# Nonlinear State Space Model Identification Using a Regularized Basis Function Expansion 

Andreas Svensson ${ }^{\star}$, Thomas B. Schön ${ }^{\star}$, Arno Solin ${ }^{\star \star}$, Simo Särkkä ${ }^{\star \star \star}$<br>* Department of Information Technology, Uppsala University, Sweden. E-mail: \{andreas.svensson, thomas.schon\} @it.uu.se<br>${ }^{* *}$ Department of Neuroscience and Biomedical Engineering, Aalto University, Finland. E-mail: arno.solin@aalto.fi<br>${ }^{\star \star \star}$ Department of Electrical Engineering and Automation, Aalto University, Finland. E-mail: simo.sarkka@aalto.fi


#### Abstract

This paper is concerned with black-box identification of nonlinear state space models. By using a basis function expansion within the state space model, we obtain a flexible structure. The model is identified using an expectation maximization approach, where the states and the parameters are updated iteratively in such a way that a maximum likelihood estimate is obtained. We use recent particle methods with sound theoretical properties to infer the states, whereas the model parameters can be updated using closed-form expressions by exploiting the fact that our model is linear in the parameters. Not to over-fit the flexible model to the data, we also propose a regularization scheme without increasing the computational burden. Importantly, this opens up for systematic use of regularization in nonlinear state space models. We conclude by evaluating our proposed approach on one simulation example and two real-data problems.


## I. Introduction

Modeling of nonlinear dynamical systems is a well-studied problem within many areas, including system identification [1,2] and time series analysis [3]. This paper is concerned with black-box identification of nonlinear state space models (SSMs), where the nonlinearities are represented using a basis function expansion. By applying the recent method [4] based on sequential Monte Carlo (SMC) and Expectation Maximization (EM), we can efficiently utilize the linear-in-its-parameters and nonlinear-in-the-state properties of our proposed model (see, e.g., [5] for discussion on such models).

The use of basis function expansions is a well-established approach within system identification, often used to identify transfer functions [6]. An early approach to use basis function expansions also for nonlinear system identification is found in [7], whereas [8, 9] are using it to identify static nonlinearities in Hammerstein and Wiener systems, respectively. The idea to combine a basis function expansion and EM to identify nonlinear SSMs dates back-to the best of our knowledgeto [10]. A recent work along the same line is [11], replacing EM with nonlinear optimization.

To avoid over-fitting to data, one remedy is to limit the number of terms in the series expansion, as discussed by [8]. Another alternative is regularization [12, 13], or equivalently, a prior on the basis function weights. The work by [14] is addressing the special case of a Gaussian process prior.

In particular, we consider identification of a nonlinear SSM

$$
\begin{align*}
x_{t+1} & =f_{x}\left(x_{t}\right)+f_{u}\left(u_{t}\right)+w_{t}  \tag{1a}\\
y_{t} & =g_{x}\left(x_{t}\right)+g_{u}\left(u_{t}\right)+e_{t} \tag{1b}
\end{align*}
$$

with $w_{t}$ and $e_{t}$ being Gaussian noise with zero mean and $\mathbb{E}\left[w_{t} w_{t}^{\top}\right]=Q, \mathbb{E}\left[e_{t} e_{t}^{\mathrm{T}}\right]=R$, and $x_{t} \in \mathbb{R}^{n_{x}}$. By making use of input-output data ( $u_{1: T}, y_{1: T}$ ) and prior knowledge of $n_{x}$, our goal is to identify a model $\mathcal{M} \triangleq\left\{f_{x}, f_{u}, g_{x}, g_{u}, Q, R\right\}$ maximizing the marginal likelihood $p\left(u_{1: T}, y_{1: T} \mid \mathcal{M}\right)$. The approach can be generalized to functions $f\left(x_{t}, u_{t}\right)$ and alike.

In practical situations, it may be too ambiguous to identify all functions in (1). It is also possible to adapt the proposed scheme to learn only parts of (1), as will be illustrated by Example IV-B.

## II. ORThogonal basis function expansions in SSMs

We restrict ourselves to consider a compact set X of $\mathbb{R}^{n_{x}}$, and assume there exists a set of orthogonal basis functions $\left\{\phi^{(k)}\right\}_{k=1}^{\infty}$ spanning the function space on X such that

$$
\begin{equation*}
f_{x}(x)=\sum_{k=1}^{\infty} \omega_{x}^{(k)} \phi^{(k)}(x) \approx \sum_{k=1}^{m} \omega_{x}^{(k)} \phi^{(k)}(x), \forall x \in \mathrm{X} \tag{2}
\end{equation*}
$$

and similar for $f_{u}(\cdot), g_{x}(\cdot)$ and $g_{u}(\cdot)$.
Example 1: The Fourier basis $\phi^{(k)}(x)=e^{\frac{i \pi k x}{L}}$ spans the space of $L^{2}$ (i.e., square-integrable) functions on $\mathrm{X}=[-L, L]$.

