# On the choice of reference in offset calibration

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#### Abstract

Sensor calibration is an indispensable task in any networked cyberphysical system. In this paper, we consider a sensor network plagued with offset errors, measuring a rank-1 signal subspace, where each sensor collects measurements under a linear model with additive zero-mean Gaussian noise. Under varying assumptions on the underlying noise covariance, we investigate the effect of using an arbitrary reference for estimating the sensor offsets, in contrast to the 'average of all the unknown offsets' as a reference. We first show that the average reference yields an efficient minimum variance unbiased estimator. If the underlying noise is homoscedastic in nature, then we prove the average reference yields a factor 2 improvement on the variance, as compared to any arbitrarily chosen reference within the network. Furthermore, when the underlying noise is independent but not identical, we derive an expression for the improvement offered by the average reference. We demonstrate our results using the problem of clock synchronization in sensor networks, and discuss directions for future work.

### 1 Introduction

Sensors play a vital role in the burgeoning fields of internet of things (IoT) [1], networked cyberphysical systems [2], and Wireless Sensor Networks (WSN) [3], with diverse applications e.g., environmental monitoring [4], multi-agent formation control [5, 6], remote sensing [7] and space systems [8, 9], to name a few. Network-wide sensor calibration is a ubiquitous challenge in these applications, which is quintessential for accurately measuring the underlying signal of interest [10, 11]. Under the assumption that the ground truth lies in a known lowerdimensional subspace of the measurement subspace, the sensor gains [12, 13, 14] and the sensor drifts [15, 16, 17] can be uniquely determined using blind calibration techniques. However, in the absence of a known reference, the offset estimation is typically infeasible, and thus leads to an ill-posed problem. In practise, in the absence of an external reference, one (or many) of the sensors within the network is chosen as a reference to make the system identifiable [18]. More recently, in pursuit of an efficient minimum variance unbiased estimator, the average of the calibration parameters (e.g., offsets) is proposed as a reference in various applications, e.g., in antenna calibration [19], in clock synchronization [20], and in sensor calibration of air-quality networks [21]. Although the average reference yields optimality for various data models, it is unclear if it yields an optimal solution for a generalized linear model for offset estimation, and if so, how much is the improvement on performance as compared to any arbitrarily chosen reference within the network?

In this curiosity driven research, we consider a sensor network plagued with offset errors, where each sensor collects measurements under a linear Gaussian model, and measures an underlying signal subspace of rank-1. If the underlying noise covariance is wide-sense stationary across the measurements, we show that the average reference is the optimal reference for estimating the unknown sensor offsets in the network. Furthermore, when the noise is i.i.d, we show that the average reference offers a factor 2 improvement in the variance of any unbiased estimator w.r.t. any arbitrarily chosen sensor reference in the network. Finally, when the noise is independent across the sensors, but not identical, we derive an expression for the improvement offered by the average reference in contrast to any other sensor within the network.

Notation and properties: The Kronecker product is indicated by  $\otimes$ , the transpose operator by  $(\cdot)^T$  and  $\equiv$  denotes equality by definition.  $\mathbf{1}_N$  and  $\mathbf{0}_N$ denote a column vector of ones and zeros respectively.  $I_N$  is an identity matrix, bdiag(a) represents a diagonal matrix with elements of vector a along the diagonal, and vec(.) indicates vectorization. then we have

$$vec(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})vec(\mathbf{B}), \tag{1}$$

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A}\mathbf{C} \otimes \mathbf{B}\mathbf{D}),$$

$$(\mathbf{A} + \mathbf{b}\mathbf{b}^{T})^{-1} = \mathbf{A}^{-1} - \mu^{-1}\mathbf{A}^{-1}\mathbf{b}\mathbf{b}^{T}\mathbf{A}^{-1},$$
(3)

$$(\mathbf{A} + \mathbf{b}\mathbf{b}^T)^{-1} = \mathbf{A}^{-1} - \mu^{-1}\mathbf{A}^{-1}\mathbf{b}\mathbf{b}^T\mathbf{A}^{-1}, \tag{3}$$

where **A**, **B**, **C** and **D** are of appropriate dimensions, and (3) holds for  $\mu =$  $(1 + \mathbf{b}^T \mathbf{A}^{-1} \mathbf{b}) \neq 0$  and if **A** is non-singular.

