

Smooth QR Decomposition of Polynomial Matrices

Faizan A. Khattak, Mohammed Bakhit, Ian K. Proudler, and Stephan Weiss

Department of Electronic & Electrical Engineering, University of Strathclyde, Glasgow G1 1XW, Scotland
{faizan.khattak,mohammed.bakhit,ian.proudler,stephan.weiss}@strath.ac.uk

Abstract—This paper presents a novel algorithm for determining a compact order QR decomposition of a polynomial matrix, where both the \mathbf{Q} and \mathbf{R} factors themselves are approximated by polynomial matrices. The QR factorisation is subject to an allpass ambiguity; existing time domain methods can lead to factorisations of high order. The proposed algorithm performs the conventional QR decomposition in the discrete Fourier transform domain. Subsequently, it establishes phase coherence between adjacent bins through a phase smoothing procedure, aimed at obtaining compact-order factors. The method is validated through experiments over an ensemble of randomized polynomial matrices and shown to outperform state-of-the-art algorithms.

I. INTRODUCTION

The QR decomposition (QRD) [1] is a well-established linear algebraic operation, which in the context of signal processing has found a wide range of applications ranging from array processing [2–4] to communications [5]. Particularly in the communication arena, for transmitting over multiple-input multiple-output (MIMO) channels, successive interference cancellation (SIC) schemes such as the vertical Bell-labs layered space-time algorithm [5, 6] have benefitted from the QRD [7]. When transitioning to broadband communication systems, the MIMO channel matrix transforms into a matrix of transfer functions, essentially becoming a polynomial matrix that depends on the complex-valued parameter z [8, 9]. To effectively operate in a broadband MIMO environment, it becomes necessary to introduce tap-delay lines and employ block processing techniques [10]. Alternatively, one can switch to polynomial matrix algebra [11] and, specifically, employ a polynomial matrix QRD (PQRD) [12–14]. PQRD has found applications in MIMO equalization, SIC [15, 16], and general PSVD algorithms [12, 17] and their deployment [18–20].

All current PQRD algorithms strive to triangularise a given polynomial matrix $\mathbf{A}(z)$ in an iterative manner such that in each iteration, a finite number of lower off-diagonal elements are eliminated via paraunitary operations [12, 13, 16]. In this iterative process, the polynomial orders of both the paraunitary matrix $\mathbf{Q}(z)$ and the approximately upper-right triangular matrix $\mathbf{R}(z)$ increase in each iteration. To achieve a complete QR decomposition, an infinite number of iterations may be required. Thus practically, only an approximate QR decomposition can be obtained as otherwise, the polynomial orders of both $\mathbf{Q}(z)$ and $\mathbf{R}(z)$ could become excessively large. This issue not only affects the accuracy of applications but also increases the computational cost of any hardware implementation. In addition, the PQRD is also reported to be

The work of F.A. Khattak was supported by Commonwealth Scholarship. M. Bakhit acknowledges funding by Mathworks and the University of Strathclyde. The work of S. Weiss was supported by the Engineering and Physical Sciences Research Council (EPSRC) Grant number EP/S000631/1 and the MOD University Defence Research Collaboration in Signal Processing.

utilized in the computation of the polynomial singular value decomposition (PSVD) in [13] where the issues of approximation error and large polynomial order are compounded by repeated PQRDs.

In principle, an at least approximate PQRD with limited support appears viable. If the matrix to be decomposed has limited support in the time domain, then its Fourier transform must be sufficiently smooth. This has previously motivated interpolation-based approaches [21, 22] for orthogonal frequency division multiplexing systems, where a QRD in a limited number of frequency bins suffice to determine the remaining QRDs at other frequencies by simple interpolation rather than by explicit QRD computations. This view is supported by perturbation analysis of the QRD, such as in [25], which shows that a small change in the matrix leads to only a limited perturbation of its QR factors.

Thus in this paper, we propose a compact order — or “smooth” — PQRD. This approach works in the DFT domain, performs independent QRDs in each frequency bin akin to [23, 24], and restores the spectral coherence by a phase smoothing operation, adopting a strategy for the extraction of analytic polynomial eigenvectors [26, 27]. However, unlike [27], the proposed method first determines the compact support of both $\mathbf{Q}[n] \circ \bullet \mathbf{Q}(z)$ and $\mathbf{R}[n] \circ \bullet \mathbf{R}(z)$, where $\mathbf{Q}[n] \circ \bullet \mathbf{Q}(z) = \sum_n \mathbf{Q}[n]z^{-n}$ denotes a transform pair, via a phase-independent support estimation method [28]. It thereafter exploits the established support length to determine the DFT size. A phase smoothing operation then completes the overall PQRD.

