# Adaptive and Dynamic Multi-Resolution Hashing for Pairwise Summations

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Abstract—In this paper, we propose Adam-Hash: an adaptive and dynamic multi-resolution hashing data-structure for fast pairwise summation estimation. Given a data-set  $X \subset \mathbb{R}^d$ , a binary function  $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ , and a point  $y \in \mathbb{R}^d$ , the Pairwise Summation Estimate  $PSE_X(y) := \frac{1}{|X|} \sum_{x \in X} f(x, y)$ . For any given data-set X, we need to design a data-structure such that given any query point  $y \in \mathbb{R}^d$ , the data-structure approximately estimates  $PSE_X(y)$  in time that is sub-linear in |X|. Prior works on this problem have focused exclusively on the case where the data-set is static, and the queries are independent. In this paper, we design a hashing-based PSE data-structure which works for the more practical *dynamic* setting in which insertions, deletions, and replacements of points are allowed. Moreover, our proposed Adam-Hash is also robust to adaptive PSE queries, where an adversary can choose query  $q_j \in \mathbb{R}^d$ depending on the output from previous queries  $q_1, q_2, \ldots, q_{j-1}$ .

#### I. INTRODUCTION

Pairwise Summation Estimation (PSE) is one of the most important problems in machine learning [1], [2], [3], [4], [5], [6], [7], [8], [9], statistics [10], [11], [12], [13], [14], [15], and scientific computing [16], [17], [18], [19], [20], [21], [22]. Given a data-set  $X \subset \mathbb{R}^d$ , a binary function  $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ , and a point  $y \in \mathbb{R}^d,$  we need the pairwise summation estimate of f(x, y) for  $x \in X$  i.e.  $PSE_X(y) = \frac{1}{|X|} \sum_{x \in X} f(x, y)$ . PSE arises naturally in ML applications: (1) Efficient training and inference of neural network: In computer vision and natural language processing, the Softmax layer with n neurons is defined as  $\frac{\exp(w_i,x)}{\sum_{j=1}^{n} \langle w_j,x \rangle}$ , where  $w_i$  is the parameter for the *i*'th neuron and *x* is the input hidden vector. A novel line of research [4], [8], [9] applies PSE to estimate  $\sum_{j=1}^{n} \langle w_j, x \rangle$ with running time sublinear in n. As a result, a sparse training and inference scheme of Softmax layer can be achieved for acceleration. (2) Fast kernel density estimation: Given a binary kernel function, we would like to estimate the density of a dataset on a query for efficient outlier detection [23], [2], [24], classification [25], [7] and clustering [26].

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#### A. The Need for Adaptive and Dynamic PSE

Recently, there has also been a lot of recent interest in developing PSE for deep learning that are robust to *adaptive* queries [27], [28], [29], [9], [30], [31], [32], [33], in which an adversary can choose a query  $q_j \in \mathbb{R}^d$  that depends on the output of our data structure to past queries  $q_1, q_2, \ldots, q_{j-1}$ . This is a natural setting in training neural networks. For instance, the input hidden vector of Softmax layer serves as a query for PSE. If we perform adversarial attacks [34] in each step, the PSE-accelerated training and inference might lead to failure in generalization. This brings new challenges for PSE because an adversary can always pick the hardest query point, and we would like to have an accuracy guarantee to hold even under this adversarial setting.

Moreover, current PSE applications in ML depend on parameters that are often changing in time and not known apriori. So it is necessary that we develop data structures which can support *dynamic* updates [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50]. One typical setting to consider is an iterative process where one data point changes for each iteration. For instance, the weights of Transformer models [9] changes significantly in the first couple of iterations and then changes smoothly in later iterations. As a result, a successful PSE algorithm for deep models should be robust to incremental updates.

#### B. Our Proposal: Adam-Hash

In this paper, we propose Adam-Hash: an adaptive and dynamic multi-resolution hashing for fast pairwise summation estimation. Formally, we start with defining the problem as follows:

**Definition I.1** (Adaptive and dynamic hashing based estimator). Given an approximation parameter  $\epsilon \in (0, 1)$  and a threshold  $\tau \in (0, 1)$ , for every convex function w, there exists a data structure which supports the operations as followed:

- INITIALIZE(). Initialize the data structure.
- QUERY(x). Given an input x, QUERY outputs an estimate  $\hat{Z}$  which approximates  $\mu = w(x)$ . even in a sequence of adaptively chosen queries
- UPDATE(i, z). Replace  $x_i$  by z into the data structure.

**Technical contributions.** Adam-Hash contain a dynamic version of Hashing-Based-Estimators (HBE) and multiresolution HBE. HBE hashes each data point into a set of hash buckets, and uses collision probability to estimate the pairwise summation function. For updating datasets, we only need to update the corresponding hash buckets. To enable our Adam-Hash to work for adversarial queries, we start from query with a single HBE in Adam-Hash of constant success probability, and boost constant success probability to high probability by obtaining the median of a set of HBE estimators. We then design a  $\epsilon_0$ -net to prove our Adam-Hash can answer a fixed set of on-net points with high probability. And finally, we generalize the results to all query points where  $||q||_2 \leq 1$  with the Lipschitz property of the target function.

*a) Roadmap.:* We introduce some related work to our paper in section II. We give a technique overview of our paper in section III. We present the dynamic version HBE in section V. Then we present the dynamic version of multi-resolution HBE in section VI. We further extend to the adaptive and dynamic version HBE in section VII. We conclude our contribution in section VIII.

#### II. RELATED WORK

*a) Pairwise summation estimation:* The pairwise summation estimation is a general formulation to a lot of machine machine problems. For instance, density estimation of kernel functions [51] is a standard PSE problem.

Recently, there is an growing direction in using hashing for PSE [52], [5], [6], [53], [15]. The general intuition is that the binary function is actually a similarity function between data point and query. As a result, we could have speedups in PSE with near neighbor search data structures.

b) Adaptive queries: In modern machine learning algorithms that involves data structures. The queries to these data structures are adaptive in two ways: (1) we have sequential queries that is non-i.i.d, such as weights each iteration of the model [29], (2) the potential threats posed by deploying algorithms in adversarial settings [54], [55], [56], [57], where an attacker could manipulate query based on the results of previous queries. Thus, current data structures should be robust in these settings so that they can be deployed in machine learning.

#### **III. TECHNIQUE OVERVIEW**

In this section, we present an overview of our techniques to realize Adam-Hash algorithm. Our introduction to Adam-Hash follows a divide-and-conquer style. We start with presenting a dynamic version of multi-resolution hashing. Next, we show how to make it robust to adaptive queries. Finally, we introduce the main algorithm in Algorithm 5. We also provide an overview of our theoretical analysis. Given an estimator Z of complexity C which is V-bounded where  $\mathbb{E}[Z] = \mu \in (0, 1]$  and threshold  $\tau \in (0, 1)$ , we first introduce the dynamic version of single-resolution HBE in Theorem V.1. The query algorithm interacts with the data structure by calling the hash function and sample a data point from the hash bucket the query falls into (Line 25 of Algorithm 1). After  $O(V((\mu)_{\tau} \log(1/\delta)))$  hash function calls, with probability at least  $1 - \delta/2$  we either have a  $(1 \pm \epsilon)$ -approximation result or QUERY outputs 0 indicating that  $\mu < \tau$ . When we want to update the dataset, we need to insert new entries or delete old entries from each hash table.

Then we present the dynamic version of multi-resolution HBE in Theorem VI.1. The estimator consists of *G* different hashing function families, and each hash function family has a weight computed from their collision probability functions (Line 24 of Algorithm 3). And the subquery estimation is more complex by combining the estimation from each hash function family at their corresponding weight (Line 10 of Algorithm 4). After interacting with the data structure for  $O(V((\mu)_{\tau} \log(1/\delta)))$  times, with high probability we either have a  $(1 \pm \epsilon)$ -approximation result or 0 is outputted indicating that  $\mu < \tau$ .

