

New Parameters of Linear Codes Expressing Security Performance of Universal Secure Network Coding

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Abstract—The universal secure network coding presented by Silva et al. realizes secure and reliable transmission of a secret message over any underlying network code, by using maximum rank distance codes. Inspired by their result, this paper considers the secure network coding based on arbitrary linear codes, and investigates its security performance and error correction capability that are guaranteed independently of the underlying network code. The security performance and error correction capability are said to be *universal* when they are independent of underlying network codes. This paper introduces new code parameters, the relative dimension/intersection profile (RDIP) and the relative generalized rank weight (RGRW) of linear codes. We reveal that the universal security performance and universal error correction capability of secure network coding are expressed in terms of the RDIP and RGRW of linear codes. The security and error correction of existing schemes are also analyzed as applications of the RDIP and RGRW.

I. INTRODUCTION

In the scenario of *secure network coding* introduced by Cai et al. [2], a source node transmits n packets from n outgoing links to sink nodes through a network that implements network coding [1,11,13], and each sink node receives n packets from n incoming links. In the network, there is a wiretapper who observes $\mu (< n)$ links. The problem is how to encode a secret message into n transmitted packets at the source node, in such a way that the wiretapper obtain no information about the message in the sense of information theoretic security.

As shown in [6], secure network coding can be seen as a generalization of the wiretap channel II [18] or secret sharing schemes based on linear codes [3,5] for network coding. Hence, in secure network coding, the secrecy is realized by introducing the randomness into n transmitted packets as follows. Suppose the message is represented by l packets S_1, \dots, S_l ($1 \leq l \leq n$). Then, the source node encodes (S_1, \dots, S_l) together with $n-l$ random packets by linear codes, and generates n transmitted packets [6,17,21].

Silva et al. [21] proposed the *universal secure network coding* that is based on maximum rank distance (MRD) codes [8]. Their scheme was universal in the sense that their scheme guarantees that over *any* underlying network code, no information about S leaks out even if any $n-l$ links are observed by a wiretapper. As shown in [21], their scheme with MRD codes is optimal in terms of security and communication rate. However, there exists some restrictions in universal secure

network coding with MRD codes. In their scheme, the network must transport packets of size $m \geq n$. The MRD code used in the scheme is defined over an $\mathbb{F}_{q^m}^n$, where \mathbb{F}_{q^m} is an m -degree field extension of a field \mathbb{F}_q with order q . Thus, the size of the field \mathbb{F}_{q^m} increases exponentially with m , and the restriction of MRD codes with $m \geq n$ invokes the large computational cost for encoding and decoding of MRD codes if n is large. It is undesirable especially in resource constraint environments.

Considering secure network coding without such a restriction, Ngai et al. [17], and later Zhang et al. [25], investigated the security performance of secure network coding based on general linear codes. They introduced a new parameter of linear codes, called the *relative network generalized Hamming weight* (RNGHW), and revealed that the security performance is expressed in terms of the RNGHW. The RNGHW depends on the set of coding vectors of the underlying network code. Hence, the RNGHW is not universal.

The aim of this paper is to investigate the security performance of universal secure network coding based on general linear codes, which is always guaranteed over *any* underlying network code, even over random network code. This paper defines the universal security performance by the following two criteria. One is called the *universal equivocation* Θ_μ that is the minimum uncertainty of the message under observation of $\mu (< n)$ links, guaranteed independently of the underlying network code. The other is called the *universal Ω -strong security*, where Ω is a performance measure such that no part of the secret message is deterministically revealed even if at most Ω links are observed. The paper [12] proposed a specific construction of the secure network coding that attains the universal $(n-1)$ -strong security, and such a scheme is called universal strongly secure network coding [20]. Namely, the definition of universal Ω -strong security given in this paper is a generalization of universal strongly secure network coding considered in [12,20] for the number of tapped links.

In order to express Θ_μ and Ω in terms of code parameters, this paper introduces two parameters of linear codes, called the *relative dimension/intersection profile* (RDIP) and the *relative generalized rank weight* (RGRW). The RGRW is a generalization of the minimum rank distance [8] of a code. We reveal that Θ_μ and Ω can be expressed in terms of the RDIP and the RGRW of the codes. Duursma et al. [5] first observed

that the *relative generalized Hamming weight* [14] exactly expresses the security performance and the error correction capability of secret sharing. Our definitions of RGRW and RDIP are motivated by their result [5].

Assume that the attacker is able not only to eavesdrop but also to inject erroneous packets anywhere in the network. Also assume that the network may suffer from the rank deficiency of the transfer matrix at a sink node. Silva et al.'s scheme based on MRD codes [21] enables to correct such errors and rank deficiency at each sink node, where its error correction capability is guaranteed over any underlying network code, i.e., universal. This paper also generalizes their result and reveals that the universal error correction capability of secure network coding based on arbitrary linear codes can be expressed in terms of the RGRW of the codes.

The remainder of this paper is organized as follows. Sect. II presents basic notations, and introduces linear network coding. Sect. III defines the universal security performance and universal error correction capability of secure network coding over wiretap network. Sect. IV defines the RDIP and RGRW of linear codes, and introduces their basic properties. In Sect. V, the universal security performance is expressed in terms of the RDIP and RGRW. The security of existing schemes [12,20,21] is also analyzed as applications of the RDIP and RGRW in Examples 17 and 21. Sect. VI gives the expression of the universal error correction capability in terms of the RGRW, and also analyze the error correction of [21] by the RGRW in Example 27.

II. PRELIMINARY

A. Basic Notations

Let $H(X)$ be the Shannon entropy for a random variable X , $H(X|Y)$ be the conditional entropy of X given Y , and $I(X; Y)$ be the mutual information between X and Y [4]. We write $|\mathcal{X}|$ as the cardinality of a set \mathcal{X} . The entropy and the mutual information are always computed by using \log_{q^m} .

Let \mathbb{F}_q stand for a finite field containing q elements and \mathbb{F}_{q^m} be an m -degree field extension of \mathbb{F}_q ($m \geq 1$). Let \mathbb{F}_q^n denote an n -dimensional row vector space over \mathbb{F}_q . Similarly, $\mathbb{F}_{q^m}^n$ stands for an n -dimensional row vector space over \mathbb{F}_{q^m} . Unless otherwise stated, we consider subspaces, ranks, dimensions, etc, over the field extension \mathbb{F}_{q^m} instead of the base field \mathbb{F}_q .

An $[n, k]$ linear code C over $\mathbb{F}_{q^m}^n$ is a k -dimensional subspace of $\mathbb{F}_{q^m}^n$. Let C^\perp denote a *dual code* of a code C . A subspace of a code is called a *subcode* [15]. For $C \subseteq \mathbb{F}_{q^m}^n$, we denote by $C|_{\mathbb{F}_q}$ a *subfield subcode* of C over \mathbb{F}_q [15]. Observe that $\dim C$ means the dimension of C as a vector space over \mathbb{F}_{q^m} whereas $\dim C|_{\mathbb{F}_q}$ is the dimension of $C|_{\mathbb{F}_q}$ over \mathbb{F}_q .

For a vector $\vec{v} = [v_1, \dots, v_n] \in \mathbb{F}_{q^m}^n$ and a subspace $V \subseteq \mathbb{F}_{q^m}^n$, we denote $\vec{v}^q = [v_1^q, \dots, v_n^q]$ and $V^q = \{\vec{v}^q : \vec{v} \in V\}$. Define a family of subspaces $V \subseteq \mathbb{F}_{q^m}^n$ satisfying $V = V^q$ by $\Gamma(\mathbb{F}_{q^m}^n) \triangleq \{\text{subspace } V \subseteq \mathbb{F}_{q^m}^n : V = V^q\}$. Also define $\Gamma_i(\mathbb{F}_{q^m}^n) \triangleq \{V \in \Gamma(\mathbb{F}_{q^m}^n) : \dim V = i\}$. For a subspace $V \subseteq \mathbb{F}_{q^m}^n$, the followings are equivalent: 1) $V \in \Gamma(\mathbb{F}_{q^m}^n)$; 2) $\dim V = \dim V|_{\mathbb{F}_q}$ [22, Lemma 1].

