Convergence of full-order observers for the slow states of a singularly perturbed system (Part II: Applications)

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Abstract— **Many natural and engineered systems exhibit a singularly perturbed structure where different time scales inherently lead to difficulties in the design of observers for the system. In our related work [1], we have shown that, under appropriate assumptions, an observer designed for the slow part of the system can be applied and results in semi-global practical asymptotical (SPA) stability of the estimation error. In this paper, we show that assumptions from [1] hold for two classes of plants and nonlinear observers. In fact, we show that the provided framework in [1] covers current results in the literature and also other cases that are not covered by existing results. Hence, we demonstrate that we generalise existing results in the literature.**

I. Introduction

In our companion paper [1], we study the performance of nonlinear observers used to estimate the slow states of singularly perturbed systems. We analyse the robustness, with respect to singular perturbations, of an observer designed using the reduced (slow) system without considering the fast states and implemented on the original plant. We provide an estimation framework for observer design for singularly perturbed systems in a sense that if the plant and the observer satisfy an appropriate set of assumptions, the design approach results in a SPA stability of the error dynamics where the estimation error is defined as the difference between the estimates and the slow states of the full system. The results in Part I are stated for a general class of plants in the following standard form

$$\dot{x} = f_s(t, x, z, u, \epsilon), \tag{1a}$$

$$\epsilon\dot{z} = f_f(t, x, z, u, \epsilon), \tag{1b}$$

$$y = h(t, x, z, u, \epsilon), \tag{1c}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ are the slow and fast state variables respectively, $y \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^r$ is the control input and $\epsilon > 0$ is the perturbation parameter of the system representing the time-scale separation.

Here, we demonstrate the results in [1] naturally cover two classes of plants and two nonlinear full-order observers. We validate that our framework applies for the class of systems and the nonlinear observer covered by current results in [2]. We show that our approach leads to SPA stability which is a stronger result than results in [2]. Whilst [2] only presents local results for a specific class of plants and one observer; we consider a range of different stability properties: local, regional and SPA stability. Even when we restrict our attention to the same class of systems considered in [2], our results are more general; furthermore, our results cover a much larger class of plants and observers than [2]. In this manuscript, we verify that [1] covers another class of systems and an observer that cannot be covered by [2]. We have also checked that our results apply for plants in which the reduced (slow) system is such that nonlinear observers in [4] - [10] can be used to estimate the slow variables. As far as we are aware, there are no existing results dealing with SPA stability. Hence, two papers (Part I and II) demonstrate and support the generality and usefulness of our design approach.

In section II, we analyse the class of plants that were considered in [2] to show that our results cover that case. We also consider the Luenberger-type nonlinear observer analysed in [2], see [3]. We show that our assumptions and results in [1] hold for that class of plants and that nonlinear observer. In section III, we consider a class of plants with fast linear dynamics and with reduced (slow) models that fit the framework of the circle criterion observer [4]. This class of plants covers a wide range of engineering problems; for instance, systems with sensors that can be modeled as fast linear dynamics [2].

Notation: The (Euclidean) norm of a vector $x \in \mathbb{R}^n$ is denoted as $|x|$. We say that $s \in \mathcal{L}_\infty$ if $||s||_\infty < \infty$, where $||s||_\infty := \operatorname{ess\,sup}_t |s(t)|$. The minimum eigenvalue of a square matrix A is denoted by $\lambda_{\min}\{A\}$.

II. Luenberger-type nonlinear observer

In this section, we demonstrate that our results in [1] cover the results reported in [2]. We show a stronger result under stronger assumptions than results in [2]. We state SPA stability and regional results. Even more if we relax some of our assumptions the results can be stated as local.

A. *Class of plants*

We consider here plants that constitute a subclass of the general class of systems (1). We study the class of plants considered in [2], such that the reduced model takes the form for which results in [3] apply. Consider the nonlinear singularly perturbed system in standard form considered in [2]

$$\dot{x} = f(x, z) \tag{2a}$$

$$\epsilon\dot{z} = M_1 x + M_2 z, \tag{2b}$$

$$y = C_1 x + C_2 z, \tag{2c}$$

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where $x \in \mathbb{R}^n$ is the slow state, $z \in \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the output, ϵ is the perturbation parameter, and C_1, C_2, M_1 and M_2 are matrices of appropriate dimensions.

