On Using Exponential Parameter Estimators with an Adaptive Controller

Parag Patre and Suresh M. Joshi

Abstract— **Typical adaptive controllers are restricted to using a specific update law to generate parameter estimates. This paper investigates the possibility of using *any* exponential parameter estimator with an adaptive controller such that the system tracks a desired trajectory. The goal is to provide flexibility in choosing any update law suitable for a given application. The development relies on a previously developed concept of controller/update law modularity in the adaptive control literature, and the use of a converse Lyapunov-like theorem. Stability analysis is presented to derive gain conditions under which this is possible, and inferences are made about the tracking error performance. The development is based on a class of Euler-Lagrange systems that are used to model various engineering systems including space robots and manipulators.**

I. INTRODUCTION

Control laws with adaptive feedforward terms have been widely used to compensate for the linear-in-the-parameter (LP) uncertainty in nonlinear systems. The update law or a parameter estimator is an integral part of an adaptive controller that generates parameter estimates used by the adaptive controller. The convergence properties of the estimator directly impact the tracking error performance, even though asymptotic tracking may still be possible without the estimates converging to their true values. Lyapunov-based techniques are commonly used to design and analyze an adaptive control law and its corresponding parameter update law, where the controller and the update law are designed in conjunction with one another such that the update law cancels certain cross terms in the stability analysis to make the overall time-derivative of the Lyapunov function negative. This restrains the design of the update law to one specific form, which typically is a gradient update law. Gradient update laws are notorious for exhibiting slower parameter convergence [1]–[3], which could lead to a slower transient performance of the tracking error in comparison to other possible adaptive update laws (e.g., least-squares or recursive least-squares update law [1], [3]–[5]). In order to overcome this issue of slow convergence, several results have been developed in the literature that aim to modify or augment the typical tracking error-based gradient update law. For example, composite adaptive update laws [1], [2], [6];

prediction error (a predicted value of the difference between the actual parameter and its estimate)-based update laws [7]–[11]; and various least-squares update laws [3], [5], [12]. In addition, several results have discussed the robustness and exponential convergence properties of estimation schemes [1], [4], [5], [13], [14]. However, the use of these update laws is still contingent upon their ability to cancel cross terms in the Lyapunov-based stability analysis. That is, only a specific type of update law can be used with the adaptive controller. This raises the question if we can have flexibility in choosing a parameter update law best suited for a given application to potentially achieve a better performance.

Intuitively, if a parameter estimator has an exponential convergence, it could be combined with the adaptive controller to achieve tracking. However, as mentioned earlier, stability cannot be guaranteed unless it is designed to cancel certain cross terms. The result in this paper is motivated by the desire to investigate if a formal proof can be presented for guaranteeing stability and tracking performance if *any* such exponential parameter estimator is used with the adaptive controller. In the existing literature, there are various results on a class of modular adaptive controllers that provide such controller/update law separation (cf. [7], [9]–[11]). One approach is to use nonlinear damping [8], [15] to yield a strong controller that is input-to-state stable (ISS) with respect to the parameter estimation error. This enables the use of any update law that yields bounded parameter estimates. A brief survey of nonlinear damping-based modular adaptive control results is provided in [7]. Another approach was investigated in [16] by designing a robust control law using the RISE (robust integral of the sign of the error [17]) feedback method, such that a generic form of an update law in terms of tracking errors could be used.

In this paper, modularity in the update law (i.e., controller/identifier separation), and tracking is achieved in three steps. First, for a system which is linear in uncertain constant parameters, an adaptive controller is developed and augmented with a nonlinear damping term. This makes the system ISS with respect to the estimation error. That is, for a bounded parameter estimation error, the tracking error is bounded. Second, a converse Lyapunov-like theorem is presented, and existence of a Lyapunov-like function is shown for systems with trajectories that are exponentially decaying to a small bound. Third, any parameter estimator that gives a parameter estimation error exponentially decaying to a small bound is used with the adaptive controller. Based on the second step, the existence of a Lyapunov-like function for such a parameter estimator is guaranteed. This combined

with the first step is used to prove a tracking result under some sufficient gain conditions such that, if the parameter estimation error is uniformly ultimately bounded (UUB), then the tracking error is also UUB.