#### 1.1 **Preliminaries**

Prior to modeling the sensor calibration problem, we briefly state the theoretical lower bound on the variance of any unbiased estimator under parametric constraints in Theorem 1, and in Theorem 2 we give the conditions for an optimal constraint set to yield an efficient estimator.

**Theorem 1** (Constrained Cramér Rao Bound (CCRB)). Consider a consistent set of k continuously differentiable constraints on  $\theta$  i.e.,  $\mathbf{c}(\theta) = 0$ , and let  $\mathbf{C}(\boldsymbol{\theta}) = \partial \mathbf{c}(\boldsymbol{\theta})/\partial \boldsymbol{\theta}^T$  be the gradient matrix which is full row rank, then the Constrained Cramér Rao lower Bound (CCRB) on the variance of any unbiased estimator exists, and is bounded by

$$\mathbb{E}\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}))(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}))^T\} \equiv \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}} \ge \mathbf{U}(\mathbf{U}^T \mathbf{F} \mathbf{U})^{-1} \mathbf{U}^T, \tag{4}$$

where U spans the null space of the gradient matrix  $C(\theta)$ , and the Fisher information matrix (FIM) is given by  $\mathbf{F} = -\mathbb{E}\{\partial^2 \log p(\mathbf{y};\boldsymbol{\theta})/\partial \boldsymbol{\theta}^2\}$ , where p(.) is the p.d.f. of the measurements y, which meets certain regularity conditions [22].

Proof. [23, Theorem 1.]

**Theorem 2** (Optimal constraint set). Any feasible set of linearly independent vectors which span the nullspace of the FIM forms an optimal constraint set.

*Proof.* Let the spectral decomposition of the FIM be  $\mathbf{F} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T$  where  $\mathbf{\Lambda}$  is a diagonal matrix containing the non-zero eigenvalues, and let  $\mathbf{V}$  contain the corresponding eigenvectors. Now, consider a constraint matrix  $\bar{\mathbf{C}}$ , such that the orthogonal basis for the null space of  $\bar{\mathbf{C}}$  spans the range of  $\mathbf{V}$ , then substituting for  $\mathbf{U}$  with  $\mathbf{V}$  in (4),we have

$$\operatorname{Tr}(\mathbf{\Sigma}_{\theta}) = \operatorname{Tr}[\mathbf{V}(\mathbf{V}^T \mathbf{F} \mathbf{V})^{-1} \mathbf{V}^T] \stackrel{(a)}{=} \operatorname{Tr}[\mathbf{\Lambda}^{-1}] \equiv \operatorname{Tr}(\mathbf{F}^{\dagger}),$$

where we substitute for  $\mathbf{F} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T$  to obtain (a), exploit the cyclic nature of the trace operator, and the orthonormal property of  $\mathbf{V}$ . Observe that the unconstrained FIM yields the lowest achievable variance for any unbiased estimator, and thus  $\bar{\mathbf{C}}$  is an optimal constraint set [20].

## 2 Data Model

Consider a network of N sensor nodes, where each sensor collects  $K \geq N$  measurements. Let the *true* signal impinging on the N sensors be given by a  $K \times N$  matrix  $\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N]$ , where the ground-truth  $\mathbf{S}$  lies in a lower dimensional subspace, say r, such that r < N. Furthermore, let  $\Gamma$  be a known  $(N-r) \times N$  projection matrix of rank r, which spans the orthogonal complement of the signal subspace, such that  $\mathbf{S}\Gamma^T = \mathbf{0}$ , then using (1), we have

$$(\mathbf{\Gamma} \otimes \mathbf{I}_K) vec(\mathbf{S}) = \bar{\mathbf{\Gamma}} \mathbf{s} = \mathbf{0}, \tag{5}$$

which is the underlying premise for blind calibration in sensor networks [10, 12, 13, 14]. In practise, if  $\Gamma$  is unknown, then the underlying subspace can be learnt over time [24]. We now consider a scenario where the N sensors are plagued with offset errors, and the measurements are corrupted with noise. The measurements of the nth sensor ( $1 \le n \le N$ ) are denoted by  $\mathbf{y}_n = \mathbf{s}_n + \theta_n \mathbf{1}_K + \eta_n$ , where  $\mathbf{s}_n$  is the true signal of length K,  $\theta_n$  is the unknown sensor offset and  $\eta_n$  is the stochastic noise on the measurements. Extending for all N sensors we have the following data model