Assumption I. *The map $f(x,z)$ is a real analytic vector function defined on $\mathbb{R}^n \times \mathbb{R}^m$. Moreover, $f(x,z)$ is globally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$ and $f(0,0) = 0$.*

Remark I. For simplicity, we assume a global Lipschitz property for $f(x,z)$ on $\mathbb{R}^n \times \mathbb{R}^m$. This assumption is justified as we intend to demonstrate that [Theorem 1, 1] leads to SPA stability rather than local stability as done in [2]. Although our conclusion is stronger under stronger assumptions than [2], our approach is such that we can easily state local results like [2] or even regional results that cannot be obtained from [2].

Assumption II. *The matrix M_2 in (2b) is a Hurwitz matrix.*

The matrix M_2 must satisfy Assumption II to guarantee the model reduction in the singular perturbation framework. Note that the linear fast dynamics (2b) covers a wide range of engineering applications [2].

B. Verification of [Assumptions 1 - 5, 1]

We now check that [Assumptions 1 - 5, 1] are satisfied for the class of plants (2). Note that [Assumption 1, 1] states that the input and its derivative belong to $\mathcal{L}_\infty$. It is observed that there are no inputs to the system (2). Therefore, it follows that [Assumption 1, 1] trivially holds relying on [Remark 2, 1].

We follow the standard singular perturbation technique to obtain the reduced and boundary layer systems. We set $\epsilon = 0$ such that the system is restricted to the slow manifold

$$M_1 x + M_2 z = 0. \tag{3}$$

Then, it follows that, $H(x) = -M_2^{-1} M_1 x$, is an isolated solution of (3). As a consequence, the system satisfies [Assumption 2, 1] which states that there exists an analytical isolated solution for the slow system. By using the isolated solution $H(x)$, we have that the reduced system is given by

$$\dot{x} = f(x, -M_2^{-1} M_1 x), \tag{4a}$$

$$y_s = Cx. \tag{4b}$$

where $C = C_1 - M_2^{-1} M_1$. We now define the change of variables $z = \xi - M_2^{-1} M_1 x$. Note that the original system (2) in the new variables (x, ξ) is given by

$$\dot{x} = f(x, \xi - M_2^{-1} M_1 x), \tag{5a}$$

$$\epsilon\dot{\xi} = M_2 \xi + \epsilon(M_2^{-1} M_1) f(x, \xi - M_2^{-1} M_1 x), \tag{5b}$$

$$y = Cx + C_2 \xi. \tag{5c}$$

By using the fast time-scale $\tau = t/\epsilon$ and setting $\epsilon = 0$, we have that the boundary layer system has the following dynamics

$$\frac{d\xi}{d\tau} = M_2 \xi, \tag{6}$$

Assumption III. *The reduced system (4) is globally exponentially stable.*

Results in [2] hold under the assumption of local exponential stability of the reduced system (4). This assumption is used to ensure closeness of solutions on the infinite time interval of the solutions of (2) with respect to the solutions of the reduced system (4) and the boundary layer system (6). Although a local version of Assumption IV is assumed in [2], we assume a global version to prove a stronger result (SPA stability) for the same class of plants and observer.

The analyticity of $f(x, -M_2^{-1} M_1 x)$ implies continuous differentiability. Moreover, it follows from [Lemma 3.3, 11] that the global Lipschitz property of $f(x, -M_2^{-1} M_1 x)$ implies that the Jacobian matrix $[\partial f / \partial x]$ is bounded on $\mathbb{R}^n$. Then, by virtue of Assumption IV, we have from [Theorem 4.14, 11] that there is a function $V_1(x)$ that satisfies the inequalities

$$c_1 |x|^2 \le V_1(x) \le c_2 |x|^2 \tag{7}$$

$$\frac{\partial V_1}{\partial x} f(x, -M_2^{-1} M_1 x) \le -c_3 |x|^2 \tag{8}$$

$$\left| \frac{\partial V}{\partial x} \right| \le c_4 |x| \tag{9}$$

for some positive constants c_1, c_2, c_3 and c_4. Therefore, it follows that [Assumption 3, 1] holds.

