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A Moment-Matching Scheme for the Passivity-Preserving Model Order Reduction of Indefinite Descriptor Systems with Possible Polynomial Parts *

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Abstract— Passivity-preserving model order reduction (MOR) of descriptor systems (DSs) is highly desired in the simulation of VLSI interconnects and on-chip passives. One popular method is PRIMA, a Krylov-subspace projection approach which preserves the passivity of positive semidefinite (PSD) structured DSs. However, system passivity is not guaranteed by PRIMA when the system is indefinite. Furthermore, the possible polynomial parts of singular systems are normally not captured. For indefinite DSs, positive-real balanced truncation (PRBT) can generate passive reduced-order models (ROMs), whose main bottleneck lies in solving the dual expensive generalized algebraic Riccati equations (GAREs). This paper presents a novel moment-matching MOR for indefinite DSs, which preserves both the system passivity and, if present, also the improper polynomial part. This method only requires solving one GARE, therefore it is cheaper than existing PRBT schemes. On the other hand, the proposed algorithm is capable of preserving the passivity of indefinite DSs, which is not guaranteed by traditional moment-matching MORs. Examples are finally presented showing that our method is superior to PRIMA in terms of accuracy.

I. Introduction

Model order reduction (MOR) has become a standard technique in the computer-aided simulation of VLSI interconnect structures and on-chip passive components. The basic idea of MOR is to replace the original huge-size model by a much smaller one, subject to little loss in the time- or frequency-domain port response. When the port response represents the admittance/impedance parameters of a passive model, system passivity is required to be preserved to ensure globally stable system-level simulation.

Moment matching schemes based on Krylov subspace [1–3] are the most popular MOR approaches in interconnect macromodeling, due to their high numerical efficiency. In these MORs, system passivity is ensured by congruence transform [3] which preserves the positive semidefinite (PSD) structures of the modified nodal analysis (MNA) equations. However, passivity cannot be preserved by any congruence transformation based scheme when the original model is not PSD structured, which usually occurs for EM extracted models, due for instance to nonsymmetric formulations, to nonsymmetric testing schemes, to discretization errors, to approximate fast matrix-vector products, and to the use of frequency dependent fullwave or substrate Green Functions. Besides, a huge numerical error may be produced when the original model is a singular descriptor system (DS) and contains a polynomial part. In this paper we will refer to the models with non-PSD structures as *indefinite* systems.

To preserve the passivity of indefinite systems, the Gramian-based positive-real balanced truncation (PRBT [4]) can be employed. In [4], the DS model is decomposed by the Weierstrass canonical form, which is prohibitively expensive and possibly unstable. After that, PRBT is performed on a standard state-space equation. Since the original *RLC* or EM-extracted models are of DS form, a more reliable approach is to perform PRBT on the original possibly singular DSs involving spectral projectors [5–8]. The main bottleneck of PRBT lies in solving the double (generalized) algebraic Riccati equations (AREs or GAREs) at the cost of $O(n^3)$. Although state-of-the-art ARE and GARE solvers [9, 10] can remarkably reduce the complexity, PRBT is still much more expensive than Krylov-subspace projections. For some standard symmetric state-space models, only one ARE is needed in the PRBT procedure [11], however such method is not applicable to asymmetric models or singular systems. Therefore, it is desirable to develop a cheaper MOR for indefinite DSs, to preserve system passivity and the possible polynomial part. The polynomial part is normally a nonzero constant term; in some cases (e.g., interconnects with strong crosstalk effects [12]), an improper part may also exist.

In this paper, we present a novel algorithm for the MOR of indefinite DSs, with preservation of both system passivity and the possible polynomial part. This work is motivated by the recent work in [13, 14], which constructs two projection matrices in MOR with one ensuring numerical accuracy and the other preserving stability. In our work, the right projection matrix is constructed by implicit moment matching, and the left one is derived from the positive real condition. The proposed MOR has some advantages listed below:

- Passivity preservation for indefinite systems.
- Polynomial-part preservation for singular systems.
- Lower complexity compared with PRBT, since only one GARE is solved.
- Higher numerical accuracy compared with popular moment-matching schemes such as PRIMA [2].

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This paper is structured as follows. Section II reviews DS passivity and points out the limitations of existing passivity-preserving MORs. Section III presents the proposed MOR for indefinite nonsingular DSs and gives the algorithmic flow. Section IV proves that the proposed algorithm preserves system passivity for indefinite models. In Section V, the implementation details for singular cases that have possibly improper parts are presented. The algorithm is then verified by the numerical examples in Section VI. This work is finally concluded in Section VII.