B. Linear Network Coding

As in [2,6,17,21,25], we consider a multicast communication network represented by a directed multigraph with unit capacity links, a single source node, and multiple sink nodes. We assume that *linear network coding* [11,13] is employed over the network. Elements of a column vector space $\mathbb{F}_q^{m \times 1}$ are called *packets*. Assume that each link in the network can carry a single \mathbb{F}_q -symbol per one time slot, and that each link transports a single packet over m time slots without delays, erasures, or errors.

The source node produces n packets $X_1, \dots, X_n \in \mathbb{F}_q^{m \times 1}$ and transmits X_1, \dots, X_n on n outgoing links over m consecutive time slots. Define the $m \times n$ matrix $X = [X_1, \dots, X_n]$. The data flow on any link can be represented as an \mathbb{F}_q -linear combination of packets $X_1, \dots, X_n \in \mathbb{F}_q^{m \times 1}$. Namely, the information transmitted on a link e can be denoted as $b_e X^T \in \mathbb{F}_q^{1 \times m}$, where $b_e \in \mathbb{F}_q^n$ is called a *global coding vector* (GCV) of e . Suppose that a sink node has N incoming links. Then, the information received at a sink node can be represented as an $N \times m$ matrix $AX^T \in \mathbb{F}_q^{N \times m}$, where $A \in \mathbb{F}_q^{N \times n}$ is the transfer matrix constructed by gathering the GCV's of N incoming links. The network code is called *feasible* if every transfer matrix to a sink node has rank n over \mathbb{F}_q . The system is called *coherent* if A is known to each sink node; otherwise, called *noncoherent*.

III. UNIVERSAL SECURITY PERFORMANCE AND UNIVERSAL ERROR CORRECTION CAPABILITY OF SECURE NETWORK CODING

This section introduces the wiretap network model with packet errors and the nested coset coding scheme in secure network coding [6,17,21,25]. Then, we define the universal security performance in terms of the *universal equivocation* and the *universal Ω -strong security* on the wiretap network model. We also define the universal error correction capability of secure network coding. From now on, only one sink node is assumed without loss of generality. In addition, we focus on the fundamental case of coherent systems in this paper due to the space constraint. But, as in [21], all analysis in this paper can be easily adapted to the case of noncoherent systems.

A. Wiretap Networks with Errors, and Nested Coset Coding

Following [2,6,17,21,25], assume that in the setup of Sect. II-B, there is a wiretapper who has access to packets transmitted on any μ links. Let \mathcal{W} be the set of $|\mathcal{W}| = \mu$ links observed by the wiretapper. Then the packets observed by the wiretapper are given by $W^T = B_{\mathcal{W}} X^T$, where rows of $B_{\mathcal{W}} \in \mathbb{F}_q^{\mu \times n}$ are the GCV's associated with the links in \mathcal{W} .

In the scenario [6,17,21,25], the source node first regards an m -dimensional column vector space $\mathbb{F}_q^{m \times 1}$ as \mathbb{F}_{q^m} , and fix l for $1 \leq l \leq n$. Let $S = [S_1, \dots, S_l] \in \mathbb{F}_q^l$ be the secret message, and assume that S_1, \dots, S_l are uniformly distributed over \mathbb{F}_q^l and mutually independent. Under the wiretapper's observation, the source node wants to transmit S without information leakage to the wiretapper. To protect S from the wiretapper, the source node encodes S to a transmitted vector $X = [X_1, \dots, X_n] \in \mathbb{F}_{q^m}^n$ of n packets by applying the *nested coset coding scheme*

[3,5,23,24] on S . In [3,5], its special case is called a *secret sharing scheme based on linear codes*.

Definition 1 (Nested Coset Coding Scheme). Let $C_1 \subseteq \mathbb{F}_{q^m}^n$ be a linear code over \mathbb{F}_{q^m} ($m \geq 1$), and $C_2 \subsetneq C_1$ be its subcode with dimension $\dim C_2 = \dim C_1 - l$ over \mathbb{F}_{q^m} . Let $\psi : \mathbb{F}_{q^m}^l \rightarrow C_1/C_2$ be an arbitrary isomorphism. For a secret message $S \in \mathbb{F}_{q^m}^l$, we choose X from a coset $\psi(S) \in C_1/C_2$ uniformly at random and independently of S .

Then, the source node finally transmit X over the network coded network. Def. 1 includes the Ozarow-Wyner coset coding scheme [18] as a special case with $C_1 = \mathbb{F}_{q^m}^n$. Hence, when we set $C_1 = \mathbb{F}_{q^m}^n$, this is the secure network coding based on Ozarow-Wyner coset coding scheme [6,17,21].

Corresponding to X transmitted from the source node, the sink node receives a vector of N packets $Y \in \mathbb{F}_{q^m}^N$. Here we extend the basic network model described in Sect. II-B to incorporate packet errors and rank deficiency of the transfer matrix $A \in \mathbb{F}_q^{N \times n}$ of the sink node. Suppose that at most t errors can occur in any of links, causing the corresponding packets to become corrupted. Then, as [19], Y can be expressed by

$$Y^T = AX^T + DZ^T,$$

where $Z \in \mathbb{F}_{q^m}^t$ is the t error packets, and $D \in \mathbb{F}_q^{N \times t}$ is the transfer matrix of Z . We define $\rho \triangleq n - \text{rank} A$ as the rank deficiency of A . In this setup, we want to decode S correctly from Y . If the network is free of errors and the network code used is feasible, X can be always reconstructed from $Y^T = AX^T$ as described in Sect. II-B. Then, the coset $\psi(S)$, and hence S , is uniquely determined from X from Def. 1.

B. Definition of Universal Security Performance

The security performance of secure network coding in the above model was measured by the following criterion [17,25].

Definition 2 (Equivocation). The minimum uncertainty θ_μ of S given $B_{\mathcal{W}}X^T$ for all possible \mathcal{W} 's ($|\mathcal{W}| = \mu$) in the network is called *equivocation*, defined as $\theta_\mu \triangleq \min_{\mathcal{W}:|\mathcal{W}|=\mu} H(S|B_{\mathcal{W}}X^T)$.

As defined in Def. 2, θ_μ depends on the underlying network code. In [17,25], θ_μ for $m = 1$ was expressed in terms of the relative network generalized Hamming weight (RNGHW) of C_1 and C_2 . The RNGHW is the value determined according to GCV's of all links in the network. Hence, the RNGHW cannot determine the equivocation over random linear network code [10]. Here, we extend Def. 2 by requiring the independence of the underlying network code, as follows.

Definition 3 (Universal Equivocation). The *universal equivocation* Θ_μ is the minimum uncertainty of S given BX^T for all $B \in \mathbb{F}_q^{\mu \times n}$, defined as

$$\Theta_\mu \triangleq \min_{B \in \mathbb{F}_q^{\mu \times n}} H(S|BX^T).$$

As defined in Def. 3, Θ_μ does not depend on the set of \mathcal{W} 's in the network. Silva et al.'s universal secure network coding scheme based on MRD codes [21] achieves $\Theta_{n-l} = H(S)$ in Def. 3 provided $m \geq n$.

Def. 3 defines the security for the whole components of a message $S = [S_1, \dots, S_l]$. Here we focus on the security for every part of S , and give the following definition.

Definition 4 (Universal Ω -Strong Security). Let $S_{\mathcal{Z}} = (S_i : i \in \mathcal{Z})$ be a tuple for a subset $\mathcal{Z} \subseteq \{1, \dots, l\}$. We say that a secure network coding scheme attains the *universal Ω -strong security* if we have

$$I(S_{\mathcal{Z}}; BX^T) = 0, \quad \forall \mathcal{Z}, \forall B \in \mathbb{F}_q^{(\Omega-|\mathcal{Z}|+1) \times n}. \quad (1)$$

As [9,16,20], a scheme with universal Ω -strong security does not leak any $|\mathcal{Z}|$ components of S even if at most $\Omega - |\mathcal{Z}| + 1$ links are observed by the wiretapper. Moreover, this guarantee holds over any underlying network code as Θ_μ . We note that if a scheme achieves the Ω -strong security, the universal equivocation Θ_μ for $\mu = \Omega - l + 1$ must be $\Theta_{\Omega-l+1} = H(S)$ as shown in Def. 4. However, the converse does not always hold.

