Code Design for Non-Coherent Detection of Frame Headers in Precoded Satellite Systems

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Abstract

In this paper we propose a simple method for generating short-length rate-compatible codes over $\mathbb{Z}_M$ that are robust to non-coherent detection for M-PSK constellations. First, a greedy algorithm is used to construct a family of rotationally invariant codes for a given constellation. Then, by properly modifying such codes we obtain codes that are robust to non-coherent detection. We briefly discuss the optimality of the constructed codes for special cases of BPSK and QPSK constellations. Our method provides an upper bound for the length of optimal codes with a given desired non-coherent distance. We also derive a simple asymptotic upper bound on the frame error rate (FER) of such codes and provide the simulation results for a selected set of proposed codes. Finally, we briefly discuss the problem of designing binary codes that are robust to non-coherent detection for QPSK constellation.

Index Terms

non-coherent detection, PLH design, precoding, rotationally invariant codes, Satellite systems

I. Introduction

In this paper we address the problem of short-length rate-compatible (RC) code design for non-coherent channels. In order to motivate the subject, we describe a specific system scenario, namely signaling in a precoded satellite system, where short codes robust to non-coherent detection are required by the system model for physical layer header (PLH). As we will see, in such systems, the phases of the received symbols are changed by an unknown random value which remains constant during the transmission of each codeword. Such a scenario can be considered as a special case of a more general model where the phases are changed at the symbol level following a stochastic process with given parameters. Both coding and modulation design for non-coherent channels have been addressed since very first days of communication theory (see for example [1] and references therein) and hence the literature on the subject is quite rich, addressing varieties of system models and strategies.

Two main approaches are presented in the literature following essentially two different strategies. The first approach is based on pairing a capacity achieving code (LDPC, Turbo, ...) with an optimized constellation space for non-coherent channel. In this approach, the main goal is to find the capacity achieving (optimal) distribution of the signal space for the given channel model. For details on some of the results obtained in this direction we refer the readers to [2]–[4]. It is important to notice that in all these works the constellation is considered to be infinite and the system performance is studied asymptotically. For finite constellation sets, the problem of finding the optimal constellation is in general an open problem. The second strategy is to fix the constellation shape and then trying to design the codebook (and possibly the constellation labeling) for non-coherent detection. This is also the strategy that we pursue in this paper. The main idea in this case is to maximize the so called *non-coherent equivalent distance* of the code[1]. The coding design problem for specific cases of M-PSK, APSK, and QAM constellations has been investigated by several authors, see for example [5]–[10] and references within. In the following, we briefly review some of the above works that are directly related to the problem we are considering and mention their differences compared to our approach.

In [5] the authors study the coding design problem for the non-coherent AWGN channel assuming a M-ary phase shift keying (PSK) modulation scheme. In particular, a coding design over $\mathbb{Z}_M$ based on exclusion of unwanted codewords is proposed and codes up to length $N = 15$ and information vector size $K = 10$ over $\mathbb{Z}_4$ and $\mathbb{Z}_8$ are constructed. In this method there exist an isomorphism between the ring over which the code is defined and the modulation scheme. This isomorphism also implies the constellation labeling. The results in [5] are mainly based on numerical approaches. F-W. Sun and H. Leib in [6] extend these results by providing a analytical framework. This is done by showing the relation between the non-coherent distance and those of Euclidean and Lee distances. In a series of papers, the authors in [7]–[9] study the block-coded modulation for MPSK, QAM and APSK constellations. In particular, the minimum non-coherent distance is obtained separately for each constellation and a given labeling.

The design strategies introduced in previously mentioned works do not perfectly match with the system scenario that we are interested in this paper. Indeed, to the best of our knowledge, no study on the optimal code lengths as a function of K and minimum distance have been presented in the literature. In this paper, we first discuss in some details the system model that we are interested in and motivate the reasons for which short-length codes with non-coherent detection are needed in such

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[1]Notice that the non-coherent distance has been defined in several ways in the literature. In this paper we adopt the definition which is in line with that of [21].

systems. Then, we propose a method to obtain such codes by modifying the generator matrix of rotationally invariant codes. A greedy algorithm is used to optimize the minimum distance of rotationally invariant codes with codeword lengths up to $N = 256$ and information vector lengths up to $K = 15$. The results are reported only for BPSK and QPSK constellations. However, the method can be extended for any M-PSK constellation. These results provide a lower bound on the minimum distance of rotationally invariant codes for a given pair $[N, K]$. Finally, we derive a simple asymptotic upper bound on the frame error rate (FER) of the designed codes. This will help us to choose a code from the table with desired FER for the PLH of the system.

The rest of this paper is organized as follows. In Section II, we describe our system model and define the problem we intend to solve. In Section III, we present the proposed design for the rotationally invariant codes and provide the tables for the optimized codes for information lengths up to $K = 14$. In Section IV we derive a simple upper bound of the error probability as a function of SNR and the minimum non-coherent equivalent distance of a code. We also explain the strategy to build such a codes using the rotationally invariant codes. This upper bound is then used to select the desired codes to design the PLH in Section V, where we also present the simulation results for a selected set of codes. Finally, we conclude the paper in Section VI and discuss some possible future research directions.

II. SYSTEM MODEL AND PROBLEM DEFINITION

Non-coherent detection is an attractive technique in several communication scenarios. Two examples are systems without carrier phase tracking and AWGN channels with flat fading channels when the effects of the phase rotation is considered independently of the amplitude variation. In this paper, we consider a precoded satellite system and explain the reasons for which non-coherent detection of PLH is needed in such system scenarios.

