with $N=3$, the Kuramoto model is thus known [9]-[11] to have at most 6 equilibria on the complete graph.

The counting the number of equilibria of the Kuramoto model with a finite number $N$ of oscillators, has a very long history and should be seen as a key structural property of the set of equilibria. Generally, it is being turned into a root counting problem for systems of polynomial equations. As we will show, there are multiple reformulations to polynomial equations, and some of the bounds on the number of roots are sensitive to the choice of the reformulation. We present strong numerical evidence that for many Kuramoto models, a bound based on the reformulation we introduce and the theory of [12], [13], and [14] (BKK) is the best known and in some sense the best possible bound for the number of complex equilibria.

Outside of the structural properties of the set of equilibria, one may be interested in the optimisation over the set. This has been particularly popular in power systems applications, where there is a cost (of power generation) associated with the equilibria, cf. [15]. Again, we demonstrate that the optimisation methods are very sensitive to the choice of the reformulation. Again, the use of the reformulation we introduce offers performance superior to those previously used.

First, in Section [II we describe a novel polynomial reformulation of the equilibrium equations of the Kuramoto model and compare it the reformulations suggested previously. Next, we review some known structural results in Section [II abd propose an upper bound on the number of equilibria based on our reformulation and the BKK theory in Section IV Next, we suggest the use of the method of moments for optimisation over the set of equilibria in Section (V) Next, we provide computational results for the Kuramoto model on benchmark graphs. We show how sensitive is the performance of both the method of moments and a numerical polynomial homotopy continuation (NPHC) method to the choice of
the reformulation. Using the NPHC method, we provide strong numerical evidence to demonstrate that our structural results are tight in certain cases. That is: we employ the numerical homotopy continuation method and demonstrate that for generically chosen natural frequencies and weights on the graph, our upper bound is always equal to the number of complex solutions. In Section VII we discuss the implications of our results and conclude.

## II. Polynomial Formulations

In studying nonlinear systems of equations like the equilibrium equations of (1), it is a common practice to first transform them into algebraic equations which would allow the use of powerful tools from algebraic geometry.

Formulation 1. Previously [5], [6], [16]-[18], the equilibrium conditions (1) were transformed into a system of polynomial equations by using the identities $\sin \left(\theta_{i}-\theta_{j}\right)=$ $\sin \theta_{i} \cos \theta_{j}-\sin \theta_{j} \cos \theta_{i}$ and then the substitution $s_{i}:=$ $\sin \theta_{i}$ and $c_{i}:=\cos \theta_{i}$ for all $i=1, \ldots, N-1$, and adding the equation $s_{i}^{2}+c_{i}^{2}-1=0$, for all $i=1, \ldots, N-1$. This way, one obtains the following system of polynomial equations:

$$
\begin{align*}
\omega_{i}-\frac{1}{N} \sum_{j=1}^{N} K_{i, j}\left(s_{i} c_{j}-s_{j} c_{i}\right) & =0  \tag{F1}\\
s_{i}^{2}+c_{i}^{2}-1 & =0
\end{align*}
$$

for $i=1, \ldots, N-1$.
Formulation 2. Alternatively, one may consider a formulation based on the so-called tangent half-angle identities [19, pages 382-383]:

$$
\begin{equation*}
\sin \theta_{i}=\frac{2 \tan \frac{\theta_{i}}{2}}{1+\tan ^{2} \frac{\theta_{i}}{2}} \text { and } \cos \theta_{i}=\frac{1-\tan ^{\frac{\theta_{i}}{2}}}{1+\tan ^{2} \frac{\theta_{i}}{2}}, \tag{2}
\end{equation*}
$$

wherein one introduces $t_{i}:=\tan \frac{\theta_{i}}{2}$. This way, one obtains the following system of polynomial equations:

$$
\begin{align*}
\omega_{i}-\frac{1}{N} \sum_{j=1}^{N} K_{i, j}\left(\frac{2 t_{i}}{1+t_{i}^{2}} \frac{1-t_{j}}{1+t_{j}^{2}}-\frac{2 t_{j}}{1+t_{j}^{2}} \frac{1-t_{i}}{1+t_{i}^{2}}\right) & =0 \\
\left(\frac{2 t_{i}}{1+t_{i}^{2}}\right)^{2}+\left(\frac{1-t_{i}}{1+t_{i}^{2}}\right)^{2}-1 & =0 \tag{F2}
\end{align*}
$$

for $i=1, \ldots, N-1$. In Figure 1 , we present an example of the first equation for $N=3$. It is clear that the degree of each equation in this algebrization increases, as one clears out the denominators to get the equations into the polynomial form, but the number of equations and the number of variables remains linear in $N$.