The scheme in [12] achieves $\Omega = n - 1$ provided $m \geq l + n$ by nested coset coding with MRD codes. The universal strongly security in [20] is a special case of Def. 4 with $\Omega = n - 1$.

C. Definition of the Universal Error Correction Capability of Secure Network Coding

In the model described in Sect. III-A, the error correction capability of secure network coding, guaranteed over any underlying network code, is defined as follows.

Definition 5 (Universally t -Error- ρ -Erasure-Correcting Secure Network Coding). A secure network coding scheme is called *universally t -error- ρ -erasure-correcting*, if

$$H(S|Y) = 0, \quad Y^T = AX^T + DZ^T,$$

$$\forall A \in \mathbb{F}_q^{N \times n} : \text{rank} A \geq n - \rho, \forall X \in \psi(S), \forall D \in \mathbb{F}_q^{N \times t}, \forall Z \in \mathbb{F}_{q^m}^t,$$

i.e., S can be uniquely determined from Y against t errors over any underlying network code with at most ρ rank deficiency.

Silva et al.'s scheme [21, Section VI] is universally t -error- ρ -erasure-correcting when the minimum rank distance [8] of C_1 is greater than $2t + \rho$.

IV. NEW PARAMETERS OF LINEAR CODES AND THEIR PROPERTIES

This section introduce the *relative dimension/intersection profile* (RDIP) and the *relative generalized rank weight* (RGRW) of linear codes. In the following sections, these parameters are used to characterize the universal security performance and the universal error correction capability of secure network coding.

A. Definition

We first define the *relative dimension/intersection profile* (RDIP) of linear codes as follows.

Definition 6 (Relative Dimension/Intersection Profile). Let $C_1 \subseteq \mathbb{F}_{q^m}^n$ be a linear code and $C_2 \subsetneq C_1$ be its subcode. Then, the i -th relative dimension/intersection profile (RDIP) of C_1

and C_2 is the greatest difference between dimensions over \mathbb{F}_{q^m} of intersections, defined as

$$K_{R,i}(C_1, C_2) \triangleq \max_{V \in \Gamma_i(\mathbb{F}_{q^m}^n)} \{\dim(C_1 \cap V) - \dim(C_2 \cap V)\}, \quad (2)$$

for $0 \leq i \leq n$.

Next, we define the *relative generalized rank weight* (RGRW) of linear codes as follows.

Definition 7 (Relative Generalized Rank Weight). Let $C_1 \subseteq \mathbb{F}_{q^m}^n$ be a linear code and $C_2 \subsetneq C_1$ be its subcode. Then, the i -th relative generalized rank weight (RGRW) of C_1 and C_2 is defined by

$$M_{R,i}(C_1, C_2) \triangleq \min \left\{ \dim V : V \in \Gamma(\mathbb{F}_{q^m}^n), \dim(C_1 \cap V) - \dim(C_2 \cap V) \geq i \right\}, \quad (3)$$

for $0 \leq i \leq \dim(C_1/C_2)$.

The relative dimension/length profile and the relative generalized Hamming weight introduced in [14] are equivalent to Eqs. (2) and (3) with $\Gamma_i(\mathbb{F}_{q^m}^n)$ and $\Gamma(\mathbb{F}_{q^m}^n)$ replaced by suitable smaller sets, respectively.

B. Basic Properties of the RDIP and the RGRW, and the Relation between the Rank Distance and the RGRW

This subsection introduces some basic properties of the RDIP and the RGRW, and also shows the relation between the RGRW and the rank distance [8]. These will be used for expressions of the universal security performance and the universal error correction capability of secure network coding.

First, we introduce the following theorem and lemma about the RDIP and the RGRW.

Theorem 8 (Monotonicity of the RDIP). Let $C_1 \subseteq \mathbb{F}_{q^m}^n$ be a linear code and $C_2 \subsetneq C_1$ be its subcode. Then, the i -th RDIP $K_{R,i}(C_1, C_2)$ is nondecreasing with i from $K_{R,0}(C_1, C_2) = 0$ to $K_{R,n}(C_1, C_2) = \dim(C_1/C_2)$, and $0 \leq K_{R,i+1}(C_1, C_2) - K_{R,i}(C_1, C_2) \leq 1$ holds.

Proof: $K_{R,0}(C_1, C_2) = 0$ and $K_{R,n}(C_1, C_2) = \dim(C_1/C_2)$, are obvious from Def. 6. Recall that

$$\Gamma_i(\mathbb{F}_{q^m}^n) = \left\{ V \subseteq \mathbb{F}_{q^m}^n : V = \{\vec{u}G : \vec{u} \in \mathbb{F}_{q^m}^i\}, G \in \mathbb{F}_q^{i \times n}, \text{rank } G = i \right\},$$

for $1 \leq i \leq n$ from [22, Lemma 1]. This implies that for any subspace $V_1 \in \Gamma_{i+1}(\mathbb{F}_{q^m}^n)$, there always exist some V_2 's satisfying $V_2 \in \Gamma_i(\mathbb{F}_{q^m}^n)$ and $V_2 \subsetneq V_1$. This yields $K_{R,i}(C_1, C_2) \leq K_{R,i+1}(C_1, C_2)$.

Next we show that the increment at each step is at most 1. Consider arbitrary subspaces $V, V' \in \Gamma(\mathbb{F}_{q^m}^n)$ such that $\dim V' = \dim V + 1$ and $V \subsetneq V'$. Let $f = \dim(C_1 \cap V) - \dim(C_2 \cap V)$; $g = \dim(C_1 \cap V') - \dim(C_2 \cap V')$. Since $\dim(C_1 \cap V) + 1 \geq \dim(C_1 \cap V') \geq \dim(C_1 \cap V)$ and $C_2 \subsetneq C_1$, we have $f+1 \geq g \geq f$ and hence $K_{R,i}(C_1, C_2) + 1 \geq K_{R,i+1}(C_1, C_2) \geq K_{R,i}(C_1, C_2)$. ■

Lemma 9. Let $C_1 \subseteq \mathbb{F}_{q^m}^n$ be a linear code and $C_2 \subsetneq C_1$ be its subcode. Then, the i -th RGRW $M_{R,i}(C_1, C_2)$ is strictly

increasing with i . Moreover, $M_{R,0}(C_1, C_2) = 0$ and

$$M_{R,i}(C_1, C_2) = \min \left\{ j : K_{R,j}(C_1, C_2) = i \right\} = \min \left\{ \dim V : V \in \Gamma(\mathbb{F}_{q^m}^n), \dim(C_1 \cap V) - \dim(C_2 \cap V) = i \right\},$$

where $0 \leq i \leq \dim(C_1/C_2)$.

