# Resource-Bounded Measure 

Jack H. Lutz<br>Department of Computer Science<br>Iowa State University<br>Ames, Iowa 50011<br>U.S.A.


#### Abstract

A general theory of resource-bounded measurability and measure is developed. Starting from any feasible probability measure $\nu$ on the Cantor space $\mathbf{C}$ (the set of all decision problems) and any suitable complexity class $\mathcal{C} \subseteq \mathbf{C}$, the theory identifies the subsets of $\mathbf{C}$ that are $\nu$-measurable in $\mathcal{C}$ and assigns measures to these sets, thereby endowing $\mathcal{C}$ with internal measure-theoretic structure. Classes $\mathcal{C}$ to which the theory applies include various exponential time and space complexity classes, the class of all decidable languages, and the Cantor space $\mathbf{C}$ itself, on which the resource-bounded theory is shown to agree with the classical theory.

The sets that are $\nu$-measurable in $\mathcal{C}$ are shown to form an algebra relative to which $\nu$-measure is well-behaved (monotone, additive, etc.). This algebra is also shown to be complete (subsets of measure 0 sets are measurable) and closed under sufficiently uniform infinitary unions and intersections, and $\nu$-measure in $\mathcal{C}$ is shown to have the appropriate additivity and monotone convergence properties with respect to such infinitary operations.

A generalization of the classical Kolmogorov zero-one law is proven, showing that when $\nu$ is any feasible coin-toss (i.e., product) probability measure on $\mathbf{C}$, every set that is $\nu$-measurable in $\mathcal{C}$ and (like most complexity classes) invariant under finite alterations must have $\nu$-measure 0 or $\nu$-measure 1 in $\mathcal{C}$.

The theory is presented here is based on resource-bounded martingale splitting operators, which are type- 2 functionals, each of which maps $\mathbb{N} \times \mathcal{D}_{\nu}$ into $\mathcal{D}_{\nu} \times \mathcal{D}_{\nu}$, where $\mathcal{D}_{\nu}$ is the set of all $\nu$-martingales. This type- 2 aspect of the theory appears to be essential for general $\nu$-measure in complexity classes $\mathcal{C}$, but the sets of $\nu$-measure 0 or 1 in $\mathcal{C}$ are shown to be characterized by the success conditions for martingales (type- 1 functions) that have been used in resource-bounded measure to date.


[^0]
## 1 Introduction

Resource-bounded measure is a complexity-theoretic generalization of classical measure theory that interacts informatively and quantitatively with many other much-studied aspects of computational complexity. Since its introduction in 1992 [6], resource-bounded measure has yielded a rapidly growing body of new results and insights across a variety of subareas of computational complexity. The recent survey papers [7, 1] summarize many of these developments, but the ongoing progress of many investigators is quickly outdating these surveys.

Notwithstanding its productivity to date, the theory developed in [6] can only be regarded as a fragment of resource-bounded measure. There are two compelling reasons for this view. First, the theory in [6] is restricted to the case where the underlying probability measure is the uniform probability measure on the Cantor space $\mathbf{C}$ of all languages, i.e., the random experiment in which a language $A \subseteq\{0,1\}^{*}$ is chosen by using an independent toss of a fair coin to decide membership of each string in $A$. Second, the theory in [6] is restricted to the sets of resource-bounded measure 0 (and their complements, the sets of resource-bounded measure 1).

Recent work of Breutzmann and Lutz [2], furthered by Kautz [4], has addressed the first of these restrictions by investigating the measure-zero fragment of resource-bounded $\nu$-measure, where the underlying probability measure $\nu$ is an arbitrary (Borel) probability measure on C. Specific results may require $\nu$ to satisfy certain complexity, independence, or positivity conditions, but these requirements arise naturally, not as artifacts of the theory.

In this paper, we address the second of the above-mentioned restrictions by developing a general theory of resource-bounded measurability and measure. Starting from any feasible probability measure $\nu$ on $\mathbf{C}$ and any suitable class $\mathcal{C} \subseteq \mathbf{C}$, we identify the subsets of $\mathbf{C}$ that are $\nu$-measurable in $\mathcal{C}$ and assign $\nu$-measures to these sets, thereby endowing $\mathcal{C}$ with internal measure-theoretic structure. Classes $\mathcal{C}$ to which the theory applies include various exponential time and space complexity classes, the class of all decidable languages, and the Cantor space $\mathbf{C}$ itself. We show that classical $\nu$-measure on $\mathbf{C}$ is precisely the case $\mathcal{C}=\mathbf{C}$ of the more general theory presented here.