The paper is organized as follows: Sec. II briefly explains the existence and uniqueness of the PQRD whereas Sec. III describes the available iterative PQRD techniques. Sec. IV presents the proposed algorithm, which is then compared with simulation results over an ensemble in Sec. V.

II. PRELIMINARIES AND MOTIVATION

A. QR Decomposition and Its Ambiguity

The QR decomposition of a matrix $\mathbf{A} \in \mathbb{C}^{M \times N}$ is not unique. Consider

$$\mathbf{A} = \mathbf{QR} = \mathbf{Q}\Phi\Phi^H\mathbf{R} = \mathbf{Q}'\mathbf{R}', \quad (1)$$

where $\mathbf{Q} \in \mathbb{C}^{M \times M}$ and $\mathbf{R} \in \mathbb{C}^{M \times N}$ are unitary and upper triangular matrices, respectively. The spurious insertion of a diagonal unitary matrix $\Phi = \text{diag}\{e^{j\phi_1}, \dots, e^{j\phi_M}\}$, s.t. $\Phi\Phi^H = \mathbf{I}$ with $\{\cdot\}^H$ the Hermitian transposition, creates new factors \mathbf{Q}' and \mathbf{R}' that are also unitary and upper triangular. Thus, the m th column of the unitary matrix \mathbf{Q} and the m th row of the upper triangular matrix \mathbf{R} are ambiguous w.r.t. a common arbitrary phase shift ϕ_m , $m = 1, \dots, M$.

B. Existence and Uniqueness of the PQRD

For a polynomial matrix $\mathbf{A}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M \times N}$, with $\mathbf{A}(z) \bullet \circ \mathbf{A}[n]$, the QRD in (1) can only achieve a suitable factorisation for one value of z , or one value of n . Therefore, the aim of the PQRD is to achieve a decomposition

$$\mathbf{A}(z) = \mathbf{Q}(z)\mathbf{R}(z), \quad (2)$$

where $\mathbf{Q}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M \times M}$ is paraunitary, such that $\mathbf{Q}(z)\mathbf{Q}^P(z) = \mathbf{I}$ with the parahermitian transposition $\mathbf{Q}^P(z) = \{\mathbf{Q}(1/z^*)\}^H$ [8] and complex conjugation $\{\cdot\}^*$, and an upper triangular $\mathbf{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M \times N}$. To use the z -domain notation, $\mathbf{A}(z)$, $\mathbf{Q}(z)$, and $\mathbf{R}(z)$ in (2) must be analytic in $z \in \mathcal{I}$, whereby the region of convergence \mathcal{I} must include at least the unit circle.

Even though proofs exist for the related polynomial eigenvalue [29–31] and polynomial singular value decompositions [31, 32], there currently is no proof that for an analytic $\mathbf{A}(z)$, the factors $\mathbf{Q}(z)$ and $\mathbf{R}(z)$ in (2) can also be analytic. Despite this lack, the factorisation (2) has been assumed in [12, 13, 15]. Analyticity implies infinite differentiability, i.e. smoothness, of $\mathbf{A}(z)|_{z=e^{j\Omega}}$ and hence motivates the considerations in [21, 22] that $\mathbf{Q}(z)$, and $\mathbf{R}(z)$ can be Laurent series. If so, then the best polynomial approximation to (2) can be achieved by shifts and truncations [27].

Assuming that (2) is valid, interesting parallels can be drawn to (1). Note that we can equivalently expand (2) as

$$\mathbf{A}(z) = \mathbf{Q}(z)\mathbf{\Phi}(z)\mathbf{\Phi}^P(z)\mathbf{R}(z) = \mathbf{Q}'(z)\mathbf{R}'(z), \quad (3)$$

with a paraunitary matrix $\mathbf{\Phi}(z) = \text{diag}\{\phi_1(z), \dots, \phi_M(z)\}$ that is diagonal and containing allpass filters $\phi_m(z)$, $m = 1, \dots, M$. Hence $\mathbf{Q}'(z) = \mathbf{Q}(z)\mathbf{\Phi}(z)$ and $\mathbf{R}'(z) = \mathbf{\Phi}^P(z)\mathbf{R}(z)$ remain paraunitary and upper right triangular matrices, and hence valid PQRD factors. Unlike for ordinary matrices, the allpass factor $\mathbf{\Phi}(z)$ determines the support of the PQRD. For example, if $\mathbf{Q}(z)$ and $\mathbf{R}(z)$ have finite order, then $\mathbf{Q}'(z)$ and $\mathbf{R}'(z)$ would have infinite order unless $\mathbf{\Phi}(z)$ takes the form of simple delays. Therefore, the impact of finding a suitable $\mathbf{\Phi}(z)$ is crucial for the order and hence for the smoothness and implementation complexity of the QR factors in (2).