In our framework, [Assumption 4, 1] states that there is a Lyapunov function for the boundary layer system satisfying certain conditions that imply uniform asymptotical stability. Since M_2 is Hurwitz, we have from [Theorem 4.6, 11] that for any given positive definite symmetric matrix Q_ξ there exists a positive definite symmetric matrix P_ξ that satisfies the following Lyapunov equation

$$P_\xi M_2 + M_2^T P_\xi = -Q_\xi. \tag{10}$$

To check [Assumption 4, 1], consider $W(\xi) = \xi^T P_\xi \xi$ as a candidate Lyapunov function for (6). It follows that

$$\frac{\partial W}{\partial \xi} M_2 \xi \le -\lambda_{\min}\{Q_\xi\} |\xi|^2, \tag{11}$$

Therefore, [Assumption 4, 1] is satisfied with $\underline{\alpha}_W(|\xi|) = \lambda_{\min}\{P_\xi\}|\xi|^2$ and $\overline{\alpha}_W(|\xi|) = \lambda_{\max}\{P_\xi\}|\xi|^2$ as the lower and upper bounds for $W(\xi)$ respectively, and with $\zeta_3 = \lambda_{\min}\{Q_2\}$ and $\alpha_W(|\xi|) = |\xi|$ as the terms satisfying (11).

The [Assumption 5, 1] gives a set of interconnection conditions to bound the terms that represent the interconnection between the reduced and the boundary layer systems. Since $f(0,0) = 0$, it follows from the global Lipschitz property of $f(x,z)$ that $|f(x, \xi - M_2^{-1} M_1 x)| \le L_1 |x| + L_2 |\xi - M_2^{-1} M_1 x|$ and $|f(x, \xi - M_2^{-1} M_1 x) - f(x, -M_2^{-1} M_1 x)| \le L_3 |\xi|$ hold for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$ for some non-negative constants L_1, L_2, and L_3. Moreover, from Converse Theorem [Theorem 4.14, 11], we know the Lyapunov function $V_1(x)$ satisfies (9).

It follows that [eq. (11), 1] is satisfied with $a_1 = 0$, $\gamma_1(\cdot) = 0$ and $b_2 = cL_3$, and that [eq. (13), 1] holds with $a_3 = 2L_2 |P_\xi| |M_2^{-1} M_1|$, $b_3 = 2(L_1 + L_2 |M_2^{-1} M_1|) |P_\xi| |M_2^{-1} M_1|$, $\gamma_3(\cdot) = 0$ and $\gamma_4(\cdot) = 0$. Note that for [eq. (12), 1] we have that $a_2 = 0$, $b_2 = 0$, and

$\gamma_2(\cdot) = 0$ since there is no input to the system and the slow and fast parts do not depend on the perturbation parameter. Hence, [Assumption 5, 1] holds.

C. Observer design and verification of [Assumptions 6 and 7, 1]

We now consider the Luenberger-type nonlinear observer introduced in [3] with the following dynamics

$$\dot{\hat{x}} = f(\hat{x}) + L(\hat{x})(y - \hat{y}), \tag{12}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer state and an estimate of $x \in \mathbb{R}^n$, and $\hat{y}$ is the estimated output. The state-dependent gain $L(\hat{x})$ is defined as

$$L(\hat{x}) = \left[\frac{\partial T}{\partial \hat{x}}\right]^{-1} B, \tag{13}$$

where $T(\hat{x})$ is a solution to a system of partial differential equations given by

$$\frac{\partial T}{\partial \hat{x}} f(\hat{x}) = AT(\hat{x}) + BC\hat{x}, \tag{14}$$

with A and B being matrices of appropriate dimensions, and A being Hurwitz.

It is shown in [2], that a coordinate transformation given by $w = T(x)$ leads to a linear error dynamics when we define the estimation error as $e = w - \hat{w}$. It can be shown that the observation error for the reduced system (4) has the following linear dynamics

$$\dot{e} = Ae. \tag{15}$$

The [Assumption 6, 1] states that there is a Lyapunov function for the error dynamics that satisfies a set of conditions implying an asymptotical stability for the estimation error. Since A is Hurwitz, it follows from [Theorem 4.6, 11] that for any given positive definite symmetric matrix Q_e there exists a positive definite symmetric matrix P_e that satisfies the following Lyapunov equation

$$P_e A + A^T P_e = -Q_e, \tag{16}$$

Consider a quadratic Lyapunov function candidate $V_2(e) = e^T P_e e$. Then, the derivative of $V_2(e)$ along the trajectories of the linear system (15) is bounded as follows

$$\frac{\partial V_2}{\partial e}(Ae) \leq -\lambda_{\min}\{Q_e\}|e|^2. \tag{17}$$

Hence, [Assumption 6, 1] holds with $\underline{\alpha}_{V_2}(|e|) = \lambda_{\min}\{P_e\}|e|^2$ and $\overline{\alpha}_{V_2}(|e|) = \lambda_{\max}\{P_e\}|e|^2$ being the lower and upper bounds for $V_2(e)$ respectively, with $\zeta_2 = \lambda_{\min}\{Q_e\}$ and $\alpha_{V_2}(|e|) = |e|$ being the elements satisfying (17), and with $\hat{\zeta}_2 = 2|P_e|$ being the constant that multiplies $\alpha_{V_2}(\cdot)$ to bound the norm of the gradient of $V_2(e)$ with respect to e.