The truncated basis function expansion (2) suggests the following approximation of (1a)

and analogously for (1b). (Note that different basis functions $\left\{\phi^{(k)}\right\}_{k=1}^{\infty}$ can be used for $f_{x}(\cdot)$ and $f_{u}(\cdot)$, although this is not reflected in the notation.) More compactly we have the following approximation of (1)

$$
\begin{align*}
x_{t+1} & =\left[\begin{array}{ll}
A & B
\end{array}\right]\left[\begin{array}{l}
\bar{\varphi}\left(x_{t}\right) \\
\bar{\varphi}\left(u_{t}\right)
\end{array}\right]+w_{t}  \tag{4a}\\
y_{t} & =\left[\begin{array}{ll}
C & D
\end{array}\right]\left[\begin{array}{l}
\bar{\varphi}\left(x_{t}\right) \\
\bar{\varphi}\left(u_{t}\right)
\end{array}\right]+e_{t} \tag{4b}
\end{align*}
$$

which is linear in the parameters, but nonlinear in the states $x_{t}$ and the inputs $u_{t}$. This structure will be exploited when we derive the maximum likelihood (ML) estimator.

```
Algorithm 1 Conditional particle filter with ancestor sampling
Input: Conditional trajectory \(x_{1: T}[k]\).
Output: Trajectory \(x_{1: T}[k+1]\) and particle system \(\left\{x_{1: T}^{(i)}, w_{T}^{(i)}\right\}_{i=1}^{N}\).
    Draw \(x_{1}^{(i)} \sim p\left(x_{1}\right)\) for \(i=1, \ldots, N-1\).
    Set \(x_{1}^{(N)}=x_{1}[k]\).
    for \(t=1, \ldots, T\) do
        Set \(w_{t}^{(i)}=\mathcal{N}\left(g_{x}\left(x_{t}^{(i)}\right)+g_{u}\left(u_{t}\right) \mid y_{t}, R\right)\).
        Draw \(a_{t}^{(i)}\) with \(\mathbb{P}\left(a_{t}^{(i)}=j\right) \propto w_{t}^{(j)}\) for \(i=1, \ldots, N-1\).
        Draw \(x_{t+1}^{(i)} \sim \mathcal{N}\left(f_{x}\left(x_{t}^{a_{t}^{(i)}}\right)+f_{u}\left(u_{t}\right), Q\right)\) for \(i=1, \ldots, N-1\).
        Set \(x_{t+1}^{(N)}=x_{t+1}[k]\).
        Draw \(a_{t}^{(N)}\) with
        \(\mathbb{P}\left(a_{t}^{(N)}=j\right) \propto w_{t}^{(j)} \mathcal{N}\left(f_{x}\left(x_{t}^{(j)}\right)+f_{u}\left(u_{t}\right) \mid x_{t+1}^{(N)}, Q\right)\).
        Set \(x_{1: t}^{i}=x_{1: t}^{a_{t}^{i}}\) for \(i=1, \ldots, N\).
    end for
```



## III. ML Identification of SSMs

The approximate model (4) allows us to cast the original problem of identifying the model $\mathcal{M}$ as the problem of identifying a (large) number of parameters $\theta \triangleq\{A, B, C, D, Q, R\}$ in (4). This reformulation of the original nonparametric model as a parametric model opens up for use of recent tools for identification of nonlinear parametric models.

## A. Nonlinear system identification using PSAEM

To identify the parameters $\theta$ in (4), several methods can be used. We will make use of the recent development [4], relying on a combination of stochastic approximation EM [15] and a conditional particle filter with ancestor sampling (CPF-AS, [16]). The CPF-AS can be interpreted as a particle smoother formulated as a Markov chain Monte Carlo (MCMC) method, and is provided in Algorithm 1. Stochastic approximation EM is compatible with MCMC-methods [17], which allows the CPF-AS to be combined into particle stochastic approximation EM (PSAEM).