This increase of degree can be partially mitigated by lifting the problem, i.e., introducing additional variables such as $a_{i}:=1+t_{i}^{2}$ and performing the appropriate substitutions. Thereby, one obtains yet different algebrizations.

We are not aware of any applications of such half-angle formulation to the Kuramoto model, although it has been used
under a variety of names in a variety of applications. It is, for example, known as the Weierstrass substitution, and traceable to the work of Euler [20, E342, Caput V, paragraph 261]. Emiris pioneered the use of this transformation in Robotics [21] and Computational Chemistry [22].

Formulation 3. In the present work, we suggest yet another transformation using the trigonometric identity [23, cf. p. 71, 4.3.1]:

$$
\begin{equation*}
\sin \left(\theta_{i}-\theta_{j}\right)=\frac{1}{2 \mathbb{I}}\left(e^{\mathbb{I}\left(\theta_{i}-\theta_{j}\right)}-e^{-\mathbb{I}\left(\theta_{i}-\theta_{j}\right)}\right) \tag{4}
\end{equation*}
$$

where $\mathbb{I}:=\sqrt{-1}$ is the imaginary unit. The equilibrium equations of (1) become

$$
\begin{equation*}
\omega_{i}-\sum_{j=1}^{N} \frac{K_{i, j}}{\mathbb{I} N}\left(e^{\mathbb{I} \theta_{i}} e^{-\mathbb{I} \theta_{j}}-e^{-\mathbb{I} \theta_{i}} e^{\mathbb{I} \theta_{j}}\right)=0 \tag{5}
\end{equation*}
$$

To formulate the equilibrium equations as an algebraic system, we let

$$
\begin{equation*}
x_{i}:=e^{\mathbb{I} \theta_{i}} \quad \text { and } \quad y_{i}:=e^{-\mathbb{I} \theta_{i}} \tag{6}
\end{equation*}
$$

for all $i=1, \ldots, N-1$. With this substitution, (5) becomes an enlarged system of $2(N-1)$ equations in $2(N-1)$ variables

$$
\begin{align*}
\sum_{j=1}^{N} \frac{K_{i, j}}{\mathbb{I} N}\left(x_{i} y_{j}-x_{j} y_{i}\right)=\omega_{i} & \text { for } i=1, \ldots, N-1  \tag{F3}\\
x_{i} y_{i}=1 & \text { for } i=1, \ldots, N-1
\end{align*}
$$

For example, with $N=3$, the system (1) becomes

$$
\begin{align*}
\frac{K_{1,2}}{3 \mathbb{I}}\left(x_{1} y_{2}-x_{2} y_{1}\right)+\frac{K_{1,3}}{3 \mathbb{I}}\left(x_{1}-y_{1}\right)-\omega_{1} & =0 \\
\frac{K_{2,1}}{3 \mathbb{I}}\left(x_{2} y_{1}-x_{1} y_{2}\right)+\frac{K_{2,3}}{3 \mathbb{I}}\left(x_{2}-y_{2}\right)-\omega_{2} & =0  \tag{7}\\
x_{1} y_{1}-1 & =0 \\
x_{2} y_{2}-1 & =0 .
\end{align*}
$$

It can be readily verified that the equilibria of (1) (with the translation symmetry removed) are in one-to-one correspondence with the special solutions of the above system (F3) that satisfy the additional restriction that $\left|x_{i}\right|=\left|y_{i}\right|=1$ for $i=1, \ldots, N-1$.