To make our data structure adaptive to adversarially chosen queries, we begin with a single HBE estimator to answer the query with  $(1\pm\epsilon)$ -approximation with a constant success probability 0.9. Then by chernoff bound we have that obtaining the median of a set of HBE estimators can boost constant success probability to high success probability. We design a  $\epsilon_0$ -net N containing  $|N| = (10/\epsilon_0)^d$  points and prove that our data structure can answer a fixed set of on-net points with high probability via union bound. Finally, we know that for each point  $q \notin N$ , there exists a  $p \in N$  such that  $||p - q||_2 \leq \epsilon_0$ . We can quantize the off-net query q to its nearest on-net query p and generalize the results to all query points where  $||q||_2 < 1$  with the k-Lipschitz property of the target function. To this end, we complete our proof for designing adaptive and dynamic data structures for multi-resolution hashing of pairwise summation estimates.

## IV. PRELIMINARY

a) Notation.: For a vector  $A \in \mathbb{R}^d$ , we define  $||A||_{\infty} = \max_{i \in [d]}(x_i)$ . We define  $||A||_2 = \sqrt{\sum_{i=1}^n x_i^2}$ . We use [n] to denote  $\{1, 2, \dots, n\}$ . For an event f(x), we define  $\mathbf{1}_{f(x)}$  such that  $\mathbf{1}_{f(x)} = 1$  if f(x) holds and  $\mathbf{1}_{f(x)} = 0$  otherwise. We use  $\Pr[\cdot]$  to denote the probability, and use  $\mathbb{E}[\cdot]$  to denote the expectation if it exists. We use  $a \in (1 \pm \epsilon) \cdot b$  to denote  $(1 - \epsilon) \cdot b \leq a \leq (1 + \epsilon) \cdot b$ .

We will make use of Hoeffding's Inequality:

**Theorem IV.1** (Hoeffding's Inequality [58]). Let  $X_1, \ldots, X_n$ be independent random variables such that  $X_i \in [a_i, b_i]$ almost surely for  $i \in [n]$  and let  $S = \sum_{i=1}^n X_i - \mathbb{E}[X_i]$ . Then, for every t > 0:

$$\Pr[S \ge t] \le \exp(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2})$$

#### V. DYNAMIC SINGLE-RESOLUTION HBE

In this section, we introduce a dynamic version of singleresolution HBE. We present our theorem for the dynamic version HBE in Theorem V.1.

**Theorem V.1** (Dynamic version of Theorem 4.2 in page 11 [52]). For a kernel w, given a V-bounded estimator Z of complexity C where  $\mathbb{E}[Z] = \mu \in (0,1]$  and parameters  $\epsilon \in (0,0.1), \tau \in (0,1), \delta \in (0,1)$ , there exists a data structure which supports the operations as followed:

- INITIALIZE $(w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+, V : \mathbb{R} \to \mathbb{R}_+, \{x_i\}_{i=1}^n \subset \mathbb{R}^d, \mathcal{H}, \epsilon \in (0, 0.1), \delta \in (0, 1), \tau \in (0, 1))$ . Given a set of data points  $\{x_i\}_{i=1}^n$ , a hashing scheme  $\mathcal{H}$ , the target function  $w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$  the relative variance function  $V : \mathbb{R} \to \mathbb{R}_+$ , accuracy parameter  $\epsilon \in (0, 0.1)$ , failure probability  $\delta \in (0, 1)$  and a threshold  $\tau \in (0, 1)$  as input, the INITIALIZE operation takes  $O(\epsilon^{-2}V(\tau)C\log(1/\delta) \cdot n)$  time.
- QUERY $(x \in \mathbb{R}^d, \alpha \in (0, 1], \tau \in (0, 1), \delta \in (0, 1))$ . Given a query point  $x \in \mathbb{R}^d$ , accuracy parameter  $\alpha \in (0, 1]$ , a threshold  $\tau \in (0, 1)$  and a failure probability  $\delta \in (0, 1)$ as input, the time complexity of QUERY operation is  $O(\epsilon^{-2}V(\tau)C\log(1/\delta))$  and the output of QUERY  $\hat{Z}$ satisfies:

$$\begin{cases} \Pr[|\widehat{Z} - \mu| \le \alpha \mu] \ge 1 - \delta, & \mu \ge \tau \\ \Pr[\widehat{Z} = 0] \ge 1 - \delta, & \mu < \tau \end{cases}$$

- INSERTX $(x \in \mathbb{R}^d)$ . Given a data point  $x \in \mathbb{R}^d$  as input, the INSERTX operation takes  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$ time to update the data structure.
- DELETEX $(x \in \mathbb{R}^d)$ . Given a data point  $x \in \mathbb{R}^d$  as input, the DELETEX operation takes  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$ time to update the data structure.

*Proof.* We can prove the theorem by combining the query correctness proof Lemma V.3 and running time lemmas including the INITIALIZE running time in Lemma V.4, the QUERY running time in Lemma V.6, the INSERTX running time in Lemma V.7 and DELETEX running time in Lemma V.8.

We remark that this statement provide a foundation of dynamic HBEs and will help the final presentation of Adam-Hash.

We present our dynamic single-resolution HBE estimator data structure in Algorithm 1 and Algorithm 2. During INI-TIALIZE, we evaluate R hash functions on all of the data points to obtain R hash tables. During SUBQUERY we leverage the hash function collision probability to compute an estimate of the pairwise summation function by

$$Z_{i,j} = \frac{1}{n} \frac{w(x,y)}{p_r(x,y)} |H_r(x)| \ \forall i \in [L], j \in [m],$$

and leverage median of means to output the final estimation. QUERY calls SUBQUERY operation for up to  $Q = \lfloor \frac{\log(\tau/(1-(c+\epsilon)))}{\log(1-\gamma)} \rfloor$  times to keep approaching the true result

and finally obtain the estimated output. INSERTX and DELE-TEX operations insert or delete corresponding hash table entries using the input data point  $x \in S^{d-1}$ .

Algorithm 1 Dynamic Single-solution HBE-Estimator Data Structure

Stri	lcture
1:	data structure
2:	members
3:	$X = \{x_i\}_{i=1}^n \subset \mathbb{R}^d \qquad \qquad \triangleright \text{ A set of data points}$
4.	$B \in \mathbb{N}$ > Number of hash functions.
5.	$\{h_{-}\}^{R}$ , $\triangleright R$ hash functions
6.	$\{H\}_{R}^{R}$ , $\triangleright$ A collection of hash tables
7.	$[n_r]_{r=1}^{r=1}$ $\mathbb{D}^d \times \mathbb{D}^d \to [0, 1]$ by The collision probability for
/.	$p_{r}_{r=1}$ . $\mathbb{N} \to [0, 1] \lor$ The conision probability for bashing schemes
0.	$u : \mathbb{P}^d \times \mathbb{P}^d \to \mathbb{P}$ . The target pointwise function
0. 0.	$W: \mathbb{R} \to \mathbb{R}^+$ $V: \mathbb{P} \to \mathbb{P}$ . The relative variance function
9. 10.	$V : \mathbb{R} \to \mathbb{R}_+$ $\lor$ The relative variance function
10:	end members
11:	<b>procedure</b> INITIALIZE $(w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+, v : \mathbb{R} \to \mathbb{R}_+)$
	$\mathbb{K}_{+}, \{x_i\}_{i=1} \subset \mathbb{K}^{*}, \epsilon \in (0, 0.1), o \in (0, 1), \tau \in (0, 1)\} $
10	Lemma V.4 $D = O(-2) + (1/5) V(-) + V = (-)^n + V = V$
12:	$R \leftarrow O(\epsilon - \log(1/\delta)V(\tau)), X \leftarrow \{x_i\}_{i=1}^{n}, w \leftarrow w, v \leftarrow$
	V
13:	Sample $\{h_r\}_{r=1}^r \sim v$ from $\mathcal{H}$ . $\{p_r\}_{r=1}^r$ are corresponding
	collision probability functions.
14:	for $r = 1 \rightarrow R$ do
15:	$H_r \leftarrow h_r(X) \triangleright$ Evaluate hash function on the dataset to
	obtain a hash table.
16:	end for
17:	end procedure
18:	<b>procedure</b> SUBQUERY $(x \in \mathbb{R}^d, V : \mathbb{R} \to \mathbb{R}_+, \mu, \epsilon \in$
	$(0,1), \delta_0 \in (0,1)$ $\triangleright$ Lemma V.2, Lemma V.5
19:	$m \leftarrow \left\lceil 6\epsilon^{-2}V(\mu) \right\rceil$
20:	$L \leftarrow \lceil 9 \log(1/\delta_0) \rceil$
21:	for $i = 1 \rightarrow L$ do
22:	Sample $r \sim [R]$
23:	for $j = 1 \rightarrow m$ do
24:	Sample a data point $y \sim H_r(x) \Rightarrow H_r(x)$ denotes
	the hash bucket where query x falls into using hash function $h_r$
25:	$Z_{i,i} \leftarrow \frac{1}{2} \frac{w(x,y)}{w(x,y)}  H_r(x)  > The hashing-based$
	estimator
26.	end for
27.	end for
28.	$Z_i \leftarrow \text{mean}\{Z_{i,1}, \dots, Z_{i,m}\}$ for $i \in [L]$
20.	$Z_i$ , mean $\{Z_1, \dots, Z_r\}$
2). 30·	return $Z$
31.	end procedure
51.	una procedure