II. Background of Passivity-Preserving MOR

A. System Passivity of LTI DSs

A DS [denoted by $\Sigma_{\text{ds}}(E, A, B, C, D)$] is described by the state-space equation

$$E\frac{dx(t)}{dt} = Ax(t) + Bu(t), \;\; y(t) = Cx(t) + Du(t). \quad (1)$$

Here $E,\ A\ \in\ \mathbb{R}^{n\times n},\ B,\ C^T\ \in\ \mathbb{R}^{n\times m}$, $x\ \in\ \mathbb{R}^n$ denotes the state vector; $u(t)$, $y(t)\ \in\ \mathbb{R}^m$ are the input and output vectors, respectively. The matrix pencil (E, A) is assumed to be regular, i.e., $\det(sE - A) \neq 0$ for some $s\ \in\ \mathbb{C}$, whereas E is not necessarily invertible, i.e., $\text{rank}(E) \leq n$. To make our results applicable to general cases, the DS is not assumed to be PSD structured as required in existing passivity-preserving congruence transformation based methods [2]. The indefinite DSs may be generated from EM extractions of on-chip parasitics [15, 16], or from the linearization or Volterra-series expression of nonlinear circuits and devices [17, 18].

There exist nonsingular matrices $T_l,\ T_r \in \mathbb{R}^{n\times n}$ that reduce (E, A) to the Weierstrass canonical form

$$E = T_l \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix} T_r \quad A = T_l \begin{bmatrix} J & 0 \\ 0 & I_{n-q} \end{bmatrix} T_r \quad (2)$$

where N is nilpotent and index-μ (i.e., $N^{\mu-1} \neq 0$ and $N^\mu = 0$). The eigenvalues of J are the finite eigenvalues of $\lambda E - A$, and also the finite poles of the DS. The pencil $\lambda E - A$ is said to be stable if all eigenvalues of J have negative real parts. (2) implies that the $m \times m$ transfer matrix can be expressed as

$$\begin{aligned} H(s) &= C(sE - A)^{-1}B + D \\ &= \underbrace{CT_r^{-1}\begin{bmatrix} (sI_q - J)^{-1} & \\ & 0 \end{bmatrix} T_l^{-1}B}_{H_{sp}(s)} + \underbrace{D + \sum_{k=0}^{\mu-1} M_k s^k}_{P(s)} \end{aligned} \quad (3)$$

with $M_k = CT_r^{-1}\begin{bmatrix} 0 & \\ & -N^k \end{bmatrix} T_l^{-1}B$. $H_{sp}(s)$ is the *strictly proper* part, and $P(s)$ denotes the *polynomial* part which might be improper (when $M_i \neq 0$ for $i \geq 1$). Denoting $\mathcal{M}_0 = D - M_0$, the *proper* part is readily given as $H_p(s) = H_{sp}(s) + \mathcal{M}_0$. The properness of $H_p(s)$ implies that we can realize it by a nonsingular DS $\Sigma_p(\mathcal{E}_0, \mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0, \mathcal{M}_0)$ with $\mathcal{E}_0$ being invertible.

If $H(s)$ represents the admittance or impedance parameters, the corresponding LTI is passive if and only if $H(s)$ is positive real [19], i.e.,

1. $H(s)$ has no poles in $\text{Re}(s) > 0$;
2. $\overline{H(s)} = H(\bar{s})$ for $s \in \mathbb{C}$;
3. $H(s) + H^*(s) \geq 0$ for all $\text{Re}(s) > 0$.

Theorem 1 [19]: *For an admittance/impedance LTI DS, the positive real condition is equivalent to: 1) $M_1 = M_1^T \geq 0$, $M_k = 0$ for $k > 1$; 2) $H_p(s)$ is positive real.*

We present a sufficient (but not necessary) condition for DS positive realness, which will be used in Section IV to verify the passivity preservation of our proposed algorithm.

Theorem 2 [19]: *if there exist $W,\ L,\ X\ \in \mathbb{R}^{n\times n}$ and $X = X^T \geq 0$ solving the linear matrix inequalities (LMIs)*

$$\begin{cases} A^T XE + E^T XA = -LL^T, \\ E^T XB - C^T = -LW, \\ D + D^T \geq W^T W \end{cases} \quad (4)$$

then the DS $\Sigma_{ds}(E, A, B, C, D)$ is positive real.