When the observer (12) is implemented on the full system, the error dynamics is not linear anymore and is given by

$$\dot{e} = Ae - BC_2\xi + \frac{\partial T}{\partial x}\Big[f(x, \xi - M_2^{-1}M_1 x) - f(x, -M_2^{-1}M_1 x)\Big]. \tag{18}$$

Without loss of generality assume that $|\partial T/\partial x| \leq L_4$ holds for all $x \in \mathbb{R}^n$ with $L_4 > 0$. As pointed out above, we have that $|f(x, \xi - M_2^{-1}M_1x) - f(x, -M_2^{-1}M_1x)| \leq L_3|\xi|$ holds for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$ with $L_3 > 0$. Then, we have that the norm of the difference between (15) and (18) is bounded as follows

$$\Big| Ae - BC_2\xi + \frac{\partial T}{\partial x}\Big[f(x, \xi - M_2^{-1}M_1 x) - f(x, -M_2^{-1}M_1 x)\Big] - Ae\Big| \leq |BC_2||\xi| + L_3L_4|\xi|.$$

Then, it follows that [Assumption 7, 1] holds with $a_4 = 0$, $a_5 = 0$, $b_4 = 2|P_e|(|BC_2| + L_3L_4)$, and $\gamma_5(\cdot) = 0$. Since all the assumptions are satisfied, we conclude that, by [Theorem 1, 1], the estimation error dynamics (18) is SPA stable.

Remark II. The global Lipschitz assumption on Assumption I can be relaxed to local Lipschitz condition and then we can still conclude SPA stability. Moreover, the bound on $|\partial T/\partial x|$ can be relaxed to a semiglobal type bound like local Lipschitz. Note that with some extra conditions the local version of Assumption I can be obtained from the analyticity of $f(x, z)$.

Remark III. Our results can be stated as local or regional if we relax our assumptions in [1].

1) Exponential stability of the error dynamics on the transformed coordinates implies asymptotical stability on the original ones: Define the estimation error in the original coordinates as $\overline{e} = x - \hat{x}$. Note that the estimation error $e = w - \hat{w}$ is defined in the transformed coordinates $w = T(x)$. Hence, we have that $e = \overline{T}(\overline{e})$ where $\overline{T}(\overline{e}) = T(x) - T(\hat{x})$. Since $T(x)$ is continuously differentiable, it follows $\overline{T}(\overline{e})$ is continuously differentiable too. Moreover, $T(0) = 0$ implies that $\overline{T}(0) = 0$. Since [Assumption 6, 1] holds, it follows that the error dynamics in terms of the transformed variables satisfy[1]

$$|e(t)| \leq \beta_e(|e_0|, t), \tag{19}$$

where $\beta_e \in \mathcal{KL}$. Since the $\overline{T}(0) = 0$ and $\overline{T}(\overline{e})$ is continuous, then there exist $\underline{\alpha}_{\overline{T}}(\cdot), \overline{\alpha}_{\overline{T}}(\cdot) \in \mathcal{K}_\infty$ such that $\underline{\alpha}_{\overline{T}}(|\overline{e}|) \leq |\overline{T}(\overline{e})| \leq \overline{\alpha}_{\overline{T}}(|\overline{e}|)$. Then, it follows from (19) that

$$|\overline{T}(\overline{e}(t))| \leq \beta_e(\overline{\alpha}_{\overline{T}}(|\overline{e}_0|), t). \tag{20}$$

Since $\underline{\alpha}_{\overline{T}}(|\overline{e}|) \leq |T(\overline{e})|$, it follows from (20) that there is a class-$\mathcal{KL}$ function $\beta_{\overline{e}}(|\overline{e}_0|, t) := \underline{\alpha}_{\overline{T}}^{-1}(\beta_e(\overline{\alpha}_{\overline{T}}(|\overline{e}_0|), t))$ such that

