Distributed Big-Data Optimization via Block Communications

Ivano Notarnicola*, Ying Sun*, Gesualdo Scutari, Giuseppe Notarstefano

Abstract

We study distributed multi-agent large-scale optimization problems, wherein the cost function is composed of a smooth possibly nonconvex sum-utility plus a DC (Difference-of-Convex) regularizer. We consider the scenario where the dimension of the optimization variables is so large that optimizing and/or transmitting the entire set of variables could cause unaffordable computation and communication overhead. To address this issue, we propose the first distributed algorithm whereby agents optimize and communicate only a portion of their local variables. The scheme hinges on successive convex approximation (SCA) to handle the nonconvexity of the objective function, coupled with a novel block-signal tracking scheme, aiming at locally estimating the average of the agents' gradients. Asymptotic convergence to stationary solutions of the nonconvex problem is established. Numerical results on a sparse regression problem show the effectiveness of the proposed algorithm and the impact of the block size on its practical convergence speed and communication cost.

I. INTRODUCTION

We consider a multi-agent system composed of N agents that cooperatively aim at solving the following (possibly nonconvex) optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & U(\mathbf{x}) \triangleq \sum_{i=1}^{N} f_i(\mathbf{x}) + \sum_{\ell=1}^{B} \underbrace{r_{\ell}^+(\mathbf{x}_{\ell}) - r_{\ell}^-(\mathbf{x}_{\ell})}_{r_{\ell}(\mathbf{x}_{\ell})} \\ \\ \text{subject to} & \mathbf{x} = [\mathbf{x}_1^\top, \dots, \mathbf{x}_B^\top]^\top \\ & \mathbf{x}_{\ell} \in \mathcal{K}_{\ell}, \quad \forall \ell \in \{1, \dots, B\}, \end{array}$$

$$(P)$$

where $\mathbf{x} \in \mathbb{R}^{dB}$ is the vector of the optimization variables, partitioned in *B* blocks, whose ℓ -th block is denoted by $\mathbf{x}_{\ell} \in \mathbb{R}^{d}$; $f_{i} : \mathbb{R}^{dB} \to \mathbb{R}$ is a smooth possibly nonconvex cost function of agent i; $r_{\ell} : \mathbb{R}^{d} \to \mathbb{R}$, $\ell \in \{1, \ldots, B\}$, is a difference of convex (DC) function commonly known by all the agents; and \mathcal{K}_{ℓ} , $\ell \in \{1, \ldots, B\}$, is a closed convex set. Function r_{ℓ} usually plays the role of a regularizer, used to promote some favorable structure on the solution \mathbf{x} ,

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such as sparsity. The DC structure of r_{ℓ} is motivated by the need of capturing in a unified formulation both convex and nonconvex regularizers, the latter being shown to achieve superior performance than their convex counterparts [1]. Problem (P) is of broad interest and models a wide range of applications including network resource allocation, target localization, as well as statistical learning problems.

Our goal is to design a distributed algorithm solving large-scale instances of (P). These problems, also referred to as big-data problems, pose the following two challenges: (i) optimizing the objective function, or even just computing the gradient with respect to all the variables, can be too costly; (ii) broadcasting over the network at each iteration all agents' local variables would incur in an unaffordable communication overhead. We are not aware of any work in the literature that can address both challenges (i) and (ii) for problem (P). In fact, as discussed next, the existing distributed algorithms either call for the optimization and transmission of the entire vector \mathbf{x} per iteration (or auxiliary variables of the same size of \mathbf{x}) or impose restrictive structures on the objective function to work.

There is a vast literature of distributed algorithms for both convex [2]–[10] and nonconvex problems [11]–[16]. Although substantially different, these methods are all based on two main steps, namely: a local optimization and then a communication step of the *entire* vector \mathbf{x} (or some related variables of the same size, e.g., multipliers). They thus fail to address challenges (i) and (ii). On the other hand, (block) coordinate descent methods [4], [17]–[19] and parallel algorithms [20]–[23] have been shown to be quite effective in handling large-scale problems by optimizing one block of the variables per time. These algorithms, however, are not readily implementable in the aforementioned distributed setting, because they assume that either all agents know the whole sum-utility or that, at each iteration, each agent has access to the current value of the other agents' variables. While these assumptions are naturally satisfied in a share-memory system (e.g., data centric architecture) or complete (graph) networks, if enforced for problem (P), they would call for an heavy message passing among the agents. We are aware of only a few distributed schemes operating on block variables, namely: [24]–[26]. They however require a certain degree of graph separability on the sum-utility function, meaning that each agent's function f_i can depend only on the variables of that agent and its neighbors, which makes them not applicable to problem (P).