III. EXISTING ITERATIVE PQRD TECHNIQUES

To date iterative PQRD algorithms are either based on the concept of the sequential second-order best rotation (SBR2) [33] or sequential matrix diagonalization (SMD) [34] algorithms, which are reviewed below.

A. SBR2-based PQRD

The SBR2 algorithm [33] is an iterative polynomial matrix EVD method that in each operation eliminates the largest off-diagonal component via an elementary paraunitary operation comprising a delay and a Givens rotation. This concept has been extended to operate as a PQRD approach by iteratively applying elementary paraunitary operations until all elements in the lower left-triangular part of the matrix are either sufficiently suppressed, or until the maximum element within that part for the matrix falls below a preset threshold.

There are two reported variants: (i) PQRD by steps (PQRD-BS) [35], and (ii) PQRD by column (PQRD-BC) [13]. The former suppresses one polynomial lower left triangular entry at a time such that all coefficients are driven below the threshold before proceeding to the next polynomial elements. The PQRD-BC variant instead aims to approximately zero a column at a time. This is performed via an iterative search and elimination of the successively largest elements in the selected column. While its convergence is faster than PQRD-BS, the search operation in PQRD-BC is comparatively more expensive [13].

B. SMD-based PQRD

The idea of the SMD-based PQRD algorithm [14] is to temporally shift as much energy as possible to a particular lag component, where then a full QR decomposition triangularizes that. The SM-PQRD performs this iteratively, until a stopping criterion similar to the SBR2-based PQRD algorithms in Sec. III-A is satisfied. As a result, SM-PQRD provides faster convergence but each iteration is more expensive than an iteration of e.g. PQRD-BC.

IV. SMOOTH QR DECOMPOSITION ALGORITHM

The PQRD algorithms in Sec. III operate in the time domain; they are proven to converge [13, 35], even though it is unclear what the ultimate allpass ambiguity according to (3) will be. Therefore, this section proposes a DFT-domain approach, where a QRD is calculated in each frequency bin. Thereafter a phase-smoothing operation has two purposes: (i) it must re-establish the phase-coherence that is lost by operating in independent frequency bins; and (ii) amongst all the possible analytic solutions for the allpass term $\mathbf{\Psi}(z)$, we want to find that solution that minimises the support and hence provides the smoothest possible solution on the unit circle.

A. QRD in Sample Points

Based on $\mathbf{A}(z) \bullet \circ \mathbf{A}[n]$, we perform a QRD independently on the sample points obtained from a K -point DFT, i.e. on $\mathbf{A}_k = \mathbf{A}(z)|_{z=e^{j\Omega_k}}$ yielding

$$\mathbf{A}_k = \mathbf{Q}_k\mathbf{R}_k, \quad k = 1, \dots, K, \quad (4)$$

where $\Omega_k = \frac{2\pi k}{K}$. Note that \mathbf{Q}_k and \mathbf{R}_k do not necessarily match up with the sample points of analytic functions in (2). In fact, we have to relate the sample points of analytic functions in (3) to its bin-wise counterpart in (4) as

$$\mathbf{Q}(e^{j\Omega_k}) = \mathbf{Q}_k\mathbf{\Theta}_k \quad (5)$$

$$\mathbf{R}(e^{j\Omega_k}) = \mathbf{\Theta}_k^H\mathbf{R}_k, \quad (6)$$

where $\mathbf{\Theta}_k = \text{diag}\{e^{j\theta_{1,k}}, \dots, e^{j\theta_{M,k}}\}$ with unknown angles $\theta_{m,k}, m = 1, \dots, M$ and $k = 0, \dots, (K-1)$.

B. Loss of Phase Coherence across DFT bins

Since QRD is performed independently in each bin, there is lack of phase coherence between the results of bin-wise QRD in (4). This lack of phase coherence is illustrated by the following example.

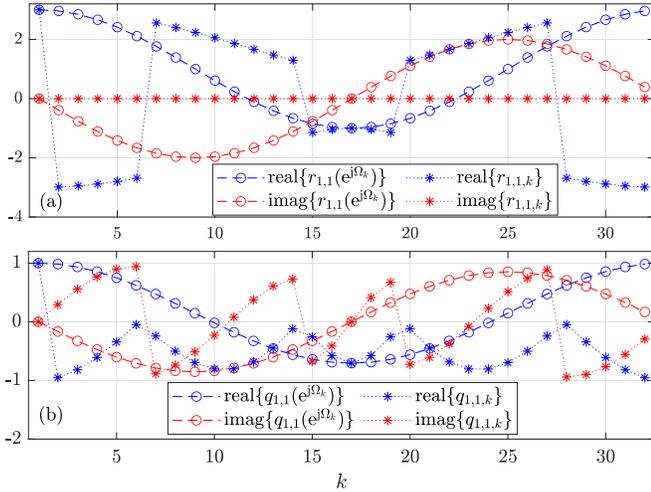


Fig. 1. Phase comparison between the ground truth PQRD and the bin-wise QRD for one element of (a) the upper right triangular matrix, i.e., $r_{1,1,k}, r_{1,1}(e^{j\Omega_k})$, and (b) the paraunitary matrix, i.e. $q_{1,1,k}, q_{1,1}(e^{j\Omega_k})$ with $K = 32$.