$$|\overline{e}(t)| \leq \beta_{\overline{T}}(|\overline{e}_0|, t). \tag{21}$$

Then, from the Converse Theorem [Theorem 4.16, 11], it follows that the there is a continuously differentiable function $V_e(\overline{e})$, different from $V_2(e)$, that satisfies the inequalities $\underline{\alpha}_{\overline{e}}(|\overline{e}|) \leq V_e(\overline{e}) \leq \overline{\alpha}_{\overline{e}}(|\overline{e}|)$, $\frac{\partial V_e}{\partial \overline{e}} f_e(x, \overline{e}) \leq -\alpha_{\overline{e}}(|\overline{e}|)$, $|\frac{\partial V}{\partial \overline{e}}| \leq \tilde{\alpha}_{\overline{e}}(|\overline{e}|)$ where $\underline{\alpha}_{\overline{e}}(\cdot), \overline{\alpha}_{\overline{e}}(\cdot), \alpha_{\overline{e}}(\cdot), \tilde{\alpha}_{\overline{e}}(|\overline{e}|) \in \mathcal{K}_\infty$, and $f_e(x, \overline{e})$ is the error dynamics in the $\overline{e}$ variable.

[1] In the sequel, $e_0 := e(0)$. The same apply for the other states.

Then, it follows that the analysis can be done either by using the error dynamics with the transformed coordinates or by using the one in the original variables.

III. Circle Criterion Observer

In this section, we consider a class of singularly perturbed plants where the reduced (slow) model takes the form in which results from [4] can be applied to design a full-order observer. This class of plants is covered by the general model (1). Note that this class of plants and the observer are not covered by [2].

A. Class of plants

Consider the class of plants with the following nonlinear singularly perturbed form

$$\dot{x} = Ax + G\gamma(Fx) + \sigma(y, u) + Bz, \tag{22a}$$

$$\epsilon\dot{z} = M_1 x + M_2 z, \tag{22b}$$

$$y = C_1 x + C_2 z, \tag{22c}$$

where the state vector $x \in \mathbb{R}^n$ corresponds to the slow state, $z \in \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the measured output variable, $u \in \mathbb{R}^r$ is the control input, ϵ is the perturbation parameter of the process, $\gamma(\cdot)$ a nondecreasing locally Lipschitz function and A, B, G, F, C_1, C_2, M_1 and M_2 are matrices of appropriate dimensions.

We require a linear dynamics in (22b) for two reasons: 1) it is easier to compute the slow manifold, and 2) with a linear fast dynamics we end up with a reduced model that exhibits a structure for which we can design a circle criterion observer [4].

Assumption IV. *The matrix M_2 in (22b) is a Hurwitz matrix.*

Assumption V. *The functions $\gamma(\cdot)$ and $\sigma(\cdot, \cdot)$ are globally Lipschitz.*

Assumption V over $\sigma(\cdot, \cdot)$ is useful to prevent the solutions of x from escaping to infinity in a finite time [4]. Later is stated through a remark that Assumption V can be relaxed.

B. Verification of [Assumptions 1 - 5, 1]

We now check the given assumptions in [1] for the class of systems represented by (22). Note that [Assumption 1, 1] requires $u, \dot{u} \in \mathcal{L}_\infty$. It is observed that no condition is needed for $\dot{u}$ because the fast dynamics does not depend on u.

Assumption VI. The input belongs to $u \in \mathcal{L}_\infty$.

It follows from the above assumption that [Assumption 1, 1] holds. To obtain the lower dimensional systems, we set $\epsilon = 0$ such that the system is restricted to the slow manifold (3). Then, [Assumption 1, 1] holds with $H(x) = -M_2^{-1}M_1 x$ which always exists. By using $H(x)$, we have that the reduced system is given by

$$\dot{x} = A_0 x + G\gamma(Fx) + \sigma(y_s, u), \tag{23a}$$

$$y_s = Cx, \tag{23b}$$

where $A_0 = A - BM_2^{-1}M_1$ and $C = C_1 - C_2 M_2^{-1}M_1$, and the pair (A, C) is detectable.

Assumption VII. *The reduced system (23) is input-to-state practical stable (ISpS), such that there exists a Lyapunov ISpS function that satisfies [Asumption 3, 1].*

We require Assumption VII because there is no need for A_0 to be Hurwitz. Assumption VII allows more generality for the matrix A_0.