## A. Correctness of Query

In this subsection, we present the lemmas to prove the correctness of SUBQUERY and QUERY.

**Lemma V.2** (Correctness of SubQuery). Given an estimator Z of complexity C which is V-bounded and  $\mathbb{E}[Z] = \mu \in (0,1]$ , taking a non-decreasing function  $V : \mathbb{R} \to \mathbb{R}_+, \mu \in (0,1)$ , a query point  $x \in \mathbb{R}^d$ , accuracy parameter  $\epsilon \in (0,1]$  and a failure probability  $\delta_0 \in (0,1)$  as input, the SUBQUERY in Algorithm 1 could obtain an estimation value Z, which satisfies:

$$\Pr[|Z - \mu| \ge \epsilon \cdot \mu] \le \delta_0$$

using  $O(\epsilon^{-2}V(\mu)\log(1/\delta))$  samples.

Algorithm 2 Dynamic Single-solution HBE-Estimator Data Structure

1: procedure QUERY $(x \in \mathbb{R}^d, \alpha \in (0, 1], \tau \in (0, 1), \delta \in (0, 1)) \triangleright$ Lemma V.6 and Lemma V.3. 2:  $\epsilon \leftarrow \frac{2}{7}\alpha, c \leftarrow \frac{\epsilon}{2}, \gamma \leftarrow \frac{\epsilon}{7}, \delta_0 \leftarrow \frac{2\alpha}{49\log(1/\tau)}$  $Q \leftarrow \left| \frac{\log(\tau/(1-(c+\epsilon)))}{1-\tau'} \right|$ 3: for  $i = -1 \xrightarrow{\log(1-\gamma)} Q$  do 4: 5:  $i \leftarrow i + 1$  $\mu_i \leftarrow (1 - \gamma)^i$ 6:  $Z_i \leftarrow \text{SUBQUERY}(x, V, \mu_i, \frac{\epsilon}{3}, \delta_0)$ 7. if  $|Z_i - \mu_i| \leq c \cdot \mu_i$  then 8: 9: break; end if 10: end for 11: if  $i < \frac{49 \log(1/\tau)}{2}$  then 12:  $return \overset{\geq}{Z_i}$ 13: 14: else return 0 15: end if 16: 17: end procedure **procedure** INSERTX $(x \in \mathbb{R}^d)$ ⊳ Lemma V.7 18: 19:  $X = X \cup x$ for  $r = 1 \rightarrow R$  do 20:  $H_r \leftarrow H_r \cup \{h_r(x)\}$  $\triangleright$  Insert x to its mapping hash 21: bucket. 22: end for 23: end procedure **procedure** DELETEX( $x \in \mathbb{R}^d$ ) ⊳ Lemma V.8 24: 25:  $X = X \setminus x$ for  $r = 1 \rightarrow R$  do 26:  $H_r \leftarrow H_r \setminus \{h_r(x)\}$ 27: 28. end for 29: end procedure 30: end data structure

The correctness of the QUERY operation is shown as follows.

**Lemma V.3** (Correctness of Query). Given an estimator Z of complexity C which is V-bounded and  $\mathbb{E}[Z] \in (0,1]$ , QUERY (in Algorithm 2) takes a query point  $x \in \mathbb{R}^d$ , accuracy parameter  $\alpha \in (0,1]$ , a threshold  $\tau \in (0,1)$  and a failure probability  $\delta \in (0,1)$  as inputs, and outputs  $Z_{est} \in \mathbb{R}$  such that:

• 
$$\Pr[|Z_{\mathsf{est}} - \mathbb{E}[Z]| \le \alpha \mathbb{E}[Z]] \ge 1 - \delta \text{ if } \mathbb{E}[Z] \ge \tau$$
  
•  $\Pr[Z_{\mathsf{est}} = 0] > 1 - \delta \text{ if } \mathbb{E}[Z] < \tau$ 

*Proof.* The query algorithm interacts with HBE data structure by invoking hash functions. HBE maintains an index of the most recent hash function invoked and increases it by one after each hash function is evaluated, which ensures that a query never computes the same hash function again so that the data points are independently sampled. When a query arrives, the query algorithm begins with the adaptive mean relaxation algorithm. Here we set  $\alpha = 1$  and probability be  $\delta/2$ . After invoking  $O(V((\mathbb{E}[Z])_{\tau} \log(1/\delta)))$  hash functions with failure probability at most  $\delta/2$ , we can have one of the following result: (1)  $\epsilon$ -approximation result, (2) QUERY outputs 0 which indicates that  $\mathbb{E}[Z] < \tau$ .

For the first scenario, we apply the SUBQUERY algorithm

which have a value that underestimates  $\mathbb{E}[Z]$  that invokes  $O\left(\epsilon^{-2}\log(1/\delta)V\left((\mathbb{E}[Z])_{\tau}\right)\right)$  more calls to the hash functions and result in an  $\epsilon$ -approximation result with failure probability at most  $\delta$  according to Lemma V.2. For the second scenario, the query algorithm outputs 0 when  $\mathbb{E}[Z] < \tau$ .

# B. Running Time

In this subsection, we prove the running time of each operation in our data structure, including: INITIALIZE, SUBQUERY, QUERY, INSERTX and DELETEX.

**Lemma V.4** (Initialize Time). Given an estimator Z of complexity C which is V-bounded and n data points, the time complexity of INITIALIZE in Algorithm 1 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot nC)$ .

*Proof.* During INITIALIZE operation, the running time is dominated by the hash function evaluations on the dataset  $\{x_i\}_{i=1}^n$ . The number of hash functions is  $R = O(\epsilon^{-2}\log(1/\delta)V(\tau))$ , so the INITIALIZE can be done in  $n \cdot RC = O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot nC)$  time.

Next, we show the SUBQUERY operation's running time.

**Lemma V.5** (SubQuery Time). Given an estimator Z of complexity C which is V-bounded, the time complexity of SUBQUERY in Algorithm 1 is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta_0)C)$ .

*Proof.* In the SUBQUERY operation, the running time is dominated by evaluating the HBE with query point  $x \in \mathbb{R}^d$  for  $mL = O(\epsilon^{-2}V_\mu \log(1/\delta_0))$  times. Because the complexity of V-bounded estimator is C, we have that the time complexity of SUBQUERY is  $O(\epsilon^{-2}V((\mu)_\tau) \log(1/\delta_0)C)$ .

We now move the QUERY operation's running time.

**Lemma V.6** (Query Time). Given an estimator Z of complexity C which is V-bounded, the time complexity of QUERY in Algorithm 2 is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta)C)$ .