In order to describe our system model we confine ourselves to the specifics of the digital video broadcasting standard (DVB-S2X) [11]. However, our assumptions are valid for a wide range of satellite communication systems (not broadcasting) where adaptive coding and modulation (ACM) is employed. In ACM schemes the forward error correction (FEC) rate and the modulation schemes are selected from a list depending on the channel state information (CSI) in order to maximize the throughput of the channel [12]. At the receiver side, before detecting/decoding the FEC frame, one needs the knowledge about the coding rate and the modulation scheme. These information, referred to as physical layer header (PLH), are encoded and sent before each FEC frame. The PLH headers in DVB-S2X and DVB-T have constant length, however, as it has been shown in [13]–[15] considerable gains on average code lengths may be obtained by employing variable length coding technique for PLH.

In DVB-S2X, in order to track the carrier phase, usually a sequence of pilot symbols are transmitted in between of blocks of data sequences. The pilot symbols are chosen from the QPSK constellation and are known to the receiver. By using techniques such as phase-locked loop (PLL) one can estimate the carrier phase for all data sequences. The residual phase noise after the PLL is usually modeled as white noise and can be further handled by, for example, optimizing the constellation space [16].

Recently, it has been shown that precoding can provide significant gains in multi-beam satellite systems with interference [17], [18]. In this paper, we consider the effect of precoding on the detection of PLH and FEC frame. The main problem is that the precoder matrix, used to reduce interference, is known at transmitter but not at the receiver. The effect is that received signal in the precoded section may be affected by a constant but unknown changes of the phase (and the amplitude) of the received signal. In such cases, the common pilots, not precoded, cannot be used to estimate the phase in the precoded parts. One way to solve this problem is to design the PLH code such that non-coherent detection is possible. In this way, after PLH detection, we can derive an estimation of the phase change due to the precoding matrix for the following FEC block of data. The PLH codes are usually short, having lengths up to few hundreds of symbols (90 symbols for normal MODCODs and 900 symbols for VL-SNR MODCODs in DVB-S2). One of the characteristics of such a system, that distinguish it from the previous system models, is the fact that not all PLH in a super frame may be precoded with the same precoder. Therefore, the phase estimation may be independent from one PLH to another. Our main problem is then to design short finite-block length codes for PLH that are robust to non-coherent detection. In the next section we provide a simple way to construct such family of codes.

The received signal can be written as $y_k = s_k e^{j\theta_k} + n_k$ where s_k is the transmitted symbol, n_k is the additive white Gaussian noise and θ_k is modeled as a random process with first order statistic uniform in $[0, 2\pi]$. Different code design approaches stem from the difference of the coherence time T_θ of the phase process θ_k and the codeword length N. Given the above discussion, in our case, we assume $T_\theta \gg N$ so that we can drop the subscript k from θ and write $y_k = s_k e^{j\theta} + n_k$.

III. RATE COMPATIBLE ROTATIONALLY INVARIANT CODES

We start the section by presenting a simple and general greedy algorithm for constructing families of RC codes with small dimension K over $\mathbb{Z}_2$. We then extend the procedure to construct codes over arbitrary groups $\mathbb{Z}_M$ and show how to particularize it to construct rotationally invariant codes for M-PSK. We also show that the designed code families are competitive with the optimal codes that are known in some cases.

Algorithm 1 Greedy Approach to Construct the Generating Matrix of a small dimension K RC Compatible Code

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1: Input: target minimum distance d_{min}^t, K
2: Initialize: Set codeword length $n = 0$, $d_{min}^0 = 0$ and $\mathbf{G}^{(0)} = ()$.
3: For all increasing n starting from 0:
4: Construct the list of input words $\mathcal{U}$ generating the minimum distance codewords (nearest neighbors):

$$\mathcal{U} = \left\{ \mathbf{u} = \arg \min \left(w_2(\mathbf{u^T G}^{(n)}) \right) \right\}.$$

5: Find a new column vector $\mathbf{g}^{(n+1)}$, to be appended to $\mathbf{G}^{(n)}$, that reduces as much as possible the number of nearest neighbors

$$\mathbf{g}^{(n+1)} \in \arg \max_{\mathbf{g}} \sum_{\mathbf{u} \in \mathcal{U}} \mathbf{g^T} \cdot \mathbf{u} \ , \tag{1}$$

where "·" denotes the scalar product (over $\mathbb{Z}_2$). If there is more than one vector satisfying (1), pick a random one.
6: Optionally one may also enforce that the new vector $\mathbf{g}^{(n+1)}$ is different from all the columns of $\mathbf{G}^{(n)}$.
7: Append the column $\mathbf{g}^{(n+1)}$ to $\mathbf{G}^{(n)}$:

$$\mathbf{G}^{(n+1)} = \left(\mathbf{G}^{(n)} | \mathbf{g}^{(n+1)} \right),$$

8: $n = n + 1$ and goto 4 if d_{min}^t is not achieved.

A. Construction of rate-compatible linear binary codes

In this part we provide a simple approach based on greedy algorithm to construct the generating matrices $\mathbf{G}^{(n)}$ of a family of good rate-compatible codes over $\mathbb{Z}_2$ with small dimension K. The step by step description of the algorithm is provided in Algorithm 1. In step 2, we initialize the generator matrix as $\mathbf{G}^{(0)}$ to the empty matrix. In step 4, $w_2()$ denotes the Hamming weight of a binary vector. The algorithm adds, at each step, a new column to the generator matrix such that the number of nearest neighbor codewords is reduced as much as possible, and hence possibly increase the minimum distance. The algorithm stops when a target desired minimum distance, d_{min}^t, or a maximum codeword size is achieved. The optional condition in step 6 guarantees that when $n = 2^K - 1$, the optimal maximum length codes (all possible columns in generating matrix) are obtained. Complexity of the algorithm is exponential with K (step 4 and 5) but linear with the code length n.