In this work, we propose BLOCK-SONATA, the first distributed algorithm for the general class of problem (P) that is able to address both challenges (i) and (ii): each agent iteratively optimizes and transmits only one block of its local variables. More specifically, BLOCK-SONATA consists of two steps, namely: 1) a local optimization step wherein agents locally solve a covexification of (P), with respect to a chosen block of their local variables; and 2) a *blockwise* consensus step, aiming at forcing an agreement among the agents' local copies. Moreover, a novel blockwise signal tracking scheme is also employed to dynamically estimate the gradient of the sum-utility function, using only local information via block-communications. Agents select the blocks to optimize/transmit in an uncoordinated fashion. Asymptotic convergence is established under mild assumptions. Compared to our recent proposal [27], BLOCK-SONATA is computationally more efficient, since it does not require at each iteration the computation of the entire gradient of the functions f_i .

II. PROBLEM SETUP

We study problem (P) under the following assumptions.

Assumption 2.1 (On Problem (P)):

- (i) $\mathcal{K}_{\ell} \neq \emptyset$ is closed and convex;
- (ii) $f_i : \mathbb{R}^{dB} \to \mathbb{R}$ is C^1 on (an open set containing) \mathcal{K} ;
- (iii) ∇f_i is L_i -Lipschitz continuous and bounded on \mathcal{K} ;
- (iv) $r_{\ell}^+ : \mathbb{R}^d \to \mathbb{R}$ is convex (possibly nonsmooth) on \mathcal{K} , with bounded subgradients on \mathcal{K} ; and $r_{\ell}^- : \mathbb{R}^d \to \mathbb{R}$ is convex on \mathcal{K} , with Lipschitz continuous bounded gradient on \mathcal{K} ;
- (v) U is coercive on \mathcal{K} , i.e., $\lim_{\mathbf{x}\in\mathcal{K},\|\mathbf{x}\|\to\infty} U(\mathbf{x}) = \infty$.

Assumption 2.1 is standard and can be easily satisfied in practice; see, e.g., [23]. Here we only remark that both the local cost f_i and the common regularizer $\sum_{\ell=1}^{B} r_{\ell}$ need not be convex.

On the communication network: The communication among the agents is modeled by a fixed, directed graph $\mathcal{G} = (\{1, \ldots, N\}, \mathcal{E})$, where $\mathcal{E} \subseteq \{1, \ldots, N\} \times \{1, \ldots, N\}$ is the set of edges. There is an edge $(i, j) \in \mathcal{E}$ if agent i can send a message to agent j. We denote by \mathcal{N}_i the set of *in-neighbors* of node i in \mathcal{G} , including itself, i.e., $\mathcal{N}_i \triangleq \{j \in \{1, \ldots, N\} \mid (j, i) \in \mathcal{E}\} \cup \{i\}$. To let information propagate over the network, we make the following assumption.

Assumption 2.2 (Network connectivity): The graph G is strongly connected.

The above setting and problem are quite general and model many applications of practical interest. An example in the context of sparse signal estimation is briefly outlined next.

Sparse regression: Consider the problem of estimating a sparse signal \mathbf{x}_0 from linear measurements $\{\mathbf{b}_i\}_{i=1}^N$, where $\mathbf{b}_i = \mathbf{D}_i \mathbf{x}_0 + \mathbf{n}_i$ with \mathbf{n}_i being the measurement noise at agent *i*'s side. The problem can be formulated as

$$\underset{\mathbf{x}\in\mathcal{K}}{\text{minimize}} \quad \sum_{i=1}^{N} \|\mathbf{b}_{i} - \mathbf{D}_{i}\mathbf{x}\|^{2} + R(\mathbf{x}),$$
(1)

where $R : \mathbb{R}^{dB} \to \mathbb{R}$ is a sparsity-promoting regularizer having the structure $R(\mathbf{x}) \triangleq \lambda \cdot \sum_{k=1}^{dB} r(x_k)$ [cf. (P)], with $\lambda > 0$. The DC structure of R is motivated by the fact that both convex regularizers (e.g., ℓ_1 , ℓ_2 , and elastic net) and the widely used nonconvex regularizers (e.g., SCAD, Log, Exp, ℓ_p norm for 0) can be written as [1]

$$r(x) \triangleq \underbrace{\eta(\theta) \cdot |x|}_{r^+(x)} - \underbrace{(\eta(\theta) \cdot |x| - r(x))}_{r^-(x)},\tag{2}$$

where $r^- : \mathbb{R} \to \mathbb{R}$ is a convex function with Lipschitz continuous derivative. Problem (1) is clearly an instance of Problem (P).

III. ALGORITHMIC DESIGN

Before describing the proposed distributed algorithm, we introduce a block-wise dynamic average consensus scheme, whereby the agents aim at cooperatively tracking the average of a time-varying signal via block-wise communications.

A. Average signal tracking via block communications

We consider the problem of tracking the average of a signal over a graph \mathcal{G} satisfying Assumption 2.2. Each agent *i* can evaluate locally a time-varying signal $\{\mathbf{u}_i^t\}_{t\in\mathbb{N}}$, and all agents aim at tracking the average signal $\bar{\mathbf{u}}^t \triangleq \frac{1}{N} \sum_{i=1}^{N} \mathbf{u}_i^t$ by exchanging information over the network. We assume that the cost of acquiring \mathbf{u}_i^t is non-negligible, e.g., \mathbf{u}_i^t can be the gradient of a function with respect to a large number of variables. Distributed tracking has been studied in [16]. However, such a scheme requires at each iteration the acquisition of the *entire* signal \mathbf{u}_i^t as well as the communication of a vector having the same size of \mathbf{u}_i^t , which is too costly. To cope with the curse of dimensionality, we develop next a signal tracking scheme that operates at the level of the blocks of signals \mathbf{u}_i^t while enabling block-communications.