Typical adaptive control designs yield asymptotic tracking without any tracking performance guarantees. In contrast, the current result proves only UUB tracking, but guarantees performance bounds on the tracking error while allowing the use of a broad class of exponential estimators. Moreover, if the parameter estimator has an exponential convergence to zero, the tracking errors also converge to zero exponentially. Compared to the previous results on controller/update law modularity, the current result can be considered as an alternative choice by the control design engineers. The result in this paper achieves modularity in the controller/update law using ISS arguments similar to [7]–[11] by including a nonlinear damping term in the torque control input. However, in [7]–[11], in order to prove tracking, additional assumptions about the update law itself had to be made. For example, it needed to be driven by a prediction error, which must be square integrable; the update law must be bounded, etc. This in turn required extra terms in the control law. Moreover, the estimator must also ensure that the estimate of the inertia matrix be positive definite. In most cases, this required the use of projection algorithms, thus leading to an increased complexity, and even though asymptotic tracking was proved, no bounds on the tracking error transient could be derived. In contrast, the result in this paper only assumes that the parameter identification scheme gives an estimation error that exponentially decreases to a small bound, and no additional assumptions are required.

Since the objective is to design an adaptive control law that utilizes parameter estimates generated by any estimator with exponential convergence, design of a specific estimator is not considered in this work. It should be noted that the construction of any such practical estimator may require additional inputs, such as bounds on measurement noise, bound on estimation error, etc. [1], [4], [5], [13], [14].

II. DYNAMIC MODEL AND PROPERTIES

The class of nonlinear dynamic systems considered in this paper can be described by the following Euler-Lagrange formulation, which can be used to model a large class of engineering systems such as space robots and manipulators [18], [19]:

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + G(q) + F(\dot{q}) = \tau(t). \qquad (1)$$

In (1), $M(q) \in \mathbb{R}^{n \times n}$ denotes the inertia matrix, $V_m(q,\dot{q}) \in \mathbb{R}^{n \times n}$ denotes the centripetal-Coriolis matrix, $G(q) \in \mathbb{R}^n$ denotes the gravity vector, $F(\dot{q}) \in \mathbb{R}^n$ denotes friction, $\tau(t) \in \mathbb{R}^n$ represents the torque input control vector, and $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^n$ denote the generalized position, velocity, and acceleration vectors, respectively. The subsequent development is based on the assumption that $q(t)$ and $\dot{q}(t)$ are measurable and that $M(q), V_m(q,\dot{q}), G(q)$, and $F(\dot{q})$ are unknown. Throughout the paper $|\cdot|$ denotes the absolute value of the scalar argument, $\|\cdot\|$ denotes the standard Euclidean

norm for a vector or the induced norm for a matrix. The following properties will be exploited in the subsequent development.

Property 1: The inertia matrix $M(q)$ is symmetric, positive definite. Therefore, it satisfies the following inequality $\forall q(t), \xi(t) \in \mathbb{R}^n$ [18], [19]:

$$m_1 \|\xi\|^2 \le \xi^T M(q)\xi \le m_2 \|\xi\|^2 \qquad (2)$$

where $m_1, m_2 \in \mathbb{R}$ are known positive constants.

Property 2: The time derivative of inertia matrix $\dot{M}(q)$ and the centripetal-Corilois matrix $V_m(q,\dot{q})$ satisfy the following skew-symmetry property $\forall q(t), \dot{q}(t), \xi(t) \in \mathbb{R}^n$ [18], [19]:

$$\xi^T \left(\frac{1}{2}\dot{M}(q) - V_m(q,\dot{q}) \right) \xi = 0. \qquad (3)$$

III. CONTROL OBJECTIVE AND DEVELOPMENT

The objective is to design an adaptive controller with the flexibility of using any parameter estimator that has an exponential parameter convergence to a small error bound, such that the system tracks a desired time-varying trajectory $q_d(t)$ despite uncertainties in the dynamic model. To quantify this objective, a position tracking error, denoted by $e_1(t) \in \mathbb{R}^n$, is defined as

$$e_1 \triangleq q_d - q. \qquad (4)$$

In addition, to facilitate the subsequent analysis, a filtered tracking error [18], $e_2(t) \in \mathbb{R}^n$, is defined as

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1 \qquad (5)$$

where $\alpha_1 \in \mathbb{R}$ denotes a positive constant. Differentiating (5) with respect to time, and premultiplying the resulting expression with $M(q)$ yields

$$M(q)\dot{e}_2 = M(q)\ddot{q}_d - M(q)\ddot{q} + M(q)\alpha_1\dot{e}_1. \qquad (6)$$