B. Passivity-Preserving MOR for PSD Models

In passivity-preserving MOR, we look for two projection matrices U and V (with $U,\ V\ \in \mathbb{R}^{n\times q}$), to generate a size-$q$ ($q \ll n$) reduced-order model (ROM)

$$E_r\frac{dz(t)}{dt} = A_r z(t) + B_r u(t), \quad y_r(t) = C_r z(t) + Du(t) \quad (5)$$

with $E_r = U^T EV$, $A_r = U^T AV$, $B_r = U^T B$ and $C_r = CV$, such that $H_r(s) = C_r(sE_r - A_r)^{-1}B_r + D$ is positive real and approximates $H(s)$.

An efficient method to generate U and V is implicit moment matching by Krylov-subspace methods such as (block) Arnoldi [2] process. An order-l block Krylov subspace $\mathcal{K}_l(F, R)$ is defined as

$$\mathcal{K}_l(F, R) = \text{colspan}\left\{R, FR, \cdots, F^{l-1}R\right\}. \quad (6)$$

Define $R = (s_0E - A)^{-1}B$ and $F = (s_0E - A)^{-1}E$, PRIMA [2] constructs the projection matrices by

$$V = \mathcal{K}_l(F, R),\ U = V. \quad (7)$$

If inductors are contained, the DS is not symmetric but still PSD structured. In this case, the block Arnoldi process [2] preserves the passivity by constructing a PSD structed ROM, which captures l moments (around s_0) of the original model. If no vectors are deleted during the projection matrix construction, the ROM size is $q = ml$ (with m being the port number).

Despite the high efficiency, Krylov-subspace projections have some well-known limitations summarized below:

1. Passivity is not preserved when the original model does not have a PSD structure. Normally, only the MNA equations for *RLC* networks have this special structure. The models from EM field solvers [15, 16] and the linear models in nonlinear circuit analysis [17, 18] are not PSD structured, and therefore their passivity can not be preserved by the above moment matching schemes.

2. Since the null space of E is normally filtered out in Krylov subspace constructions [4], the resulting ROM is normally

nonsingular which misses the M_i ($i = 0, 1, \cdots$) term in the original model. Therefore, the polynomial term $P(s)$ cannot be preserved by existing Krylov-subspace projections.

C. Positive-Real Balanced Truncation for Indefinite Systems

One way to preserve the positive realness of an indefinite system involves using PRBT by solving a pair of Lur'e equations [4] or algebraic Riccati equations (AREs) [9]. For a strict positive real DS-form nonsingular system Σ_p, the generalized AREs (GAREs) are defined as

$$\mathcal{A}Q_c\mathcal{E}^T + \mathcal{E}Q_c\mathcal{A}^T + \mathcal{E}Q_c\mathcal{C}^T\mathcal{C}Q_c\mathcal{E}^T + \mathcal{B}\mathcal{B}^T = 0 \quad (8a)$$

$$\mathcal{A}^TQ_o\mathcal{E} + \mathcal{E}^TQ_o\mathcal{A} + \mathcal{E}^TQ_o\mathcal{B}\mathcal{B}^TQ_o\mathcal{E} + \mathcal{C}^T\mathcal{C} = 0 \quad (8b)$$

where $\mathcal{M}_0 + \mathcal{M}_0^T > 0$, $\mathcal{E} = \mathcal{E}_0$, $\mathcal{B} = \mathcal{B}_0\left(\mathcal{M}_0 + \mathcal{M}_0^T\right)^{-\frac{1}{2}}$, $\mathcal{C} = \left(\mathcal{M}_0 + \mathcal{M}_0^T\right)^{-\frac{1}{2}}\mathcal{C}_0$, $\mathcal{A} = \mathcal{A}_0 - \mathcal{B}\mathcal{C}$. The symmetric positive semidefinite (SPSD) matrices Q_c and Q_o, called the positive-real controllability and observability Gramians, respectively, are the unique stabilizing solution to the GAREs.

Since Q_c, $Q_o \geq 0$, there exist factors L_c and L_o such that $Q_c = L_cL_c^T$ and $Q_o = L_oL_o^T$. We compute the singular value decomposition (SVD) of $L_o^T\mathcal{E}_0L_c$:

$$L_o^T\mathcal{E}_0L_c = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}\begin{bmatrix} V_1 & V_2 \end{bmatrix}^T, \quad (9)$$

where $\Sigma_1 = \text{diag}(\sigma_1, ..., \sigma_q)$, $\Sigma_2 = \text{diag}(\sigma_{q+1}, ..., \sigma_n)$ and $\sigma_1 \geq ... \geq \sigma_q \geq \sigma_{q+1} \geq ... \geq \sigma_n$ are the Hankel singular values. Finally, the projection matrices are constructed as

$$U = L_oU_1\Sigma_1^{-\frac{1}{2}}, \ \ V = L_cV_1\Sigma_1^{-\frac{1}{2}}. \quad (10)$$