Despite its simplicity, this algorithm provides families of RC codes with performances close to those of optimal codes. Such a comparison is done in Table I. In this table we report the required length needed to satisfy a target minimum distance for given K. The columns labeled **B** correspond to the optimal codes. The optimal minimum distances are known for all values of $K = 2, \ldots, 15$ and $N = 2, \ldots, 250$. Routines to construct optimal codes can be found in [19]. The results for the codes obtained using the greedy algorithm 1 are presented under the column labeled **2**.

It is important to notice that for $K = 2$ the Cordaro-Wagner codes are optimal and the greedy approach indeed results in the same codes. Also for $K = 3$ the solution obtained by our approach is always optimal independently from the value of d_{min}. By increasing K we diverge from the optimal codes, however, for $K \leq 7$ the results are still surprisingly close to the optimal codes. This observation, encourages us to use the same algorithm to design also rotationally invariant codes as explained in the next sections. As a final note, it is important to notice that the results in Table I provide us with a *family* of rate compatible codes, as a good code of length N is obtained by adding a column to the generator matrix of a good code with length $N - 1$.

On the other hand, if one is not interested in the rate compatibility constraint, one may run multiple time the greedy algorithm and pick the optimum designed codes for any desired pair $(K, d_{\min})$. The random selection of the column in step 5 of the algorithm guarantees that each run of the greedy algorithm yields different results. We show the best results for 100 runs in Table II. The improvement compared to a single run is quite significant for $K = 8, 9$ and 10.

B. Construction of rate-compatible linear codes over $\mathbb{Z}_M$

The previous greedy algorithm for construction of good rate compatible linear binary codes can be generalized to construct linear codes over $\mathbb{Z}_M$. In this section we consider codes over $\mathbb{Z}_4$ while extension to larger values of M is straightforward.

Any (K, N) linear code $\mathcal{C}$ over $\mathbb{Z}_4$ (sub-module) can be generated with a generator matrix of the form:

$$\mathbf{G} = \begin{pmatrix} A \\ 2B \end{pmatrix},$$

where A is a $(k_1 \times N/2)$ matrix with elements in $\mathbb{Z}_4$ and B is a $(k_2 \times N/2)$ matrix with elements in $\mathbb{Z}_2$. The code, which has dimension $K = 2k_1 + k_2$, is generated by multiplying $\mathbf{G}$ by a vector $\mathbf{u}^T = (\mathbf{u}_1, \mathbf{u}_2)$ with the first k_1 components in $\mathbb{Z}_4$ and the last k_2 components in $\mathbb{Z}_2$. Multiple choice of the pair k_1, k_2 for the same K originate different code types.

If the considered distance between group elements is such that it can be computed by applying a proper weight function w_4 to their $\mathbb{Z}_4$ sum

$$d(a, b) = w_4(a + b),$$

TABLE I
CODE LENGTH FOR GIVEN K AND TARGET MINIMUM DISTANCE $d_{\min}$. VALUES ARE REPORTED FOR BEST BINARY CODES (**B**) AND DESIGNED RC CODES WITH GREEDY ALGORITHM: BINARY (**2**), BINARY AND RI FOR BPSK (**2 RI2**), BINARY AND ROBUST TO NC DETECTION FOR QPSK (**2 NC4**), OVER $\mathbb{Z}_4$ (**4**),OVER $\mathbb{Z}_4$ AND RI FOR QPSK (**4 RI4**). ALL CODES ARE OBTAINED WITH A SINGLE RUN OF THE GREEDY ALGORITHM.

	B	2	2 RI2	2 NC4	4	4 RI4	B	2	2 RI2	2 NC4	4	4 RI4	B	2	2 RI2	2 NC4	4	4 RI4
$K\backslash d_{\min}$			2						4						10			
2	**3**	3	4	4	4	4	**6**	6	8	8	6	8	**15**	15	20	20	16	20
3	**4**	4	4	4	6	4	**7**	7	8	8	8	8	**18**	18	20	20	20	20
4	**5**	5	6	6	6	6	**8**	9	9	8	10	8	**20**	20	22	22	22	22
5	**6**	6	6	7	8	8	**10**	10	10	11	10	12	**21**	22	22	23	24	24
6	**7**	7	8	8	8	8	**11**	11	12	12	14	14	**23**	23	24	24	26	24
7	**8**	8	8	8	10	10	**12**	12	14	14	16	14	**24**	25	26	26	28	28
8	**9**	9	10	10	10	10	**13**	14	14	14	16	16	**26**	27	28	28	30	30
9	**10**	10	10	11	12	12	**14**	15	16	16	18	16	**27**	28	28	30	32	32
10	**11**	11	12	12	12	12	**15**	16	18	16	18	18	**28**	30	30	31	34	34
11	**12**	12	12	12	14	14	**16**	17	18	18	20	18	**30**	31	32	32	38	36
12	**13**	13	14	14	14	14	**18**	18	18	19	20	20	**31**	33	34	34	36	36
13	**14**	14	14	15	16	16	**19**	19	20	20	22	22	**32**	35	36	36	38	38
14	**15**	15	16	16	16	16	**20**	20	21	20	24	24	**34**	35	36	36	42	40