Each agent *i* maintains a local variable $\mathbf{x}_{(i)}^t$, whose ℓ -th block is denoted by $\mathbf{x}_{(i,\ell)}^t$, with $\ell \in \{1, \ldots, B\}$. At iteration *t*, each agent *i* picks a block-index, say ℓ_i^t , and broadcasts the block $\mathbf{x}_{(i,\ell_i^t)}^t$ to its neighbors. Based on the information (blocks) received from its neighbors and the acquired block of the local signal \mathbf{u}_i^t , agent *i* updates block-wise its entire vector $\mathbf{x}_{(i)}^t$ (according to the mechanism that we will introduce shortly). Since there is no coordination among the agents, they will likely transmit blocks associated with different indices. This implies that blocks with different index will "travel" on different communication graphs, which in general do not coincide with \mathcal{G} : agent $j(\neq i)$ is an in-neighbor of *i* if $j \in \mathcal{N}_i$ and agent *j* sends block ℓ to *i* at iteration *t*. This naturally suggests the adoption of block-dependent communication graphs, one per block ℓ . Specifically, $\mathcal{G}_\ell^t \triangleq (\{1, \ldots, N\}, \mathcal{E}_\ell^t)$, which is a *time-varying* subgraph of \mathcal{G} associated to block ℓ at iteration *t*, whose edge set is defined as $\mathcal{E}_\ell^t \triangleq \{(j, i) \in \mathcal{E} \mid j \in \mathcal{N}_{i,\ell}^t, i \in \{1, \ldots, N\}\}$, where $\mathcal{N}_{i,\ell}^t$ is the in-neighborhood of agent *i* associated with the block-index ℓ , $\mathcal{N}_{i,\ell}^t \triangleq \{j \in \mathcal{N}_i \mid \ell_j^t = \ell\} \cup \{i\} \subseteq \mathcal{N}_i$.

Using block-dependent graphs one can solve the tracking problem block-wise. Therefore, in the following, we focus only on block ℓ , without loss of generality. The task reduces to developing a tracking algorithm over the time-varying directed graph $\{\mathcal{G}_{\ell}^t\}_{t\in\mathbb{N}}$. Building on [16], we propose the following adapt-then-combine scheme:

$$\mathbf{v}_{(i,\ell)}^{t} = \mathbf{x}_{(i,\ell)}^{t} + \frac{1}{\phi_{(i,\ell)}^{t}} (\mathbf{u}_{i,\ell}^{t+1} - \mathbf{u}_{i,\ell}^{t})$$

$$\phi_{(i,\ell)}^{t+1} = \sum_{j \in \mathcal{N}_{i,\ell}^{t}} a_{ij\ell}^{t} \phi_{(j,\ell)}^{t}, \quad \phi_{(i,\ell)}^{0} = 1, \quad \ell \in \{1, \dots, B\},$$

$$\mathbf{x}_{(i,\ell)}^{t+1} = \frac{1}{\phi_{(i,\ell)}^{t+1}} \sum_{j \in \mathcal{N}_{i,\ell}^{t}} a_{ij\ell}^{t} \phi_{(j,\ell)}^{t} \mathbf{v}_{(j,\ell)}^{t},$$
(3)

where $\{a_{ij\ell}^t\}_{ij}$ is a set of weights that need to be properly chosen. Collecting these weights in a matrix $\mathbf{A}_{\ell}^t \triangleq [a_{ij\ell}^t]_{ij}$, we make the following standard assumptions on \mathbf{A}_{ℓ}^t .

Assumption 3.1 (On the Weighting Matrix \mathbf{A}_{ℓ}^{t}): For all $\ell \in \{1, \ldots, B\}$ and t > 0, matrix \mathbf{A}_{ℓ}^{t} satisfies the following conditions:

- (i) $a_{ii\ell}^t \ge \vartheta > 0$, for all $i \in \{1, \ldots, N\}$;
- (ii) $a_{ij\ell}^t \ge \vartheta > 0$, for all $(j, i) \in \mathcal{E}_{\ell}^t$;
- (iii) \mathbf{A}_{ℓ}^{t} is column stochastic, i.e., $\mathbf{1}^{\top}\mathbf{A}_{\ell}^{t} = \mathbf{1}^{\top}$.

Roughly speaking, the block-tracking scheme in (3), can be interpreted as follows: each agent first updates its local estimate towards the current signal $\mathbf{u}_{i,\ell}^{t+1}$, and then averages it with the local updates of its neighbors. The scalar variable $\phi_{(i,\ell)}$ is introduced to obtain a convex combination of the received $\mathbf{v}_{(i,\ell)}$'s through the equivalent weights $(a_{ij\ell}^t \phi_{(j,\ell)}^t)/\phi_{(i,\ell)}^{t+1}$ (recall that, by Assumption 3.1, \mathbf{A}_{ℓ}^t is column stochastic, but in general is not row stochastic).