$$y = s + H\theta + \eta, \tag{6}$$

where  $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \dots, \mathbf{y}_N^T]^T$ ,  $\mathbf{s} = [\mathbf{s}_1^T, \mathbf{s}_2^T, \dots, \mathbf{s}_N^T]^T$ ,  $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_N]^T$ ,  $\mathbf{H} \equiv \mathbf{I}_N \otimes \mathbf{1}_K$ , and  $\boldsymbol{\eta} = [\boldsymbol{\eta}_1^T, \boldsymbol{\eta}_2^T, \dots, \boldsymbol{\eta}_N^T]^T$ . Furthermore, let the noise on the system (6) be zero-mean Gaussian, where the covariance function is homogeneous i.e., wide sense stationary across the K measurements i.e.,  $\boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma} \otimes \mathbf{I}_K)$ . Then exploiting the assumption (5), multiplying (6) by the known  $\bar{\boldsymbol{\Gamma}}$ , and rearranging the terms we have

$$\bar{\Gamma} \mathbf{y} \sim \mathcal{N} (\bar{\Gamma} \mathbf{H} \boldsymbol{\theta}, \bar{\Gamma} (\mathbf{\Sigma} \otimes \mathbf{I}_K) \bar{\Gamma}^T).$$
 (7)

The Fisher information (FIM) for the above data model is straightforward [22], and is given by

$$\mathbf{F} = \mathbf{H}^T \bar{\mathbf{\Gamma}}^T (\mathbf{\Gamma} \mathbf{\Sigma} \mathbf{\Gamma}^T \otimes \mathbf{I}_K)^{-1} \bar{\mathbf{\Gamma}} \mathbf{H} = K \mathbf{\Gamma}^T (\mathbf{\Gamma} \mathbf{\Sigma} \mathbf{\Gamma}^T)^{-1} \mathbf{\Gamma}$$
(8)

where we substitute for  $\bar{\Gamma} = \Gamma \otimes \mathbf{I}_K$ ,  $\mathbf{H} = \mathbf{I}_N \otimes \mathbf{1}_K$  and use the relation (2). Since  $\Gamma$  is rank deficient, note that the FIM is also rank deficient. Let  $\bar{\mathbf{C}}$  span the nullspace of the FIM, then from Theorem 2, the optimal parametric constraint set is given by  $\bar{\mathbf{C}}^T \boldsymbol{\theta} = \mathbf{d}$ , where  $\mathbf{d}$  is a known response vector of length N - r. If sufficient data is collected i.e.,  $K \geq N$ , and if the covariance  $\Sigma$  is known, then  $\bar{\mathbf{C}}$  can be designed, which would lead to a *data-driven* reference.

### 2.1 Single source

In the following section, we look at a special case of (7), where the sensors measure an identical signal e.g., densely deployed sensor array measuring air quality [25]. In these scenarios, the signal subspace for the kth measurement ( $\forall k \leq K$ ) across all N nodes, spans a column vector of ones i.e.,  $\mathbf{1}_N$ . Subsequently, the projection matrix takes the form  $\mathbf{\Gamma} \equiv \mathbf{U}_2^T = \begin{bmatrix} -\mathbf{1}_{N-1} & \mathbf{I}_{N-1} \end{bmatrix}^T$ , and (7) and (8) simplify to

$$\mathbf{z} \equiv \bar{\mathbf{U}}_{2}^{T} \mathbf{y} \sim \mathcal{N}(\bar{\mathbf{U}}_{2}^{T} \mathbf{H} \boldsymbol{\theta}, \mathbf{U}_{2}^{T} \boldsymbol{\Sigma} \mathbf{U}_{2} \otimes \mathbf{I}_{K}), \tag{9}$$

$$\mathbf{F} = K\mathbf{U}_2(\mathbf{U}_2^T \mathbf{\Sigma} \mathbf{U}_2)^{-1} \mathbf{U}_2^T. \tag{10}$$

where  $\bar{\mathbf{U}}_2 = \mathbf{U}_2 \otimes \mathbf{I}_K$ . Observe that since  $\mathbf{U}_2$  is rank deficient,  $\mathbf{F}$  is also rank deficient by least r = 1, which is intuitively expected, since at least 1 reference is needed to uniquely estimate all the sensor offsets.