We now define the change of variables $z = \xi - M_2^{-1}M_1 x$. Then, the original system (22) in the (x, ξ) variables is given by

$$\dot{x} = Ax + G\gamma(Fx) + \sigma(y, u) + B(\xi - M_2^{-1}M_1 x), \tag{24a}$$

$$\begin{aligned} \epsilon\dot{\xi} = {} & M_2\xi + \epsilon(M_2^{-1}M_1)[Ax + G\gamma(Fx) + \sigma(y, u) \\ & + B(\xi - M_2^{-1}M_1 x)], \end{aligned} \tag{24b}$$

$$y = Cx + C_2\xi \tag{24c}$$

By expressing (24) in the fast time-scale $\tau = t/\epsilon$, we have that the boundary layer system at $\epsilon = 0$ is given by (6). It follows, under the same study as in Section II-B, that [Assumption 4, 1] holds with $\underline{\alpha}_W(|\xi|) = \lambda_{\min}\{P_\xi\}|\xi|^2$ and $\overline{\alpha}_W(|\xi|) = \lambda_{\max}\{P_\xi\}|\xi|^2$ being the lower and upper bounds for $W(\xi)$ respectively, and with $\zeta_3 = -\lambda_{\min}\{Q_\xi\}$ and $\alpha_W(|\xi|) = |\xi|$ being the terms bounding the derivative of $W(\xi)$ as in (11).

Assumption VIII. *The full system (24) satisfies the interconnection conditions in [Assumption 5, 1].*

The functions upper bounding the interconnection conditions come from [Assumptions 3 and 4, 1]. We need Assumption VIII due to the generality of the Lyapunov function in Assumption VII.

C. Observer design and verification of [Assumptions 6 and 7, 1]

We now consider the circle criterion observer proposed in [4], for the reduced order model (23), with the following dynamics

$$\dot{\hat{x}} = A_0\hat{x} + L(C\hat{x} - y) + G\gamma(F\hat{x} + K(C\hat{x} - y)) + \sigma(y, u), \tag{25}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer's state and an estimate of the state, K and L are gain matrices of appropriate dimensions which must be designed. By following the approach described in [1], the observer (25) must be designed for the reduced system (23), and then implemented on the full singularly perturbed system (24). Therefore, we design an observer for (23) by using the approach in [1]. Define the estimation error as $e := x - \hat{x}$. It follows that the error dynamics is given by

$$\dot{e} = (A_0 + LC)e + G[\gamma(Fx) - \gamma(F(x - e) - KCe)]. \tag{26}$$

To check [Assumption 6, 1], we consider the Lyapunov function $V_2(e) = e^T P_3 e$, where $P_3 = P_3^T > 0$. The matrix P_3 is obtained by solving the following LMI from [4]

$$\begin{bmatrix} \overline{A}^T P_3 + P_3\overline{A} + \hat{\nu} & P_3 G + (F + KC)^T\Lambda \\ G^T P_3 + \Lambda(F + KC) & 0 \end{bmatrix} \leq 0, \tag{27}$$

where $\Lambda > 0$ is a diagonal matrix and an observer design parameter, $\hat{\nu} > 0$ is also an observer design parameter and

$\overline{A} = (A_0 + LC)$. When the LMI in (27) is satisfied, it follows that [4]

$$\frac{\partial V_2}{\partial e} f_e(x,e) \leq -\hat{\nu}|e|^2, \tag{28}$$

with $f_e(x,e) = (A_0 + LC)e + G[\gamma(Fx) - \gamma(F(x - e) - KCe)]$. Then, [Assumption 6, 1] holds with $\underline{\alpha}_{V_2}(|e|) = \lambda_{\min}\{P_3\}|e|^2$ and $\overline{\alpha}_{V_2}(|e|) = \lambda_{\max}\{P_3\}|e|^2$ being the lower and upper bounds for $V_2(|e|)$ respectively, with $\zeta_2 = \hat{\nu}$ and $\alpha_{V_2}(|e|) = |e|$ being the elements that satisfy the bound in (28), and with $\hat{\zeta}_2 = 2|P_3|$ being the constant that multiplies $\alpha_{V_2}(\cdot)$ to bound the norm of the gradient of $V_2(e)$ with respect to e.

When the observer (25) is implemented on the full system (24), the error dynamics becomes

$$\begin{aligned} \dot{e} &= (A_0 + LC)e + G\gamma(Fx) + B\xi + LC_2\xi \\ &\quad - G\gamma(F(x-e) - K(Ce + C_2\xi)). \end{aligned} \tag{29}$$

By considering (26) and (29), we have that [Assumption 7, 1] holds with $a_4 = 0$, $a_5 = 0$, $b_4 = 2(|P_1||B + LC_2| + L_1|P_1G||KC_2|)$, and $\gamma_5(\cdot) = 0$.