In the above PRBT, $\mathcal{E} = \mathcal{E}_0$ is assumed to be nonsingular. The recent literatures [5, 6] show that PRBT can be performed directly on a singular DS via solving a pair of projected GAREs which involve spectral projectors [c.f. (22)]. In Section V, we will show that with spectral projectors the proper part $H_p(s)$ can always be realized by a nonsingular DS, and then one GARE is enough to guarantee system passivity.

The most significant advantage of PRBT is its passivity-preservation nature regardless of system structures. However, two GAREs need to be solved in the PRBT flow. Although the cost could be reduced to $O(n^2)$ for sparse systems by Newton's iteration [6, 10] and although a generalized Lyapunov equation solver has been recently developed based on LR-GADI [20], it is still desirable to develop a cheaper routine for passive MOR of indefinite systems.

III. Proposed MOR Scheme

Similar to [13], we begin with the nonsingular DS $\Sigma_p(\mathcal{E}_0, \mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0, \mathcal{M}_0)$, which does not necessarily have a PSD structure and therefore passivity is not guaranteed by existing Krylov-subspace projection methods. This section presents the passivity-preserving MOR flow, whose passivity-preservation property and implementation for singular systems will be covered in Sections IV & V.

The proposed MOR consists of four steps as follows.

- **Step 1:** Construct the right projection matrix V. This matrix can be constructed by various existing methods to guarantee numerical accuracy. Here we use the block Arnodi algorithm as in PRIMA [2], which leads to $V \in \mathbb{R}^{n\times q}$, $q \approx ml$ if vector deflation is omitted. V can also be obtained by multi-point Krylov subspace if wide-band accuracy is required.

- **Step 2:** Solve $Q_o \geq 0$ from the GARE (8b). Here, Newton's method can be used [6, 10], and in each iteration a generalized Lyapunov function can be solved by LR-GADI [20] at the complexity of $O(n^2)$. Alternatively, Σ_p can be first converted to a standard state-space model by absorbing $\mathcal{E}_0$ into $\mathcal{A}_0$ and $\mathcal{B}_0$, followed by computing the corresponding positive-real observability Gramian Q_o^{ss}, and finally getting $Q_o = \mathcal{E}_0^{-T}Q_o^{ss}\mathcal{E}_0^{-1}$. The standard ARE can be solved efficiently by QADI [9] without Newton iterations, but huge numerical errors may be introduced in matrix inversion if $\mathcal{E}_0$ is ill-conditioned. Another problem is that matrix sparsity is normally destroyed by the matrix inversion.

- **Step 3:** Analogous to [14], we construct the left projection matrix U by

$$U^T = (V^T\mathcal{E}_0^TQ_o\mathcal{E}_0V)^{-1}V^T\mathcal{E}_0^TQ_o. \quad (11)$$

  Since $V \in \mathbb{R}^{n\times q}$, this step only costs $O(q^3)$, which is very cheap. Here we have assumed that $V^T\mathcal{E}_0^TQ_o\mathcal{E}_0V$ is invertible (similar to the situation in [13]), which is normally true in practice. Note that an additional condition $Q_o > 0$ is required if we attempt to guarantee the theoretical invertability. Because $\mathcal{E}_0$ is nonsingular, and all column vectors of V are linearly independent, when the Gramian is positive definite ($Q_o > 0$), the $q \times q$ matrix $V^T\mathcal{E}_0^TQ_o\mathcal{E}_0V > 0$ and the matrix inversion is well posed.

- **Step 4:** Construct the ROM Σ_{pr} ($\mathcal{E}_{0r}$, $\mathcal{A}_{0r}$, $\mathcal{B}_{0r}$, $\mathcal{C}_{0r}$, $\mathcal{M}_{0r}$) by $\mathcal{E}_{0r} = U^T\mathcal{E}_0V$, $\mathcal{A}_{0r} = U^T\mathcal{A}_0V$, $\mathcal{B}_{0r} = U^T\mathcal{B}_0$, $\mathcal{C}_{0r} = \mathcal{C}_0V$ and $\mathcal{M}_{0r} = \mathcal{M}_0$. Because $\mathcal{E}_{0r} = (V^T\mathcal{E}_0^TQ_o\mathcal{E}_0V)^{-1}V^T\mathcal{E}_0^TQ_o\mathcal{E}_0V = I$, the obtained ROM is a standard state-space model.