Using (1), (4), (5) in (6), we get

$$M(q)\dot{e}_2 = Y\theta - \tau(t) - V_m(q,\dot{q})e_2 \qquad (7)$$

where the following property is used:

Property 3: A part of the dynamics in (1) can be linearly parameterized as [18], [20]

$$\begin{aligned} Y\theta &= M(q)\ddot{q}_d + V_m(q,\dot{q})(\dot{q}_d + \alpha_1 e_1) + G(q) \\ &\quad + F(\dot{q}) + M(q)\alpha_1\dot{e}_1 \end{aligned} \qquad (8)$$

where $\theta \in \mathbb{R}^p$ contains the constant unknown system parameters (e.g., component masses, moments of inertia, damping and friction coefficients, etc.), and $Y(q,\dot{q},\dot{q}_d,\ddot{q}_d) \in \mathbb{R}^{n \times p}$ is a regression matrix that contains known nonlinear functions of the known generalized coordinates $q(t), \dot{q}(t)$ and desired trajectories.

Based on the open-loop error system in (7), the torque control input is designed as

$$\tau(t) = Y\hat{\theta}(t) + k_1 e_2(t) + k_n \|Y\|^2 e_2(t) \qquad (9)$$

where $k_1, k_n \in \mathbb{R}$ are positive constants, $\hat{\theta}(t) \in \mathbb{R}^p$ is an adaptive estimate of θ. Nonlinear damping terms, similar to $k_n \|Y\|^2 e_2(t)$ in the control input, have been previously

used in results such as [8], [10]. Substituting (9) in to (7), we get the following closed-loop error system:

$$M(q)\dot{e}_2 = Y\tilde{\theta} - k_1 e_2 - k_n \|Y\|^2 e_2 - V_m(q,\dot{q})e_2 \quad (10)$$

where $\tilde{\theta}(t) \in \mathbb{R}^p$ is the parameter estimation error defined as

$$\tilde{\theta}(t) = \theta - \hat{\theta}(t).$$

IV. INPUT-TO-STATE STABILITY

In this section, we show the ISS property of the system with respect to the parameter estimation error $\tilde{\theta}(t)$. This allows controller/update law modularity, and thus, any update law that generates bounded parameter estimates can be used with the adaptive controller. We now have the theorem:

Theorem 1: The closed-loop error system in (10) is input-to-state stable with respect to $\tilde{\theta}(t)$, i.e., if $\left\|\tilde{\theta}(t)\right\| \in \mathcal{L}_\infty$, then $\|e_1(t)\| \in \mathcal{L}_\infty$.

Proof: Define a Lyapunov function candidate

$$V_{iss} = \frac{1}{2}e_2^T M(q) e_2. \quad (11)$$

Using (2), $V_{iss}(t)$ can be lower and upper bounded as

$$\frac{1}{2}m_1 \|e_2\|^2 \leq V_{iss} \leq \frac{1}{2}m_2 \|e_2\|^2. \quad (12)$$

Taking the time derivative of (11), and substituting from (10), we get

$$\dot{V}_{iss} = e_2^T[Y\tilde{\theta} - k_1 e_2 - k_n \|Y\|^2 e_2 - V_m(q,\dot{q})e_2] \quad (13)$$
$$+ \frac{1}{2}e_2^T \dot{M}(q) e_2.$$

Using the skew-symmetry property (3), the expression in (13) can be upper bounded as

$$\dot{V}_{iss} \leq -k_1 \|e_2\|^2 + \|e_2\| \|Y\| \left\|\tilde{\theta}\right\| - k_n \|e_2\|^2 \|Y\|^2.$$

Completing the squares for the last two terms

$$\dot{V}_{iss} \leq -k_1 \|e_2\|^2 - k_n \left(\|Y\| \|e_2\| - \frac{1}{2k_n}\left\|\tilde{\theta}\right\|\right)^2 + \frac{1}{4k_n}\left\|\tilde{\theta}\right\|^2$$

therefore,

$$\dot{V}_{iss} \leq -k_1 \|e_2\|^2 + \frac{1}{4k_n}\left\|\tilde{\theta}\right\|^2. \quad (14)$$