#### 2.2 Constraints

A conventional solution to estimate the offset is to arbitrarily assume one of the sensors as a reference, which without a loss of generality, we choose as sensor 1. The corresponding parametric constraint, the gradient vector and the orthonormal bases for the nullspace of the gradient vector are given as

$$\theta_1 = 0, \quad \mathbf{c}_1 = \begin{bmatrix} 1, \mathbf{0}_{N-1}^T \end{bmatrix}^T, \quad \mathbf{U}_1 = \begin{bmatrix} \mathbf{0}_{N-1}^T \\ \mathbf{I}_{N-1} \end{bmatrix},$$
 (11)

respectively, where we use the subscript 1 to denote a *single* reference. Alternatively, the average of all unknown offsets could be used as a reference, which has been proposed in various applications [19, 20, 21]. The constraint, the corresponding gradient and the bases for the nullspace of the gradient are then given as

$$\frac{1}{N} \sum_{n=1}^{N} \theta_n = 0, \quad \mathbf{c}_2 = N^{-1} \mathbf{1}_N, \quad \mathbf{U}_2 = \begin{bmatrix} -\mathbf{1}_{N-1}^T \\ \mathbf{I}_{N-1} \end{bmatrix}, \tag{12}$$

respectively, where the subscript 2 is used to denote the *average* reference. Increasing the number of constraints would further improve the performance

of any estimator, however, we limit our discussion in this paper to a single constraint, which is sufficient for identifiabilty.

#### 2.3 Effect on sensor bias

If the network is calibrated, then observe that  $\theta_n = 0 \,\,\forall\, n \leq N$ . Now, by choosing a reference e.g., sensor 1, we force the corresponding offset to 0 i.e.,  $\theta_1 = 0$ , and hence implicitly introduce a bias  $\phi_1$  into our final estimator, which reflects the unidentifiable offset of sensor 1. Along similar lines, (12) introduces a bias, which is the mean of  $\phi_n \,\,\forall\, n \leq N$ , where  $\phi_n$  is the implicit bias of the nth sensor. Now, if the mean of the underlying offsets are centered around 0, then as  $N \to \infty$ , observe that (12) minimizes the overall network bias. On the contrary, by using the single reference (11), we rely entirely on the performance of sensor 1, which leads to a single point of failure. In the next section we study the effect of these constraints on the variance of an unbiased estimator for  $\boldsymbol{\theta}$ .

### 3 Optimal reference

We now aim to compare the performance of any constrained unbiased estimator for (9), under varying assumptions of the covariance  $\Sigma$ . More concretely, let  $\Sigma_1$  and  $\Sigma_2$  denote the lower bound on the variance of any unbiased estimator under the constraints (11) and (12) respectively, then using (4), we aim to evaluate and analyze

$$Tr(\mathbf{\Sigma}_1) = Tr[(\mathbf{U}_1^T \mathbf{F} \mathbf{U}_1)^{-1}], \tag{13a}$$

$$\operatorname{Tr}(\mathbf{\Sigma}_2) = \operatorname{Tr}\left[ (\mathbf{U}_2^T \mathbf{F} \mathbf{U}_2)^{-1} \mathbf{U}_2^T \mathbf{U}_2 \right], \tag{13b}$$

where we exploit the cyclic nature of the trace operator, and the property  $\mathbf{U}_1^T \mathbf{U}_1 = \mathbf{I}$  in (13a). To this end, we have the following result.

**Theorem 3** (Optimal reference). Consider an network of N sensors, where each sensor collects K measurements based on the data model (9), then the following statements hold.

- (a) Heteroscedasticity: The optimal reference for estimating the unknown sensor offsets is the average reference (12)
- (b) Homoscedasticity: If  $\Sigma = \sigma^2 \mathbf{I}$ , the optimal reference for estimating the unknown offsets, outperforms any arbitrarily chosen reference in the network, by a factor 2.
- (c) Independent, but not identical: In the special case of

$$\Sigma = \mathrm{bdiag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_N^2), \tag{14}$$

the variance of the optimal unbiased estimator outperforms any arbitrarily chosen sensor reference by

$$\frac{N}{K} \frac{\sigma_i^2 \sum_{n=1, n \neq i}^N \sigma_n^2}{\sigma_i^2 + \sum_{n=1, n \neq i}^N \sigma_n^2},\tag{15}$$

where  $\sigma_i^2$  is the variance of the chosen sensor reference.