Example 1: Consider the simple case where

$$\mathbf{R}(z) = \begin{bmatrix} 1 + 2z^{-1} & 2 + 3z^{-1} \\ 0 & 1 + 3z^{-1} \end{bmatrix}, \quad (7)$$

and $\mathbf{Q}(z)$ is defined by an elementary paraunitary operation [36]

$$\mathbf{Q}(z) = [\mathbf{q}_1(z), \mathbf{q}_2(z)] = \mathbf{I} - \mathbf{e}\mathbf{e}^H + \mathbf{e}\mathbf{e}^H z^{-1}, \quad (8)$$

where $\mathbf{e} = [2 + 5j, 2 + j]^T / \sqrt{34}$. The resulting polynomial matrix $\mathbf{A}(z) = \mathbf{Q}(z)\mathbf{R}(z)$ is of order 2. To visualize the lack of phase coherence, a 32-point DFT of $\mathbf{A}[n]$ is performed and a QRD is independently calculated in each bin according to (4). Similarly, the available ground-truth $\mathbf{Q}(z)$ from (8) and $\mathbf{R}(z)$ from (7) is evaluated on the unit circle at 32 equidistant sample points i.e. for $z = e^{j\Omega_k}, \Omega_k = \frac{2\pi k}{32}$. We now compare the real and imaginary parts of $q_{1,1}(e^{j\Omega_k})$ and $r_{1,1}(e^{j\Omega_k})$ with their bin-wise counterparts $q_{1,1,k}$ and $r_{1,1,k}$ in Fig 1. It can be seen that both real and imaginary parts of ground-truth samples evolve smoothly across the frequency points but that the QRD factors obtained independently in each bin do not.

C. Phase Smoothing

Phase smoothing can be performed either on the elements of the columns of \mathbf{Q}_k or rows of \mathbf{R}_k . Here, we apply it to the columns of \mathbf{Q}_k to make it easier to adopt the phase smoothing procedure reported in [27]. The aim of this method is to find a maximally smooth interpolation over all components of each column of \mathbf{Q}_k , for $k = 0, \dots, (K-1)$, by adjusting the phase shifts in $\boldsymbol{\theta}_k$ in (5). For this reason, we perform a Dirichlet interpolation [37, 38] over $\mathbf{u}_{i,m} \in \mathbb{C}^K$,

$$\mathbf{u}_{i,m} = [u_{i,m,1}, \dots, u_{i,m,K}]^T = \text{diag}\{\mathbf{w}_m\} \mathbf{q}_{i,m}, \quad (9)$$

where $\mathbf{q}_{i,m} = [q_{i,m,1}, \dots, q_{i,m,K}]^T$ holds the K sample points of elements in the i th row and m th column of \mathbf{Q}_k and $\mathbf{w}_m = [e^{j\theta_{m,1}}, \dots, e^{j\theta_{m,K}}]^T$ contains the yet to be determined

phase shifts. Over the phase-corrected sample points $\mathbf{u}_{i,m}$, the resulting Dirichlet interpolation $I_{i,m}^{(K)}(e^{j\Omega})$ will be [27]

$$\begin{aligned} I_{i,m}^{(K)}(e^{j\Omega}) &= \frac{1}{K} \sum_{n=0}^{K-1} e^{-j\Omega n} \sum_{k=0}^{K-1} u_{i,m,k} e^{j2\pi n k / K} \\ &= \frac{1}{\sqrt{K}} \mathbf{e}_K^H(e^{j\Omega}) \cdot \mathbf{W}_K^H \cdot \mathbf{u}_{i,m}, \end{aligned} \quad (10)$$

where $\mathbf{e}_K^H = [1, e^{j\Omega}, \dots, e^{-j\Omega(K-1)}]$ and \mathbf{W}_K^H is a K -point unitary DFT matrix.