Using (12) in (14), the upper bound on $\dot{V}_{iss}(t)$ can be expressed as

$$\dot{V}_{iss} \leq -2\lambda_{iss}V_{iss} + \frac{1}{4k_n}\left\|\tilde{\theta}\right\|^2 \quad (15)$$

where

$$\lambda_{iss} = \frac{k_1}{m_2}.$$

After solving the differential inequality in (15), we get

$$V_{iss}(t) \leq V_{iss}(0)e^{(-2\lambda_{iss}t)} + \frac{1}{8\lambda_{iss}k_n}\sup_{0\leq\tau\leq t}\left\|\tilde{\theta}(\tau)\right\|^2. \quad (16)$$

Hence, from (12) and (16)

$$\|e_2(t)\|^2 \leq \frac{m_2}{m_1}\|e_2(0)\|^2 e^{(-2\lambda_{iss}t)} \quad (17)$$
$$+ \frac{1}{4m_1\lambda_{iss}k_n}\sup_{0\leq\tau\leq t}\left\|\tilde{\theta}(\tau)\right\|^2.$$

Using the fact that $a^2 \leq b^2 + c^2 \Rightarrow a \leq b + c$ for all positive scalars a, b, c, (17) can be expressed as

$$\|e_2(t)\| \leq \sqrt{\frac{m_2}{m_1}}\|e_2(0)\| e^{(-\lambda_{iss}t)} \quad (18)$$
$$+ \frac{1}{2\sqrt{m_1\lambda_{iss}k_n}}\sup_{0\leq\tau\leq t}\left\|\tilde{\theta}(\tau)\right\|.$$

The bound in (18) indicates input-to-state stability (ISS) (cf. Def 4.7, pp. 175 [21]). That is, if $\sup_{0\leq\tau\leq t}\left\|\tilde{\theta}(\tau)\right\|$ is bounded, then $\|e_2(t)\|$ is bounded, and therefore, from (5), the tracking error $\|e_1(t)\|$ is bounded. $\square$

Any update law that guarantees a bounded $\left\|\tilde{\theta}\right\|$ can be used. Further, if $\left\|\tilde{\theta}(t)\right\|$ is UUB, then subsequently it will be shown (Theorem 3) that the tracking error is UUB. First we prove the following converse Lyapunov-like theorem.

V. CONVERSE LYAPUNOV-LIKE THEOREM

The following can be proved directly by extending the result in Theorem 4.14 of [21].

Theorem 2: Consider a nonlinear system

$$\dot{x}(t) = f(t,x) \quad (19)$$

where $f : [0,\infty) \times D \to \mathbb{R}^n$ is continuously differentiable, $D = \{x \in \mathbb{R}^n \mid \|x\| < r\}$ for some r, and the Jacobian matrix $[\partial f/\partial x]$ is uniformly bounded on D. Let $x = 0$ be an equilibrium point for the system in (19). Let k, λ, ε, and r_0 be positive constants with $r_0 < r/k$. Let $D_0 = \{x \in \mathbb{R}^n \mid \|x\| < r_0\}$. Assume that $\forall x(t_0) \in D_0$, $\forall t \geq t_0 \geq 0$, the trajectories of the system in (19) satisfy

$$\|x(t)\| \leq k\|x(t_0)\| e^{-\lambda(t-t_0)} + \varepsilon.$$

Then, there exists a function $V : [0,\infty) \times D_0 \to \mathbb{R}$ that satisfies the inequalities

$$c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2 + c_3 \|x\| + c_4$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) \leq -c_5 \|x\|^2 + c_6 \|x\| + c_7$$

for some positive constants $c_1, c_2, \ldots, c_7$. Moreover, if $r = \infty$ and the origin is globally uniformly ultimately bounded, then $V(t,x)$ is defined and satisfies the aforementioned inequalities on $\mathbb{R}^n$. Furthermore, if the system is autonomous, V can be chosen independent of t.