While the tracking scheme (3) unlocks block-communications, it still requires, at each iteration, the acquisition of the entire signal \mathbf{u}_i^t . To cope with this issue, we propose to replace \mathbf{u}_i^t with a surrogate local variable, denoted by $\hat{\mathbf{u}}_i^t$, initialized as $\hat{\mathbf{u}}_i^0 = \mathbf{u}_i^0$. At iteration t, agent i acquires only a block of signal \mathbf{u}_i^t , say block ℓ_i^t for notation simplicity, and updates $\hat{\mathbf{u}}_i^t$ as

$$\widehat{\mathbf{u}}_{i,\ell}^{t} = \begin{cases} \mathbf{u}_{i,\ell}^{t}, & \text{if } \ell = \ell_{i}^{t}, \\ \widehat{\mathbf{u}}_{i,\ell}^{t-1}, & \text{if } \ell \neq \ell_{i}^{t}, \end{cases}$$
(4)

where $\mathbf{u}_{i,\ell}^t$ [resp. $\hat{\mathbf{u}}_{i,\ell}^t$] denotes the ℓ -th block of \mathbf{u}_i^t [resp. $\hat{\mathbf{u}}_i^t$]. That is, vector $\hat{\mathbf{u}}_i^t$ collects agent *i*'s most recent information on \mathbf{u}_i^t .

To summarize, the proposed block-tracking scheme reads as (3), where $\mathbf{u}_{i,\ell}^t$ [resp. $\mathbf{u}_{i,\ell}^{t+1}$] is replaced by $\hat{\mathbf{u}}_{i,\ell}^t$ [resp. $\hat{\mathbf{u}}_{i,\ell}^{t+1}$], defined in (4). To ensure convergence–i.e., $\lim_{t\to\infty} \|\mathbf{x}_{(i,\ell)}^t - \hat{\mathbf{u}}_{\ell}^t\| = 0$, for all ℓ -we need the following assumptions on the connectivity of $\{\mathcal{G}_{\ell}^t\}_{t\in\mathbb{N}}$, which is widely used in the literature of push-sum-like algorithms.

Assumption 3.2: For all $\ell \in \{1, \ldots, B\}$, there exists a finite integer T > 0 such that the graph sequence $\{\mathcal{G}_{\ell}^t\}_{t \in \mathbb{N}}$ is *T*-strongly connected, i.e., the union graph $(\{1, \ldots, N\}, \bigcup_{s=t}^{t+T-1} \mathcal{E}_{\ell}^s)$ is strongly connected, for all t > 0. Since each digraph \mathcal{G}_{ℓ}^t is induced by the adopted block-selection rule, its connectivity clearly depends on it. A key question, addressed next, is then: how to design, in a distributed and uncoordinated way, agents' block-selection rules and \mathbf{A}_{ℓ}^t that fulfill Assumption 3.1 and 3.2?

By the definition of \mathcal{G}_{ℓ}^{t} , all the edges in the underlying graph \mathcal{G} leaving node *i* will be also edges of \mathcal{G}_{ℓ}^{t} if agent *i* sends block ℓ at time *t*. Since \mathcal{G} is strongly connected (cf. Assumption 2.2), \mathcal{G}_{ℓ}^{t} is *T*-strongly connected if, starting from any time t > 0, all agents send block ℓ within *T* iterations, which translates in the following essentially cyclic block-selection rule.

Assumption 3.3 (Block-selection Rule): For each agent $i \in \{1, ..., N\}$, there exists a (finite) constant $T_i > 0$ such that $\bigcup_{s=0}^{T_i-1} \{\ell_i^{t+s}\} = \{1, ..., B\}$, for all $t \ge 0$.

Note that the above rule does not impose any coordination among the agents: at each iteration, different agents may update different blocks. It is not difficult to show that, under Assumptions 2.2 and 3.3, there exits a $0 < T \leq \max_{i \in \{1,...,N\}} T_i$, such that $\bigcup_{s=0}^{T-1} \mathcal{G}_{\ell}^{t+s}$, $\ell \in \{1,...,B\}$, is strongly connected, for all $t \geq 0$. We show next how agents can *locally* build a matrix \mathbf{A}_{ℓ}^t satisfying Assumption 3.1. Observe that at iteration t,