*Proof.* Because in QUERY operations, SUBQUERY is called for at most fixed  $Q = \lfloor \frac{\log(\tau/(1-(c+\epsilon)))}{\log(1-\gamma)} \rfloor$  times, the time complexity of QUERY operation is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta)C)$ .

Next, we present the running time for the INSERTX operation.

**Lemma V.7** (InsertX Time). *Given an estimator Z of complexity C which is V-bounded, the time complexity of* INSERTX *in Algorithm 2 is*  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$ .

*Proof.* During INSERTX operation, the running time is dominated by the hash function evaluations on the inserted data point  $x \in \mathbb{R}^d$ . The number of hash functions is  $R = O(\epsilon^{-2}\log(1/\delta)V(\tau))$ , so the INSERTX can be done in  $RC = O(\epsilon^{-2}\log(1/\delta)V(\tau) \cdot C)$  time.

We present the running time for the DELETEX operation as follows.

**Lemma V.8** (DeleteX Time). Given an estimator Z of complexity C which is V-bounded, the time complexity of DELETEX in Algorithm 2 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$ .

*Proof.* During DELETEX operation, the running time is dominated by the hash function evaluations on the to be deleted data point  $x \in \mathbb{R}^d$ . The number of hash functions is  $R = O(\epsilon^{-2} \log(1/\delta)V(\tau))$ , so the DELETEX can be done in  $RC = O(\epsilon^{-2} \log(1/\delta)V(\tau) \cdot C)$  time.

## VI. DYNAMIC MULTI-RESOLUTION HBE

In this section, we extend the dynamic correction to multiresolution hashing. We present our theorem for the dynamic version multi-resolution HBE in Theorem VI.1.

**Theorem VI.1** (Our results, Dynamic version of Theorem 5.4 in page 17 [6]). *Given an approximation parameter*  $\epsilon \in (0, 1)$  *and a threshold*  $\tau \in (0, 1)$ *, for a convex function w, we have a data structure the allows operations as below:* 

- INITIALIZE ( $w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+, V : \mathbb{R} \to \mathbb{R}_+, \{x_i\}_{i=1}^n \subset S^{d-1}, \{\mathcal{H}_g\}_{g=1}^G, \epsilon \in (0,0.1), \delta \in (0,1), \tau \in (0,1)$ ). Given a set of data points  $\{x_i\}_{i=1}^n \subset S^{d-1}$ , a collection of hashing scheme  $\{\mathcal{H}_g\}_{g=1}^G$ , the target function  $w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ , the relative variance function  $V : \mathbb{R} \to \mathbb{R}_+$ , accuracy parameter  $\epsilon \in (0,0.1)$ , failure probability  $\delta \in (0,1)$  and a threshold  $\tau \in (0,1)$  as input, the INITIALIZE operation takes  $O(\epsilon^{-2}V(\tau)C\log(1/\delta) \cdot n)$  time.
- QUERY $(x \in S^{d-1}, \alpha \in (0, 1], \tau \in (0, 1), \delta \in (0, 1))$ . Given a query point  $x \in S^{d-1}$ , accuracy parameter  $\alpha \in (0, 1]$ , a threshold  $\tau \in (0, 1)$  and a failure probability  $\delta \in (0, 1)$  as input, the time complexity of QUERY operation is  $O(\epsilon^{-2}V(\tau)C\log(1/\delta))$  and the output of QUERY  $\hat{Z}$  satisfies:

$$\begin{cases} \Pr[|\widehat{Z} - \mu| \le \alpha \mu] \ge 1 - \delta, & \mu \ge \tau \\ \Pr[\widehat{Z} = 0] \ge 1 - \delta, & \mu < \tau \end{cases}$$

- INSERTX $(x \in S^{d-1})$ . Given a data point  $x \in S^{d-1}$  as input, the INSERTX operation takes  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$  time to update the data structure.
- DELETEX( $x \in S^{d-1}$ ). Given a data point  $x \in S^{d-1}$  as input, the DELETEX operation takes  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$  time to update the data structure.

*Proof.* We can prove the theorem by combining the running time lemmas including the INITIALIZE running time in Lemma VI.4, the QUERY running time in Lemma VI.6, the INSERTX running time in Lemma VI.7 and DELETEX running time in Lemma VI.8, and query correctness proof Lemma VI.3.

We remark that as for now, we have a dynamic data structure for PSE with potential applications in neural network training and kernel density estimation.

We present the dynamic multi-resolution HBE estimator data structure in Algorithm 3 and Algorithm 4. During INI-TIALIZE, we evaluate R hash functions for G hash function

families on all of the data points to obtain  $G \cdot R$  hash tables. During SUBQUERY we leverage different hash function collision probabilities of G hash function families to compute an estimate of the pairwise summation function by  $\forall i \in [L], j \in [m]$ 

$$Z_{i,j} = \frac{1}{|X|} \sum_{g=1}^{G} \frac{\widetilde{w}_{r,g}(x, y_g) \cdot w(x, y_g)}{p_{r,g}(x, y_g)} |H_{r,g}(x)|$$

and leverage median of means to output the final estimation. QUERY calls SUBQUERY operation for up to  $Q = \lfloor \frac{\log(\tau/(1-(c+\epsilon)))}{\log(1-\gamma)} \rfloor$  times to keep approaching the true result and finally obtain the estimated output. INSERTX and DELETEX operations insert or delete corresponding hash table entries for all hash function families using the input data point  $x \in S^{d-1}$ .

#### A. Correctness of Query

In this section, we present the lemmas for correctness of SUBQUERY and QUERY operation in the dynamic multiresolution HBE in Lemma VI.3.

**Lemma VI.2** (Correctness of SubQuery). Given a multiresolution HBE estimator Z of complexity C which is Vbounded and  $\mathbb{E}[Z] = \mu \in (0, 1]$ , taking a non-decreasing function  $V : \mathbb{R} \to \mathbb{R}_+$ ,  $\mu \in (0, 1)$ , a query point  $x \in \mathbb{R}^d$ , accuracy parameter  $\epsilon \in (0, 1]$  and a failure probability  $\delta_0 \in (0, 1)$  as input, the SUBQUERY in Algorithm 4 can get an estimate Z such that

$$\Pr[|Z - \mu| \ge \epsilon \cdot \mu] \le \delta_0$$

using  $O(\epsilon^{-2}V(\mu)\log(1/\delta))$  samples.

We now move to the correctness for the QUERY operation.

**Lemma VI.3** (Correctness of Query). Given a multi-resolution HBE estimator Z of complexity C which is V-bounded and  $\mathbb{E}[Z] \in (0,1]$ , QUERY Z<sub>est</sub> (in Algorithm 4) takes a query point  $x \in \mathbb{R}^d$ , accuracy parameter  $\alpha \in (0,1]$ , a threshold  $\tau \in (0,1)$  and a failure probability  $\delta \in (0,1)$  as inputs, and outputs Z<sub>est</sub>  $\in \mathbb{R}$  such that:

•  $\Pr[|Z_{\mathsf{est}} - \mathbb{E}[Z]| \le \alpha \mathbb{E}[Z]] \ge 1 - \delta \text{ if } \mathbb{E}[Z] \ge \tau$ 

• 
$$\Pr[Z_{\mathsf{est}} = 0] \ge 1 - \delta \text{ if } \mathbb{E}[Z] < \eta$$

*Proof.* The query algorithm invokes hash function calls to sample data points from the data structure. The data structure increases the index of the most recent hash function invoked by one after each hash function is computed, which enables a query to never evaluate the same hash function again and sample data points independently. When a query arrives, the query algorithm begins with the adaptive mean relaxation algorithm with  $\alpha = 1$  and probability  $\delta/2$ . After interacting with the data structure for  $O(V((\mathbb{E}[Z])_{\tau} \log(1/\delta)))$  times, we either obtain an  $\epsilon$ -approximation result or QUERY outputs 0 which indicates that  $\mathbb{E}[Z] < \tau$  with probability at least  $1 - \delta/2$ .