Proof: Let $\phi(\tau;t,x)$ denote the solution of the system starting at (t,x); i.e., $\phi(t;t,x) = x$. For all $x \in D_0$, $\phi(\tau;t,x) \in D$ for all $\tau \geq t$. Let

$$V(t,x) = \int_t^{t+\delta} \phi^T(\tau;t,x)\phi(\tau;t,x)d\tau \quad (20)$$

where δ is a positive constant to be chosen. Due to exponentially decaying bound on the trajectories, we have

$$
\begin{aligned}
V(t,x) &= \int_t^{t+\delta} \|\phi(\tau;t,x)\|^2 \, d\tau \\
&\leq \int_t^{t+\delta} (k^2 \|\phi(t;t,x)\|^2 \, e^{-2\lambda(\tau-t)} \\
&\quad + 2\varepsilon k \|\phi(t;t,x)\| \, e^{-\lambda(\tau-t)} + \varepsilon^2) d\tau
\end{aligned}
$$

Using the fact $\phi(t;t,x) = x$,

$$
V(t,x) \leq \frac{k^2}{2\lambda} \left(1 - e^{-2\lambda\delta}\right) \|x\|^2 + \frac{2\varepsilon k}{\lambda} \left(1 - e^{-\lambda\delta}\right) \|x\| + \varepsilon^2\delta. \tag{21}
$$

Since, the Jacobian matrix $[\partial f/\partial x]$ is uniformly bounded on D, let

$$
\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq L, \quad \forall \, x \in D.
$$

Then,

$$
\|f(t,x)\| \leq L \|x\| \tag{22}
$$

and $\phi(\tau;t,x)$ satisfies the lower bound (see Appendix, Lemma 1)

$$
\|\phi(\tau;t,x)\|^2 \geq \|x\|^2 \, e^{-2L(\tau-t)}. \tag{23}
$$

Hence, using (23) in (20), $V(t,x)$ satisfies the lower bound

$$
V(t,x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau \|x\|^2 = \frac{1}{2L}\left(1 - e^{-2L\delta}\right) \|x\|^2. \tag{24}
$$

Thus, from (21) and (24), $V(t,x)$ satisfies the first inequality of the theorem with

$$
\begin{aligned}
c_1 &= \frac{1}{2L}\left(1 - e^{-2L\delta}\right), \quad c_2 = \frac{k^2}{2\lambda}\left(1 - e^{-2\lambda\delta}\right) \tag{25} \\
c_3 &= \frac{2\varepsilon k}{\lambda}\left(1 - e^{-\lambda\delta}\right), \quad c_4 = \varepsilon^2\delta.
\end{aligned}
$$

To calculate the derivative of $V(t,x)$ along the system trajectories, the following sensitivity functions are defined

$$
\phi_t(\tau;t,x) = \frac{\partial}{\partial t}\phi(\tau;t,x); \quad \phi_x(\tau;t,x) = \frac{\partial}{\partial x}\phi(\tau;t,x).
$$

Then,

$$
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) \\
&\quad -\phi^T(t;t,x)\phi(t;t,x) \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_t(\tau;t,x)\,d\tau \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_x(\tau;t,x)\,d\tau \\
&\quad \times f(t,x) \\
&= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau;t,x)[\phi_t(\tau;t,x) \\
&\quad + \phi_x(\tau;t,x)f(t,x)]d\tau - \|x\|^2.
\end{aligned}
$$

Using the fact that (see Appendix, Lemma 2)

$$
\phi_t(\tau;t,x) + \phi_x(\tau;t,x)f(t,x) \equiv 0, \quad \forall \, \tau \geq t \tag{26}
$$

we get

$$
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) &= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) - \|x\|^2 \\
&\leq (k\|x\|e^{-\lambda\delta} + \varepsilon)^2 - \|x\|^2 \\
&\leq -\left(1 - k^2 e^{-2\lambda\delta}\right)\|x\|^2 \\
&\quad + 2\varepsilon k e^{-\lambda\delta}\|x\| + \varepsilon^2.
\end{aligned}
$$

Therefore, the second inequality of the theorem is satisfied with

$$
c_5 = 1 - k^2 e^{-2\lambda\delta}, \quad c_6 = 2\varepsilon k e^{-\lambda\delta}, \quad c_7 = \varepsilon^2. \tag{27}
$$

$\square$

VI. Parameter Estimation and Tracking

Theorem 3: Suppose there exists a parameter estimator such that the parameter estimation error $\tilde{\theta}(t)$ satisfies

$$
\left\|\tilde{\theta}(t)\right\| \leq k\left\|\tilde{\theta}(0)\right\|e^{-\lambda t} + \varepsilon \tag{28}
$$

then, the control law (9) in conjunction with the parameter estimator yields globally uniformly ultimately bounded (GUUB) tracking stability for the system in (1) and all closed-loop signals are bounded.