Since the underlying aim is a smooth function, we compute the power in its p th derivative to define a smoothness metric [27, 37, 38]

$$\chi_m^{(K),p} = \sum_{i=1}^M \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\partial^p I_{i,m}^{(K)}(e^{j\Omega})}{\partial \Omega^p} \right|^2 d\Omega, \quad (11)$$

where the vectorial form of the derivative part is

$$\frac{\partial^p I_{i,m}^{(K)}(e^{j\Omega})}{\partial \Omega^p} = \frac{1}{\sqrt{K}} \mathbf{e}_K^H(e^{j\Omega}) \mathbf{D}_K^p \cdot \mathbf{W}_K^H \cdot \mathbf{u}_{i,m},$$

with $\mathbf{D}_K = \text{diag}\{0, -j, -2j, \dots, -(K-1)j\}$. This allows the formulation of (11) as

$$\chi_m^{(K),p} = \frac{1}{K} \sum_{\mu=1}^M \|\mathbf{D}_K^p \cdot \mathbf{W}_K^H \cdot \mathbf{u}_{i,m}\|_2^2, \quad (12)$$

which can be related to the phase-shifts as

$$\chi_m^{(K),p} = \mathbf{w}_m^H \mathbf{C}_{K,p} \mathbf{w}_m \quad (13)$$

with

$$\mathbf{C}_{K,p} = \sum_{i=1}^M \text{diag}\{\mathbf{q}_{i,m}^*\} \mathbf{W}_K \mathbf{D}_K^{2p} \mathbf{W}_K^H \text{diag}\{\mathbf{q}_{i,m}\}.$$

Now the objective is to minimize (13) w.r.t. $\boldsymbol{\theta}_m = [\theta_{m,1}, \dots, \theta_{m,K}]^T$ through an iterative gradient method. The iterative update for $\boldsymbol{\theta}_m$ can be given as

$$\boldsymbol{\theta}_m[j+1] = \boldsymbol{\theta}_m[j] - \mu \mathbf{H}^{-1} \frac{\partial \chi_m^{(K),p}(\boldsymbol{\theta}_m[j])}{\partial \boldsymbol{\theta}_m}, \quad (14)$$

where

$$\frac{\partial \chi_m^{(K),p}}{\partial \boldsymbol{\theta}_m} = 2 \text{Im}\{\text{diag}\{\mathbf{w}^*\} \mathbf{C}_{K,p} \mathbf{w}\} \quad (15)$$

and Hessian matrix [39] is given by

$$\begin{aligned} \mathbf{H} &= 2 \text{Re}\{\text{diag}\{\mathbf{w}^*\} \mathbf{C}_{K,p} \text{diag}\{\mathbf{w}\} \\ &\quad - \text{diag}\{\text{diag}\{\mathbf{w}^*\} \mathbf{C}_{K,p} \mathbf{w}\}\}, \end{aligned} \quad (16)$$

which might not always be positive definite. The second part i.e. $\text{diag}\{\text{diag}\{\mathbf{w}^*\} \mathbf{C}_{K,p} \mathbf{w}\}$ is shown to be less significant compared to the first part [39]; if ignored, the resulting approximate Hessian is positive semi-definite [39].

Once a stationary point $\mathbf{w}_{m,\min}$ has been found, Theorem 5 in [27] states that a small modulation applied to the minimum point as

$$\mathbf{w}'_m = \text{diag}\left\{1, e^{j\frac{2\pi}{K}\ell}, \dots, e^{j\frac{2\pi}{K}(K-1)\ell}\right\} \mathbf{w}_{m,\min}, \quad (17)$$

where $\ell \in \mathbb{Z}$, approximately leads to a new stationary point. Since the statement is only true for small $\ell \ll K$, we vary it from 1 to 5 in the current method. The optimization process is continued with different shifts until an overall smallest point is found.

D. Minimum DFT Size

The phase smoothing procedure must be carried out at a DFT size greater than the support of $\mathbf{q}_m[n] \circ \bullet \mathbf{q}_m(z)$. To avoid phase smoothing at DFT size smaller than the support of $\mathbf{q}_m[n]$, the support of $\mathbf{q}_m[n]$ can be estimated using the method reported in [28]. This support estimation method first determines the autocorrelation function of the component of each column of $\mathbf{Q}[n]$ through its bin-wise DFT magnitudes because

$$X_{i,m}[k] = |q_{i,m}(e^{j\Omega_k})|^2 = |q_{i,m,k}|^2, \quad (18)$$

where $X_{i,m}[k]$ is the DFT of $x_{i,m}[v] = \sum_n q_{i,m}[n] q_{i,m}^*[v-n]$, the auto-correlation of $q_{i,m}[n] \circ \bullet q_{i,m}(z)$. Once the auto-correlation is determined with minimum time-domain aliasing, the support of $q_{i,m}[n]$ is considered as the half of its auto-correlation function. This way the support of each component of $\mathbf{Q}[n]$ can be determined upon which the minimum DFT size for the phase smoothing can be decided. For instance, if \hat{N}_Q is the support size, the minimum DFT size must be equal to \hat{N}_Q .