For the first scenario, we apply the SUBQUERY algorithm which results in an value that underestimates of  $\mathbb{E}[Z]$ . Moreover it invokes  $O\left(\epsilon^{-2}\log(1/\delta)V\left((\mathbb{E}[Z])_{\tau}\right)\right)$  more hash

## Algorithm 3 Multi-Resolution HBE-Estimator Data Structure

1: data structure 2: members  $X = \{x_i\}_{i=1}^n \subset \mathbb{R}^d$ ▷ A set of data points 3:  $R \in \mathbb{N}$ 4: ▷ Number of estimators. 5:  $G \in \mathbb{N}$  $\triangleright$  Number of hash schemes.  $\{\mathcal{H}_g\}_{g=1}^G \triangleright A$  collection of hashing schemes per estimator. 6:  $\mathcal{H}_q = \{ f : \mathbb{R}^d \to \mathbb{R} \}$  $\{\{h_{r,g}\}_{g=1}^G\}_{r=1}^R$ 7: ▷ A collection of hash functions  $\{ \{H_{r,g}\}_{g=1}^{G} \}_{r=1}^{R} \qquad \triangleright A \ control \{ \{p_{r,g}\}_{g=1}^{G} \}_{r=1}^{R} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0,1]$  $\triangleright$  A collection of hash tables 8: ▷ The collision <u>9</u>. probability for hashing functions  $w: \mathbb{R}^{d} \times \mathbb{R}^{d} \to \mathbb{R}_{+} \qquad \rhd \text{ The} \\ \{\{\widetilde{w}_{r,g}\}_{g=1}^{G}\}_{r=1}^{R}: \mathbb{R}^{d} \times \mathbb{R}^{d} \to \mathbb{R}_{+} \end{cases}$ ▷ The target pairwise function 10: 11:  $\triangleright$  A collection of weight functions  $V: \mathbb{R} \to \mathbb{R}_+$ ▷ The relative variance function 12: 13: end members 14: **procedure** INITIALIZE $(w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+, V : \mathbb{R} \to \mathbb{R}_+, \{x_i\}_{i=1}^n \subset \mathbb{R}^d, \{\mathcal{H}_g\}_{g=1}^G, \epsilon \in (0, 0.1), \delta \in (0, 1), \tau \in \mathbb{R}^d$ ⊳ Lemma VI.4 (0,1) $R \leftarrow O(\epsilon^{-2} \log(1/\delta)V(\tau))$ 15:  $\begin{aligned} G \leftarrow \lfloor \frac{\log(\frac{1-|\rho_{+}|}{1-|\rho_{-}|})}{\log(1+\sqrt{\frac{\epsilon}{8|\ell_{\min}|}})} \rfloor \\ X \leftarrow \{x_{i}\}_{i=1}^{n}, w \leftarrow w, V \leftarrow V \end{aligned}$ 16: 17: for  $g = 1 \rightarrow G$  do 18: Sample  $\{h_{r,g}\}_{r=1}^R \sim v$  from  $\mathcal{H}_g$ .  $\{p_{r,g}\}_{r=1}^R$  are corre-19: sponding collision probability functions. 20end for for  $g=1 \to G~\mathrm{do}$ 21: for  $r = 1 \rightarrow R$  do 22:  $H_{r,g} \leftarrow h_{r,g}(X)$ ▷ Evaluate hash function on the 23: dataset to obtain a hash table,  $_{,g}(x,y)$  $\triangleright \sum_{i=1}^{G} \widetilde{w}_{r,i} = 1$ 24:  $\widetilde{w}_{r,g}(x,y) \leftarrow$  $\overline{\sum_{i=1}^G p_{r,i}^2(x,y)}$ end for 25: end for 26: 27: end procedure **procedure** INSERTX( $x \in S^{d-1}$ ) ⊳ Lemma VI.7 28. 29:  $X \leftarrow X \cup x$ 30. for  $r = 1 \rightarrow R$  do for  $g = 1 \rightarrow G$  do 31:  $H_{r,g} \leftarrow H_{r,g} \cup \{h_{r,g}(x)\} \mathrel{\triangleright} \text{Insert } x \text{ to its mapping}$ 32: hash bucket. end for 33: end for 34: 35: end procedure 36: **procedure** DELETEX( $x \in S^{d-1}$ ) ⊳ Lemma VI.8 37:  $X \leftarrow X \setminus x$ for  $r = 1 \rightarrow R$  do 38: for  $g = 1 \rightarrow G$  do 39:  $H_{r,g} \leftarrow H_{r,g} \setminus \{h_{r,g}(x)\} \triangleright \text{Insert } x \text{ to its mapping}$ 40: hash bucket. end for 41: 42: end for 43: end procedure

function calls and obtains an  $\epsilon$ -approximation result with success probability of at least  $1 - \delta$  according to Lemma V.2. For the second scenario, the query algorithm outputs 0 when  $\mathbb{E}[Z] < \tau.$ 

## Algorithm 4 Multi-Resolution HBE-Estimator Data Structure

1: procedure SUBQUERY $(x \in \mathbb{R}^d, V : \mathbb{R} \to \mathbb{R}_+, \mu \in (0, 1), \epsilon \in$  $(0,1), \delta_0 \in (0,1)$ ▷ Lemma VI.2. Lemma VI.5

 $m \leftarrow [6\epsilon^{-2}V(\mu)]$ 2:  $L \leftarrow \lceil 9 \log(1/\delta_0) \rceil$ 3:

for  $i = 1 \rightarrow L$  do 4:

Sample  $r \sim [R]$ 5:

for  $j = 1 \rightarrow m$  do 6·

for  $q = 1 \rightarrow G$  do 7:

8: Sample a data point  $y_q \sim H_{r,q}(x)$  $H_{r,q}(x)$  denotes the hash bucket where query x falls into using hash function  $h_{r,g}$ 

9: end for  $1 \quad \mathbf{\nabla} G \quad \widetilde{w}_{--}(x, y_{-}) \cdot w(x, y_{-}) + \dots$ 

10: 
$$Z_{i,j} \leftarrow \frac{1}{|X|} \sum_{g=1}^{G} \frac{w_{r,g}(x,y_g) - w(x,y_g)}{p_{r,g}(x,y_g)} |H_{r,g}(x)| \triangleright \text{ The multi-resolution hashing-based estimator}$$

end for 11:

end for 12:

 $Z_i \leftarrow \operatorname{mean}\{Z_{i,1}, \cdots, Z_{i,m}\}$  for  $i \in [L]$ 13:

 $Z \leftarrow \text{median}\{Z_1, \cdots, Z_L\}$ 14:

return Z 15:

16: end procedure

17: procedure QUERY $(x \in \mathbb{R}^d, \alpha \in (0, 1], \tau \in (0, 1), \delta \in (0, 1)) \triangleright$ Lemma VI.3 and Lemma VI.6 20

18: 
$$\epsilon \leftarrow \frac{2}{7}\alpha, c \leftarrow \frac{1}{2}, \gamma \leftarrow \frac{1}{7}, \delta_0 \leftarrow \frac{2\alpha}{49\log(1/\tau)}$$
  
19:  $Q \leftarrow \lfloor \frac{\log(\tau/(1-(c+\epsilon)))}{(c+\epsilon)} \rfloor$ 

 $Q \leftarrow \lfloor \frac{\log(1-\gamma)}{\log(1-\gamma)}$  for  $i = -1 \xrightarrow{} Q$  do 20: 21:

$$i \leftarrow i + 1$$

 $\begin{array}{l} \mu_i \leftarrow (1 - \gamma) \\ Z_i \leftarrow \text{SUBQUERY}(x, V, \mu_i, \frac{\epsilon}{3}, \delta_0) \end{array}$ 

if 
$$|Z_i - \mu_i| \leq c \cdot \mu_i$$
 then

break; end if

end for 27: 28:

22:

23:

24:

25:

26:

29:

if  $i \leq \frac{49 \log(1/\tau)}{2\alpha}$  then return  $Z_i^{\alpha}$ 

30: else

```
31:
           return 0
32.
       end if
```

```
33: end procedure
```

```
34: end data structure
```

# B. Running Time

In this section, we prove the time complexity of INITIALIZE, SUBQUERY, INSERTX and DELETEX operations.