Proof: Theorem 2 implies that there exists a function $V_{clf}\left(t,\tilde{\theta}\right)$ such that

$$
c_1\left\|\tilde{\theta}\right\|^2 \leq V_{clf} \leq c_2\left\|\tilde{\theta}\right\|^2 + c_3\left\|\tilde{\theta}\right\| + c_4 \tag{29}
$$

and

$$
\dot{V}_{clf} \leq -c_5\left\|\tilde{\theta}\right\|^2 + c_6\left\|\tilde{\theta}\right\| + c_7. \tag{30}
$$

Define

$$
V_s = V_{iss} + V_{clf} \tag{31}
$$

where V_{iss} was defined in (11), which satisfies the following lower and upper bounds:

$$
V_s \geq \frac{1}{2}m_1\|e_2\|^2 + c_1\left\|\tilde{\theta}\right\|^2 \tag{32}
$$

$$
V_s \leq \frac{1}{2}m_2\|e_2\|^2 + c_2\left\|\tilde{\theta}\right\|^2 + c_3\left\|\tilde{\theta}\right\| + c_4. \tag{33}
$$

Taking the time derivative of (31), and using the results of Theorems 1 and 2

$$
\dot{V}_s \leq -k_1\|e_2\|^2 + \frac{1}{4k_n}\left\|\tilde{\theta}\right\|^2 - c_5\left\|\tilde{\theta}\right\|^2 + c_6\left\|\tilde{\theta}\right\| + c_7. \tag{34}
$$

Using the upper bound of (33) in (34)

$$
\begin{aligned}
\dot{V}_s &\leq -2\lambda_{iss}V_s - \left(c_5 - 2\lambda_{iss}c_2 - \frac{1}{4k_n}\right)\left\|\tilde{\theta}\right\|^2 \\
&\quad + (2\lambda_{iss}c_3 + c_6)\left\|\tilde{\theta}\right\| + 2\lambda_{iss}c_4 + c_7. \tag{35}
\end{aligned}
$$

Let $\gamma_1 = c_5 - 2\lambda_{iss}c_2 - \frac{1}{4k_n}$; $\gamma_2 = 2\lambda_{iss}c_3 + c_6$; $\gamma_3 = 2\lambda_{iss}c_4 + c_7$, where γ_1, γ_2, γ_3 are positive constants such that the following sufficient conditions should be satisfied

$$
k_n > \frac{1}{4(c_5 - 2\lambda_{iss}c_2)}, \quad \lambda_{iss} < \frac{c_5}{2c_2}. \tag{36}
$$

Completing the squares for the second and the third terms in (35)

$$\dot{V}_s \leq -2\lambda_{iss}V_s + \gamma \tag{37}$$

where the positive constant γ is defined as

$$\gamma = \frac{\gamma_2^2}{4\gamma_1} + \gamma_3. \tag{38}$$

After solving the differential inequality of (37), we get

$$V_s(t) \leq V_s(0)e^{-2\lambda_{iss}t} + \frac{\gamma}{2\lambda_{iss}}\left(1 - e^{-2\lambda_{iss}t}\right). \tag{39}$$

From (32), we have

$$\frac{1}{2}m_1\|e_2\|^2 \leq \frac{1}{2}m_1\|e_2\|^2 + c_1\left\|\tilde{\theta}\right\|^2 \leq V_s.$$

Then, the following bound on the filtered tracking error $e_2(t)$ can be obtained:

$$\|e_2(t)\|^2 \leq \frac{2V_s(0)}{m_1}e^{-2\lambda_{iss}t} + \frac{\gamma}{m_1\lambda_{iss}}. \tag{40}$$

Therefore,

$$\|e_2(t)\| \leq \sqrt{\frac{2V_s(0)}{m_1}}e^{-\lambda_{iss}t} + \sqrt{\frac{\gamma}{m_1\lambda_{iss}}}. \tag{41}$$

Using (5), a similar bound for $\|e_1(t)\|$ can be derived, and hence, $q(t)$, $\dot{q}(t)$ are bounded and the control torque (9) is bounded. $\square$

Remark 1: Substituting for c_2 and c_5 from (25) and (27) in to the gain condition for λ_{iss} in (36), we have

$$\lambda_{iss} < \lambda\frac{\left(1 - k^2 e^{-2\lambda\delta}\right)}{k^2\left(1 - e^{-2\lambda\delta}\right)}. \tag{42}$$