We have checked that all assumptions in [1] hold for plants in the form of (22) and the circle criterion observer (25). Therefore, we conclude that the error dynamics (29) are SPA stable when the observer is implemented on the original system, see [Theorem 1, 1].

Remark IV. The global Lipschitz assumption on $\gamma(\cdot)$ and $\sigma(\cdot,\cdot)$ can be relaxed to local Lipschitz condition and then we can still conclude SPA stability.

Remark V. If the matrix A_0 is Hurwitz, [Assumption 3, 1] holds with $V_1(x)$ being a quadratic Lyapunov function. Moreover, it is straightforward to find the functions and constants for which [Assumption 5, 1] is satisfied.

D. Example: A class of chemical systems with a fast linear sensor

In this subsection, we study a class of chemical systems with a linear fast sensor covered by the class of system in the form of (22). First, we show that the system satisfies [Assumptions 1 - 5, 1]. Then, we design the circle criterion observer (25) for the reduced system, and check [Assumptions 6 and 7, 1]. Simulations results are presented too.

1) Model of the system: Consider the model for two chemical species given by

$$\dot{x}_1 = \frac{1}{t_r}(x_2 - x_1), \tag{30a}$$

$$\dot{x}_2 = -x_2^3 + \mu x_2 - \lambda - kx_1, \tag{30b}$$

where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ corresponds to the rate reactions of two chemical species, the constants μ, λ and k are parameters of the system, and t_r is the relaxation time for x_1 to approach x_2. The model in (30) represents a number of chemical systems, particularly those containing autocatalytic reactions, when the reaction is carried out in a continuous-flow stirred tank reactor [12].

Assumption IX. *The species x_1 evolve in the same time-scale of the species x_2, i.e., $t_r = 1$.*

Reactions that occur at the same rate are common in chemical processes as the SCR catalyst [13], and also in biological systems as in the predator-pray system [12], [14].

Assumption X. *The output of the system (30) is measured by using a sensor with a linear fast dynamics given by*

$$\epsilon\dot{z} = x_1 - z. \tag{31}$$

Sensors with a fast linear dynamics are common in chemical and biological reactors [2]. Under Assumptions IX and X we have that the full singularly perturbed system is

$$\dot{x}_1 = x_2 - x_1, \tag{32a}$$

$$\dot{x}_2 = -x_2^3 + \mu x_2 - \lambda - kx_1, \tag{32b}$$

$$\epsilon\dot{z} = x_1 - z, \tag{32c}$$

$$y = z. \tag{32d}$$

Note that we can rewrite (32) in the form of (22) with

$$A = \begin{bmatrix} -1 & 1 \\ -k & \mu \end{bmatrix},\ G = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ F = \begin{bmatrix} 0 & 1 \end{bmatrix},\ B = 0,$$

$M_1 = 1$, $M_2 = -1$, $C_1 = 0$, $C_2 = 1$ and $\sigma(y,u) = [0, -\lambda]^T$.

2) Verification of [1, Assumption 1 - 5]: We found that it is more convenient to check [Assumption 1 - 5, 1] by studying the system in its state space representation given by (32). Note that λ in (32b) can be seen as a constant input, so [Assumption 1, 1] is satisfied.

We set $\epsilon = 0$ so that we obtain from (32c) that the slow manifold is given by: $x_1 - z = 0$. We have that the isolated solution of the algebraic equation is $H(x) = x_1$, which implies that the system (32) satisfies [Assumption 2, 1]. Then, the reduced system is given by

$$\dot{x}_1 = x_2 - x_1, \tag{33a}$$

$$\dot{x}_2 = -x_2^3 + \mu x_2 - \lambda - kx_1, \tag{33b}$$

$$y_s = x_1. \tag{33c}$$

To check [Assumption 3, 1], consider the Lyapunov function $V_1(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ for the slow system (33) . We take the derivative of $V_1(x)$ along the solutions of the reduced system (33). Then, we apply completion of squares and perform algebraic manipulations to obtain

$$\frac{\partial V_2}{\partial x} f(x) \leq -\zeta_1 \alpha_{V_1}^2(|(x_1,x_2)|) + \delta_{V_1}, \tag{34}$$

with $f(x) = [x_2 - x_1, -x_2^3 + \mu x_2 - \lambda - kx_1]^T$ and where $\alpha_{V_1}(|(x_1,x_2)|) = |(x_1,x_2)|$, $\zeta_1 = 1/8$ and $\delta_{V_1} = k^4 + \frac{1}{4}\lambda^2 + 2(\mu^2+1) + \frac{3}{8}$. Then, it follows that [Assumption 3, 1] is satisfied with the given Lyapunov function $V_1(x)$.