Remark: As U is constructed by (11), we have colspan$\{U\} \subset$ colspan$\{Q_o\}$, which is similar to the case in PRBT [c.f. (9) & (10)]. Because U is in the range of the observability Gramian Q_o, the proposed MOR has superior accuracy over PRIMA which simply sets $U = V$. This property will be verified by the numerical examples in Section VI.

IV. Passivity Preservation

In this section, we show that the proposed MOR scheme generates passive ROMs. Since the obtained ROM is a standard state-space model, Theorem 2 implies that this ROM is passive if there exist $W_r, \ L_r, \ X_r \ \in \mathbb{R}^{q\times q}$ and $X_r = X_r^T \geq 0$ solving the LMIs

$$\begin{cases} \mathcal{A}_{0r}^TX_r + X_r\mathcal{A}_{0r} = -L_rL_r^T, \\ X_r\mathcal{B}_{0r} - \mathcal{C}_{0r}^T = -L_rW_r, \\ \mathcal{M}_0 + \mathcal{M}_0^T \geq W_r^TW_r. \end{cases} \quad (12)$$

Lemma 1: *When the ROM is constructed as in Section III, there exists a symmetric $Q_{or} \geq 0$ that solves the ARE*

$$\mathcal{A}_r^T Q_{or} + Q_{or}\mathcal{A}_r + Q_{or}\mathcal{B}_r\mathcal{B}_r^T Q_{or} + \mathcal{C}_r^T\mathcal{C}_r = 0 \tag{13}$$

Here $\mathcal{A}_r = U^T\mathcal{A}V$, $\mathcal{B}_r = U^T\mathcal{B}$ and $C_r = \mathcal{C}V$, with U defined in (11).

Proof: We define the quadratic matrix function

$$\Psi(\mathcal{X}) = \mathcal{A}_r^T\mathcal{X} + \mathcal{X}\mathcal{A}_r + \mathcal{X}\mathcal{B}_r\mathcal{B}_r^T\mathcal{X} + \mathcal{C}_r^T\mathcal{C}_r \tag{14}$$

with $\mathcal{X} = \mathcal{X}^T$. Lemma 1 holds if we can find an $\mathcal{X} \geq 0$ such that

$$\Psi(\mathcal{X}) = 0. \tag{15}$$

In fact, a feasible solution to (15) is $\bar{\mathcal{X}} = V^T\mathcal{E}^T Q_o\mathcal{E}V \geq 0$, where V is the right projection matrix used in the proposed MOR scheme, and $Q_o \geq 0$ is the readily obtained stabilizing solution to (8b). To see this, we rewrite $\Psi(\bar{\mathcal{X}})$ as

$$\begin{aligned}\Psi(\bar{\mathcal{X}}) = {} & V^T\mathcal{A}^T UV^T\mathcal{E}^T Q_o\mathcal{E}V + V^T\mathcal{E}^T Q_o\mathcal{E}VU^T\mathcal{A}V \\ & + V^T\mathcal{E}^T Q_o\mathcal{E}VU^T\mathcal{B}\mathcal{B}^T UV^T\mathcal{E}^T Q_o\mathcal{E}V + V^T\mathcal{C}\mathcal{C}^T V.\end{aligned} \tag{16}$$

According to (11) and $\mathcal{E} = \mathcal{E}_0$, we have

$$\begin{aligned}UV^T\mathcal{E}^T Q_o\mathcal{E}V &= Q_o\mathcal{E}V(V^T\mathcal{E}^T Q_o\mathcal{E}V)^{-1}V^T\mathcal{E}^T Q_o\mathcal{E}V \\ &= Q_o\mathcal{E}V\end{aligned} \tag{17}$$

which reduces (16) to

$$\Psi(\bar{\mathcal{X}}) = V^T\left(\mathcal{A}^T Q_o\mathcal{E} + \mathcal{E}^T Q_o\mathcal{A} + \mathcal{E}^T Q_o\mathcal{B}\mathcal{B}^T Q_o\mathcal{E} + \mathcal{C}\mathcal{C}^T\right)V. \tag{18}$$

Since Q_o solves (8b), we have $\Psi(\bar{\mathcal{X}}) = 0$ and $Q_{or} = V^T\mathcal{E}^T Q_o\mathcal{E}V \geq 0$ solves the ARE (13).