Proof: First we have

$$\begin{aligned} & \min \left\{ j : K_{R,j}(C_1, C_2) \geq i \right\} \\ &= \min \left\{ j : \exists V \in \Gamma_j(\mathbb{F}_{q^m}^n), \text{ such that } \dim(C_1 \cap V) - \dim(C_2 \cap V) \geq i \right\} \\ &= \min \left\{ \dim V : V \in \Gamma(\mathbb{F}_{q^m}^n), \dim(C_1 \cap V) - \dim(C_2 \cap V) \geq i \right\} \\ &= M_{R,i}(C_1, C_2). \end{aligned}$$

From Theorem 8, we have $\left\{ j : K_{R,j}(C_1, C_2) = i \right\} \cap \left\{ j : K_{R,j}(C_1, C_2) \geq i+1 \right\} = \emptyset$. We thus have

$$\begin{aligned} M_{R,i}(C_1, C_2) &= \min \left\{ j : K_{R,j}(C_1, C_2) \geq i \right\} \\ &= \min \left\{ j : K_{R,j}(C_1, C_2) = i \right\}. \end{aligned}$$

Therefore the RGRW is strictly increasing with i and thus

$$M_{R,i}(C_1, C_2) = \min \left\{ \dim V : V \in \Gamma(\mathbb{F}_{q^m}^n), \dim(C_1 \cap V) - \dim(C_2 \cap V) = i \right\},$$

is established. ■

Next, we show the relation between the rank distance [8] and the RGRW. Let $\phi_m : \mathbb{F}_{q^m} \rightarrow \mathbb{F}_q^{m \times 1}$ be an \mathbb{F}_q -linear isomorphism that expands an element of \mathbb{F}_{q^m} as a column vector over \mathbb{F}_q with respect to some fixed basis for \mathbb{F}_{q^m} over \mathbb{F}_q . Then, we define the *rank over \mathbb{F}_q* of a vector $\vec{x} = [x_1, \dots, x_n] \in \mathbb{F}_{q^m}^n$, denoted by $\text{rank}_{\mathbb{F}_q}(\vec{x})$, as the rank of $m \times n$ matrix $[\phi_m(x_1), \dots, \phi_m(x_n)]$ over \mathbb{F}_q . The rank distance [8] between two vectors $\vec{x}, \vec{y} \in \mathbb{F}_{q^m}^n$ is given by $d_R(\vec{x}, \vec{y}) \triangleq \text{rank}_{\mathbb{F}_q}(\vec{y} - \vec{x})$. The minimum rank distance [8] of a code C is given as $d_R(C) \triangleq \min\{d_R(\vec{x}, \vec{y}) : \vec{x}, \vec{y} \in C, \vec{x} \neq \vec{y}\} = \min\{d_R(\vec{x}, \vec{0}) : \vec{x} \in C, \vec{x} \neq \vec{0}\}$. For a subspace $V \subseteq \mathbb{F}_{q^m}^n$, we define by $V^* \triangleq \sum_{i=0}^{m-1} V^{q^i}$ the sum of subspaces $V, V^q, \dots, V^{q^{m-1}}$.

Lemma 10. For a subspace $V \subseteq \mathbb{F}_{q^m}^n$ with $\dim V = 1$, we have $\dim V^* = d_R(V)$.

Proof: Let $\vec{b} = [b_1, \dots, b_n] \in V$ be a nonzero vector, which implies $\text{rank}_{\mathbb{F}_q}(\vec{b}) = d_R(V)$. Let $M \triangleq [a_{i,j}]_{i,j=1}^{m,n} \in \mathbb{F}_{q^m}^{m \times n}$, $a_{i,j} = b_j^{q^{i-1}}$. Each vector in V^* is represented by an \mathbb{F}_{q^m} -linear combination of $\vec{b}, \vec{b}^q, \dots, \vec{b}^{q^{m-1}}$, and hence $\dim V^* = \text{rank } M$.

For $\alpha_1, \alpha_2 \in \mathbb{F}_q, \beta_1, \beta_2 \in \mathbb{F}_{q^m}$, we have $\alpha_1 \phi_m(\beta_1) + \alpha_2 \phi_m(\beta_2) = \phi_m(\alpha_1 \beta_1 + \alpha_2 \beta_2)$. This implies that there always exists some $P \in \mathbb{F}_q^{n \times n}$ with $\text{rank } P = n$ satisfying

$$\vec{b}^q P = [g_1, \dots, g_{d_R(V)}, 0, \dots, 0] \in \mathbb{F}_{q^m}^n, g_j \neq 0, \quad (4)$$

where $g_1, \dots, g_{d_R(V)}$ are linearly independent over \mathbb{F}_q , and note that P represents the elementary column operation on $[\phi_m(b_1), \dots, \phi_m(b_n)]$. Also for $\alpha_1, \alpha_2 \in \mathbb{F}_q, \beta_1, \beta_2 \in \mathbb{F}_{q^m}$, we have $\alpha_1 \beta_1^{q^i} + \alpha_2 \beta_2^{q^i} = (\alpha_1 \beta_1 + \alpha_2 \beta_2)^{q^i}$ ($0 \leq i \leq m-1$). Hence, for $P \in \mathbb{F}_q^{n \times n}$ satisfying Eq. (4), we also have $\vec{b}^{q^i} P = [g_1^{q^i}, \dots, g_{d_R(V)}^{q^i}, 0, \dots, 0] \in \mathbb{F}_{q^m}^n$ for all $0 \leq i \leq m-1$. Thus, by

the elementary column operation on M over \mathbb{F}_q , represented by P , we get MP . By eliminating zero columns from MP , we obtain a matrix $M' = [f_{i,j}]_{i,j=1}^{m,d_R(V)}$, $f_{i,j} = g_j^{q^{i-1}}$, where $\text{rank } M' = \text{rank } M$. Let $M'_k \in \mathbb{F}_q^{k \times d_R(V)}$ ($1 \leq k \leq d_R(V)$) be the submatrix consisting of the first k rows of M' . Since $d_R(V) \leq \min\{m, n\}$ and $g_1, \dots, g_{d_R(V)}$ are linearly independent, M'_k is the generator matrix of $[d_R(V), k]$ Gabidulin code and $\text{rank } M'_k = k$ [8]. Thus, $M'_{d_R(V)}$ is nonsingular, and hence we have $\text{rank } M'_{d_R(V)} = \text{rank } M' = d_R(V)$. Therefore, $\dim V^* = \text{rank } M = \text{rank } M' = d_R(V)$. ■

Lemma 11. For a code $C_1 \subseteq \mathbb{F}_{q^m}^n$ and its subcode $C_2 \subsetneq C_1$, the first RGRW can be represented as $M_{R,1}(C_1, C_2) = \min\{d_R(\vec{x}, \vec{0}) : \vec{x} \in C_1 \setminus C_2\}$.

Proof: $M_{R,1}(C_1, C_2)$ can be represented as

$$\begin{aligned} & M_{R,1}(C_1, C_2) \\ &= \min\{\dim W : W \in \Gamma(\mathbb{F}_{q^m}^n), \dim(C_1 \cap W) - \dim(C_2 \cap W) \geq 1\} \\ &= \min\{\dim W : W \in \Gamma(\mathbb{F}_{q^m}^n), \\ & \exists V \subseteq W \text{ such that } V \subseteq (C_1 \cap W), V \not\subseteq (C_2 \cap W), \dim V \geq 1\}. \end{aligned} \quad (5)$$

For any subspace $V \subseteq \mathbb{F}_{q^m}^n$ with $\dim V \geq 1$, there always exists some $W \in \Gamma(\mathbb{F}_{q^m}^n)$ satisfying $W \supseteq V$, because we have $V^* \in \Gamma(\mathbb{F}_{q^m}^n)$ and $V^* \supseteq V$. Also, for subspaces W and $V \subseteq W$ with $\dim V \geq 1$, if W is the smallest space in $\Gamma(\mathbb{F}_{q^m}^n)$ including V , then $W = V^*$ [22]. Thus Eq. (5) can be rewritten as

$$\begin{aligned} & \min\{\dim W : V \subseteq \mathbb{F}_{q^m}^n, \dim V \geq 1 \\ & \exists W \supseteq V, W \in \Gamma(\mathbb{F}_{q^m}^n), \text{ such that } V \subseteq (C_1 \cap W), V \not\subseteq (C_2 \cap W)\} \\ &= \min\{\dim V^* : V \subseteq \mathbb{F}_{q^m}^n, V \subseteq (C_1 \cap V^*), V \not\subseteq (C_2 \cap V^*), \dim V \geq 1\} \\ &= \min\{\dim V^* : V \subseteq C_1, V \not\subseteq C_2, \dim V \geq 1\}, \end{aligned} \quad (6)$$

where the last equality of Eq. (6) is obtained by $V \subseteq (C_1 \cap V^*) \Leftrightarrow V \subseteq C_1$, and $V \not\subseteq (C_2 \cap V^*) \Leftrightarrow V \not\subseteq C_2$ from $V^* \supseteq V$. For subspaces V and $V' \supseteq V$, we have $\dim V^* \leq \dim V'^*$. Therefore, Eq. (6) can be rewritten as follows.

$$\begin{aligned} & \min\{\dim V^* : V \subseteq C_1, V \not\subseteq C_2, \dim V \geq 1\} \\ &= \min\{\dim V^* : V \subseteq C_1, V \not\subseteq C_2, \dim V = 1\} \\ &= \min\{d_R(V) : V \subseteq C_1, V \not\subseteq C_2, \dim V = 1\} \text{ (by Lemma 10)} \\ &= \min\{d_R(\vec{x}, \vec{0}) : \vec{x} \in C_1 \setminus C_2\}. \end{aligned} \quad \blacksquare$$

Lemma 11 immediately yields the following corollary.