	B	2	2 RI2	2 NC4	4	4 RI4	B	2	2 RI2	2 NC4	4	4 RI4	B	2	2 RI2	2 NC4	4	4 RI4
$K\backslash d_{\min}$			20						30						50			
2	**30**	30	40	40	30	40	**45**	45	60	60	46	60	**75**	75	100	100	76	100
3	**35**	35	40	40	36	40	**53**	53	60	60	54	60	**88**	88	100	100	88	100
4	**38**	39	40	40	40	40	**57**	57	62	62	60	62	**95**	95	102	102	96	102
5	**40**	41	42	43	42	44	**59**	61	62	63	62	64	**99**	100	102	103	104	104
6	**42**	43	44	44	46	46	**60**	63	66	64	68	64	**101**	103	104	104	110	106
7	**43**	45	46	47	48	50	**62**	66	66	64	70	70	**102**	107	106	108	112	110
8	**45**	49	50	48	52	52	**65**	69	71	68	76	72	**105**	110	110	110	118	112
9	**47**	50	50	52	54	54	**67**	71	72	72	74	72	**107**	112	114	115	118	116
10	**48**	52	54	54	56	56	**68**	74	76	76	80	78	**109**	116	118	118	122	120
11	**50**	55	54	56	62	56	**70**	77	78	78	84	80	**113**	120	120	120	128	126
12	**52**	56	56	58	64	60	**72**	78	80	80	86	84	**116**	122	124	124	130	128
13	**53**	58	60	60	62	64	**74**	80	82	84	88	86	**118**	124	126	127	134	130
14	**54**	60	60	60	62	64	**76**	82	84	84	88	88	**119**	128	128	128	134	130

TABLE II
CODE LENGTH FOR GIVEN K AND TARGET MINIMUM DISTANCE $d_{\min}$. VALUES ARE REPORTED FOR BEST BINARY CODES (**B**) AND SEVERAL DESIGNED RC CODES: BINARY (**2**), BINARY AND RI FOR BPSK (**2 RI2**), BINARY AND ROBUST TO NC DETECTION FOR QPSK (**2 NC4**), OVER $\mathbb{Z}_4$ (**4**),OVER $\mathbb{Z}_4$ AND RI FOR QPSK (**4 RI4**). THE VALUES OF BEST CODES WITH 100 RUNS OF THE GREEDY ALGORITHM ARE REPORTED.

	B	2	2 RI2	2 NC4	4	4 RI4	B	2	2 RI2	2 RI4	4	4 RI4	Best	2	2 RI2	2 RI4	4	4 RI4
$K\backslash d_{min}$			2						4						10			
2	**3**	3	4	4	4	4	**6**	6	8	8	6	8	**15**	15	20	20	16	20
3	**4**	4	4	4	4	4	**7**	7	8	8	8	8	**18**	18	20	20	18	20
4	**5**	5	6	6	6	6	**8**	9	8	8	10	8	**20**	20	22	22	22	22
5	**6**	6	6	7	6	8	**10**	10	10	11	10	12	**21**	21	22	23	22	24
6	**7**	7	8	8	8	8	**11**	11	12	12	12	12	**23**	23	24	24	24	24
7	**8**	8	8	8	8	8	**12**	12	13	13	12	14	**24**	25	26	26	26	26
8	**9**	9	10	10	10	10	**13**	14	14	14	14	16	**26**	26	28	28	28	28
9	**10**	10	10	11	10	12	**14**	15	15	15	14	16	**27**	28	28	28	30	28
10	**11**	11	12	12	12	12	**15**	16	16	16	18	16	**28**	29	30	30	32	30

	B	2	2 RI2	2 NC4	4	4 RI4	B	2	2 RI2	2 RI4	4	4 RI4	B	2	2 RI2	2 RI4	4	4 RI4
$K\backslash d_{min}$			20						30						50			
2	**30**	30	40	40	30	40	**45**	45	60	60	46	60	**75**	75	100	100	76	100
3	**35**	35	40	40	36	40	**53**	53	60	60	54	60	**88**	88	100	100	88	100
4	**38**	38	40	40	38	40	**57**	57	62	62	58	62	**95**	95	102	102	96	102
5	**40**	40	42	43	42	44	**59**	59	62	62	60	62	**99**	99	102	103	100	104
6	**42**	42	44	44	44	44	**60**	62	62	62	64	62	**101**	102	104	104	104	104
7	**43**	45	44	46	46	46	**62**	65	64	64	68	62	**102**	105	106	106	108	106
8	**45**	47	48	48	50	48	**65**	68	68	68	70	68	**105**	109	108	108	114	110
9	**47**	49	50	51	52	52	**67**	70	72	72	74	72	**107**	112	114	114	116	114
10	**48**	51	52	52	54	52	**68**	73	74	74	78	76	**109**	115	116	116	120	118

linearity of code ($\forall \mathbf{a}, \mathbf{b} \in \mathcal{C} \rightarrow \mathbf{a} + \mathbf{b} \in \mathcal{C}$) implies that distance spectrum from any codeword coincides with the weight spectrum of the code. Considering these generalizations, step 4 of the greedy algorithm 1 should be substituted to

- Construct the list of input words $\mathcal{U} \in \mathbb{Z}_4^{k_1} \times \mathbb{Z}_2^{k_2}$ generating the minimum weight codewords (nearest neighbors):

$$\mathcal{U} = \left\{ \mathbf{u} = \arg\min \left(w_4(\mathbf{u}^{\mathbf{T}} \mathbf{G}^{(n)}) \right) \right\}$$

and step 5 with

- Find a new column vector $\mathbf{g}^{(n+1)}$, to be appended to $\mathbf{G}^{(n)}$, that reduces as much as possible the number of nearest neighbors:

$$\mathbf{g}^{(n+1)} \in \arg\min_{\mathbf{g}} \sum_{\mathbf{u} \in \mathcal{U}} \mathbb{I}(0 = w_4(\mathbf{g}^{\mathbf{T}} \cdot \mathbf{u})) \ , \tag{2}$$

where $\mathbb{I}()$ is the indicator function, returning 1 if its argument is true.