□

Lemma 2: *If the ROM is constructed as in Section III, there exist $W_r,\ L_r,\ X_r \in \mathbb{R}^{q\times q}$ and $X_r = X_r^T \geq 0$ solving the LMIs in (12).*

Proof: We set $\mathcal{M}_0 + \mathcal{M}_0^T = W_r^T W_r$, then W_r can be solved as $W_r = \left(\mathcal{M}_0 + \mathcal{M}_0^T\right)^{\frac{1}{2}}$. We further set

$$X_r\mathcal{B}_{0r} - \mathcal{C}_{0r}^T = -L_r W_r \tag{19}$$

which gives

$$\begin{aligned}L_r &= -\left(X_r\mathcal{B}_{0r} - \mathcal{C}_{0r}^T\right)\left(\mathcal{M}_0 + \mathcal{M}_0^T\right)^{-\frac{1}{2}} \\ &= -X_r\mathcal{B}_r + \mathcal{C}_r^T.\end{aligned} \tag{20}$$

Then the LMI problem reduces to looking for an SPSD matrix X_r such that

$$\mathcal{A}_r^T X_r + X_r\mathcal{A}_r + X_r\mathcal{B}_r\mathcal{B}_r^T X_r + \mathcal{C}_r^T\mathcal{C}_r = 0. \tag{21}$$

Lemma 1 implies that a feasible solution to (21) is $X_r = Q_{or} = V^T\mathcal{E}^T Q_o\mathcal{E}V \geq 0$.

□

According to Lemmas 1 & 2, as well as Theorem 2, the passivity of the obtained ROM is guaranteed.

V. Passive MOR for Singular Models

This section applies the proposed MOR to singular DSs, to preserve both passivity and the possible polynomial part. For a passive singular DS, its transfer matrix can be written as $H(s) = H_p(s) + sM_1$ with $M_1 \geq 0$ and $H_p(s)$ positive real. The improper part sM_1 may appear, for example, when strong crosstalk effects exist [12]. Although the block Arnoldi process preserves the PSD structures for MNA equations, M_0 [c.f. the definition under (3)] and sM_1 are normally lost, and therefore the polynomial part $P(s)$ cannot be accurately captured. For indefinite singular models from EM solvers, neither passivity can be preserved, nor $P(s)$ can be captured by existing Krylov-subspace moment matchings.

For a singular DS, if (E, A) is regular, there exist the left and right spectral projectors defined as

$$P_l = T_l\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}T_l^{-1}, \quad P_r = T_r^{-1}\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}T_r \tag{22}$$

which can be formed by canonical projector technique [7, 8] without computing the expensive and possibly unstable Weierstrass canonical form. For MNA equations, P_l and P_r can be obtained by their closed forms if the circuit topology is given [5, 6]. If we do not know the circuit topology or the DS is generated by EM solvers, thus often indefinite, spectral projectors can be constructed at the cost of $O(n)$ to $O(n^2)$ by exploiting the matrix sparsity [21]. With P_l and P_r, two projected GAREs can be solved to balance and truncate the original singular DS [5, 6], with passivity and the polynomial part preserved.

Here we present a novel and more efficient method: we use spectral projectors to extract the improper part and reconstruct the proper part by a nonsingular DS; after that the proper-part model is reduced by the proposed MOR via solving only one GARE. The framework is summarized below.

- **Step 1:** Projector-based system decomposition. With $\alpha > 0$, we extract the proper subsystem by

$$\begin{aligned}&\mathcal{E}_0 = EP_r - \alpha A(I - P_r);\ \ \mathcal{A}_0 = A, \\ &\mathcal{C}_0 = CP_r,\ \mathcal{B}_0 = B,\ \ \mathcal{M}_0 = G(0).\end{aligned} \tag{23}$$

  Here $G(s) = C(I - P_r)(sE - A)^{-1}B + D$. Meanwhile, M_1 can be extracted by

$$M_1 = \frac{G(s_1) - G(s_2)}{s_1 - s_2} \tag{24}$$

  with $s_1, s_2 \in \mathbb{R}^+$ randomly selected. It is straightforward to prove that $\Sigma_p(\mathcal{E}_0, \mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0, \mathcal{M}_0)$ is a realization of the proper and positive real function $H_p(s)$. Meanwhile (23) ensures that $\mathcal{E}_0$ is nonsingular.

- **Step 2:** Reduce the proper subsystem $\Sigma_p(\mathcal{E}_0, \mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0, \mathcal{M}_0)$. Use the MOR scheme proposed in Section III to reduce the proper subsystem, obtaining a ROM $\mathcal{E}_{0r}, \mathcal{A}_{0r}, \mathcal{B}_{0r}, \mathcal{C}_{0r}, \mathcal{M}_0$. In this step, the improper part is unchanged.