Now, consider the change of variables $z = \xi + x_1$ such that, from (32c), we have $\epsilon\dot{\xi} = -\xi - \epsilon(x_2 - x_1)$. Then, in the fast time scale $\tau = t/\epsilon$, the boundary layer system is given by $d\xi/d\tau = -\xi$. In order to verify that the boundary layer system satisfies [Assumption 4, 1], we consider the Lyapunov function $W(\xi) = \frac{1}{2}\xi^2$. By computing the derivative of $W(\xi)$, in the fast time scale, along the solutions of the boundary layer system, we have that

$$\frac{\partial W}{\partial \xi}\xi \leq -\zeta_3 \alpha_W^2(|\xi|), \tag{35}$$

where $\alpha_W(|\xi|) = |\xi|$ and $\zeta_3 = 1$. As a consequence, [Assumption 4, 1] holds.

We now check the interconnection conditions in [Assumption 5, 1]. It can be verified that [eq. (11), 1] is satisfied with $a_1 = 0$, $b_1 = 0$ and $\gamma_1(\cdot) = 0$. We have that [eq. (12), 1] holds with $a_2 = 0$, $b_2 = 0$, and $\gamma_2(\cdot) = 0$, and [eq. (13), 1] is satisfied with $a_3 = 0$, $b_3 = 2$, $\gamma_3(\cdot) = 0$, and $\gamma_4(\cdot) = 0$. Then, [Assumption 5, 1] holds.

3) Observer design and verification of [1, Assumptions 6 and 7]: Let $\mu = 2$, $\lambda = 1$, and $k = 1$. We now aim to design a circle criterion observer (25) for the reduced system (33) which can be rewritten in the form of (23) with

$$A_0 = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}, \; G = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \; H = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and $\sigma(y,u) = [0,-1]^T$. By setting $\nu = 1$ and solving the LMI (27), we obtain that the gain matrices for the observer designed for the reduced system (33) are given by $K = -2.4905$, $L = \begin{bmatrix} -3.2359 \\ -13.5228 \end{bmatrix}$, and that the symmetric matrix $P_2 = \begin{bmatrix} 101.05 & -28.10 \\ -28.10 & 11.28 \end{bmatrix}$. Note that we can conclude from results in Section III-C that [Assumptions 6 and 7, 1] hold.

4) Simulation results: The simulation results for the circle criterion observer implemented on (32) are shown in Figure 1. We present the estimation error performance for three different values of ϵ. It is observed that the estimation error converges to a region around the origin.

IV. Conclusions

This manuscript complements our theoretical results given in our companion paper [1]. We have shown through two classes of systems and nonlinear observers that the given assumptions and results in [1] cover existing results in the literature [2]. Moreover, they apply to a new class of systems not previously covered in the literature when the reduced order model allows for a circle criterion observer.

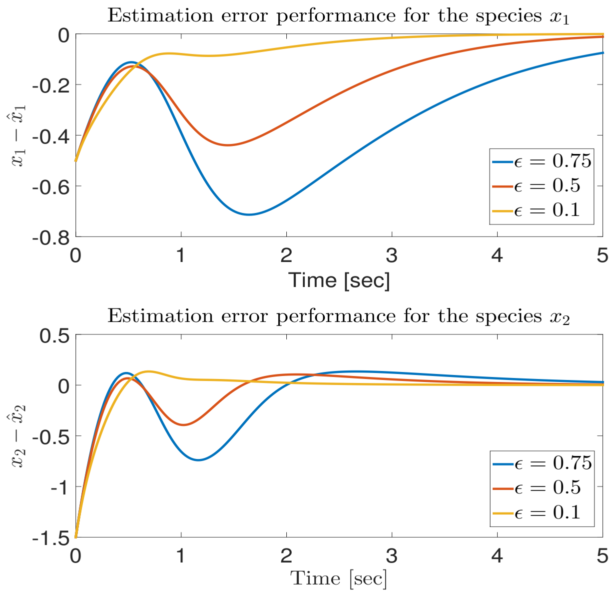

Fig. 1. Estimation error performance when the observer (25) designed for the reduced system (33) is implemented on the original system (32). We considered $[x_1(0), x_2(0)]^T = [1, 0.5]^T$, $z(0) = 0.8$ $[\hat{x}_1(0), \hat{x}_2(0)]^T = [1.5, 2]^T$.

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