Lemma VI.4 (Initialize Time). Given a V-bounded multiresolution HBE estimator of complexity C and n data points, the time complexity of INITIALIZE in Algorithm 3 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta)\cdot nC).$ 

*Proof.* During INITIALIZE operation, the running time is dominated by the hash function evaluations on the dataset  $\{x_i\}_{i=1}^n$ . The number of hash functions is R = $O(\epsilon^{-2}\log(1/\delta)V(\tau))$ , so the INITIALIZE can be done in  $n \cdot RC = O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot nC)$  time.

The SUBQUERY time for our data structure is the following.

Lemma VI.5 (SubQuery Time). Given a V-bounded multiresolution HBE estimator of complexity C, the time complexity of SUBQUERY in Algorithm 4 is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta_0)C)$ .

Proof. In the SUBQUERY operation, the running time is dominated by computing the multi-resolution HBE with query point  $x \in \mathbb{R}^d$  for  $mL = O(\epsilon^{-2}V_\mu \log(1/\delta_0))$ times. Because the complexity of V-bounded estimator is C, we have that the time complexity of SUBQUERY is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta_0)C).$ 

We present the QUERY time as follows.

Lemma VI.6 (Query Time). Given a V-bounded multiresolution HBE estimator of complexity C, the time complexity of QUERY in Algorithm 4 is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta)C)$ .

Proof. Because in QUERY operations, SUBQUERY is called for at most fixed  $Q = \lfloor \frac{\log(\tau/(1-(c+\epsilon)))}{\log(1-\gamma)} \rfloor$  times, the time complexity of QUERY operation is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta)C)$ . 

The INSERT time for the data structure is the following.

Lemma VI.7 (Insert Time). Given a V-bounded multiresolution HBE estimator of complexity C, the time complexity of INSERTX in Algorithm 3 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$ .

Proof. During INSERTX operation, the running time is dominated by the hash function evaluations on the inserted data point  $x \in \mathbb{R}^d$ . The number of hash functions is R = $O(\epsilon^{-2}\log(1/\delta)V(\tau))$ , so the INSERTX can be done in RC = $O(\epsilon^{-2}\log(1/\delta)V(\tau)\cdot C)$  time. 

Now we present the DELETE time for the data structure.

Lemma VI.8 (Delete Time). Given a V-bounded multiresolution HBE estimator of complexity C, the time complexity of DELETEX in Algorithm 3 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$ .

*Proof.* During DELETEX operation, the running time is dominated by the hash function evaluations on the to be deleted data point  $x \in \mathbb{R}^d$ . The number of hash functions is  $R = O(\epsilon^{-2} \log(1/\delta) V(\tau))$ , so the DELETEX can be done in  $RC = O(\epsilon^{-2} \log(1/\delta)V(\tau) \cdot C)$  time. 

#### VII. ADAM-HASH: ADAPTIVE AND DYNAMIC HBE

In this section, we present Adam-Hash: an adaptive and dynamic multi-resolution hashing-based estimator data structure design in Algorithm 5, and give the corresponding theorem in Theorem VII.1. Then we present the lemmas and proofs for correctness of query in section VII-A. We present the lemmas and proofs for the time complexity of operations in our dynamic multi-resolution HBE data structure in section VII-B.

Theorem VII.1 (Our results, Adaptive and dynamic hashing based estimator). Let the approximation parameter be  $\epsilon \in$ (0,1) and a threshold  $\tau \in (0,1)$ . Given any convex function w, we show that we can have a data structure allows operations as below:

• INITIALIZE $(w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+, V : \mathbb{R} \to \mathbb{R}_+, \{x_i\}_{i=1}^n \subset S^{d-1}, \{\mathcal{H}_g\}_{g=1}^G, \epsilon \in (0, 0.1), \delta \in (0, 1), \tau \in (0, 1)).$  Given a set of data points  $\{x_i\}_{i=1}^n \subset S^{d-1}, \{x_i\}_{i=1}^n \in \mathbb{R}$  $\mathcal{S}^{d-1}$ , a collection of hashing scheme  $\{\mathcal{H}_g\}_{g=1}^{G}$ , the k-Lipschitz target function  $w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ , the relative

## Algorithm 5 Adam-Hash Data Structure

1: data structure

2: members  $\{\mathrm{HBE}_j\}_{j=1}^L$ ▷ A set of HBE estimators 3:

 $\tilde{L} \in \mathbb{N}$ ▷ Number of HBE estimators 4: 5:  $n \in \mathbb{N}$ 

 $\triangleright$  Number of data points.

6: end members

7: procedure INITIALIZE $(w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+, V : \mathbb{R} \to \mathbb{R}^d$  $\mathbb{R}_{+}, \{x_i\}_{i=1}^n \subset \mathbb{R}^d, \{\mathcal{H}_g\}_{g=1}^G, \epsilon \in (0, 0.1), \delta \in (0, 1), \tau \in (0, 1)$ ⊳ Lemma VII.10 (0,1) $L \leftarrow O(\log((10k/\epsilon\tau)^d/\delta))$ 8:

```
9:
        n \gets n
```

```
for j = 1 \rightarrow L do
10:
```

11: 
$$\operatorname{HBE}_{i}$$
.INITIALIZE $(w, V, \{x_i\}_{i=1}^{n}, \{\mathcal{H}_q\}_{q=1}^{G}, \epsilon, \delta, \tau)$ 

```
12:
       end for
```

13: end procedure

- 14: procedure QUERY $(q \in \mathbb{R}^d, \epsilon \in (0, 1], \tau \in (0, 1), \delta \in (0, 1))$   $\triangleright$ Lemma VII.2, Lemma VII.11
- Let N denote the  $\epsilon_0$ -net of  $\{x \in \mathbb{R}^d \mid ||x||_2 \leq 1\}$ . 15:
- Find a point  $p \in N$  which is the closest to q. 16:
- 17: for  $k \in [L]$  do
  - $y_k = \text{HBE}_k.\text{QUERY}(p, \epsilon, \tau, \delta)$
- end for 19:

18:

20:  $\widetilde{z} \leftarrow \text{Median}(\{y_k\}_{k=1}^L)$ 

21: return  $\widetilde{z}$ 

22: end procedure

23: procedure INSERTX( $x \in \mathbb{R}^d$ ) ▷ Lemma VII.12  $n \leftarrow n+1$ 

```
24:
25:
          for k = 1 \rightarrow L do
```

26:  $\text{HBE}_j$ .INSERTX(x)▷ Insert the data point into each HBE data structure.

▷ Lemma VII.13

27: end for

28: end procedure

29: procedure DELETEX( $x \in \mathbb{R}^d$ )

30:  $n \leftarrow n-1$ 31:

for  $k = 1 \rightarrow L$  do

 $HBE_i.DELETEX(x)$ 32:

33: end for 34: end procedure

> variance function  $V : \mathbb{R} \to \mathbb{R}_+$ , accuracy parameter  $\epsilon \in (0, 0.1)$ , failure probability  $\delta \in (0, 1)$  and a threshold  $\tau \in (0,1)$  as input, the INITIALIZE operation takes  $O(\epsilon^{-2}V(\tau)C\log(1/\delta) \cdot n \cdot \log((10k/\epsilon\tau)^d/\delta))$  time.