- **Step 3:** Singular ROM reconstruction. Construct a ROM for the original singular system

$$E_r = \begin{bmatrix} \mathcal{E}_{0r} & & \\ & 0 & I_m \\ & 0 & 0 \end{bmatrix}, A_r = \begin{bmatrix} \mathcal{A}_{0r} & & \\ & I_m & \\ & & I_m \end{bmatrix},$$
$$B_r = \begin{bmatrix} \mathcal{B}_{0r} \\ 0 \\ M_1 \end{bmatrix}, C_r = \begin{bmatrix} \mathcal{C}_{0r} & -I_m & 0 \end{bmatrix}, \; D_r = \mathcal{M}_0. \tag{25}$$

One can prove that the resulting transfer matrix is $H_r(s) = C_r(sE_r - A_r)^{-1}B_r + \mathcal{M}_0 + sM_1$.

From the results in Section IV, we know that $H_{pr}(s) = C_r(sE_r - A_r)^{-1}B_r + \mathcal{M}_0$ is positive real. Since $M_1 = M_1^T \geq 0$, according to Theorem 1, the final ROM is passive. The proposed scheme has some advantages over the existing algorithms:

- Only one GARE is required compared with the the DS-form PRBT, which requires solving two GAREs and is directly performed on singular DSs [5, 6].
- We only need to approximate the strictly proper part $H_{sp}(s)$. Because $H_{sp}(s)$ monolithically decays in the high-frequency band, we only need to match its moments in the low-frequency band, and therefore significantly fewer expansion points are needed compared with existing moment-matching schemes.
- The polynomial part is preserved without any numerical error, therefore the obtained ROM is very accurate in the high-frequency band. Because the low-frequency-band response can be easily captured by moment matching, the proposed algorithm has very good global accuracy.

VI. Numerical Examples

This section verifies the proposed MOR using an order-1505 coupled *RLC* interconnect example. This MNA model has a singular E matrix, $D = 0$, and the port number is 5. Due to the strong crosstalk effects, this DS model has an improper part which cannot be captured by conventional moment matchings. Note that although this model is PSD structured, the extracted proper subsystem is indefinite. All algorithms are implemented in Matlab and executed in a 2.66 GHz desktop with 2 GB of RAM .

Since this *RLC* model is a singular DS, in the first step we use the right spectral projector to extract the proper and improper parts. This step (projector construction plus system decomposition) costs 0.015 seconds, which is negligible in the MOR process. The extracted proper subsystem is a *nonsingular indefinite* DS, which is reduced to an order-50 ROM by PRIMA [2], a DS-form PRBT [5, 6] and the MOR scheme proposed in this paper. The CPU times are listed in Table I. For fairness, we use the DC point as the single expansion point in both PRIMA and in our MOR scheme. PRIMA is the fastest, which is expected due to its sparse matrix-vector operations. Compared with PRBT, the proposed algorithm is more than $2\times$ faster since only one GARE is solved and it does not require the matrix factorization in (9).

TABLE I
Comparison of different MORs for the indefinite proper DS.

Model Size	Number of Port	PRIMA (sec)	PRBT (sec)	Proposed (sec)
1505	5	1.76	507.8	243.1