Corollary 12. For a linear code C , $d_R(C) = M_{R,1}(C, \{\vec{0}\})$ holds.

This shows that $M_{R,1}(\cdot, \{\vec{0}\})$ is a generalization of $d_R(\cdot)$. Now we present the following proposition that generalizes the Singleton-type bound of the rank distance [8].

Proposition 13 (Generalization of Singleton-Type Bound). Let $C_1 \subseteq \mathbb{F}_{q^m}^n$ be a linear code and $C_2 \subsetneq C_1$ be its subcode. Then, the RGRW of C_1 and C_2 is upper bounded by

$$M_{R,i}(C_1, C_2) \leq \min\left\{1, \frac{m}{n - \dim C_2}\right\} (n - \dim C_1) + i, \quad (7)$$

for $1 \leq i \leq \dim(C_1/C_2)$.

Proof: We can consider that C_2 is a systematic code without loss of generality. That is, the first $\dim C_2$ coordinates of each basis of C_2 is one of canonical bases of $\mathbb{F}_{q^m}^{\dim C_2}$. Let $S \subsetneq \mathbb{F}_{q^m}^n$ be a linear code such that C_1 is a direct sum of C_2 and S . Then, after suitable permutation of coordinates, a basis of S can be chosen such that its first $\dim C_2$ coordinates are zero. Then, the effective length [7] of a code S is less than or equal to $n - \dim C_2$. Hence we have

$$\begin{aligned} d_R(S) &\leq \min\left\{1, \frac{m}{n - \dim C_2}\right\} (n - \dim C_2 - \dim S) + 1, \\ &= \min\left\{1, \frac{m}{n - \dim C_2}\right\} (n - \dim C_1) + 1, \end{aligned} \quad (8)$$

from the Singleton-type bound for rank metric [8].

Here we write $\kappa = \min\{1, m/(n - \dim C_2)\}$ for the sake of simplicity. Recall that $d_R(S) = M_{R,1}(S, \{\vec{0}\})$ from Corol. 12, and $M_{R,1}(S, \{\vec{0}\}) \leq \kappa(n - \dim C_1) + 1$ holds from Eq. (8).

We shall use the mathematical induction on t . We see that Eq. (9) is true for $t = 1$. Assume that for some $t \geq 1$,

$$M_{R,t}(S, \{\vec{0}\}) \leq \kappa(n - \dim C_1) + t, \quad (9)$$

is true. Then, by the monotonicity shown in Prop. 9,

$$M_{R,t+1}(S, \{\vec{0}\}) \leq M_{R,t}(S, \{\vec{0}\}) + 1 \leq \kappa(n - \dim C_1) + t + 1,$$

holds. Thus, it is proved by mathematical induction that Eq. (9) holds for $1 \leq t \leq \dim(C_1/C_2)$.

Lastly, we prove Eq. (7) by the above discussion about the RGRW of S and $\{\vec{0}\}$. For an arbitrary fixed subspace $V \subseteq \mathbb{F}_{q^m}^n$, we have $\dim(C_1 \cap V) \geq \dim(S \cap V) + \dim(C_2 \cap V)$, because C_1 is a direct sum of S and C_2 . Hence, $\dim(C_1 \cap V) - \dim(C_2 \cap V) \geq \dim(S \cap V)$ holds, and we have $M_{R,i}(C_1, C_2) \leq M_{R,i}(S, \{\vec{0}\})$ for $1 \leq i \leq \dim(C_1/C_2)$ from Def. 7. Therefore, from the foregoing proof, we have

$$M_{R,i}(C_1, C_2) \leq M_{R,i}(S, \{\vec{0}\}) \leq \kappa(n - \dim C_1) + i,$$

for $1 \leq i \leq \dim(C_1/C_2)$, and the proposition is proved. ■

Prop. 13 immediately yields the following corollary.

Corollary 14. For a linear code $C \subseteq \mathbb{F}_{q^m}^n$, $M_{R,i}(C, \{\vec{0}\}) \leq \min\{1, m/n\}(n - \dim C) + i$ for $1 \leq i \leq \dim C$. The equality holds for all i if and only if C is an MRD code.

V. UNIVERSAL SECURITY PERFORMANCE ON WIRETAP NETWORKS

In this section, we express Θ_μ and Ω given in Sect. III-B in terms of the RDIP and RGRW. From now on, we use the following definition.

Definition 15. For $B \in \mathbb{F}_q^{\mu \times n}$, we define $V_B \triangleq \{\vec{u}B : \vec{u} \in \mathbb{F}_{q^m}^\mu\} \subseteq \mathbb{F}_{q^m}^n$.

Recall that if an \mathbb{F}_{q^m} -linear space $V \subseteq \mathbb{F}_{q^m}^n$ admits a basis in \mathbb{F}_q^n then $V \in \Gamma(\mathbb{F}_{q^m}^n)$ [22], which implies

$$V_B \in \Gamma(\mathbb{F}_{q^m}^n). \quad (10)$$

First, we give the following theorem for the universal equivocation Θ_μ given in Def. 3

Theorem 16. Consider the nested coset coding in Def. 1. Then, the universal equivocation Θ_μ of C_1, C_2 is given by

$$\Theta_\mu = l - K_{R,\mu}(C_2^\perp, C_1^\perp).$$

Proof: Let $B \in \mathbb{F}_q^{\mu \times n}$ be an arbitrary matrix. By the chain rule [4], we have the following equation for the conditional entropy of S given BX^T :

$$\begin{aligned} H(S|BX^T) &= H(S, X|BX^T) - H(X|S, BX^T) \\ &= H(X|BX^T) + H(S|X, BX^T) - H(X|S, BX^T) \\ &= H(X|BX^T) - H(X|S, BX^T). \end{aligned} \quad (11)$$

Then, from [25, Proof of Lemma 4.2], we have

$$\begin{aligned} H(X|BX^T) &= n - \dim C_1^\perp - \dim V_B + \dim(C_1^\perp \cap V_B), \\ H(X|S, BX^T) &= n - \dim C_2^\perp - \dim V_B + \dim(C_2^\perp \cap V_B). \end{aligned}$$

By substituting these equations into Eq. (11), we have

$$\begin{aligned} H(S|BX^T) &= \dim C_2^\perp - \dim C_1^\perp - \dim(C_2^\perp \cap V_B) + \dim(C_1^\perp \cap V_B) \\ &= l - \dim(C_2^\perp \cap V_B) + \dim(C_1^\perp \cap V_B). \end{aligned} \quad (12)$$

By Eq. (10) we have

$$\left\{ V_B : B \in \mathbb{F}_q^{\mu \times n} \right\} = \bigcup_{i \leq \mu} \Gamma_i(\mathbb{F}_{q^m}^n). \quad (13)$$

Thus, by Eq. (12) and Def. 6, the universal equivocation Θ_μ is given as follows.

$$\begin{aligned} \Theta_\mu &= \min_{B \in \mathbb{F}_q^{\mu \times n}} H(S|BX^T) \\ &= l - \max_{B \in \mathbb{F}_q^{\mu \times n}} \left\{ \dim(C_2^\perp \cap V_B) - \dim(C_1^\perp \cap V_B) \right\} \\ &= l - \max_{V \in \bigcup_{i \leq \mu} \Gamma_i(\mathbb{F}_{q^m}^n)} \left\{ \dim(C_2^\perp \cap V) - \dim(C_1^\perp \cap V) \right\} \text{ (by Eq. (13))} \\ &= l - \max_{V \in \Gamma_\mu(\mathbb{F}_{q^m}^n)} \left\{ \dim(C_2^\perp \cap V) - \dim(C_1^\perp \cap V) \right\} \text{ (by Thm. 8)} \\ &= l - K_{R,\mu}(C_2^\perp, C_1^\perp). \quad \blacksquare \end{aligned}$$

Example 17. The existing schemes [12,20,21] used MRD codes as C_1^\perp and C_2^\perp , where $m \geq n$. By Corol. 12, we have $\dim(V \cap C_2^\perp) = 0$ for any $V \in \Gamma_{\dim C_2}(\mathbb{F}_{q^m}^n)$. This implies $K_{R,\mu}(C_2^\perp, C_1^\perp) = K_{R,\mu}(C_2^\perp, \{\vec{0}\}) = 0$ for $0 \leq \mu \leq \dim C_2$.