It is not clear, however, which of these formulations to use in which applications. In this paper, we consider two criteria related to two applications.

## III. Bounds on the Number of EQuilibria

Via the transformations given in (F1) or (F3), the problem of counting equilibria of the Kuramoto model is turned into a root counting problem for systems of polynomial equations. This approach has a long history, going back to [?], [9], [10], who have shown that for $N=3$, the Kuramoto model on a complete graph of 3 nodes has at most 6 equilibria. Perhaps the best-known bound on the number of roots of a system of polynomial equations comes from the theorem of Bézout.

$$
\begin{align*}
& K_{1,2}\left(\frac{2 t_{1}}{1+t_{1}^{2}} \frac{1-t_{2}}{1+t_{2}^{2}}-\frac{2 t_{2}}{1+t_{2}^{2}} \frac{1-t_{1}}{1+t_{1}^{2}}\right)+K_{1,3}\left(\frac{2 t_{1}}{1+t_{1}^{2}} \frac{1-t_{3}}{1+t_{3}^{2}}-\frac{2 t_{3}}{1+t_{3}^{2}} \frac{1-t_{1}}{1+t_{1}^{2}}\right)=3 \omega_{1}  \tag{3}\\
& K_{2,1}\left(\frac{2 t_{2}}{1+t_{2}^{2}} \frac{1-t_{1}}{1+t_{1}^{2}}-\frac{2 t_{1}}{1+t_{1}^{2}} \frac{1-t_{2}}{1+t_{2}^{2}}\right)+K_{2,3}\left(\frac{2 t_{2}}{1+t_{2}^{2}} \frac{1-t_{3}}{1+t_{3}^{2}}-\frac{2 t_{3}}{1+t_{3}^{2}} \frac{1-t_{2}}{1+t_{2}^{2}}\right)=3 \omega_{1}
\end{align*}
$$

Figure 1. A reformulation of the example using $t_{i}:=\tan \frac{\theta_{i}}{2}$.

Bézout's bound is simply the product of the degrees of all the equations. In the example shown in (7), since each of the four equations is quadratic (degree 2), the highest possible number of isolated solutions as given by Bézout's bound is therefore $2^{4}=16$. In general, Bézout's bound for the system (F3) is $2^{2(N-1)}$. Bézout's bound is a basic result in intersection theory [24], the study of how the varieties defined by an ideal given by an algebraic equation intersect one another.

Bi-homogeneous Bézout's bound for the equilibrium equations for (1) can be derived [6], [8], [25] as $\binom{2(N-1)}{N-1}$. In a wide variety of special cases of the Kuramoto model, there are case-specific bounds as well. These include complete graphs [26], nearest-neighbour coupling on one-dimensional lattice graphs [16], [26], [27], e.g.], two-dimensional lattice graphs [16], [17], [28]-[31], and three-dimensional lattice graphs [31], [32], and the homogeneous frequencies case [6].

Notice that these bounds relate only to isolated roots. Casetti et al. [26] have shown that even after fixing the trivial zero mode by setting $\theta_{N}=0$, there may exist infinitely many equilibria, known as incoherent manifolds [33]. Surprisingly, such infinite families of equilibria were also shown to exist in the one-, two- and three-dimensional lattice model with nearest neighbour interaction in [16], [30], [34], where it was demonstrated that the number of infinite families of equilibria grows exponentially in $N$.

## IV. The BKK Bound

In the present contribution, we establish an upper bound on the number of equilibria for (1) using a novel polynomial formulation and the theory of Bernstein [12], Kushnirenko [13], and Khovanski [14], to whom we refer to using the acronym BKK. The advantage of this bound over existing bounds is that it takes into consideration of the sparsity of the connections in the underlying network. This marks a significant leap forward from the recent studies of the Kuramoto model and the closely related load flow equations for electric-power networks [5], [8], [35] from algebraic view points.