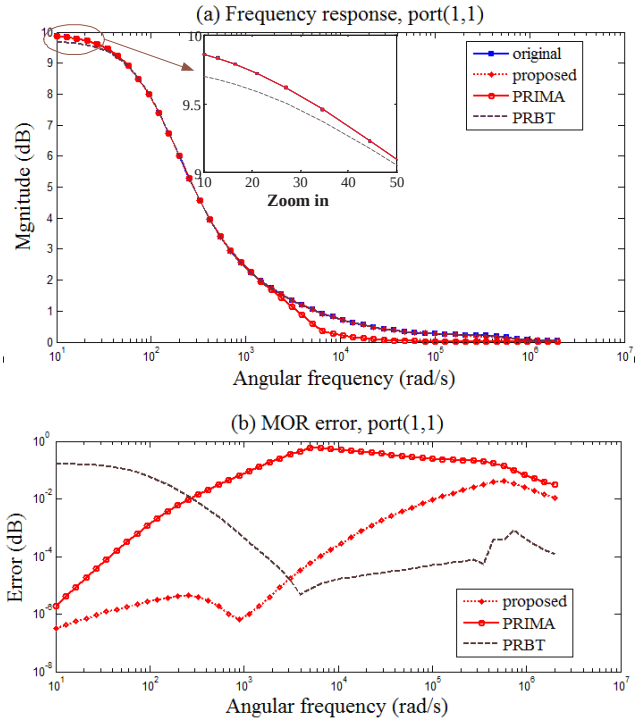

Fig. 1. Accuracy comparison of various MOR schemes for the indefinite strictly proper $H_{sp}(s)$. (a) The frequency response of port (1,1); (b) the numerical errors of the ROMs obtained by different algorithms. We can see that PRBT is not accurate in the low-frequency band, and PRIMA is inaccurate in the high-frequency band, but the proposed MOR result is indistinguishable with the original one in the whole frequency band.

We compare the accuracy of different MORs, by plotting the frequency response of the strictly proper part (i.e., $H_{sp}(s)$ which approaches 0 as $s \to \infty$). As shown in Fig. 1, PRBT has higher accuracy over PRIMA in the high-frequency band, but lower accuracy in the low-frequency band. Compared with PRIMA, the proposed MOR has higher global accuracy. If expansion points in the high-frequency band are used, the accuracy of the projection-based MORs can be further improved.

To verify the passivity of the proper part (i.e., $H_p(s) = H_{sp}(s) + \mathcal{M}_0$) of the obtained ROMs, we compute the generalized eigenvalues of the passivity test matrix pencils adopted in generalized Hamiltonian method (GHM [22, 23]). As shown in Fig. 2, GHM finds many purely imaginary results for the ROM of PRIMA, which implies that PRIMA cannot preserve passivity for this indefinite DS. On the other hand, the ROM from our proposed algorithm is passive, since no purely imaginary generalized eigenvalues are obtained by GHM.

Finally, we compare our method and PRIMA directly on the original singular *RLC* model (ROM size: 50). Again, we compare their accuracy in approximating the original transfer matrix. As plotted in Fig. 3, the proposed MOR can accurately capture the polynomial part and the approximated re-

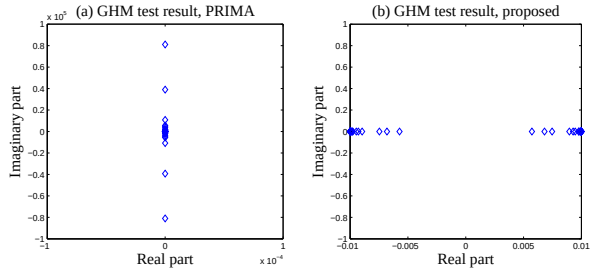

Fig. 2. (a) Passivity test results for the ROM by PRIMA (nonpassive); (b) test results for the ROM from the proposed MOR (passive).

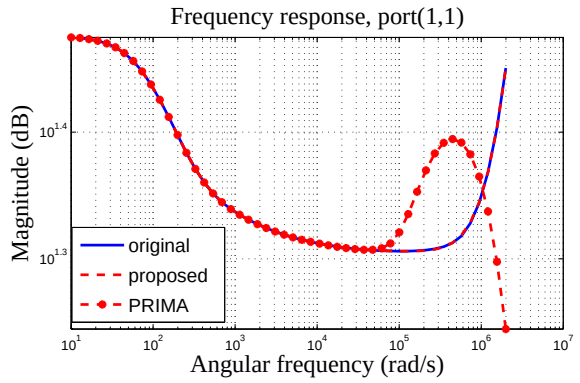

Fig. 3. MOR results for the original singular *RLC* model that contains a polynomial part. The response from the proposed approach is basically overlapping with the original one in the entire frequency band.

sult matches the original one with little error, whereas PRIMA produces a strictly proper system which is very inaccurate in the high-frequency band. Note that PRIMA not only misses the improper part, but it also misses the constant term M_0 in the polynomial part. For the proposed MOR, since the proper part is reduced with passivity preservation and the improper part is kept unchanged, the resulting ROM for the whole system is also passive. Similar to the proposed algorithm, in DS-form PRBT, we can also add the improper part to the approximated proper transfer matrix. Therefore, with spectral projectors PRBT can also preserve the polynomial part.

VII. Conclusion

In this paper we have presented a new algorithm for the MOR of indefinite DSs. Our approach can preserve system passivity regardless of the system structure. With spectral projectors, the proposed flow can preserve the possible polynomial part of a singular DS model. Since only one GARE is solved, the proposed MOR is more efficient than PRBT. Compared with PRIMA, our method provides superior numerical accuracy.

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