E. Overall Procedure

The proposed algorithm estimates the support of $\mathbf{Q}[n]$, denoted \hat{N}_Q , to execute phase smoothing at the smallest feasible DFT size K . During the phase smoothing process, it evaluates error in time-domain aliasing and paraunitarity as

$$\sum_{n \in \mathbb{Z}} \|\hat{\mathbf{Q}}^H[-n] * \hat{\mathbf{Q}}[n] - \delta[n] \mathbf{I}\|_{\mathbb{F}}^2,$$

where $\hat{\mathbf{Q}}[n]$ is obtained via a K -point IFFT of $\text{diag}\{e^{j\theta_{1,k}}, \dots, e^{j\theta_{M,k}}\} \mathbf{Q}_k$. If the paraunitarity error is lower than a predefined small threshold $\epsilon > 0$, the selected DFT size K is deemed sufficient, and the phase smoothing procedure can be concluded. However, if the error does not meet the threshold even after the maximum allowed iterations, the DFT size is doubled. Both the execution time and the accuracy of the decomposition obtained through the proposed method depend on the value of ϵ . For practical experimentation and applications, setting $\epsilon \leq 10^{-5}$ is typically considered an appropriate threshold. Note that since both $\hat{\mathbf{Q}}(z)$ and $\hat{\mathbf{R}}(z)$ are phase coupled, only one phase smoothing needs to be performed per DFT size.

V. SIMULATIONS AND RESULTS

A. Performance Metrics

In order to compare the performance of the different PQRD approaches, we define a triangularization metric

$$\xi_t = 1 - \frac{\sum_{i,j \geq i,n} \|\hat{r}_{i,j}[n]\|^2}{\sum_n \|\mathbf{A}[n]\|_{\mathbb{F}}^2}, \quad (19)$$

where $\hat{r}_{i,j}[n]$ is the (i,j) th element of $\hat{\mathbf{R}}[n]$. Additionally, the execution time and the order of the resulting paraunitary

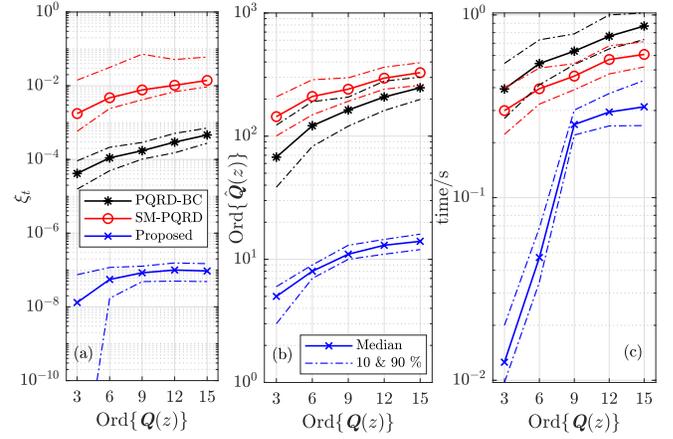


Fig. 2. Ensemble test results showing (a) triangularization ratio, (b) order of resulting $\hat{\mathbf{Q}}(z)$ and, (c) execution time for the proposed PQRD algorithm, as well as for PQRD-BC [13] and SM-PQRD [14].

matrix $\mathbf{Q}(z)$ and the upper triangular matrix $\mathbf{R}(z)$ are assessed as metrics when comparing PQRD methods.

B. Ensemble Test

The proposed algorithm is tested over an ensemble of matrices $\mathbf{A}(z) \in \mathbb{C}^{4 \times 4}$ with known ground-truth PQRD factorisation consisting of a paraunitary $\mathbf{Q}(z)$ and an upper right triangular $\mathbf{R}(z)$. The ensemble results are computed by jointly varying the orders $\mathbf{Q}(z)$ and $\mathbf{R}(z)$, $\text{Ord}\{Q(z)\}$ and $\text{Ord}\{R(z)\}$, from 3 to 15 in steps of 3, and by generating 100 instantiations of matrices for each order. For this, the coefficients of $\mathbf{R}(z)$ are drawn from a complex Gaussian distribution, while $\mathbf{Q}(z)$ is generated through a sequence of elementary paraunitary operations [36].

The resulting triangularization ratio ξ_t , illustrated in Fig. 2(a), of the proposed method is orders of magnitude lower than the state-of-the-art PQRD-BC and SM-PQRD. Moreover, this greater degree of triangularisation is achieved at significantly lower order paraunitary matrix $\hat{\mathbf{Q}}(z)$ as shown in Fig. 2(b). Lastly, the execution time of the proposed method on average is lowest of all three methods. The sharp rise in the execution time curve for the proposed method from $\text{Ord}\{Q(z)\} = 3$ to 9 is due to step changes in the DFT size i.e. the DFT sizes at these points are 8, 16, 32, respectively.