With the natural M-PSK mapping $m \leftrightarrow \frac{1}{\sqrt{2}}e^{j2\pi m/M}$ the induced weight correspondent to the Euclidean distance is $w_4(0,1,2,3) = (0,1,2,1)$ so that eq. (2) is not equivalent to eq. (1). When multiple columns exist satisfying eq. (2) one additional requirement may be that of minimizing the number of scalar products with weight 1.

The results for constructed codes over $\mathbb{Z}_4$ are presented in tables I and II under the column label **4**. For each value of K we report the results for the code type with the largest value of k_1, i.e. $k_1 = \lfloor K/2 \rfloor$. Results for differents type are not reported as they do not provide significant differences.

Notice that the length in bits N of codes over $\mathbb{Z}_4$ is always an even number. For small values of K the codes $\mathbb{Z}_4$ have the same length as their binary counterparts (with exception of 1 bit due to the even length constraint). On the other hand, for large K a small increase can be seen.

C. Construction of rotationally invariant codes

A codebook $\mathcal{S}$ for M-PSK constellation is rotationally invariant when all the rotated versions of any codeword belong to the codebook.

For M-PSK constellation and linear codes over $\mathbb{Z}_M$ generating group, this condition is equivalent to impose that the all-one codeword (denoted by $\bar{1}$) belong to the codebook $\mathcal{C}$. This property can be enforced constraining the first row of the generating matrix $\mathbf{G}$ to $\bar{1}$. This additional constraint in turn can be easily incorporated into the greedy Algorithm 1 to generate families of good rate compatible and rotational invariant codes.

The results for rotationally invariant codes over $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are presented in tables I and II with column labels **2 RI2** and **4 RI4** respectively. In principle the length of rotational invariant codes for a given minimum distance and information bit must be larger or equal to those codes without any constraints. In our tables in few cases when $K \leq 7$ the rotational invariant codes have smaller length compared to optimized codes. This is due to two facts. First, the greedy algorithm does not always provides the optimal solution and second, the search space is smaller for rotationally invariant codes (for any given K) and therefore the greedy algorithm performs slightly better.

In Figure 1 we report the minimum distance growth as a function of K and N of the generated codes for BPSK and QPSK constellations with 100 runs of the greedy algorithm. The main observation is that as K increases, both the optimized codes and rotationally invariant codes show the same minimum distance growth as a function of code length.

Generator matrices of a good rate compatible family corresponding to the row $K = 8$ of table I are reported in the appendix.

IV. FROM ROTATIONALLY INVARIANT CODES TO CODES FOR NON-COHERENT DETECTION

In this section we describe how to construct codes robust to non-coherent detection starting from RI codes. First we present a simple upper bound on the FER of the RI codes which motivates our construction of the codes for non-coherent detection

A. Union bound for non-coherent detection

Simple upper bounds (UBs) to frame error probability of codes, P_E, can be obtained by applying the union bound based on the pairwise error probability (PEP) $P(\mathbf{s} \to \mathbf{s}')$:

$$P_E \leq \frac{1}{2^K} \sum_{\mathbf{s}} \sum_{\mathbf{s}' \neq \mathbf{s}} P(\mathbf{s} \to \mathbf{s}'). \tag{3}$$

For coherent detection and AWGN channel PEP is upper bounded by

$$P(\mathbf{s} \to \mathbf{s}') \leq \exp\left(-\frac{\|\mathbf{s} - \mathbf{s}'\|^2}{4N_0} \right),$$

where $\mathbf{s}, \mathbf{s}'$ are the constellation sequences in the codebook.

When codebook is constructed by applying a proper mapping $\mathbf{s}(\mathbf{c})$ to codewords $\mathbf{c}$ of linear code $\mathcal{C}$ over a *generating group* of the considered constellation the PEP can be simplified as

$$P(\mathbf{s}(\mathbf{c}) \to \mathbf{s}(\mathbf{c}')) = P(\mathbf{s}(\mathbf{0}) \to \mathbf{s}(\mathbf{c}' - \mathbf{c})).$$

In this case outer sum in (3) can be eliminated (geometrical uniform codes) and UB can be written as:

$$P_E^c(\mathcal{C}) \leq \sum_{\mathbf{c} \in \mathcal{C} \neq \mathbf{0}} P(\mathbf{s}(\mathbf{0}) \to \mathbf{s}(\mathbf{c})). \tag{4}$$

When using non-coherent detection on M-PSK, it can be demonstrated (see for example [22]) that the PEP is asymptotically upper bounded (for large code lengths) by

$$P(\mathbf{s} \to \mathbf{s}') \tilde{\leq} \exp\left(-\frac{d_{\text{eq}}^2(\mathbf{s}, \mathbf{s}')}{4N_0} \right), \tag{5}$$

(a) Optimized code over $\mathbb{Z}_2$ (**2**)

(b) Optimized code over $\mathbb{Z}_2$, RI for BPSK constellation (**2RI2**)

(c) Optimized code over $\mathbb{Z}_4$ (**4**)