First, let us briefly review the BKK work [12]-[14] on the number of isolated non-zero complex solutions which is a refinement of the Bézout bound that takes into consideration the monomials that appear in the polynomial system: Given a polynomial, each of its terms give rise to an exponent vector. For instance, for the term $x^{3} y^{2} z^{1}$, the exponent vector is simply the vector whose entries are the exponents of
$x, y$ and $z$, respectively, i.e., $(3,2,1)$. The choice of this ordering is inconsequential as long as it is kept the same for each equation. The set of all exponent vectors derived from the non-zero terms of an polynomial equation is called the support of that equation. For example, if we arrange the variables in the order of $\left(x_{1}, y_{1}, x_{2}, y_{2}\right)$, then the supports of the four equations in (7) are

$$
\begin{aligned}
& \{(1,0,0,1),(0,1,1,0),(1,0,0,0),(0,1,0,0),(0,0,0,0)\} \\
& \{(0,1,1,0),(1,0,0,1),(0,0,1,0),(0,0,0,1),(0,0,0,0)\} \\
& \{(1,1,0,0),(0,0,0,0)\} \\
& \{(0,0,1,1),(0,0,0,0)\}
\end{aligned}
$$

A convex set is a set of points in which the line segment connecting any pair of points in the set also lie in that set. The convex hull of a set is the minimal convex set containing that set. For a polynomial, the convex hull of its support is known as the Newton polytope of that polynomial. In the study of convex polytopes, the mixed volume of several polytopes is an important concept. which can be considered as a generalization of the concept of volume into the context of several polytopes. Given $n$ convex polytopes $Q_{1}, \ldots, Q_{n} \subset \mathbb{R}^{n}$ and positive real numbers $\lambda_{1}, \ldots, \lambda_{n}$ Minkowski's Theorem states that the $n$-dimensional volume of the Minkowski sum $\lambda_{1} Q_{1}+\cdots+\lambda_{n} Q_{n}$, defined as

$$
\left\{\lambda_{1} q_{1}+\cdots+\lambda_{n} q_{n} \mid q_{i} \in Q_{i} \text { for } i=1, \ldots, n\right\}
$$

is a homogeneous polynomial of degree $n$ in the variables $\lambda_{1}, \ldots, \lambda_{n}$. The coefficient associated with the monomial $\lambda_{1} \cdots \lambda_{n}$ in this polynomial is known as the mixed volume of the polytopes $Q_{1}, \ldots, Q_{n}$. In the simplest case, the mixed volume of two line segments on the plane is precisely the area of the parallelogram spanned by translations of these two line segments. With these definitions, one can state:

Theorem 1 (Bernstein [12]). Given a system of n polynomial equations in $n$ variables, the number of isolated complex solutions for which no variable is zero is bounded above by the mixed volume of the Newton polytopes of the equations.

Recall that via the change of variables given in (6), each (real) equilibrium of (11) corresponds to a unique non-zero complex solution of (F3). The BKK bound given above hence provides an upper bound to the number of isolated (real) equilibria.

Bernstein [12] has also shown that the BKK bound is generically exact: when the coefficients in the polynomial
system are chosen at random, with probability one, the number of isolated complex solutions for which no variable is zero is exactly the BKK bound. In the polynomial formulation of the Kuramoto model given in (F3), if certain relations are imposed on the coefficients (e.g., the coefficients of $x_{1} y_{2}$ and $x_{2} y_{1}$ in (7) must be the same) the generic exactness still holds true under a mild additional condition. This result of [12], translated to the language of Kuramoto model is thus:

Theorem 2. If there exists a choice of $K_{i, j}$ 's and $\omega_{i}$ 's for which the number of non-zero complex solutions of (F3) is the BKK bound, then for almost all choices of complex $K_{i, j}$ 's and $\omega_{i}$, the number of non-zero complex solutions of (F3) will be the BKK bound.

In other words, among the systems (F3) for all possible choices of the $K_{i, j}$ 's and $\omega_{i}$ 's, if the BKK bound is attainable then it must also be generically exact. In Section VI, we shall compute the BKK bound for the polynomial system (F3) induced by a number of graphs. Then, the attainability and hence the generic exactness of the BKK bound in each case is verified by solving the system (F3) for some specific chosen set of $K_{i, j}$ 's and $\omega_{i}$ 's.