On the other hand, $K_{R,\dim C_1}(C_2^\perp, \{\vec{0}\}) = \dim C_1 - \dim C_2$ by Corol. 14. Since $\dim(V \cap C_1^\perp) = 0$ for any $V \in \Gamma_{\dim C_1}(\mathbb{F}_{q^m}^n)$ by Corol. 12, we have $K_{R,\dim C_1}(C_2^\perp, C_1^\perp) = \dim C_1 - \dim C_2$. By Theorem 8, $K_{R,\mu}(C_2^\perp, C_1^\perp) = \mu - \dim C_2$ for $\dim C_2 \leq \mu \leq \dim C_1$.

By Theorem 16, we see that $\Theta_\mu = l - \max\{0, \mu - \dim C_2\}$ for $0 \leq \mu \leq \dim C_1 (= l + \dim C_2)$ in the schemes [12,20,21].

We then have the following corollary by the RGRW. Corol. 18 shows that the wiretapper obtain no information of S from any $M_{R,1}(C_2^\perp, C_1^\perp) - 1$ links.

Corollary 18. Consider the nested coset coding in Def. 1. Then, the wiretapper must observe at least $M_{R,j}(C_2^\perp, C_1^\perp)$ links to obtain the mutual information j ($1 \leq j \leq l$) between S and observed packets.

Proof: From Eq. (12), the smallest number μ of tapped links satisfying $I(S; BX^T) = j$ ($1 \leq j \leq l$) is

$$\begin{aligned} \min \left\{ \mu : \exists B \in \mathbb{F}_q^{\mu \times n}, I(S; BX^T) = j \right\} \\ &= \min \left\{ \mu : \exists B \in \mathbb{F}_q^{\mu \times n}, l - H(S|BX^T) = j \right\} \\ &= \min \left\{ \mu : \exists B \in \mathbb{F}_q^{\mu \times n}, \dim(C_2^\perp \cap V_B) - \dim(C_1^\perp \cap V_B) = j \right\}. \end{aligned}$$

From [22, Lemma 1] and Lemma 9, this equation can be rewritten as follows.

$$\begin{aligned} \min \left\{ \mu : \exists B \in \mathbb{F}_q^{\mu \times n}, \dim(C_2^\perp \cap V_B) - \dim(C_1^\perp \cap V_B) = j \right\} \\ &= \min \left\{ \dim V : V \in \Gamma(\mathbb{F}_{q^m}^n), \dim(C_2^\perp \cap V) - \dim(C_1^\perp \cap V) = j \right\} \\ &= M_{R,j}(C_2^\perp, C_1^\perp). \quad \blacksquare \end{aligned}$$

Although the message S has been assumed to be uniformly distributed over $\mathbb{F}_{q^m}^l$ in Sect. III-A, the following proposition reveals that the wiretapper still obtain no information of S from any $M_{R,1}(C_2^\perp, C_1^\perp) - 1$ links even if S is arbitrarily distributed.

Proposition 19. Fix the transfer matrix B to the wiretapper. Suppose that the wiretapper obtain no information of S from BX^T when S is uniformly distributed over $\mathbb{F}_{q^m}^l$ as described in Sect. III-A. Then, even if S is chosen according to an arbitrary distribution over $\mathbb{F}_{q^m}^l$, the wiretapper still obtain no information of S from BX^T , that is, $I(S; BX^T) = 0$.

Proof: When we assume that S is arbitrarily distributed over $\mathbb{F}_{q^m}^l$, $H(X|S, BX^T)$ is upper bounded as follows from [21, Proof of Lemma 6] and [25, Proof of Lemma 4.2].

$$H(X|S, BX^T) \leq n - \dim C_2^\perp - \dim V_B + \dim(C_2^\perp \cap V_B).$$

Also, since X is uniformly distributed over a coset $\psi(S) \in C_1/C_2$ for fixed S , we have $H(X|S) = \dim C_2 = n - \dim C_2^\perp$. For the dimension of a subspace $\{BX^T : X \in C_1\}$, we have

$$\begin{aligned} \dim \{BX^T : X \in C_1\} &= \text{rank } BG^T = \text{rank } GB^T \\ &= \dim \{G\vec{v}^T : \vec{v} \in V_B\} = \dim V_B - \dim(C_1^\perp \cap V_B), \end{aligned}$$

where $G \in \mathbb{F}_{q^m}^{\dim C_1 \times n}$ is a generator matrix of C_1 . Hence we have $H(BX^T) \leq \dim V_B - \dim(C_1^\perp \cap V_B)$. We thus have

$$\begin{aligned} I(S; BX^T) &= I(S, X; BX^T) - I(X; BX^T|S) \\ &= H(BX^T) - H(X|S) + H(X|S, BX^T) \\ &\leq \dim(C_2^\perp \cap V_B) - \dim(C_1^\perp \cap V_B) \end{aligned} \quad (14)$$

for any distribution of S . By $I(S; BX^T) = H(S) - H(S|BX^T)$ and Eq. (12) we can see that the equality holds if S is uniformly distributed. Therefore, for fixed B , if $I(S; BX^T) = 0$ holds for uniformly distributed S , then the right hand side of Eq. (14) is zero, which implies that $I(S; BX^T) = 0$ also holds for arbitrarily distributed S from the nonnegativity of mutual information [4]. \blacksquare

Lastly, we express Ω in Def. 4 in terms of the RGRW. For a subset $\mathcal{J} \subseteq \{1, \dots, N\}$ and a vector $\vec{c} = [c_1, \dots, c_N] \in \mathbb{F}_{q^m}^N$, let $P_{\mathcal{J}}(\vec{c})$ be a vector of length $|\mathcal{J}|$ over \mathbb{F}_{q^m} , obtained by removing the t -th components c_t for $t \notin \mathcal{J}$. For example for $\mathcal{J} = \{1, 3\}$ and $\vec{c} = [1, 1, 0, 1]$ ($N = 4$),

we have $P_{\mathcal{J}}(\vec{c}) = [1, 0]$. The *punctured code* $P_{\mathcal{J}}(C)$ of a code $C \in \mathbb{F}_{q^m}^N$ is given by $P_{\mathcal{J}}(C) \triangleq \{P_{\mathcal{J}}(\vec{c}) : \vec{c} \in C\}$. The *shortened code* $C_{\mathcal{J}}$ of a code $C \subseteq \mathbb{F}_{q^m}^N$ is defined by $C_{\mathcal{J}} \triangleq \{P_{\mathcal{J}}(\vec{c}) : \vec{c} = [c_1, \dots, c_N] \in C, c_i = 0 \text{ for } i \notin \mathcal{J}\}$. For example for $C = \{[0, 0, 0], [1, 1, 0], [1, 0, 1], [0, 1, 1]\}$ ($N = 3$) and $\mathcal{J} = \{2, 3\}$, we have $C_{\mathcal{J}} = \{[0, 0], [1, 1]\}$. We then have the following theorem for the universal Ω -strong security defined in Def. 4.