*Proof.* (Proof of Theorem 3(a)) Observe from (10) and (12) that  $\mathbf{Fc}_2 = \mathbf{0}$ , since  $\mathbf{1}_N$  spans the nullspace of  $\mathbf{U}_2^T$ , and hence using Theorem 2, the *average* (12) is the optimal reference. In other words, (12) yields a minimum variance unbiased estimator (MVUE) for the data model (9).

In the following sections, we present the proofs of Theorem 3(b) and Theorem 3(c).

### 3.1 Homoscedasticity

*Proof.* (Proof of Theorem 3(b)) The optimal reference for estimating the sensor offsets is the *average* of all the unknown offsets as proved in Theorem 3(a). Hence, when  $\Sigma = \sigma^2 \mathbf{I}$ , it suffices to show

$$\delta \equiv \text{Tr}(\mathbf{\Sigma}_2)/\text{Tr}(\mathbf{\Sigma}_1) = 0.5,\tag{16}$$

for a homoscedastic system, where  $\text{Tr}(\Sigma_1)$  and  $\text{Tr}(\Sigma_2)$  are given in (13). In this scenario, the FIM in (10) simplifies to

$$\mathbf{F} = \sigma^{-2} K \mathbf{U}_2 (\mathbf{U}_2^T \mathbf{U}_2)^{-1} \mathbf{U}_2^T = \sigma^{-2} K \mathbf{U}_2 \mathbf{\Psi}^{-1} \mathbf{U}_2^T, \tag{17}$$

where we define

$$\mathbf{\Psi} \equiv \mathbf{U}_2^T \mathbf{U}_2 = \mathbf{I}_{N-1} + \mathbf{1}_{N-1} \mathbf{1}_{N-1}^T, \tag{18a}$$

$$\mathbf{\Psi}^{-1} \equiv (\mathbf{U}_2^T \mathbf{U}_2)^{-1} = \mathbf{I}_{N-1} - N^{-1} \mathbf{1}_{N-1} \mathbf{1}_{N-1}^T, \tag{18b}$$

and use the Sherman-Morrison formula (3) to obtain (18b). Now, substituting the FIM (17) and  $U_1$  (11) in (13a), the CCRB for a single reference is

$$\operatorname{Tr}(\mathbf{\Sigma}_{1}) = \frac{\sigma^{2}}{K} \operatorname{Tr} \left[ (\mathbf{U}_{1}^{T} \mathbf{U}_{2} \mathbf{\Psi}^{-1} \mathbf{U}_{2}^{T} \mathbf{U}_{1})^{-1} \right] \stackrel{(a)}{=} \frac{\sigma^{2}}{K} \operatorname{Tr}(\mathbf{\Psi}),$$

$$\stackrel{(b)}{=} \frac{\sigma^{2}}{K} \operatorname{Tr}[\mathbf{I}_{N-1} + \mathbf{1}_{N-1} \mathbf{1}_{N-1}^{T}] = \frac{2\sigma^{2}}{K} (N-1). \tag{19}$$

where we use the property  $\mathbf{U}_1^T\mathbf{U}_2 = \mathbf{I}$  in (a), and substitute for  $\mathbf{\Psi}$  (18a) to obtain (b) in (19)). Along similar lines, the CCRB for the average reference (12) is obtained by substituting for the FIM (17) in (13b), which yields

$$\operatorname{Tr}(\mathbf{\Sigma}_{2}) = \frac{\sigma^{2}}{K} \operatorname{Tr}\left[\left(\mathbf{U}_{2}^{T}\left(\mathbf{U}_{2}\mathbf{\Psi}^{-1}\mathbf{U}_{2}^{T}\right)\mathbf{U}_{2}\right)^{-1}\mathbf{U}_{2}^{T}\mathbf{U}_{2}\right],$$

$$= \frac{\sigma^{2}}{K} \operatorname{Tr}(\mathbf{\Psi}) = \frac{\sigma^{2}}{K}(N-1), \tag{20}$$

where we use the definitions (18). Finally, from (19) and (20), we have (16), and hence proved.  $\Box$ 

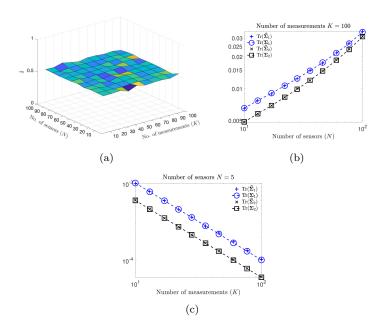


Figure 1: Figure (a) shows an estimate of (16) i.e.,  $\hat{\delta} = \hat{\Sigma}_2/\hat{\Sigma}_1$ , for varying K and N, illustrating the factor 2 improvement using the average reference in contrast to a single reference in the network, under i.i.d Gaussian noise. Under the assumption of independent but not identical Gaussian noise, (b) and (c) show that the estimator performance, for varying K (fixed N) and varying N (fixed K) respectively.