(d) Optimized code over $\mathbb{Z}_4$, RI for QPSK constellation (**4RI4**)

Fig. 1. Minimum distance growth as a function of code length for best codes obtained by greedy algorithm (100 runs). Several values of K are reported.

where

$$d_{\text{eq}}^2(\mathbf{s}, \mathbf{s}') = \min_{i=0,\ldots,M-1} ||\mathbf{s} - e^{j\frac{2\pi i}{M}}\mathbf{s}'||^2,$$

and $\tilde{\leq}$ indicates the asymptotic inequality as $N \to \infty$. Equation (5) in turn can be further upper bounded as follows

$$P(\mathbf{s} \to \mathbf{s}') \tilde{\leq} \sum_{i=0}^{M-1} \exp\left(-\frac{||\mathbf{s} - e^{j2\pi i/M}\mathbf{s}'||^2}{4N_0}\right).$$

The union bound for non-coherent detection takes then the following approximate form:

$$P_E^{nc} \quad \tilde{\leq} \quad \frac{1}{2^K} \sum_{\mathbf{s}} \sum_{\mathbf{s}' \neq \mathbf{s}} \sum_{i=0}^{M-1} \exp\left(-\frac{||\mathbf{s} - e^{j2\pi i/M}\mathbf{s}'||^2}{4N_0}\right). \tag{6}$$

Now consider a $(N, K-m)$ code $\mathcal{C}$ constructed from a linear (N, K) RI code $\mathcal{C}''$ over $\mathbb{Z}_M$ by eliminating the all-one row of its generating matrix. Its union bound reads:

$$
\begin{aligned}
P_E^{nc}(\mathcal{C}) \quad &\tilde{\leq} \quad \frac{1}{2^{K-m}} \sum_{\mathbf{c}} \sum_{\mathbf{c}' \neq \mathbf{c}} \sum_{i=0}^{M-1} \exp\left(-\frac{||\mathbf{s}(\mathbf{c}) - e^{j2\pi i/M}\mathbf{s}(\mathbf{c}')||^2}{4N_0}\right) \\
&= \quad \frac{1}{2^{K-m}} \sum_{\mathbf{c}} \sum_{\mathbf{c}' \neq \mathbf{c}} \sum_{i=0}^{M-1} \exp\left(-\frac{||\mathbf{s}(\mathbf{c}) - \mathbf{s}(\mathbf{c}' + \mathbf{i})||^2}{4N_0}\right).
\end{aligned} \tag{7}
$$

The second identity stems from the fact that rotating the constellation sequence by $2\pi i/M$ corresponds to adding the all-i sequence ($\mathbf{i}$) to the codeword. The two inner sums in (7) enumerate all the codewords of $\mathcal{C}''$ not equal to a rotated version of $\mathbf{s}(\mathbf{c})$, so that:

$$P_E^{nc}(\mathcal{C}) \;\tilde{\leq}\; \frac{1}{2^{K-m}} \sum_{\mathbf{c}} \sum_{\mathbf{c}''\in\mathcal{C}'',\mathbf{c}''\neq\mathbf{c}+\mathbf{i}} \exp\left(-\frac{||\mathbf{s}(\mathbf{c})-\mathbf{s}(\mathbf{c}'')||^2}{4N_0}\right). \tag{8}$$

Now we can exploit the geometrical uniformity of $\mathcal{C}$ to simplify the UB (8) removing the outer sum:

$$P_E^{nc}(\mathcal{C}) \;\tilde{\leq}\; \sum_{\mathbf{c}''\in\mathcal{C}''\neq\mathbf{i}} \exp\left(-\frac{||\mathbf{s}(\mathbf{0})-\mathbf{s}(\mathbf{c}'')||^2}{4N_0}\right) \leq P_E^c(\mathcal{C}'') \tag{9}$$

An approximation of the upper bound to FER with non-coherent detection for the $(N, K-m)$ code $\mathcal{C}$ is then upper bounded by the upper bound to FER with coherent detection of the correspondent RI code $\mathcal{C}''$. The difference being that all M rotated sequences $\mathbf{i}$ are excluded from the sum (9) instead of just the all zero sequence as in (4).

The generating matrix of a *good* $(K-m, N)$ code for non-coherent detection of 2^m-PSK can then be obtained starting with a *good* $K \times N$ generating matrix of a rotationally invariant code constructed as described in section III-C by eliminating the first all-one row. The new codebook is such that the distance from any codeword to any other codeword *and all its rotated versions* is large.

B. Codes on $\mathbb{Z}_2$ for non-coherent detection of QPSK constellations

Even though linear codes built over $\mathbb{Z}_M$ can be represented in binary, they are not binary linear codes. Focusing on QPSK constellation, our previous construction results in a linear code over $\mathbb{Z}_4$. If a linear *binary* code is desired one can use the alternative construction presented in [21]. In this case one construct a *binary* $K \times N$ generator matrix where the two first rows are fixed to be $\overline{10}$ and $\overline{01}$ codewords using the above mentioned greedy approach. This forces the binary codebook to have $\overline{1}$, $\overline{10}$ and $\overline{01}$ as codewords. However, since $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$ are not isomorphic, no direct mapping can be found between the rotations of QPSK constellation and the codebook algebra. Therefore, contrary to what has been reported previously in [21], these codes are not rotationally invariant. Nevertheless, code suitable for non-coherent detection can still be obtained by this construction after eliminating the first two rows. Our simulations indicate that such codes perform quite well with the non-coherent detector. The results based on this construction are reported in Tables I and II under the column name **2 NC4**.