The expression on the RHS is of the order of $\mathcal{O}(\lambda)$. Thus, the condition in (42) indicates that the decay rate of the tracking error is less than that of the estimator. In other words, the parameter estimator should have a fast convergence in order to ensure a good tracking performance. Similarly, from (40) using (38), we have

$$\frac{\gamma}{m_1\lambda_{iss}} = \frac{1}{m_1\lambda_{iss}}\left(\frac{\gamma_2^2}{4\gamma_1} + \gamma_3\right)$$

$$= \frac{1}{m_1\lambda_{iss}}\left(\frac{(2\lambda_{iss}c_3 + c_6)^2}{4(c_5 - 2\lambda_{iss}c_2 - 1/4k_n)}\right)$$

$$+ \frac{1}{m_1\lambda_{iss}}(2\lambda_{iss}c_4 + c_7).$$

Substituting $c_1, c_2, \ldots, c_7$ from (25) and (27),

$$\frac{\gamma}{m_1\lambda_{iss}} = \frac{4\varepsilon^2 k^2\left(2\lambda_{iss}\left(1 - e^{-\lambda\delta}\right) + \lambda e^{-\lambda\delta}\right)^2}{\lambda\left(\lambda - \lambda k^2 e^{-2\lambda\delta} - \lambda_{iss}k^2\left(1 - e^{-2\lambda\delta}\right) - \frac{\lambda}{4k_n}\right)}$$

$$\times\frac{1}{m_1\lambda_{iss}} + \frac{\varepsilon^2}{m_1\lambda_{iss}}(2\lambda_{iss}\delta + 1) \tag{43}$$

which is of the order of $\mathcal{O}\left(\varepsilon^2\right)$, and therefore, from (41), the ultimate bound on $\|e_2(t)\|$ is $\mathcal{O}(\varepsilon)$.

Remark 2: If the estimation error $\tilde{\theta}(t)$ in (28) exponentially converges to zero (i.e., $\varepsilon = 0$), then from (43), the tracking error also exponentially converges to a zero steady-state.

VII. SIMULATION STUDY

In order to validate the theoretical development, a computer simulation was performed on a space robot system. Control of space robots is important for various space applications such as material transport, cooperative autonomous construction, maintenance of space stations and satellites, etc. Specifically, we consider a space robot consisting of a base (mass m_0), and a two-DOF revolute manipulator with equal link lengths l and point masses m_1 and m_2 at the end of each link. Although the proposed method can control the full system, for simplicity the proposed adaptive controller is applied only for the motion control of the arm. It is assumed that the attitude of the base can be effectively controlled by other control methods. Neglecting the gravity term in (1), the system dynamics can be derived in a manner similar to [22] so that $M = p_1R_1 + p_2R_2 + p_3R_3$, where $p_1 = \frac{m_0 m_1}{m_c}$, $p_2 = \frac{m_1 m_2}{m_c}$, $p_3 = \frac{m_0 m_2}{m_c}$, where $m_c = m_0 + m_1 + m_2$, and

$$R_1 = \begin{bmatrix} l^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} l^2 & l^2 \\ l^2 & l^2 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 2(1+c_2)l^2 & (1+c_2)l^2 \\ (1+c_2)l^2 & l^2 \end{bmatrix}$$

where $c_2 = \cos(q_2)$. $V_m = p_3R_4$, where

$$R_4 = \begin{bmatrix} -2l^2 s_2\dot{q}_2 & -l^2 s_2\dot{q}_2 \\ l^2 s_2\dot{q}_1 & 0 \end{bmatrix}, \text{ where } s_2 = \sin(q_2).$$