We show next how agents can *locally* build a matrix \mathbf{A}_{ℓ}^{t} satisfying Assumption 3.1. Observe that at iteration t, if agent j selects block ℓ , it sends $\mathbf{v}_{(j,\ell)}^{t}$ to any agent i that is its out-neighbor; or send it to no one, otherwise. In addition, $a_{jj\ell}^{t}$ must be nonzero by Assumption 3.1. Consequently, the j-th column of \mathbf{A}_{ℓ}^{t} , denoted by $\mathbf{A}_{\ell}^{t}(:,j)$, can only have the following two possible sparsity patterns: (i) all $a_{ij\ell}^{t}$, with $i \in \{\{1,\ldots,B\}: (j,i) \in \mathcal{E}\}$, is nonzero if $\ell_{j}^{t} = \ell$; (ii) only $a_{jj\ell}^{t}$ is nonzero if $\ell_{j}^{t} \neq \ell$. To meet the requirement that \mathbf{A}_{ℓ}^{t} is column stochastic, agent j thus either select a stochastic vector $\mathbf{A}_{\ell}^{t}(:,j)$ matching the sparsity pattern described in case (i), if $\ell_{j}^{t} = \ell$; or set $\mathbf{A}_{\ell}^{t}(:,j)$ to

be the *j*-th vector of the canonical basis, if $\ell_j^t \neq \ell$. It is not difficult to check that such weights can be constructed locally by each agent, with no coordination with the others.

We conclude this section, noting that the proposed block-trackig scheme can be used also to solve the average consensus problem wherein agents aim to estimate the average of their initial estimates, i.e., $(1/N) \sum_{i=1}^{N} \mathbf{x}_{(i)}^{0}$. Specifically, by reinterpreting the consensus problem as tracking of the average of the constant signal $\mathbf{x}^{0} \triangleq [\mathbf{x}_{(1)}^{0\top}, \dots, \mathbf{x}_{(N)}^{0\top}]^{\top}$, it is enough to set $\mathbf{u}_{i}^{t} \equiv \mathbf{x}^{0}$, $t \ge 0$, and absorb the **v**-variable, which leads to the following *block-consensus* algorithm:

$$\phi_{(i,\ell)}^{t+1} = \sum_{j \in \mathcal{N}_{i,\ell}^t} a_{ij\ell}^t \phi_{(j,\ell)}^t, \\
\mathbf{x}_{(i,\ell)}^{t+1} = \frac{1}{\phi_{(i,\ell)}^{t+1}} \sum_{j \in \mathcal{N}_{i,\ell}^t} a_{ij\ell}^t \phi_{(j,\ell)}^t \mathbf{x}_{(j,\ell)}^t, \qquad \forall \ell \in \{1,\dots,B\}.$$
(5)

B. BLOCK-SONATA: A constructive approach

We are now in the position to introduce our algorithmic framework. Observe that what couples the functions f_i in Problem (P) is the common vector variable **x**. To decouple the problem a natural step is then introducing for each agent *i* a local copy $\mathbf{x}_{(i)}$ of **x**. Yet, agent *i* faces the following challenges: (i) the dimension of $\mathbf{x}_{(i)}$ is large; (ii) f_i and $-r_{\ell}^-$ are nonconvex; and (iii) $\sum_{j \neq i} f_j$ is unknown. To cope with these issues, we introduce BLOCK-SONATA (cf. Algorithm 1), an iterative scheme leveraging SCA techniques, coupled with a parallel blockwise consensus/tracking step based on (3) and (5), as detailed next.

Local optimization: At iteration t, agent i selects and optimizes a block of $\mathbf{x}_{(i)}^t$, say ℓ_i^t [this addresses challenge (i)]. To deal with the nonconvexity of f_i and $-r_{\ell_i^t}^-$ [challenge (ii)], we approximate f_i with a strongly convex surrogate \tilde{f}_{i,ℓ_i^t} and $-r_{\ell_i^t}^-$ by its linearization at $\mathbf{x}_{(i,\ell_i^t)}^t$. The unknown term $\sum_{j\neq i} f_j$ is replaced by a linear function whose coefficient $\tilde{\pi}_{(i,\ell_i^t)}^t$ aims to track $\sum_{j\neq i} \nabla f_{j,\ell_i^t}(\mathbf{x}_{(i)}^t)$ [challenge (iii)]. The resulting problem (7) is thus a strongly convex approximation of problem (P) and admits a unique solution $\tilde{\mathbf{x}}_{(i,\ell_i^t)}^t$. Agent *i* then updates its local copy $\mathbf{x}_{(i,\ell_i^t)}^t$ along direction $\tilde{\mathbf{x}}_{(i,\ell_i^t)}^t - \mathbf{x}_{(i,\ell_i^t)}^t$, with step-size γ^t , see (8). Note that agent *i* does not optimize blocks $\ell \neq \ell_i^t$, hence we let $\mathbf{v}_{(i,\ell)}^t = \mathbf{x}_{(i,\ell)}^t$, $\forall \ell \neq \ell_i^t$. Agent *i* then broadcasts $\mathbf{v}_{(i,\ell_i^t)}^t$ to its neighbors.

Blockwise consensus/gradient tracking: To force consensus on $\mathbf{x}_{(i)}^t$, agent *i* update in parallel all its blocks $\mathbf{x}_{(i,\ell)}^t$, $\ell \in \{1, \ldots, B\}$, based on the received variables $\mathbf{v}_{(j,\ell_j^t)}^t$. Leveraging the block consensus scheme (5), the aforementioned updates read (9)–(10).