V. EXAMPLE OF PLH CODE CONSTRUCTION FOR BPSK AND QPSK CONSTELLATIONS

In this section we report an example of PLH code construction for $K = 8$ and target FER=10^{-8} for SNR=0, 5 and 10 dB.

The first step is that of establishing the required minimum distance of codes for achieving the desired FER at the target SNR. In order to do so we use the following conservative upper bound to FER

$$FER \leq (2^K - 1)\frac{1}{2}\mathrm{erfc}\left(\sqrt{Ad_{\min}\frac{E_s}{N_0}}\right). \tag{10}$$

with $A = 1$ for BPSK and $A = 1/2$ for QPSK. This simple bound provides a conservative value for the minimum distance using BPSK and QPSK. The required values are reported in 2nd and 5th column of table III.

The required generating matrices are then selected depending on the type of detection. When using coherent detection (CD) one has to pick the generating matrix of a $(N, 8)$ code with minimal length N achieving the required minimum distance. When using non-coherent detection (NCD) one has to pick the generating matrix of a $(N, 8 + m)$ RI code over $\mathbb{Z}_{2^m}$ with minimal length and remove the first row. The required lengths (in symbols) using the rate-compatible code families constructed as described in section III are reported in table III. One can see that, as expected, the additional requirement of robustness to non-coherent detection has a marginal impact on the required code lengths. One can also observe that using QPSK gives a small advantage in terms of required PLH length in symbols.

TABLE III
REQUIRED CODE LENGTHS (SYMBOLS) FOR PLH ACHIEVING FER=10^{-8} AT 0, 5 AND 10 DB. **CD**: COHERENT DETECTION. **NCD**: NON-COHERENT DETECTION.

SNR	BPSK			QPSK		
	$d_{\min}$	CD	NCD	$d_{\min}$	CD	NCD
0	22	51	54	43	50	52
5	7	21	22	14	19	20
10	3	13	14	5	9	10

We simulated the FER performance of codes constructed in this way when using coherent ML detection and noncoherent ML detection. In Figures 2 to 5 we report the performances of designed codes and comparison with bounds (4) or (9) (dashed lines) or the simpler bound (10) (solid lines).

Fig. 2. Comparison of simulation results and bounds for codes designed for BPSK with coherent detection (2). Target SNR at FER=10^{-8} is 0, 5 and 10 dB, corresponding to required $d_{\min}$ of 22,7, and 3.

To obtain the performance results we used the ML detection rules:

$$y_k = s_k + n_k \qquad\qquad \mathbf{c}_{ML}^c = \mathrm{argmax}_{\hat{\mathbf{c}}}\Re(\mathbf{y}\cdot\mathbf{s}^*(\hat{\mathbf{c}})) \;\; \text{Coherent}$$
$$y_k = s_k e^{j\theta} + n_k \qquad\qquad \mathbf{c}_{ML}^{nc} = \mathrm{argmax}_{\hat{\mathbf{c}}}|\mathbf{y}\cdot\mathbf{s}^*(\hat{\mathbf{c}})| \;\; \text{Non}-\text{Coherent}$$

Notice that bound (9), is valid only for large values of code length, a situation far from true for the considered short length codes. This fact is responsible of the mismatch between bounds and simulations observed in Figures 5 and 3.

VI. Conclusions and Further Research

In this paper we presented a strategy to design short-length rate-compatible codes robust to non-coherent detection. We discussed in some details a system scenario, i.e., the coding for PLH in precoded satellite broadcast systems, where such codes are necessary to correctly detect the PLH. A greedy algorithm has been proposed and used to construct the desired codes. Our method provides an upper bound of the code length needed to assure a given non-coherent distance between the codewords. The research can be extended in several ways. For example, it will be interesting to provide some theoretical upper and lower bounds for the finite-block length coding design with a desired non-coherent distance. Another possible study, is to change the constellation space instead of using non-coherent detectable codes. As an example, by adopting a 1-3-APSK constellation one may be able to design shorter codes due to the fact that the non-coherent detectability as defined earlier may not be needed any more. In general, designing the constellation space by optimizing an opportunistic objective function (using for example the techniques proposed in [16] and [23]) to reduce the code length is an interesting research topic. Some works in this direction are undergoing.

Appendix

In tables IV and V we report the generator matrices for the rate-compatible code obtained using the greedy algorithm. The maximum code length is $N = 256$ and the code size is $K = 8$. Each rows of the table reports one line of the generating matrix in hexadecimal notation. Each hexadecimal number has its LSB on the right, so the binary representation should be read from right to left.

Fig. 3. Comparison of simulation results and bounds for codes designed for BPSK with non-coherent detection (**2RI2**). Target SNR at FER=10^{-8} is 0, 5 and 10 dB, corresponding to required $d_{\min}$ of 22,7, and 3 respectively.

TABLE IV
BPSK CODES. GENERATOR MATRICES OF GOOD RATE-COMPATIBLE CODES FOR COHERENT AND NON-COHERENT DETECTION. $K = 8$ AND MAXIMUM
CODE LENGTH $N = 256$.