Theorem 20. Let $\bar{\{i\}} \triangleq \{1, \dots, l+n\} \setminus \{i\}$. Fix C_1, C_2 and ψ in Def. 1 and consider the corresponding nested coset coding scheme in Def. 1. By using C_1, C_2 and ψ , define

$$C'_1 \triangleq \{[S, X] : S \in \mathbb{F}_{q^m}^l \text{ and } X \in \psi(S)\} \subseteq \mathbb{F}_{q^m}^{l+n}.$$

For each index $1 \leq i \leq l$, we define a punctured code $\mathcal{D}_{1,i}$ of C'_1 as $\mathcal{D}_{1,i} \triangleq P_{\bar{\{i\}}}(C'_1) \subseteq \mathbb{F}_{q^m}^{l+n-1}$, and a shortened code $\mathcal{D}_{2,i}$ of C'_1 as $\mathcal{D}_{2,i} \triangleq (C'_1)_{\bar{\{i\}}} \subseteq \mathbb{F}_{q^m}^{l+n-1}$. Then, the value Ω in Def. 4 is given by

$$\Omega = \min \left\{ M_{R,1}(\mathcal{D}_{2,i}^\perp, \mathcal{D}_{1,i}^\perp) : 1 \leq i \leq l \right\} - 1. \quad (15)$$

Proof: Define $C'_2 \triangleq \{[\vec{0}, \vec{c}_2] : \vec{c}_2 \in C_2\} \subseteq \mathbb{F}_{q^m}^{l+n}$. Since $C_2 \not\subseteq C_1$, C'_2 is also a subcode of C'_1 . Thus, in terms of C'_1 and C'_2 , we can see that the vector $[S, X] \in \mathbb{F}_{q^m}^{l+n}$ is generated by a nested coset coding scheme of C'_1 and C'_2 from S . Then, from the definition of C'_1 and C'_2 , we can see that $\mathcal{D}_{2,i}$ is a subcode of $\mathcal{D}_{1,i}$ with dimension $\dim \mathcal{D}_{2,i} = \dim \mathcal{D}_{1,i} - 1 = \dim C_1 - 1$ over \mathbb{F}_{q^m} for each $i \in \{1, \dots, l\}$.

Let $\mathcal{L} \triangleq \{1, \dots, l\}$ and $S_{\mathcal{L} \setminus \{i\}} \triangleq [S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_l]$ for each $1 \leq i \leq l$. For $S_i \in \mathbb{F}_{q^m}$ define a coset

$$\phi(S_i) \triangleq \{[S_{\mathcal{L} \setminus \{i\}}, X] : S_{\mathcal{L} \setminus \{i\}} \in \mathbb{F}_{q^m}^{l-1} \text{ and } X \in \psi(S)\} \in \mathcal{D}_{1,i} / \mathcal{D}_{2,i}.$$

Here we define $Z_{\bar{\{i\}}} \triangleq P_{\bar{\{i\}}}([S, X]) = [S_{\mathcal{L} \setminus \{i\}}, X] \in \mathcal{D}_{1,i}$. Recall that S_1, \dots, S_l are mutually independent and uniformly distributed over \mathbb{F}_{q^m} . Thus, considering a nested coset coding scheme that generates $Z_{\bar{\{i\}}}$ from a secret message $S_i \in \mathbb{F}_{q^m}$ with $\mathcal{D}_1, \mathcal{D}_2$, we can see that $Z_{\bar{\{i\}}} \in \phi(S_i) \in \mathcal{D}_{1,i} / \mathcal{D}_{2,i}$ is chosen uniformly at random from $\phi(S_i)$. Therefore, we have $I(S_i; DZ_{\bar{\{i\}}}^\top) = 0$ for any $D \in \mathbb{F}_q^{\mu \times (n+l-1)}$ whenever $\mu < M_{R,1}(\mathcal{D}_{2,i}^\perp, \mathcal{D}_{1,i}^\perp)$ from Corol. 18.

For an arbitrary subset $\mathcal{R} \subseteq \mathcal{L} \setminus \{i\}$, define a matrix $F_{\mathcal{R}}$ that consists of $|\mathcal{R}|$ rows of an $(l-1) \times (l-1)$ identity matrix, satisfying $[S_j : j \in \mathcal{R}]^\top = F_{\mathcal{R}} S_{\mathcal{L} \setminus \{i\}}^\top$. For an arbitrary matrix $B \in \mathbb{F}_q^{k \times n}$ ($0 \leq k \leq n$), set $D = \begin{bmatrix} F_{\mathcal{R}} & 0 \\ 0 & B \end{bmatrix}$. Then, from the foregoing proof, we have

$$\begin{aligned} 0 &= I(S_i; DZ_{\bar{\{i\}}}^\top) = I(S_i; S_{\mathcal{R}}, BX^\top) = H(S_i | S_{\mathcal{R}}) - H(S_i | BX^\top, S_{\mathcal{R}}) \\ &= H(S_i) - H(S_i | BX^\top, S_{\mathcal{R}}) = I(S_i; BX^\top | S_{\mathcal{R}}), \end{aligned}$$

whenever $|\mathcal{R}| + k < M_1(\mathcal{D}_{2,i}^\perp, \mathcal{D}_{1,i}^\perp)$. Since $I(S_i; BX^\top | S_{\mathcal{R}}) = 0$ is equivalent to Eq. (1) from [20, Prop. 5], we have Eq. (15) by selecting the minimum value of $M_{R,1}(\mathcal{D}_{2,i}^\perp, \mathcal{D}_{1,i}^\perp) - 1$ for $1 \leq i \leq l$. ■

Example 21. The scheme proposed in [12] used a systematic MRD code as C'_1 (not C_1), where $m \geq l+n$. We proved $\min \{M_{R,1}(\mathcal{D}_{2,i}^\perp, \mathcal{D}_{1,i}^\perp) : 1 \leq i \leq l\} = n$ in [12, Proof of Theorem

4]. By Theorem 20, we see that the scheme [12] attains the universal $(n-1)$ -strong security in the sense of Def. 4, while [12] proved it by adapting the proof argument in [20].

As shown in Prop. 19, no information of S is leaked from less than $M_{R,1}(C_2^\perp, C_1^\perp)$ tapped links even if S is arbitrarily distributed. In contrast, S must be uniformly distributed over $\mathbb{F}_{q^m}^l$ to establish Theorem 20. This is because elements of S need to be treated as extra random packets, as in strongly secure network coding schemes [9, 16, 20].

VI. UNIVERSAL ERROR CORRECTION CAPABILITY OF SECURE NETWORK CODING

This section derives the universal error correction capability by the approach of [19, Section III]. Recall that the received packets Y is given by $Y^\top = AX^\top + DZ^\top$ in the setup of Sect. III-A, and that X is chosen from the coset $\psi(S) \in C_1/C_2$ corresponding to S by the nested coset coding in Def. 1. From now on, we write $\mathcal{X} \triangleq \psi(S)$ for the sake of simplicity.

First, we define the *discrepancy* [19] between \mathcal{X} and Y by $\Delta_A(\mathcal{X}, Y) \triangleq \min \{r \in \mathbb{N} : D \in \mathbb{F}_q^{N \times r}, Z \in \mathbb{F}_{q^m}^r, X \in \mathcal{X}, Y^\top = AX^\top + DZ^\top\}$

$$= \min \{d_R(XA^\top, Y) : X \in \mathcal{X}\}, \quad (16)$$

where the second equality is derived from [19, Lemma 4]. This definition of $\Delta_A(\mathcal{X}, Y)$ represents the minimum number r of error packets Z required to be injected in order to transform at least one element of \mathcal{X} into Y , as [20, Eq. (9)].

Next, we define the Δ -distance [19] between \mathcal{X} and \mathcal{X}' , induced by $\Delta_A(\mathcal{X}, Y)$, as

$$\delta_A(\mathcal{X}, \mathcal{X}') \triangleq \min \{\Delta_A(\mathcal{X}, Y) + \Delta_A(\mathcal{X}', Y) : Y \in \mathbb{F}_{q^m}^N\}, \quad (17)$$

for $\mathcal{X}, \mathcal{X}' \in C_1/C_2$.