Finally, we need to introduce the update of $\widetilde{\pi}_{(i,\ell)}^t$ so that $\lim_{t\to\infty} \|\widetilde{\pi}_{(i,\ell)}^t - \sum_{j\neq i} \nabla_\ell f_j(\mathbf{x}_{(i)}^t)\| = 0$. To this end, we rewrite $\sum_{j\neq i} \nabla_\ell f_j(\mathbf{x}_{(i)}^t)$ as

$$\sum_{j \neq i} \nabla_{\ell} f_j(\mathbf{x}_{(i)}^t) = N \cdot \underbrace{\frac{1}{N} \sum_{j=1}^N \nabla_{\ell} f_j(\mathbf{x}_{(i)}^t)}_{\overline{\nabla_{\ell} f}(\mathbf{x}_{(i)}^t)} - \nabla_{\ell} f_i(\mathbf{x}_{(i)}^t)}$$

Since $\nabla_{\ell} f_i(\mathbf{x}_{(i)}^t)$ can be evaluated locally by agent *i*, the task boils down to estimate the average gradient $\overline{\nabla_{\ell} f}(\mathbf{x}_{(i)}^t)$, $\forall \ell \in \{1, \dots, B\}$. We can then readily invoke the blockwise tracking scheme (3) with ℓ_i^{t+1} selected according to the essentially cyclic rule (Assumption 3.3) and $\mathbf{u}_i^t \triangleq \nabla f_i(\mathbf{x}_{(i)}^t)$, leading to the updates (11)-(12).

Algorithm 1: BLOCK-SONATA

Set t = 0, $\phi_{(i)}^0 = \mathbf{1}$, $\widehat{\mathbf{g}}_i^0 = \mathbf{y}_{(i)}^0 = \nabla f_i(\mathbf{x}_{(i)}^0)$, $\ell_i^0 \in \{1, \dots, B\}$.

Local Optimization:

$$\widetilde{\boldsymbol{\pi}}_{(i,\ell_i^t)}^t = N \cdot \mathbf{y}_{(i,\ell_i^t)}^t - \nabla_{\ell_i^t} f_i(\mathbf{x}_{(i)}^t), \tag{6}$$

$$\widetilde{\mathbf{x}}_{(i,\ell_i^t)}^t \triangleq \underset{\mathbf{x}_{\ell_i^t} \in \mathcal{K}_{\ell_i^t}}{\operatorname{argmin}} r_{\ell_i^t}^+(\mathbf{x}_{\ell_i^t}) + \widetilde{f}_{i,\ell_i^t}(\mathbf{x}_{\ell_i^t};\mathbf{x}_{(i)}^t) + (\widetilde{\boldsymbol{\pi}}_{(i,\ell_i^t)}^t - \nabla r_{\ell_i^t}^-(\mathbf{x}_{(i,\ell_i^t)}^t))^\top (\mathbf{x}_{\ell_i^t} - \mathbf{x}_{(i,\ell_i^t)}^t),$$
(7)

$$\mathbf{v}_{(i,\ell_i^t)}^t = \mathbf{x}_{(i,\ell_i^t)}^t + \gamma^t (\widetilde{\mathbf{x}}_{(i,\ell_i^t)}^t - \mathbf{x}_{(i,\ell_i^t)}^t);$$
(8)

Broadcast $\mathbf{v}_{(i,\ell_i^t)}^t$, $\phi_{(j,\ell)}^t$, $\mathbf{y}_{(j,\ell_i^t)}^t$ to the out-neighbors;

Averaging and Gradient Tracking:

For $\ell \in \{1, \ldots, B\}$: receive $\phi_{(j,\ell)}^t, \mathbf{v}_{(j,\ell)}^t$ from $j \in \mathcal{N}_{i,\ell}^t$, and set

$$\phi_{(i,\ell)}^{t+1} = \sum_{j \in \mathcal{N}_{i,\ell}^t} a_{ij\ell}^t \, \phi_{(j,\ell)}^t, \tag{9}$$

$$\mathbf{x}_{(i,\ell)}^{t+1} = \frac{1}{\phi_{(i,\ell)}^{t+1}} \sum_{j \in \mathcal{N}_{i,\ell}^t} a_{ij\ell}^t \, \phi_{(j,\ell)}^t \mathbf{v}_{(j,\ell)}^t; \tag{10}$$

Select $\ell_i^{t+1} \in \{1, \dots, B\}$ and update

$$\widehat{\mathbf{g}}_{i,\ell}^{t+1} = \begin{cases} \nabla_{\ell_i^{t+1}} f_i(\mathbf{x}_{(i)}^{t+1}), & \text{if } \ell = \ell_i^{t+1}, \\ \widehat{\mathbf{g}}_{i,\ell}^t, & \text{otherwise;} \end{cases}$$
(11)

For $\ell \in \{1, \ldots, B\}$: receive $\phi_{(j,\ell)}^t \mathbf{y}_{(j,\ell)}^t + \widehat{\mathbf{g}}_{j,\ell}^{t+1} - \widehat{\mathbf{g}}_{j,\ell}^t$ from $j \in \mathcal{N}_{i,\ell}^t$, and set

$$\mathbf{y}_{(i,\ell)}^{t+1} = \frac{1}{\phi_{(i,\ell)}^{t+1}} \sum_{j \in \mathcal{N}_{i,\ell}^t} a_{ij\ell}^t \left(\phi_{(j,\ell)}^t \mathbf{y}_{(j,\ell)}^t + \widehat{\mathbf{g}}_{j,\ell}^{t+1} - \widehat{\mathbf{g}}_{j,\ell}^t \right).$$
(12)

Remark 3.4: Note that the block selected in the tracking step (11) needs not to be the same as the one used in the optimization step (6). However, in BLOCK-SONATA we let them be equal so that to perform the two aforementioned steps only one block-gradient computation is needed.