## V. The Optimisation Problems

One can also optimise over the system obtained using either of the three substitutions, possibly intersected with additional polynomial inequalities.

Let us denote the algebraic set, which is obtained by intersecting the reformulation of (1) with $r$ additional polynomial inequalities as $K$. Let us consider a polynomial objective function $f(x), x \in K$ and its global minimum $f^{*}$ achieved at one or more $x^{*} \in K$. Then, it is easy to see:

Theorem 3 (Asymptotic Convergence). Whenever $K$ is nonempty and there exists an $M>0$ such that $\|x\|_{\infty}<M$ for all $x \in K$, there exists a hierarchy of semidefinite programming relaxations $\left[S D P_{r}\right]$ their respective duals $\left[S D P_{r}\right]^{*}$ such that the following holds:
(a) $\inf \left[S D P_{r}\right] \nearrow f^{*}$ as $r \rightarrow \infty$,
(b) sup of semidefinite-programming duals of $\left.S D P_{r}\right] \nearrow$ $f^{*}$ as $r \rightarrow \infty$,
(c) if there exists a unique global minimizer $x^{*} \in K$, with respect to $f$, then as $r$ tends to infinity the components of the optimal solution of $\left[S D P_{r}\right]$ corresponding to the linear terms converge to $x^{*}$.

The proofs follow from the seminal work of Lasserre [36]. Although Theorem 3 states just the existence of such a hierarchy, there are readily available algorithms [37] for constructing the hierarchy and computing an arbitrarilyaccurate approximation of $\left[\mathrm{SDP}_{r}\right]$. Albeit non-trivial, the algorithms have been implemented successfully [15], [38, e.g.]. Under slightly stronger assumptions [39], one can also show finite convergence.

Notice that the inequalities are often bounds on the phasedifference of adjacent oscillators:

$$
\begin{equation*}
\left|\theta_{i}-\theta_{j}\right| \leq z_{i j} \tag{8}
\end{equation*}
$$

for some constant $z_{i j}$. In power-systems applications, for instance, such inequalities bound losses, thermal limits, and allow for a certain realism of the equilibria. In the traditional reformulation, the inequality (8) becomes

$$
\begin{equation*}
\left(s_{i} c_{j}-s_{j} c_{i}\right) \leq \arcsin \left(z_{i j}\right) \tag{9}
\end{equation*}
$$

where $\arcsin \left(z_{i j}\right)$ is clearly a constant, which can be precomputed. In the second reformulation, the constraint (8) becomes a bound on $x_{i}-y_{i}$. In the third reformulation, the constraint (8) becomes:

$$
\begin{equation*}
t_{i} \leq \arctan \left(z_{i j}\right) \tag{10}
\end{equation*}
$$

with care needed to consider the appropriate orthant, but again with a constant right-hand side. Still, this leaves the question as to which reformulation to use open.

## VI. Computational Results

First, we compare the BKK bound for both formulations (F1) and (F3) on the sparsest connected graphs, known as the path graphs. In a path graph, the $i$-th node for $1<i<N-1$ is connected to two of its neighbors: the $(i-1)$-th node and the $(i+1)$-th node forming a path. The results are presented in Table [

In particular, the generic root count of Table $\mathbb{\square}$ presents the results of experiments using a numerical polynomial homotopy continuation method (NPHC). The NPHC method guarantees that one will obtain all isolated complex solutions for a system of polynomial equations by following the following strategy [40], [41]: to solve a system of polynomial equations, one starts with an upper bound on the number of complex solutions of the system. Then, another system is created such that the system has exactly the same number of complex solutions as the upper bound, and it is easy to solve. Finally, each solution of this new system is evolved over a single parameter towards the system to be solved. In particular, we have used the computational packages HOM4PS-3.0 of [42] and [43], as well as Bertini of [44], [45].

Second, we compare the performance of optimisation methods suggested in Theorem 3 on formulations (F1) and (F3) in Table [II) (F3) turns out to perform better than (F1), due to the numbers of variables being similar in (F3) and (F1), while the degrees of some of the monomials in (F3) are lower than in (F1).