PSAEM will generate a sequence $\theta[1], \theta[2], \ldots$ converging to a stationary point of $p_{\theta}\left(u_{1: T}, y_{1: T}\right)$. Algorithmically, it starts from an arbitrary initial parameter $\theta[0]$ and a state space trajectory $x_{1: T}[0]$, and then iterates the following two steps until convergence:
(E) run CPF-AS (Algorithm 1) with $\theta[k-1]$ as model to obtain $\left\{x_{1: T}^{(i)}, w_{T}^{(i)}\right\}_{i=1}^{N}$,
(M) update the parameters $\theta[k-1] \mapsto \theta[k]$ as the maximizing argument to an auxiliary function $\mathbb{Q}_{k}(\theta)$ (detailed below).
The M-step in the PSAEM algorithm amounts to maximizing the auxiliary function

$$
\begin{equation*}
\mathbb{Q}_{k}(\theta)=\left(1-\gamma_{k}\right) \mathbb{Q}_{[k-1]}(\theta)+\gamma_{k}\left(w_{T}^{(j)} \sum_{j=1}^{N} \log p\left(x_{1: T}^{(j)}, y_{1: T} \mid \theta\right)\right) \tag{5}
\end{equation*}
$$

with $\left\{\gamma_{k}\right\}_{k \geq 1}$ being a sequence fulfilling $\gamma_{1}=1, \gamma_{k+1}<\gamma_{k}$, $\sum_{k} \gamma_{k}=\infty$ and $\sum_{k} \gamma_{k}^{2}<\infty$. Due to the structure of our problem there is a closed-form solution (see Theorem 1) available for the problem $\theta[k]=\arg \max _{\theta} \mathbb{Q}_{k}(\theta)$. Here, $\mathbb{E}_{\theta}[\cdot]$ denotes expectation under the model $\theta$.

```
Algorithm 2 ML identification of nonlinear SSM using PSAEM
    Initialize \(\theta[0]\) arbitrarily.
    Set \(x_{1: T}[0]\) arbitrarily (conditional trajectory in CPF-AS).
    for \(k \geq 1\) do
        Run Algorithm 1 with conditional trajectory \(x_{1: T}[k-1]\) and \(f_{x}(x)=\)
        \(A[k-1] \bar{\varphi}(x), f_{u}(x)=B[k-1] \bar{\varphi}(x)\), etc.
        Compute \(A[k], B[k]\), etc. according to (7).
    end for
```

Theorem 1 (Maximizing $\mathbb{Q}$ ): Consider a model on the form

$$
\begin{equation*}
\zeta_{t}=\Gamma z_{t}+v_{t} \tag{6}
\end{equation*}
$$

with $\mathbb{E}\left[v_{t}^{\top} v_{t}\right]=\Pi$ (cf. (4): $\zeta_{t}=x_{t+1}, \Gamma=\left[\begin{array}{ll}A & B\end{array}\right]$, $\left.\Pi=Q, z_{t}^{\top}=\left[\bar{\varphi}\left(x_{t}\right)^{\top} \bar{\varphi}\left(u_{t}\right)^{\mathrm{\top}}\right]\right)$. Further, $\zeta_{t}, z_{t}$ are given and $\theta \triangleq\{\Gamma, \Pi\}$ is to maximize. Then $\theta[k]=\arg \max _{\theta} \mathbb{Q}_{k}(\theta)$ is

$$
\begin{align*}
\Gamma[k] & =\Psi[k] \Sigma[k]^{-1}  \tag{7a}\\
\Pi[k] & =\Phi[k]-\Psi[k] \Sigma[k]^{-1} \Psi[k] \tag{7b}
\end{align*}
$$

where

$$
\begin{align*}
& \Phi[k]=\left(1-\gamma_{k}\right) \Phi[k-1]+\frac{1}{T} \Sigma_{t=1}^{T} \mathbb{E}_{\theta[k-1]}\left[\zeta_{t}^{\top} \zeta_{t} \mid y_{1: T}\right],  \tag{7c}\\
& \Psi[k]=\left(1-\gamma_{k}\right) \Psi[k-1]+\frac{1}{T} \Sigma_{t=1}^{T} \mathbb{E}_{\theta[k-1]}\left[\zeta_{t}^{\top} z_{t} \mid y_{1: T}\right],  \tag{7d}\\
& \Sigma[k]=\left(1-\gamma_{k}\right) \Sigma[k-1]+\frac{1}{T} \Sigma_{t=1}^{T} \mathbb{E}_{\theta[k-1]}\left[z_{t}^{\top} z_{t} \mid y_{1: T}\right] . \tag{7e}
\end{align*}
$$

In case $\zeta_{t}$ or $z_{t}$ are given by a particle system $\left\{z_{1: T}^{(i)}, w_{T}^{(i)}\right\}_{i=1}^{N}$, the expectations are computed as

$$
\begin{align*}
& \mathbb{E}_{\theta[k-1]}\left[z_{t}^{\top} z_{t} \mid y_{1: T}\right]=\int z_{t}^{\top} z_{t} p_{\theta[k-1]}\left(z_{1: T} \mid y_{1: T}\right) d z_{t} \\
& \approx \sum_{i=1}^{N} w_{T}^{(i)} z_{t}^{(i)^{\top}} z_{t}^{(i)} . \tag{8}
\end{align*}
$$

Proof of Theorem 1: The proof is omitted due to space limitation, but follows from [4, 15, 18].