VI. CONCLUSION

We have proposed a novel approach for a compact order QRD of a polynomial matrix. The proposed algorithm finds a smooth association between the adjacent bins' conventional QRD through a phase smoothing algorithm for overall compact time-domain support. The comparison over an ensemble of matrices of variable length shows that the proposed algorithm outperforms state-of-the-art algorithms by an order of magnitude in order and accuracy of the decomposition. Moreover, the proposed method also promises reduced execution time. Hence, overall the proposed PQRD approach can be executed in shorter time, and yields polynomial factors of lower order that are less expensive to implement compared to state-of-the-art algorithms.

REFERENCES

- [1] G.H. Golub and C.F. Van Loan, *Matrix Computations*, 3rd ed., The Johns Hopkins University Press, 1996.
- [2] I.K. Proudler, T. Shepherd, and J.G. McWhirter, "Computationally efficient QRD-based wide-band beamforming," in *International Conference on Acoustics, Speech, and Signal Processing*, vol. 3, pp. 1799–1802, 1990.
- [3] A. Sayed and T. Kailath, "A state-space approach to adaptive rls filtering," *IEEE Signal Processing Magazine*, vol. 11, no. 3, pp. 18–60, 1994.
- [4] I. Proudler, J. McWhirter, M. Moonen, and G. Hekstra, "Formal derivation of a systolic array for recursive least squares estimation," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 43, no. 3, pp. 247–254, 1996.
- [5] A. Paulraj, R. Nabar, and D. Gore, *Introduction to space-time wireless communications*, 1st ed., Cambridge University Press, May 2008.
- [6] G.D. Golden, C.J. Foschini, R.A. Valenzuela, and P.W. Wolniansky, "Detection algorithm and initial laboratory results using V-BLAST space-time communication architecture," *Electronics Letters*, vol. 35, pp. 14–16, Jan. 1999.
- [7] D. Wübben, R. Bohnke, J. Rinas, V. Kuhn, and K. D. Kammeyer, "Efficient algorithm for decoding layered space-time codes," *Electronics Letters*, vol. 37, pp. 1348–1350, Oct. 2001.
- [8] P.P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs: Prentice Hall, 1993.
- [9] K.J. Pope and R.E. Bogner, "Blind signal separation II. Linear, convolutive combinations," *Digital Signal Processing*, vol. 6, no. 1, pp. 17–28, Jan. 1996.
- [10] C.Y. Chen and P.P. Vaidyanathan, "Precoded FIR and redundant V-BLAST systems for frequency-selective MIMO channels," *IEEE Transactions on Signal Processing*, vol. 55, pp. 3390–3404, July 2007.
- [11] V. Neo, S. Redif, J. McWhirter, J. Pestana, I. Proudler, S. Weiss, and N. P.A., "Polynomial eigenvalue decomposition for multichannel broadband signal processing," *IEEE Signal Processing Magazine*, to appear.
- [12] J.A. Foster, J.G. McWhirter, M.R. Davies, and J.A. Chambers, "An algorithm for calculating the QR and singular value decompositions of polynomial matrices," *IEEE Transactions on Signal Processing*, vol. 58, no. 3, pp. 1263–1274, Mar. 2010.
- [13] —, "An algorithm for calculating the qr and singular value decompositions of polynomial matrices," *IEEE Transactions on Signal Processing*, vol. 58, no. 3, pp. 1263–1274, 2010.
- [14] F. Coutts, J. Corr, K. Thompson, S. Weiss, I.K. Proudler, and J.G. McWhirter, "Multiple shift QR decomposition for polynomial matrices," in *IMA International Conference on Mathematics in Signal Processing*, Birmingham, UK, Dec. 2016.
- [15] M. Davies, S. Lambotaran, J. Foster, J. Chambers, and J.G. McWhirter, "Polynomial matrix QR decomposition and iterative decoding of frequency selective MIMO channels," in *IEEE Wireless Communications and Networking Conference*, pp. 1–6, Apr. 2009.
- [16] D. Hassan, S. Redif, S. Lambotaran, and I.K. Proudler, "Sequential polynomial QR decomposition and decoding of frequency selective MIMO channels," in *26th European Signal Processing Conference*, pp. 460–464, Sep. 2018.
- [17] M. Bakhit, F.A. Khattak, I.K. Proudler, S. Weiss, and G. Rice, "Compact order polynomial singular value decomposition of a matrix of analytic functions," in *IEEE 9th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Costa Rica, Dec. 2023.
- [18] C.H. Ta and S. Weiss, "A jointly optimal precoder and block decision feedback equaliser design with low redundancy," in *15th European Signal Processing Conference*, Poznan, Poland, pp. 489–492, Sep. 2007.