### 3.2 Independent, but not identical

We now consider the scenario (14), where without loss of generality, we assume  $\sigma_1^2 \leq \sigma_2^2 \leq \ldots \leq \sigma_N^2$ . Following immediately, sensor 1 (with the lowest variance) is an appropriate reference for calibration. To this end, the constraints (11) hold, and thus proving Theorem 3(c) is equivalent to showing

$$Tr(\Sigma_1) - Tr(\Sigma_2) = \frac{N}{K} \frac{\sigma_1^2 \sum_{n=2}^{N} \sigma_n^2}{\sigma_1^2 + \sum_{n=2}^{N} \sigma_n^2}.$$
 (21)

*Proof.* (Proof of Theorem 3(c)) The CCRB for the single reference scenario is obtained by substituting for the FIM (10) and  $U_1$  (11) in (13a), which yields

$$\operatorname{Tr}(\mathbf{\Sigma}_{1}) = K^{-1}\operatorname{Tr}\left[\left(\mathbf{U}_{1}^{T}\mathbf{U}_{2}(\mathbf{U}_{2}^{T}\mathbf{\Sigma}\mathbf{U}_{2})^{-1}\mathbf{U}_{2}^{T}\mathbf{U}_{1}\right)^{-1}\right]$$
$$= K^{-1}\operatorname{Tr}(\mathbf{U}_{2}^{T}\mathbf{\Sigma}\mathbf{U}_{2}) = K^{-1}\operatorname{Tr}(\mathbf{\Omega}), \tag{22}$$

where we exploit  $\mathbf{U}_1^T \mathbf{U}_2 = \mathbf{I}$  and introduce

$$\mathbf{\Omega} \equiv \mathbf{U}_2^T \mathbf{\Sigma} \mathbf{U}_2 = \bar{\mathbf{\Sigma}} + \sigma_1^2 \mathbf{1}_{N-1} \mathbf{1}_{N-1}^T. \tag{23}$$

The CCRB for the average reference (12), using (13b) is,

$$\operatorname{Tr}(\mathbf{\Sigma}_{2}) = \operatorname{Tr}\left[\left(\mathbf{U}_{2}^{T}(\mathbf{U}_{2}(\mathbf{U}_{2}^{T}\mathbf{\Sigma}\mathbf{U}_{2})^{-1}\mathbf{U}_{2}^{T}\right)\mathbf{U}_{2}\right)^{-1}\mathbf{U}_{2}^{T}\mathbf{U}_{2}\right]$$
$$=K^{-1}\operatorname{Tr}\left[\left(\mathbf{\Psi}\mathbf{\Omega}^{-1}\mathbf{\Psi}\right)^{-1}\mathbf{\Psi}\right] = K^{-1}\operatorname{Tr}\left(\mathbf{\Omega}\mathbf{\Psi}^{-1}\right), \tag{24}$$

where we use the definitions (18) and (23). Subtracting (24) from (22), we have  $\operatorname{Tr}(\mathbf{\Sigma}_1) - \operatorname{Tr}(\mathbf{\Sigma}_2) = NK^{-1}\operatorname{Tr}(\mathbf{\Omega}\mathbf{1}_{N-1}\mathbf{1}_{N-1}^T)$ . Substituting for  $\mathbf{\Omega}$  from (23), and further manipulations yield (21), and hence proved.