The identification procedure is summarized in Algorithm 2.

## B. Regularization

The model (4) is very flexible. When it comes to identification this flexibility can cause problems, such as overly complex models over-fitted to the data or the lack of unique solutions.
Regularization is one remedy to avoid such problems [12]. Intuitively, regularization amounts to solving the problem $\arg \max _{\theta} p_{\theta}\left(u_{1: T}, y_{1: T}\right)$ under the additional constraint of 'keeping $\theta$ as small as possible'. We will extend our approach to incorporate $L^{2}$ regularization, or equivalently, assigning Gaussian priors to the weights $\omega$.

By assigning a zero-mean Gaussian prior with precision matrix $P$ to the basis function weights, that is, $p\left(\left[\omega^{(1)} \cdots \omega^{(m)}\right]\right) \sim \mathcal{N}\left(0, P^{-1}\right)$, the developments in [19] can be used to derive the alternative version of (7a)-(7b) for regularized identification

$$
\begin{align*}
\Gamma[k] & =\Psi[k]\left(\Sigma[k]+\frac{1}{T} P\right)^{-1},  \tag{9a}\\
\Pi[k] & =\Phi[k]-\Psi[k]\left(\Sigma[k]+\frac{1}{T} P\right)^{-1} \Psi[k] . \tag{9b}
\end{align*}
$$

It is clear that the use of a prior with infinite variance $(P=0)$ retrieves the non-regularized identification algorithm.


Fig. 1. The first example, with three different settings: $m=6$ basis functions (top), $m=100$ basis functions (middle) and $m=100$ basis functions with regularization (bottom). The model with $m=6$ is not flexible enough to describe the 'steep' part of $f$, but results in a sensible, albeit not perfect, model. The second model is very flexible with its 101 parameters, and becomes a typical case of over-fitting to the data points (cf. the distribution of the data at the very bottom), causing numerical problems and a useless model. The regularization in the third case is a clear remedy to this problem, still maintaining the high flexibility of the model.

A natural question is indeed how to choose the prior precision $P$. As stated by [12], the optimal choice (in terms of mean square error) is $P_{\mathrm{opt}}^{-1}=\mathbb{E}\left[\left[\omega^{(1)} \cdots \omega^{(m)}\right]^{\top}\left[\omega^{(1)} \cdots \omega^{(m)}\right]\right]$, if we think of $\omega^{(1)}, \ldots, \omega^{(m)}$ as being random variables. As an example, with the natural assumption of $f_{x}(\cdot)$ being smooth, the diagonal elements of $P$ should be larger with increasing order of the Fourier basis functions. The special case of assuming $f_{x}(\cdot)$ to be a sample from a Gaussian process is addressed by [14].

Other regularization schemes, such as $L^{1}$, are possible but will not result in closed-form expressions such as (9).

## C. Computational aspects

Let $N$ denote the number of particles in the CPF-AS, $m$ the numer of terms used in the basis function expansion, $T$ the number of data points and $K$ the numer of iterations used in Algorithm 2. The computational load is then $\mathcal{O}(m T K N)+$ $\mathcal{O}\left(m^{3}\right)$. In practice, $N$ and $m$ can be chosen fairly small (e.g., $N=5$ and $m=10$ for a 1D model).

## D. Convergence

The convergence properties of PSAEM are not yet fully understood, but it can under certain assumptions be shown to converge to a stationary point of $p_{\theta}\left(u_{1: T}, y_{1: T}\right)$ by [17, Theorem 1]. We have not experienced practical problems with the convergence, although it is sensitive to initialization when the dimension of $\theta$ is large (e.g., 1000 parameters).

TABLE I
Results for the Hammerstein-Wiener Benchmark

| Experiment with $T=2000$ |  |
| ---: | :--- |
| Mean simulation error | 0.0005 V |
| Standard deviation of simulation error | 0.020 V |
| RMS simulation error | 0.020 V |
| Run time | 13 min |

## IV. NUMERICAL EXAMPLES

We demonstrate our proposed method on a series of numerical examples. The source code is available via the web site of the first author.