Having introduced the algorithm, the remaining question is how to choose the surrogate functions \tilde{f}_i and the step-size γ^t . Convergence of BLOCK-SONATA is guaranteed under the following assumptions.

Assumption 3.5 (On the Surrogate Functions): Given Problem (P) under Assumption 2.1, each surrogate function $\tilde{f}_{i,\ell} : \mathcal{K}_{\ell} \times \mathcal{K} \to \mathbb{R}$ satisfies:

- (i) $\tilde{f}_{i,\ell}(\bullet; \mathbf{x})$ is uniformly strongly convex on \mathcal{K}_{ℓ} ;
- (ii) $\nabla \tilde{f}_{i,\ell}(\mathbf{x}_{\ell};\mathbf{x}) = \nabla_{\ell} f_i(\mathbf{x})$, for all $\mathbf{x} \in \mathcal{K}$;
- (iii) $\nabla \tilde{f}_{i,\ell}(\mathbf{x}_{\ell}; \bullet)$ is uniformly Lipschitz continuous on \mathcal{K} ;

where $\nabla \tilde{f}_{i,\ell}$ denotes the partial gradient of $\tilde{f}_{i,\ell}$ with respect to its first argument.

Assumption 3.6 (On the step-size): The sequence $\{\gamma^t\}$, with each $0 < \gamma^t \le 1$, satisfies: (i) $\sum_{t=0}^{\infty} \gamma^t = \infty$ and

 $\sum_{t=0}^{\infty} (\gamma^t)^2 < \infty; \text{ (ii) } \gamma^t / \eta \le \gamma^{t+1} \le \gamma^t, \text{ for all } t \ge 0 \text{ and some } \eta \in (0,1).$

Assumption 3.5 states that \tilde{f}_i should be regarded as a (simple) strongly convex approximation of f_i that preserves its first order properties. Several valid choices for \tilde{f}_i are available; see, e.g., [15], [23]. Assumption 3.6 is the standard diminishing step-size rule (i) with the extra requirement (ii), which ensures all the blocks contribute "equally" to the optimization. Condition (ii) can be met easily in practice [28], [29]; an example is given in Sec. IV. The convergence of BLOCK-SONATA is given in the following theorem, whose proof is omitted due to space limitation; see [30].

Theorem 3.7: Let $\{(\mathbf{x}_{(i)}^t)_{i=1}^N\}_{t\in\mathbb{N}}$ be the sequence generated by BLOCK-SONATA, and let $\bar{\mathbf{x}}^t \triangleq (1/N) \sum_{i=1}^N \mathbf{x}_{(i)}^t$. Suppose Assumptions 2.1, 2.2, 3.1, 3.2, 3.5, and 3.6 are satisfied; then there hold:

(i) consensus: $\|\mathbf{x}_{(i)}^t - \bar{\mathbf{x}}^t\| \to 0$ as $t \to \infty$, for all $i \in \{1, \dots, N\}$;

(ii) convergence: $\{\bar{\mathbf{x}}^t\}_{t\in\mathbb{N}}$ is bounded and every of its limit points is a stationary solution of Problem (P).

BLOCK-SONATA enjoys the property that at each iteration agents not only solve a low-dimensional optimization problem, but also transmit a limited amount of information. Moreover, compared to our previous scheme in [27], in BLOCK-SONATA, the gradient of f_i are computed only with respect to one block rather than the whole variable, and this further saves local computation cost.

IV. NUMERICAL SIMULATIONS

In this section we test BLOCK-SONATA on an instance of the sparse regression problem (1), where \mathcal{K} is a box constraint set, and R is chosen to be the logarithmic function [31], with $\lambda = 0.1$; in its DC reformulation in (2) we set $\theta = 10$ (the specific expression of $\eta(\theta)$ and thus r^+ and r^- therein can be found in [1]). Finally, let $f_i(\mathbf{x}) = \|\mathbf{b}_i - \mathbf{D}_i \mathbf{x}\|^2 + R(\mathbf{x}).$