More specifically, Table II presents the dimensions of SDP relaxations obtained using SparsePOP of [38] at the first applicable level of the hierarchy of Theorem 3 for generic parameter-values for the Kuramoto model on the path graph. Note that at $r=1$, the hierarchies of [38] and [36] coincide. The $m \times n$ constraint matrix $A$ of the

Table I
COMPARISON AMONG DIFFERENT BOUNDS ON THE NUMBER OF EQUILIBRIA AND THE ACTUAL NUMBER OF COMPLEX SOLUTIONS FOR GENERIC PARAMETER-VALUES FOR THE KURAMOTO MODEL ON THE PATH GRAPH.

| Nodes | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Bézout's | 16 | 64 | 256 | 1024 | 4096 | 16384 | 65536 | 262144 | 1048576 | 4194304 | 16777216 |
| Bi-h. Bézout's | 6 | 20 | 70 | 252 | 924 | 3432 | 12870 | 48620 | 184756 | 705432 | 2704156 |
| BKK for (F1) | 8 | 24 | 80 | 256 | 832 | 2688 | 8704 | 28160 | 91136 | 294912 | 954368 |
| BKK for (F3) | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
| Generic root count | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |

SDP relaxation is described by the product $m n$ (dim. of $A$ ) and the number of non-zero entries therein (nnz. of $A$ ). These measures influence the memory requirements of any solver. Additionally, we list the maximum $n_{i}$ among $n_{i} \times n_{i}$ positive semi-definite blocks (max. PSD block), which influence the run-time of primal-dual interior-point methods such as SeDuMi [46].

## VII. Discussion and Conclusion

In this article, we have reformulated the stationary equations of the Kuramoto model to polynomial equations in three different ways. One of the reformulations is novel. The so-called half-angle transform has been used across several fields, but has not been applied to the Kuramoto model, as far as we know. Finally, one of the reformulatins is implicit in much related work on the Kuramoto model. All three allow for the use of results from semi-algebraic and algebraic geometry.

In terms of structural results, we have provided a prescription to compute an upper bound on the number of equilibria of the model for a given graph topology, which we called the BKK bound. For the complete graph with arbitrary (and inhomogeneous) coupling strengths and natural frequencies, this bound matches the best previously available upper bound, $\binom{2(N-1)}{N-1}$. We have demonstrated, however, for sparser graphs such as path graphs, the BKK bound for a new polynomial formulation is significantly lower than bihomogeneous Bézout's bound, as well as the BKK bound for the traditional polynomial formulation.

We also provide constructive results. We show how to use the reformulations with the so-called method of moments, which has been developed in semi-algebraic geometry, and which makes it possible to optimise over the stationary equations and a variety of further equalities and inequalities. We also demonstrate the computational trade-offs of using the three reformulations. This may often be preferable to the use of homotopy continuation methods. Where homotopy continuation methods are used, the BKK bound can be considered as means of constructing a starting system in solving the stationary equations. The bound also provides a concrete stopping criterion to any stochastic method for solving the non-linear equation.

Considering that systems with sine and cosine of angles and difference of angles are not unique to the Kuramoto
model, this can have far-reaching implications. For instance, we point out a remarkable parallel between upper bounds for the equilibria of the Kuramoto model and those for the equilibria of the complete power flow equations [47], [48]. We should also like to point out that BKK-like results have been shown for affine spaces as well [49]-[52], which could perhaps be used.

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Table II
COMPARISON AMONG THE DIMENSIONS OF SDP INSTANCES OBTAINED AT THE FIRST APPLICABLE LEVEL OF THE HIERARCHY OF THEOREM 3 FOR GENERIC PARAMETER-VALUES FOR THE KURAMOTO MODEL ON THE PATH GRAPH.

| Nodes | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Dim. of $A$ for $\overline{F 1}$ | 200 | 450 | 800 | 1250 | 1800 | 2450 | 3200 | 4050 | 5000 | 6050 | 7200 | 8450 |
| Nnz. of $A$ for $\overline{F 1}$ | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 130 |
| F1 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

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