• QUERY $(x \in S^{d-1}, \epsilon \in (0, 0.1), \tau \in (0, 1), \delta \in (0, 1)).$ Given a query point  $x \in S^{d-1}$ , accuracy parameter  $\epsilon \in (0, 0.1)$ , a threshold  $\tau \in (0, 1)$  and a failure probability  $\delta \in (0, 1)$  as input, the time complexity of QUERY operation is  $O(\epsilon^{-2}V(\tau)C\log(1/\delta) \cdot \log((10k/\epsilon\tau)^d/\delta))$ and the output of QUERY  $\widehat{Z}$  satisfies:

$$\begin{cases} \Pr[|\widehat{Z} - \mu| \le \epsilon \mu] \ge 1 - \delta, & \mu \ge \tau \\ \Pr[\widehat{Z} = 0] \ge 1 - \delta, & \mu < \tau \end{cases}$$

even when the queries are adaptive.

- INSERTX $(x \in S^{d-1})$ . Given a data point  $x \in S^{d-1}$  as input, the INSERTX operation takes  $O(\epsilon^{-2}V(\tau)\log(1/\delta)$ .  $C \cdot \log((10k/\epsilon\tau)^d/\delta))$  time to update the data structure.
- DELETEX $(x \in S^{d-1})$ . Given a data point  $x \in \mathcal{S}^{d-1}$  as input, the DELETEX operation takes  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C \cdot \log((10k/\epsilon\tau)^d/\delta))$  time to update the data structure.

*Proof.* We can prove the theorem by combining the running time lemmas including Lemma VII.10, Lemma VII.11, Lemma VII.12 and Lemma VII.13, and query correctness proof Lemma VII.2.

## A. Correctness of Query

The goal of this section is to prove the correctness for QUERY in Algorithm 5 in Lemma VII.2,

**Lemma VII.2** (Correctness of Query). Given an estimator Z of complexity C which is V-bounded and  $\mathbb{E}[Z] = \mu \in (0, 1]$ , QUERY  $\hat{Z}$  in Algorithm 5 takes a query point  $x \in \mathbb{R}^d$ , accuracy parameter  $\epsilon \in (0, 1]$ , a threshold  $\tau \in (0, 1)$  and a failure probability  $\delta \in (0, 1)$  as inputs, and outputs  $\hat{Z} \in \mathbb{R}$  such that:

•  $\Pr[|\widehat{Z} - \mu| \le \epsilon \mu] \ge 1 - \delta$  if  $\mu \ge \tau$ 

• 
$$\Pr[Z=0] \ge 1 - \delta \text{ if } \mu < \tau$$

*Proof.* When  $\mu \geq \tau$ , for QUERY in Algorithm 5, first we prove each HBE estimator can answer the query with constant success probability in Lemma VII.3, second we prove the median of query results from *L* HBE estimators can achieve  $\epsilon$  approximation with high probability in Lemma VII.4, third we prove that for all query points on a  $\epsilon$ -net can be answered with  $\epsilon$  approximation with high probability in Lemma VII.6 and fourth we prove that for all query points  $||q||_2 \leq 1$ , QUERY in Algorithm 5 can give an answer with  $\epsilon$  approximation with high probability in Lemma VII.6

When  $\mu < \tau$ , the QUERY in Algorithm 5 returns 0 with probability  $1 - \delta$ .

1) Starting with Constant Probability: First we need to prove the HBE estimator can answer the query approximately with a constant success probability.

**Lemma VII.3** (Constant probability). Given  $\epsilon \in (0, 0.1), \tau \in (0, 1)$ , a query point  $q \in \mathbb{R}^d$  and a set of data points  $X = \{x_i\}_i^n \subset \mathbb{R}^d$ , let  $Z(q) := \frac{1}{|X|} \sum_{x \in X} w(x, q)$  an estimator HBE can answer the query which satisfies:

$$\text{HBE.QUERY}(q,\epsilon,0.1) \in (1\pm\epsilon) \cdot Z(q)$$

with probability 0.9.

2) Boost the Constant Probability to High Probability: Then we want to boost the constant success probability to high success probability via obtaining the median of L queries.

**Lemma VII.4** (Boost the probability). We write the failure probability as  $\delta_1 \in (0, 0.1)$  and accuracy parameter as  $\epsilon \in (0, 0.1)$ . Given  $L = O(\log(1/\delta_1))$  estimators  $\{\text{HBE}_j\}_{j=1}^L$ . For each fixed query point  $q \in \mathbb{R}^d$ , the median of queries from L estimators satisfies that:

Median({HBE<sub>j</sub>.QUERY(q,  $\epsilon$ , 0.1)} $_{j=1}^{L}$ )  $\in$  (1  $\pm \epsilon$ )  $\cdot Z(q)$ 

with probability  $1 - \delta_1$ .

*Proof.* From Lemma VII.3 we know each estimator  $HBE_j$  can answer the query that satisfies:

$$\text{HBE.QUERY}(q, \epsilon, 0.1) \in (1 \pm \epsilon) \cdot Z(q)$$

with probability 0.9.

From the chernoff bound we know the median of  $L = O(\log(1/\delta_1))$  queries from  $\{\text{HBE}_j\}_{j=1}^{L}$  satisfies:

$$Median(\{HBE_j.QUERY(q,\epsilon,0.1)\}_{i=1}^L) \in (1 \pm \epsilon) \cdot Z(q)$$

with probability  $1 - \delta_1$ .

Therefore, we complete the proof.

*3)* From each Fixed Point to All the Net Points: In this section, we present Fact VII.5 and generalize from each fixed point to all on-net points in Lemma VII.6.

**Fact VII.5.** Let N be the  $\epsilon_0$ -net of  $\{x \in \mathbb{R}^d \mid ||x||_2 \le 1\}$ . Let |N| denote the size of N. Then  $|N| \le (10/\epsilon_0)^d$ .

**Lemma VII.6** (From for each fixed to for all net points). Let N denote the  $\epsilon_0$ -net of  $\{x \in \mathbb{R}^d \mid ||x||_2 \leq 1\}$ . Let |N| denote size of N. Given  $L = \log(|N|/\delta)$  estimators  $\{\text{HBE}_j\}_{i=1}^L$ .

With probability  $1 - \delta$ , we have: for all  $q \in N$ , the median of queries from L estimators satisfies that:

Median({HBE<sub>j</sub>.QUERY(q,  $\epsilon$ , 0.1)} $_{i=1}^{L}$ )  $\in$  (1  $\pm \epsilon$ )  $\cdot Z(q)$ .

*Proof.* There are |N| points on the d dimension  $\epsilon_0$ -net when  $||q||_2 \leq 1$ . From Lemma VII.4 we know that for each query point q on N, we have :

Median({HBE<sub>j</sub>.QUERY(q,  $\epsilon$ , 0.1)} $_{j=1}^{L}$ )  $\in$  (1  $\pm \epsilon$ )  $\cdot Z(q)$ 

with failure probability  $\delta/|N|$ .

Next, we could union bound all |N| points on N and obtain the following:

$$\forall \|q\|_2 \leq 1 :$$
  
Median $(\{\text{HBE}_j.\text{QUERY}(q,\epsilon,0.1)\}_{j=1}^L) \in (1 \pm \epsilon) \cdot Z(q)$ 

with probability  $1 - \delta$ .

4) From Net Points to All Points: With Lemma VII.6, we are ready to prove all query points  $||q||_2 \le 1$  can be answered approximately with high probability.

**Lemma VII.7** (*k*-Lipschitz). Given a dataset  $X \subset \mathbb{R}^d$ , and a query vector  $q \in \mathbb{R}^d$ , the *k*-Lipschitz target function w(q, x), then the summation  $Z(q) = \frac{1}{|X|} \sum_{x \in X} w(q, x)$  is *k*-Lipschitz. *Proof.* Based on the assumption that the target function w(q, x) is *k*-Lipschitz, we have:

$$\forall x \in X : |w(q_1, x) - w(q_2, x)| \le k \cdot ||q_1 - q_2||_2 \quad (1)$$

To prove Z(q) is k-Lipschitz, we have:

$$|Z(q_1) - Z(q_2)|$$
  
=  $|\frac{1}{|X|} \sum_{x \in X} w(q_1, x) - \frac{1}{|X|} \sum_{x \in X} w(q_2, x)|$   
=  $|\frac{1}{|X|} \sum_{x \in X} (w(q_1, x) - w(q_2, x))|$ 

$$\leq \frac{1}{|X|} \sum_{x \in X} (k \cdot ||q_1 - q_2||_2)$$
  
=  $k \cdot ||q_1 - q_2||_2$ 

where the first step comes from the definition of Z(q), the second step comes from merging the summation, the third step comes from Eq. (1). Therefore, we have that Z(q) is k-Lipschitz.