Lemma 22. For $\mathcal{X}, \mathcal{X}' \in C_1/C_2$, we have

$$\delta_A(\mathcal{X}, \mathcal{X}') = \min \{d_R(XA^\top, X'A^\top) : X \in \mathcal{X}, X' \in \mathcal{X}'\}. \quad (18)$$

Proof: First we have

$$\begin{aligned} \delta_A(\mathcal{X}, \mathcal{X}') &= \min \{\Delta_A(\mathcal{X}, Y) + \Delta_A(\mathcal{X}', Y) : Y \in \mathbb{F}_{q^m}^N\} \\ &= \min \left\{ \min \{d_R(XA^\top, Y) : X \in \mathcal{X}\} \right. \\ &\quad \left. + \min \{d_R(X'A^\top, Y) : X' \in \mathcal{X}'\} : Y \in \mathbb{F}_{q^m}^N \right\} \\ &= \min \{d_R(XA^\top, Y) + d_R(X'A^\top, Y) : X \in \mathcal{X}, X' \in \mathcal{X}', Y \in \mathbb{F}_{q^m}^N\}. \quad (19) \end{aligned}$$

The rank distance satisfies the triangle inequality $d_R(XA^\top, X'A^\top) \leq d_R(XA^\top, Y) + d_R(X'A^\top, Y)$ for $\forall Y \in \mathbb{F}_{q^m}^N$ [8]. This lower bound can be achieved by choosing, e.g., $Y = XA^\top$. Therefore, from Eq. (19), we have Eq. (18). ■

The next lemma shows that $\Delta_A(\mathcal{X}, Y)$ is *normal* [19, Definition 1].

Lemma 23. For all $\mathcal{X}, \mathcal{X}' \in C_1/C_2$ and all $0 \leq i \leq \delta_A(\mathcal{X}, \mathcal{X}')$, there exists some $Y \in \mathbb{F}_{q^m}^N$ such that $\Delta_A(\mathcal{X}, Y) = i$ and $\Delta_A(\mathcal{X}', Y) = \delta_A(\mathcal{X}, \mathcal{X}') - i$.

Proof: Let $\mathcal{X}, \mathcal{X}' \in C_1/C_2$ and let $0 \leq i \leq d = \delta_A(\mathcal{X}, \mathcal{X}')$. Then, $d = \min \{d_R(XA^\top, X'A^\top) : X \in \mathcal{X}, X' \in \mathcal{X}'\}$ from Lemma 22. Let $\bar{X} \in \mathcal{X}$ and $\bar{X}' \in \mathcal{X}'$ be vectors

satisfying $d = d_R(\bar{X}A^T, \bar{X}'A^T)$. From the proof of [19, Theorem 6], we can always find two vectors $W, W' \in \mathbb{F}_q^m$ such that $W + W' = (\bar{X}' - \bar{X})A^T$, $\text{rank}_{\mathbb{F}_q}(W) = i$ and $\text{rank}_{\mathbb{F}_q}(W') = d - i$. Taking $\bar{Y} = \bar{X}A^T + W = \bar{X}'A^T - W'$, we have $d_R(\bar{X}A^T, \bar{Y}) = i$ and $d_R(\bar{X}'A^T, \bar{Y}) = d - i$. We thus obtain $\Delta_A(\mathcal{X}, \bar{Y}) \leq i$ and $\Delta_A(\mathcal{X}', \bar{Y}) \leq d - i$ from Eq.(16). On the other hand, since $\delta_A(\mathcal{X}, \mathcal{X}') = d$, we have $\Delta_A(\mathcal{X}, Y) + \Delta_A(\mathcal{X}', Y) \geq d$ for any $Y \in \mathbb{F}_q^m$ from Eq.(17). Therefore, $\Delta_A(\mathcal{X}, \bar{Y}) = i$ and $\Delta_A(\mathcal{X}', \bar{Y}) = d - i$ hold. ■

Let $\delta_A(C_1/C_2)$ be the minimum Δ -distance given by

$$\delta_A(C_1/C_2) \triangleq \min\{\delta_A(\mathcal{X}, \mathcal{X}') : \mathcal{X}, \mathcal{X}' \in C_1/C_2, \mathcal{X} \neq \mathcal{X}'\}.$$

As [19, Theorem 7], from Lemma 23 and [19, Theorem 3], we have the following proposition.

Proposition 24. A nested coset coding scheme with C_1, C_2 is guaranteed to determine the unique coset \mathcal{X} against any t packet errors for any fixed A if and only if $\delta_A(C_1/C_2) > 2t$. ■

Here we note that if \mathcal{X} is uniquely determined, S is also uniquely determined from Def. 1.

Lemma 25. $\delta_A(C_1/C_2) = \min\{d_R(XA^T, X'A^T) : X, X' \in C_1, X' - X \notin C_2\}$.

Proof:

$$\begin{aligned} \delta_A(C_1/C_2) &= \min\{\delta_A(\mathcal{X}, \mathcal{X}') : \mathcal{X}, \mathcal{X}' \in C_1/C_2, \mathcal{X} \neq \mathcal{X}'\} \\ &= \min\{\min\{d_R(XA^T, X'A^T) : X \in \mathcal{X}, X' \in \mathcal{X}'\} : \mathcal{X}, \mathcal{X}' \in C_1/C_2, \mathcal{X} \neq \mathcal{X}'\} \\ &= \min\{d_R(XA^T, X'A^T) : X \in \mathcal{X} \in C_1/C_2, X' \in \mathcal{X}' \in C_1/C_2, \mathcal{X} \neq \mathcal{X}'\} \\ &= \min\{d_R(XA^T, X'A^T) : X, X' \in C_1, X' - X \notin C_2\}. \quad \blacksquare \end{aligned}$$

Theorem 26. Consider the nested coset coding in Def. 1. Then, the scheme is a universally (i.e., simultaneously for all $A \in \mathbb{F}_q^{N \times n}$ with rank deficiency at most ρ) t -error- ρ -erasure-correcting secure network coding if and only if $M_{R,1}(C_1, C_2) > 2t + \rho$.

Proof: For the rank deficiency $\rho = n - \text{rank} A$, we have $d_R(X, X') - \rho \leq d_R(XA^T, X'A^T)$, and there always exists $A \in \mathbb{F}_q^{N \times n}$ depending on (X, X') such that the equality holds. Thus, from Lemma 25, we have

$$\begin{aligned} \min_{\substack{A \in \mathbb{F}_q^{N \times n} \\ \text{rank} A = n - \rho}} \delta_A(C_1/C_2) &= \min\{d_R(X, X') : X, X' \in C_1, X' - X \notin C_2\} - \rho \\ &= \min\{d_R(X, \vec{0}) : X \in C_1, X \notin C_2\} - \rho \\ &= M_{R,1}(C_1, C_2) - \rho. \quad (\text{by Lemma 11}) \end{aligned}$$

Therefore, we have $\min_{A: \text{rank} A = n - \rho} \delta_A(C_1/C_2) < \min_{A: \text{rank} A = n - \rho'} \delta_A(C_1/C_2)$ for $\rho > \rho'$, and hence we obtain $\min_{A: \text{rank} A \geq n - \rho} \delta_A(C_1/C_2) = \min_{A: \text{rank} A = n - \rho} \delta_A(C_1/C_2) = M_{R,1}(C_1, C_2) - \rho$. ■

Example 27. The existing scheme [21] used MRD codes as C_1, C_2 , where $m \geq n$. Then, by Corol.14, we have $M_{R,1}(C_1, \{\vec{0}\}) = n - \dim C_1 + 1$. Since $\dim(V \cap C_2) = 0$ for any $V \in \Gamma_{\dim C_2}^{\perp}(\mathbb{F}_q^m)$ by Corol.12 and $\dim C_2^{\perp} > n - \dim C_1$, we have $M_{R,1}(C_1, C_2) = M_{R,1}(C_1, \{\vec{0}\})$. Thus, by Theorem 26

and Corol.12, the scheme is universally t -error- ρ -erasure-correcting when $M_{R,1}(C_1, \{\vec{0}\}) = d_R(C_1) > 2t + \rho$, as shown in [21, Theorem 11].

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