A continuous friction model similar to [23] is used, $F = \gamma_1 R_5 + \gamma_2 R_6 + \gamma_3\dot{q}$, where $R_5 = \tanh(0.1\dot{q}) - \tanh(0.2\dot{q})$, $R_6 = \tanh(0.15\dot{q})$. It is assumed that the link length l is known, while the masses m_0, m_1, m_2 (equivalently, p_1, p_2, p_3) and the friction coefficients γ_1, γ_2, γ_3 are unknown. Defining $\zeta_1 = \ddot{q}_d + \alpha_1\dot{e}_1$, $\zeta_2 = \dot{q}_d + \alpha_1 e_1$, and comparing with (8), we have $Y = \begin{bmatrix} R_1\zeta_1 & R_2\zeta_2 & (R_3\zeta_1 + R_4\zeta_2) & R_5 & R_6 & \dot{q} \end{bmatrix}$ and the unknown parameter $\theta = \begin{bmatrix} p_1 & p_2 & p_3 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}^T$. An arbitrary parameter estimator is constructed such that the bound in (28) is satisfied. The parameter estimates are initialized to half of their true values and the resulting parameter estimation error is shown in Fig. 1. The adaptive control law in (9) with the control gains chosen by trial and error is employed to track a desired sinusoidal trajectory. The position tracking error converges to a small bound (shown in Fig. 2) as proved by the stability analysis.

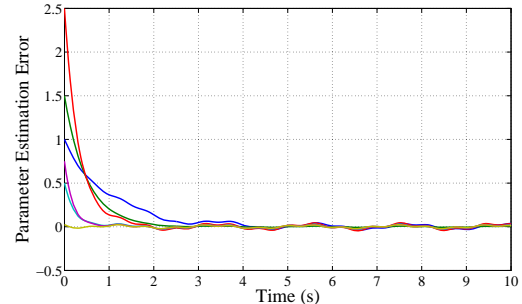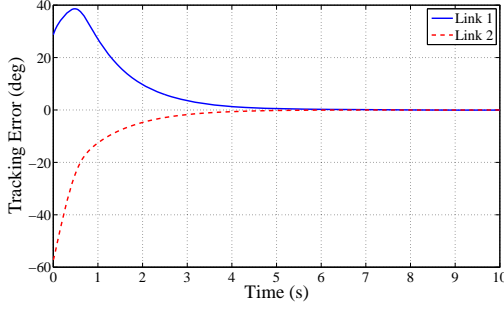Fig. 1. $\tilde{\theta}(t)$ exponentially converges to a small bound.

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Fig. 2. Position tracking error.

VIII. Conclusion and Future Work

Using a converse Lyapunov-like theorem, sufficient gain conditions were derived under which any exponential parameter estimator can be used with an adaptive controller to achieve tracking with a performance bound on the tracking error. The adaptive controller is augmented with a nonlinear damping term which makes the closed-loop system input-to-state stable enabling controller/update law separation. The current development is based on Euler-Lagrange dynamic systems. Future work will focus on extending the result to more general systems with parametric uncertainty.

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IX. Appendix

Lemma 1: In (23), $\|\phi(\tau;t,x)\|^2 \geq \|x\|^2 e^{-2L(\tau-t)}$.
Proof: From (19) and (22)

$$
\begin{aligned}
\left| \frac{d}{dt} \left[x^T(t) x(t) \right] \right| & = \left| 2x^T(t) \dot{x}(t) \right| \\
& = \left| 2x^T(t) f(t,x) \right| \leq 2L \|x\|^2.
\end{aligned}
$$

Therefore,

$$
-2L \|x\|^2 \leq \frac{d}{dt} \left[x^T(t) x(t) \right] \leq 2L \|x\|^2. \tag{44}
$$

Solving the differential inequality in (44) such that $\phi(\tau;t,x)$ denotes the solution of the system starting at (t,x) (i.e., $\phi(t;t,x) = x$)

$$
\|x\| e^{-L(\tau-t)} \leq \|\phi(\tau;t,x)\| \leq \|x\| e^{L(\tau-t)}.
$$

Hence, (23) holds. $\square$

Lemma 2: In (26), $\phi_t(\tau;t,x) + \phi_x(\tau;t,x) f(t,x) \equiv 0, \quad \forall\, \tau \geq t$.
Proof: The solution $\phi(\tau;t,x)$ starting at (t,x) can be expressed as

$$
\begin{aligned}
\phi(\tau;t,x) & = \phi(t;t,x) + \int_t^\tau f(s,\phi(s))\,ds \\
& = x + \int_t^\tau f(s,\phi(s))\,ds.
\end{aligned}
$$

Therefore,

$$
\phi_t(\tau;t,x) = \frac{\partial x}{\partial t} - f(t,\phi(t)) = -f(t,x) \tag{45}
$$

and

$$
\phi_x(\tau;t,x) = \frac{\partial x}{\partial x} = 1. \tag{46}
$$

From (45) and (46), it can be verified that (26) holds. $\square$