The following fact shows that if the elements of an array shift by a bounded value  $\epsilon$ , then the median of the array shifts by a bounded value  $3\epsilon$ .

**Fact VII.8** (folklore). Given two list of numbers such that  $|a_i - b_i| \le \epsilon$ , for all  $i \in [n]$ . Then we have

$$|\operatorname{Median}(\{a_i\}_{i\in[n]}) - \operatorname{Median}(\{b_i\}_{i\in[n]})| \leq 3\epsilon$$

**Lemma VII.9** (From net points to all points). Given  $L = O(\log((10k/\epsilon\tau)^d/\delta))$  estimators  $\{\text{HBE}_j\}_{j=1}^L$ , with probability  $1 - \delta$ , for all query points  $||q||_2 \le 1$ , there exits a point  $p \in N$  which is the closest to q, we have the median of queries from L estimators satisfies that:

 $\forall \|q\|_2 \le 1:$ Median({HBE<sub>j</sub>.QUERY(p, \epsilon, 0.1)}\_{i=1}^L) \in (1 \pm \epsilon) \cdot Z(q).

*Proof.* We define an event  $\xi$  to be the following,

 $\forall p \in N,$ Median({HBE<sub>j</sub>.QUERY(p,  $\epsilon, 0.1$ )}<sup>L</sup><sub>j=1</sub>)  $\in (1 \pm \epsilon) \cdot Z(p)$ 

Using Lemma VII.6 with  $L = \log(|N|/\delta)$ , we know that

$$\Pr[\text{ event } \xi \text{ holds }] \geq 1 - \delta$$

Using Fact VII.5, we know that

$$L = \log(|N|/\delta) = \log((10/\epsilon_0)^d/\delta) = \log((10k/\epsilon\tau)^d/\delta)$$

where the last step follows from  $\epsilon_0 \leq \epsilon \tau / k$ .

We condition the above event  $\xi$  to be held. (Then the remaining proof is not depending on any randomness, for each and for all becomes same.) For each point  $q \notin N$ , there exists a  $p \in N$  such that

$$\|p - q\|_2 \le \epsilon_0 \tag{2}$$

For each  $q \notin N$ , we quantize off-net query q to its nearest on-net query p. we know

$$|Z(q) - Z(p)| \le k \cdot ||q - p||_2 \le k\epsilon_0 \le \epsilon\tau \tag{3}$$

where the first step follows that  $Z(\cdot)$  is k-Lipschitz, the second step comes from Eq. (2) and the third step comes from  $\epsilon_0 \leq \epsilon \tau/k$ . Using the on-net query p to answer the off-net quantized query q, we have:

$$Median(\{HBE_j.QUERY(p, \epsilon, 0.1)\}_{j=1}^L) \in (1 \pm \epsilon) \cdot Z(p).$$

Using Eq. (3), we can obtain that

 $Median(\{HBE_j.QUERY(p,\epsilon,0.1)\}_{j=1}^L) \in (1 \pm \epsilon) \cdot (Z(q) \pm \epsilon \tau).$ 

Using  $\forall j \in [L] : \text{HBE}_j.\text{QUERY}(p, \epsilon, 0.1) \geq \tau$ , we have

$$Median(\{HBE_j.QUERY(p,\epsilon,0.1)\}_{j=1}^L) \in (1 \pm \epsilon)^2 Z(q).$$

We know that  $(1 - \epsilon)^2 \ge (1 - 3\epsilon)$  and  $(1 + \epsilon)^2 \le (1 + 3\epsilon)$ when  $\epsilon \in (0, 0.1)$ . Rescaling the  $\epsilon$  completes the proof.  $\Box$ 

#### B. Running Time

In this section, we provide several lemmas to prove the time complexity of each operation in our data structure. We remark that the running time complexity corresponds to potential energy consumption in practice. We hope that our guidance in theory would help improves the energy efficiency in the running of our algorithm.

1) Initialization Time:

**Lemma VII.10** (Initialize Time). Given an estimator Z of complexity C which is V-bounded and n data points, the time complexity of INITIALIZE in Algorithm 3 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot nC)$ .

*Proof.* Because in INITIALIZE operation in Algorithm 5, HBE.INITIALIZE is called for  $L = O(\log((10k/\epsilon\tau)^d/\delta))$  times to initialize L HBE data structures, and each HBE.INITIALIZE takes  $O(\epsilon^{-2}V(\tau)C\log(1/\delta) \cdot n)$  time to complete. Therefore, the overall time complexity of INITIAL-IZE operation in Algorithm 5 is  $O(\epsilon^{-2}V(\tau)C\log(1/\delta) \cdot n \cdot \log((10k/\epsilon\tau)^d/\delta))$ 

2) *Query Time:* We prove the time complexity of QUERY in Lemma VII.11.

**Lemma VII.11** (Query Time). Given an estimator Z of complexity C which is V-bounded, the time complexity of QUERY in Algorithm 5 is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta)C \cdot \log((10k/\epsilon\tau)^d/\delta))$ .

*Proof.* Because in QUERY operation in Algorithm 5, HBE.QUERY is called for  $L = O(\log((10k/\epsilon\tau)^d/\delta))$ times, and each HBE.QUERY takes  $O(\epsilon^{-2}V(\tau)C\log(1/\delta))$ time to complete. As a result, the total time of QUERY operation in Algorithm 5 is  $O(\epsilon^{-2}V((\mu)_{\tau})\log(1/\delta)C \cdot \log((10k/\epsilon\tau)^d/\delta))$ .

*3) Maintenance Time:* We provide the time complexity of INSERTX in Lemma VII.12.

**Lemma VII.12** (Insert Time). Given an estimator Z of complexity C which is V-bounded, the time complexity of INSERTX in Algorithm 5 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C \cdot \log((10k/\epsilon\tau)^d/\delta))$ .

*Proof.* Because in INSERTX operation in Algorithm 5, HBE.INSERTX is called for  $L = O(\log((10k/\epsilon\tau)^d/\delta))$  times, and each HBE.INSERTX takes  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$  time to complete. Therefore, the overall time complexity of IN-SERTX operation in Algorithm 5 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C \cdot \log((10k/\epsilon\tau)^d/\delta))$ .

We provide the time complexity of DELETEX in Lemma VII.13.

**Lemma VII.13** (Delete Time). Given an estimator Z of complexity C which is V-bounded, the time complexity of DELETEX in Algorithm 5 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C \cdot \log((10k/\epsilon\tau)^d/\delta))$ .

*Proof.* Because in DELETEX operation in Algorithm 5, HBE.DELETEX operation is called for  $L = O(\log((10k/\epsilon\tau)^d/\delta))$  times, and each HBE.DELETEX operation takes  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C)$ time to Therefore, the overall time complete. complexity of DELETEX operation operation in Algorithm 5 is  $O(\epsilon^{-2}V(\tau)\log(1/\delta) \cdot C \cdot \log((10k/\epsilon\tau)^d/\delta)).$ 

## VIII. CONCLUSION

Pairwise Summation Estimation (PSE) is an important yet challenging task in machine learning. In this paper, we present Adam-Hash: the first provable adaptive and dynamic multiresolution hashing for PSE. In an iterative process, the data set changes by a single data point per iteration, and our data structure outputs an approximation of the pairwise summation of a binary function in sub-linear time in the size of the data set. Our data structure also works for the adaptive setting where an adversary can choose a query based on previous query results. We hope our proposal would shed lights on joint innovations of data structures and machine learning.

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