	0−31	32−63	64−95	96−127	128−159	160−191	192−223	224−255
	Coherent detection ($\mathbb{Z}_2$)							
1	B60CC170	C06D5BD9	3792D386	0C019BF6	A2AC8EA4	5A6655A6	0F09AFAF	A2E06F0C
2	8740E3C4	EBF583EA	40D580B3	0FDB00DF	EE798298	69F03FF3	3FFFA955	960395A9
3	2CB158A6	3F9EC328	134ACC79	C265AB06	F5554BFF	95FF363F	CFFFC965	F6F0CFF5
4	D7A5A915	527217DA	A3FFED21	E6B15804	F61AC94F	F0009C33	C05C3609	995C5C96
5	ADD81F34	A1813BF1	B187ED2A	E66B02B1	65528FD8	C553CC0C	9AA553F0	9AC03F9A
6	CFAABDA7	C0D26C37	A2C19ED8	FF03E402	1B2CF598	6F050CA0	3F0C9F63	0C9F030C
7	9E3FFC0F	9FF5238C	8BF334D3	D4EA816C	8FB01B03	A60F6CA9	A6A3CA53	F74A936A
8	CA4DA8F3	661EBE67	CB9558BE	726B3285	8B54C955	0AC0FA09	036FA3A9	378306CA
	Non Coherent detection of BPSK ($\mathbb{Z}_2$)							
−	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	F999750E	F18E7072	60BB790A	703FCE66	ECAABE56	6096E920	A0FA9039	6A5369AA
2	6A66FECB	853B0E6B	3CD19CCF	733328D4	188B2829	6566FB34	99C5C355	33659AC0
3	650F1B00	F23732A8	E6B8E6CA	FC159543	EC35B180	FC636E85	0C93FF09	A900330A
4	CA9BEF81	D397FD46	64D49C39	17DA5A4D	BC102B14	5999F46D	555C03C3	03CA6FA0
5	0FF15456	49982A0E	9F9620F0	71B330E6	C7E43FD6	C03334D8	5F509663	3F035A60
6	6F602B84	627C1B11	06D153AC	17F33F14	E9594E70	FF3C6E64	0A063590	3A366C0C
7	5336D973	CCAE71BF	45384AF0	4C3F18C1	2468DBEA	953577B0	FAC6636C	90A9C59C
8	559B1E57	2C354DBA	94C21C9C	16BFE8CE	AF328170	A0968680	9003FC36	F530F995

Fig. 4. Comparison of simulation results and bounds for codes designed for QPSK with coherent detection (**4**). Target SNR at FER=10^{-8} is 0, 5 and 10 dB, corresponding to required $d_{\min}$ of 43,14, and 5.

TABLE V
QPSK CODES.GENERATOR MATRICES OF GOOD RATE-COMPATIBLE CODES FOR COHERENT AND NON-COHERENT DETECTION. $K = 8$ AND MAXIMUM CODE LENGTH $N = 256$.

	0–31	32–63	64–95	96–127	128–159	160–191	192–223	224–255
	Coherent detection (codes over $\mathbb{Z}_4$)							
1–2	5CA272B7	8BB1BEEB	B645D089	E8DD708A	B16F2C92	BD6C24B3	390F4A8D	632DCE31
3–4	9E0592A6	0E8D8D45	99DDECC8	3A06F70D	0866F421	EE8F409C	CEE1D497	E7AD3D2D
5–6	68B29C05	9EB80F05	B6C6B50E	F443A875	787406A9	6C017FD0	5A4CB78E	0BBFC41C
7–8	570C66E7	2C003E2C	A05678FA	5F23FE74	A8016DB0	FC76395D	3E6ED680	677BAFD8
	Non Coherent detection of QPSK (codes over $\mathbb{Z}_4$)							
–	55555555	55555555	55555555	55555555	55555555	55555555	55555555	55555555
1–2	773F8EF4	A840C467	9C9E7E57	70762CD3	1214EA38	AFC0B6CA	C68A58F0	FF77C946
3–4	EF9E5FF2	916302C1	D5D27978	880B090D	EF82BA05	956A5CD2	5F7DD5D2	89B64BC5
5–6	6CF632A6	3A8ACF2B	572D64A5	5FF77D0D	AFF031F6	CD07AF80	35D00DF0	E8DFB604
7–8	C02A2B67	839C432D	3DB1875F	E7F7C875	449A58A7	9FFEBF6B	4647DD82	341C8002
	Non Coherent detection of QPSK (codes over $\mathbb{Z}_2$)							
–	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
–	AAAAAAAA	AAAAAAAA	AAAAAAAA	AAAAAAAA	AAAAAAAA	AAAAAAAA	AAAAAAAA	AAAAAAAA
1	33450EDA	82F99BB4	656CA503	19B61663	36501FC3	6A69F93F	741E26C5	5A0003E2
2	F8252756	1AD5E6D5	C939916E	726C4BCC	C96ABD0F	FA000A9C	4FB93E07	ABB33000
3	9150254A	B4E249C5	6A699FEE	D44BC8A9	AF30B67B	E3533006	6B127392	0FA0AEBB
4	765CC63C	9C41E7C8	A5059438	8EBE5E93	395C0616	453FF366	2E1BAB96	CE2A3011
5	C99A7FB9	625D30D7	C09F09AC	5990AB03	F96AA05E	A5636CA3	5B3BCCB4	98805987
6	FFAA59A6	A1F655A0	C3CA37D4	58D94600	9A395B36	7A56AAAA	83D1A6BB	CC3C0FD1
7	55461830	B433128F	5ACA3CE4	F1E5979A	59F534D1	4F5066A0	E276999B	9BB991B4
8	8F192E8D	52DB486A	959A6A0E	9FF6399A	C3F9EAD0	F90F9AC5	419BB5EF	3E133CCF

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Fig. 5. Comparison of simulation results and bounds for codes designed for QPSK with non-coherent detection (**4RI4**). Target SNR at FER=10^{-8} is 0, 5 and 10 dB, corresponding to required $d_{\min}$ of 43,14, and 5.

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