## A. Simulated example

As a first simple numerical example, consider an autonomous system (i.e., no $u_{t}$ ) defined by

$$
\begin{equation*}
x_{t+1}=\frac{-10 x_{t}}{1+3 x_{t}^{2}}+w_{t}, \quad y_{t}=x_{t}+e_{t} \tag{10}
\end{equation*}
$$

where $w_{t} \sim \mathcal{N}(0,0.1)$ and $e_{t} \sim \mathcal{N}(0,0.5)$. We identify $f(\cdot)$ and $Q$ from $T=1000$ simulated measurements $y_{1: T}$, while assuming $g(\cdot)$ and $R$ to be known. We consider three different settings with $m=6$ basis functions, $m=100$ basis functions and $m=100$ basis functions with regularization, respectively, all using the Fourier basis. To encode the a priori assumption of $f(\cdot)$ being a smooth function, we choose the regularization as a Gaussian prior of $w_{k}$ with standard deviation inversely proportional to $k$. The results are shown in Figure 1, where the over-fitting problem for $m=100$, and how regularization helps, is apparent.

## B. Hammerstein-Wiener benchmark

To illustrate how to adapt our approach to problems with a given structure, we apply it to the real-data HammersteinWiener system identification benchmark by [20]. We will use a subset with 2000 data points from the original data set for estimation. Based on the domain knowledge provided by [20] (two third order linear systems in a cascade with a static nonlinearity between), we identify a model with the structure

$$
\begin{align*}
{\left[\begin{array}{l}
x_{t+1}^{1} \\
x_{t+1}^{2} \\
x_{t+1}^{3}
\end{array}\right] } & =A_{1}\left[\begin{array}{l}
x_{t}^{1} \\
x_{t}^{2} \\
x_{t}^{3}
\end{array}\right]+B u_{t},  \tag{11a}\\
{\left[\begin{array}{l}
x_{t+1}^{4} \\
x_{t+1}^{5} \\
x_{t+1}^{6}
\end{array}\right] } & =A_{2}\left[\begin{array}{l}
x_{t}^{4} \\
x_{t}^{5} \\
x_{t}^{6}
\end{array}\right]+\left[\begin{array}{c}
\Sigma_{k} \omega^{(k)} \phi^{(k)}\left(x_{t}^{3}\right) \\
0 \\
0
\end{array}\right],  \tag{11b}\\
y_{t} & =C\left[x_{t}^{4} x_{t}^{5} x_{t}^{6}\right], \tag{11c}
\end{align*}
$$

where the superindex on the state denotes a particular component of the state vector. Furthermore, we have omitted all noise terms for notational brevity. There is only one nonlinear function, but the linear parts can be seen as the special case where $\left\{\phi^{(k)}(x)\right\}_{k=1}^{m}=\{x\}$, which can directly be incorporated into the presented framework.

We present the results in Table I (all metrics are with respect to the evaluation data from the original data set). We refer to [21] for a thorough evaluation of alternative methods.


Fig. 2. Modeling a motorized camera with 6 inputs and 2 outputs.

## C. Nonlinear MIMO model

We now consider black-box identification of a nonlinear MIMO system, used as an example by the System Identification Toolbox [22]. The data consists of 188 measurements from a motorized camera. The input vector contains 6 variables: 3 translational velocity components and 3 rotational velocity components. The output vector contains two variables: the position in the image of a fixed (in 3D space) point.

We model this using a two-dimensional state space and a known linear measurement function $y_{t}=I_{2} x_{t}+e_{t}$, and a simple regularization, with a prior precision proportional to the order of the basis functions. The results are displayed in Figure 2, where it can be seen that the simulated output follows the true output very closely. The resulting RMSE is 0.52 pixels, whereas the best results reported by [22] for a nonlinear ARX (using a wavelet network) is an RMSE of 2.22 pixels, and 2.13 for a rather complex Hammerstein model.

## V. CONCLUSIONS AND FURTHER WORK

We have presented a model and algorithm for black box identification of nonlinear state-space models (1). Regularization is a key for its use in practice, providing a systematic approach to tune the model complexity.

Within linear system identification it has recently [12, 13] been realized that the use of regularization with carefully selected Gaussian process priors can enforce system properties such as smoothness and stability. Our model construction opens up for similar developments also for nonlinear systems and a more systematic approach to designing the regularization priors (e.g, akin to the Gaussian process case [14]) constitutes future work. Besides this, some other topics for further investigation would be the use of alternative basis functions (e.g., wavelets and Legendre functions), the initialization of Algorithm 2, and to replace the particle filter with a smoother based on basis function expansions akin to [23].

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