### 4 Simulations: Clock synchronization

We consider a sensor network of N clocks, where each clock is plagued with an offset, and collects K measurements [26, 27]. Let  $\mathbf{t} = [t_1, t_2, \dots, t_K]$  denote the true time over K instances, and let  $\mathbf{T}_i = [T_{i,1}, T_{i,2}, \dots, T_{i,K}]^T$  be the K measurements at the ith clock, then we have  $\mathbf{T}_i = \mathbf{t} + \theta_i \mathbf{1}_K + \boldsymbol{\eta}_i$ , where  $\theta_i$  and  $\boldsymbol{\eta}_i$  are the clock offset and the stochastic noise plaguing the ith clock respectively. Let  $\mathbf{T} = [\mathbf{T}_1, \mathbf{T}_2, \dots, \mathbf{T}_N]^T$  populate all the time measurements from the N sensors, then observe that the data model (9) holds where  $\mathbf{y} = vec(\mathbf{T})$ . Subsequently, to estimate the unknown sensor offsets, we aim to minimize the constrained weighted least squares formulation

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\mathbf{W}\bar{\mathbf{P}}(\mathbf{y} - \mathbf{H}\boldsymbol{\theta})\|_{2}^{2} \quad \text{s.t. } \mathbf{c}(\boldsymbol{\theta}),$$
(25)

where **W** is a weighting matrix,  $\bar{\mathbf{P}} = \mathbf{P} \otimes \mathbf{I}_K$ , where **P** is a  $N \times N$  centering matrix, **c** is the gradient of the parametric constraint. The cost function has a closed form solution for a non-empty constraint set **c**, such that  $\begin{bmatrix} \mathbf{H}^T \bar{\mathbf{P}}^T & \mathbf{c}^T \end{bmatrix}^T$  is full rank [28]. The estimator (25) under constraints (11) and (12) are denoted by  $\hat{\boldsymbol{\theta}}_1$  and  $\hat{\boldsymbol{\theta}}_2$  respectively. Observe that, with  $\mathbf{W} = \boldsymbol{\Sigma}^{-1/2}$ , (25) yields the optimal estimator i.e., the minimum variance unbiased estimator for (9), independently for the respective constraints (11) and (12).

We perform experiments to validate the derived lower bounds, using the performance of the optimal estimators (25) under parametric constraints (11) and (12), for varying assumptions on the noise covariance matrix. In case of  $\Sigma = \sigma^2 \mathbf{I}$ , where  $\sigma^2 = 10^{-3}$ , we perform a Monte carlo experiment by linearly varying both the number of sensors and measurements from 10 to 100. We evaluate an estimate of (16) i.e.,  $\hat{\delta} = \hat{\Sigma}_2/\hat{\Sigma}_1$  for each parameter set, where  $\hat{\Sigma}_1$  and  $\hat{\Sigma}_2$  are the estimated covariances under constraints (11) and (12) respectively, over  $N_{exp} = 500$  Monte Carlo runs. Figure 1(a) visually illustrates the factor 2 improvement, which is invariant over the number of measurements or sensors, as stated in Theorem 3(b).

When the underlying noise is independent, but not identical (14), we consider two independent simulations i.e., a fixed number of sensors N=5, and increasing number of measurements from K=10 to 100 in Figure 1(b), and a fixed number of measurements K=100, and varying the number of sensors

from N=10 to 100 in Figure 1(c). Note that the estimated variances meet the CCRBs, as expected. Furthermore, observe from (22) and (24), that for a fixed N, the CCRB linearly decreases with increasing K. However, for varying N with a fixed K, the distribution of variances impacts the performance of the estimator. If the probability distribution of the variances is available, then further deductions can be made, using Theorem 3(c).

Note that in the absence of a known reference, our data model in the presence of constraints (11) and (12), will implicitly introduce a bias which depends on the offsets as discussed in Section 2.3. The average reference will minimize this bias for larger N, or if the underlying sensor offset distribution is symmetrical e.g., standard normal distribution. In more practical scenarios, despite smaller number of sensors e.g., N = 5, the average reference has shown to outperform a single reference for limited datasets [21].

### 5 Conclusion

In this paper, we investigated the effect of the choice of reference in offset estimation of a sensor network, where sensor measurements are plagued with zero-mean Gaussian noise. We show that the average of all the unknown offsets as a parametric constraint i.e., the average reference, yields the MVUE. In the particular case of homoscedasticity, the improvement on the variance of the MVUE is by a factor 2, as compared to any single reference, which is not readily intuitive at the outset of the problem formulation. In our future work, we aim to investigate an optimal constraint set for other (non-linear) data models and probability distributions, including a Bayesian inference framework when the signal subspace is unknown, and where the choice of the prior on the unknown parameters would play a key role in the optimality of the constrained estimator.

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