- [19] S. Weiss, S.J. Schlecht, O. Das, and E. De Sena, "Polynomial Procrustes problem: Paraunitary approximation of matrices of analytic functions," in *European Signal Processing Conference*, Helsinki, Finland, Sep. 2023.
- [20] N. Moret, A. Tonello, and S. Weiss, "MIMO precoding for filter bank modulation systems based on PSVD," in *IEEE 73rd Vehicular Technology Conference*, May 2011.
- [21] D. Cescaio and H. Bölcskei, "QR decomposition of Laurent polynomial matrices sampled on the unit circle," *IEEE Transactions on Information Theory*, vol. 56, no. 9, pp. 4754–4761, Sep. 2010.
- [22] —, "Algorithms for interpolation-based QR decomposition in MIMO-OFDM systems," *IEEE Transactions on Signal Processing*, vol. 59, no. 4, pp. 1719–1733, Apr. 2011.
- [23] S. Weiss, I.K. Proudler, F.K. Coutts, and J. Pestana, "Iterative approximation of analytic eigenvalues of a parahermitian matrix EVD," in *IEEE International Conference on Acoustics, Speech and Signal Processing*, Brighton, UK, May 2019.
- [24] S. Weiss, I. K. Proudler, and F. K. Coutts, "Eigenvalue decomposition of a parahermitian matrix: extraction of analytic eigenvalues," *IEEE Transactions on Signal Processing*, vol. 69, pp. 722–737, 2021.
- [25] G.W. Stewart, "Perturbation bounds for the QR factorization of a matrix," *SIAM Journal on Numerical Analysis*, vol. 14, no. 3, pp. 509–518, 1977.
- [26] S. Weiss, I. Proudler, F. Coutts, and J. Deeks, "Extraction of analytic eigenvectors from a parahermitian matrix," in *International Conference on Sensor Signal Processing or Defence*, Edinburgh, UK, Sep. 2020.
- [27] S. Weiss, I.K. Proudler, F.K. Coutts, and F.A. Khattak, "Eigenvalue decomposition of a parahermitian matrix: extraction of analytic eigenvectors," *IEEE Transactions on Signal Processing*, vol. 71, pp. 1642–1656, Apr. 2023.
- [28] F.A. Khattak, I.K. Proudler, and S. Weiss, "Support estimation of analytic eigenvectors of parahermitian matrices," in *International Symposium on Recent Advances in Electrical Engineering & Computer Sciences (RAEE & CS)*, 2022.
- [29] S. Weiss, J. Pestana, and I. K. Proudler, "On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix," *IEEE Transactions on Signal Processing*, vol. 66, no. 10, pp. 2659–2672, May 2018.
- [30] S. Weiss, J. Pestana, I.K. Proudler, and F.K. Coutts, "Corrections to 'On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix'," *IEEE Transactions on Signal Processing*, vol. 66, no. 23, pp. 6325–6327, Dec. 2018.
- [31] G. Barbarino and V. Noferini, "On the Rellich eigendecomposition of para-Hermitian matrices and the sign characteristics of *-palindromic matrix polynomials," *Linear Algebra and its Applications*, vol. 672, pp. 1–27, Sep. 2023.
- [32] S. Weiss, I.K. Proudler, G. Barbarino, J. Pestana, and J.G. McWhirter, "On properties and structure of the analytic singular value decomposition," *IEEE Transactions on Signal Processing*, submitted 2023.
- [33] J. G. McWhirter, P. D. Baxter, T. Cooper, S. Redif, and J. Foster, "An EVD algorithm for para-Hermitian polynomial matrices," *IEEE Transactions on Signal Processing*, vol. 55, no. 5, pp. 2158–2169, 2007.
- [34] S. Redif, S. Weiss, and J.G. McWhirter, "Sequential matrix diagonalization algorithms for polynomial EVD of parahermitian matrices," *IEEE Transactions on Signal Processing*, vol. 63, no. 1, pp. 81–89, 2015.
- [35] J. Foster, J. McWhirter, and J. Chambers, "A polynomial matrix QR decomposition with application to MIMO channel equalisation," in *41st Asilomar Conference on Signals, Systems and Computers*, pp. 1379–1383, Nov. 2007.
- [36] P. Vaidyanathan, "Theory of optimal orthonormal subband coders," *IEEE Transactions on Signal Processing*, vol. 46, no. 6, pp. 1528–1543, June 1998.
- [37] S. Weiss and M.D. Macleod, "Maximally smooth dirichlet interpolation from complete and incomplete sample points on the unit circle," in *IEEE International Conference on Acoustics, Speech, and Signal Processing*, Brighton, UK, May 2019.
- [38] S. Weiss, J. Selva, and M.D. Macleod, "Measuring smoothness of trigonometric interpolation through incomplete sample points," in *28th European Signal Processing Conference*, Amsterdam, Netherlands, pp. 2319–2323, Jan. 2021.
- [39] M. Tohidian, H. Amindavar, and A.M. Reza, "A DFT-based approximate eigenvalue and singular value decomposition of polynomial matrices," *EURASIP Journal on Advances in Signal Processing*, vol. 2013, no. 1, pp. 1–13, 2013.