As surrogate $\tilde{f}_{i,\ell}$ of f_i (cf. Assumption 3.5), we use the linearization of f_i at the current iterate, i.e.,

$$\begin{aligned} \hat{f}_{i,\ell}(\mathbf{x}_{(i,\ell)}; \mathbf{x}_{(i)}^{t}) \\ &= \left(2\mathbf{D}_{i,\ell}^{\top}(\mathbf{D}_{i} - \mathbf{b}_{i}) \right)^{\top} (\mathbf{x}_{(i,\ell)} - \mathbf{x}_{(i,\ell)}^{t}) + \frac{\tau_{i}}{2} \|\mathbf{x}_{(i,\ell)} - \mathbf{x}_{(i,\ell)}^{t}\|^{2} \\ &- \lambda \cdot \sum_{k=1}^{d} \left(\frac{dr^{-}((\mathbf{x}_{(i,\ell)}^{t})_{k})}{dx} (\mathbf{x}_{(i,\ell)} - \mathbf{x}_{(i,\ell)}^{t})_{k} \right), \end{aligned}$$
(13)

where $(\mathbf{x}_{(i,\ell)}^t)_k$ denotes the k-th scalar component of $\mathbf{x}_{(i,\ell)}^t$, and $dr^-((\mathbf{x}_{(i,\ell)}^t)_k)/dx$ is the derivative of r^- evaluated at $(\mathbf{x}_{(i,\ell)}^t)_k$ It is worth noting that (13) admits a unique minimizer, whose expression is omitted because of the space limit.

We simulated a network of N = 50 agents communicating over a fixed undirected graph \mathcal{G} , generated using an Erdős-Rényi random model. We compared two extreme topologies: a densely and a poorly-connected one, with algebraic connectivity equal to 45 and 5, respectively. There are 500 optimization variables, and we set $\mathcal{K} \triangleq [-10, 10]^{500}$. The components of the ground-truth signal \mathbf{x}_0 are generated independently according to the Gaussian distribution $\mathcal{N}(0, 1)$. To impose sparsity on \mathbf{x}_0 , we set the smallest 80% of the entries of \mathbf{x}_0 to zero. Each agent *i* has a measurement matrix $\mathbf{D}_i \in \mathbb{R}^{50 \times 500}$ with i.i.d. $\mathcal{N}(0, 1)$ distributed entries (with ℓ_2 -normalized rows), and the observation noise \mathbf{n}_i has entries i.i.d. distributed according to $\mathcal{N}(0, 0.5)$. The diminishing step-size γ^t follows the rule $\gamma^t = \gamma^{t-1}(1 - \mu\gamma^{t-1})$, with $\gamma^0 = 0.1$ and $\mu = 10^{-4}$. The proximal parameter is $\tau_i = 5$ for the poorly connected example and $\tau_i = 1$ for the densely connected one. To evaluate the algorithmic performance, we use two merit functions. The first one-given by $J^t \triangleq \|\bar{\mathbf{x}}^t - \mathcal{P}_{\mathcal{K}}\left(SS_{\eta\lambda}\left(\bar{\mathbf{x}}^t - \left(\sum_{i=1}^N \nabla f_i(\bar{\mathbf{x}}^t) - \lambda \cdot \sum_{k=1}^{dB} dr^-((\bar{\mathbf{x}}^t)_k)/dx\right)\right)\right)\|_{\infty}$ -measures the distance from stationarity of the average of the agents' iterates $\bar{\mathbf{x}}^t$ while the second one- $D^t \triangleq \max_{i \in \{1,...,N\}} \|\mathbf{x}_{(i)}^t - \bar{\mathbf{x}}^t\|$ -quantifies the consensus disagreement at each iteration.

We compare our algorithm with a non-block-wise distributed gradient algorithm; we adapted the gradient-push in [2] to a constrained nonconvex problem according to the protocol proposed in [32] (no formal proof of convergence for such scheme is available in the nonconvex setting). The performance of BLOCK-SONATA for different choices of the block dimension are reported in Fig. 1 (a). Recalling that t is the iteration counter used in Algorithm 1, to fairly compare the algorithm runs for different block sizes, we plot J^t and D^t versus the normalized number of iterations t/B.

The figure shows that for all runs (with different block sizes), both consensus and stationarity are achieved by BLOCK-SONATA within 100 normalized iterations, while the plain gradient scheme using all the blocks is much slower. Let t_{end} be the completion time up to a tolerance of 10^{-3} , i.e., the iteration counter of the distributed algorithm such that $J^{t_{end}} < 10^{-3}$. Fig. 1 (b) shows the normalized completion time t_{end}/B versus the number of blocks B. This highlights how the communication cost reduces by increasing the number of blocks.



Figure 1: (a) optimality measurement J^t (solid) and consensus error D^t (dashed) versus the normalized iteration for several choices of blocks *B*: Algebraic connectivity equal to 5. (b) Completion time required to obtain $J^t < 10^{-3}$ versus the number of blocks *B*.

V. CONCLUSION

In this paper we studied non-convex distributed big-data optimization problems and proposed BLOCK-SONATA to solve them. Leveraging on SCA techniques and a novel block-tracking/consensus mechanism, the proposed distributed scheme is the fist one unlocking local block-optimization and block-communications. Asymptotic convergence to a stationary point of the problem was established, and numerical tests on the sparse regression problem demonstrated the effectiveness of algorithm.

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