We show that the sets that are $\nu$-measurable in $\mathcal{C}$ form an algebra relative to which $\nu$-measure in $\mathcal{C}$ is well-behaved (monotone, additive, etc.). We also show that this algebra is complete (subsets of measure 0 sets are measurable), that it is closed under sufficiently uniform infinitary unions and intersections, and that $\nu$-measure in $\mathcal{C}$ has the appropriate additivity and monotone convergence properties with respect to these infinitary operations.

We prove a resource-bounded generalization of the classical Kolmogorov zero-one law [5]. This states that, when $\nu$ is a feasible coin-toss (i.e., product) probability measure on
$\mathbf{C}$, every set that is $\nu$-measurable in $\mathcal{C}$ and is (like most complexity classes) invariant under finite alterations must have $\nu$-measure 0 in $\mathcal{C}$ or $\nu$-measure 1 in $\mathcal{C}$.

The theory presented here is not a straightforward extension of the measure-zero fragment of resource-bounded measure that has been investigated to date. The basic objects in our development here are resource-bounded martingale splitting operators, each of which is a type-2 functional mapping $\mathbb{N} \times \mathcal{D}_{\nu}$ into $\mathcal{D}_{\nu} \times \mathcal{D}_{\nu}$, where $\mathcal{D}_{\nu}$ is the set of all $\nu$-martingales. This approach, which can be regarded as a martingale-based, complexity-theoretic generalization of the classical Carathéodory definition of measure, entails a variety of new proof techniques. This type- 2 aspect of the theory appears to be essential in order to achieve the algebraic properties mentioned above in complexity classes. However, we show that the sets of $\nu$ measure 0 or 1 in $\mathcal{C}$ are characterized by the same success conditions for martingales (type- 1 functions) that have been used in resource-bounded measure to date.

The theory presented here extends and deepens the relationship between resource-bounded measure and classical measure theory. We expect this to open the way for new applications in computational complexity by enabling the adaptation of powerful techniques from measure-theoretic probability theory. The theory also opens the way for the investigation of complexity classes using probability measures that are not subject to the Kolmogorov zeroone law, thereby accruing the quantitative benefit of measures throughout the interval $[0,1]$. This may lead to new applications of the resource-bounded probabilistic method, especially in situations where the application itself may dictate the use of such a probability measure. Finally, we believe that the work presented here highlights the importance of the continuing investigation of higher-type computational complexity.

## 2 Notation and Functionals

In this paper, $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{R}$ is the set of real numbers.

We write $\{0,1\}^{*}$ for the set of all (finite, binary) strings, and we write $|x|$ for the length of a string $x$. The empty string, $\lambda$, is the unique string of length 0 . The standard enumeration of $\{0,1\}^{*}$ is the sequence $s_{0}=\lambda, s_{1}=0, s_{2}=1, s_{3}=00, \ldots$, ordered first by length and then lexicographically. For $x, y \in\{0,1\}^{*}$, we write $x<y$ if $x$ precedes $y$ in this standard enumeration. For $n \in \mathbb{N},\{0,1\}^{n}$ denotes the set of all strings of length $n$, and $\{0,1\} \leq n$ denotes the set of all strings of length at most $n$.

If $x$ is a string or an (infinite, binary) sequence, and if $0 \leq i \leq j<|x|$, then $x[i . . j]$ is the string consisting of the $i^{\text {th }}$ through $j^{\text {th }}$ bits of $x$. In particular, $x[0 . . i-1]$ is the $i$-bit prefix of $x$. We write $x[i]$ for $x[i . . i]$, the $i^{\text {th }}$ bit of $x$. (Note that the leftmost bit of $x$ is $x[0]$, the $0^{\text {th }}$ bit of $x$.)

If $w$ is a string and $x$ is a string or sequence, then we write $w \sqsubseteq x$ if $w$ is a prefix of $x$, i.e., if there is a string or sequence $y$ such that $x=w y$.

The Boolean value of a condition $\phi$ is $\llbracket \phi \rrbracket=$ if $\phi$ then 1 else 0 .
We use the discrete logarithm

$$
\log n=\min \left\{k \in \mathbb{N} \mid 2^{k} \geq n\right\}
$$

Note that $\log 0=0$.
As in [6], for each $i \in \mathbb{N}$ we define a class $G_{i}$ of functions from $\mathbb{N}$ into $\mathbb{N}$ as follows.

$$
\begin{aligned}
G_{0} & =\{f \mid(\exists k) f(n) \leq k n\} \\
G_{i+1} & =2^{G_{i}(\log n)}=\left\{f \mid\left(\exists g \in G_{i}\right) f(n) \leq 2^{g(\log n)}\right\}
\end{aligned}
$$

We also define the functions $\hat{g}_{i} \in G_{i}$ by $\hat{g}_{0}(n)=2 n, \quad \hat{g}_{i+1}(n)=2^{\hat{g}_{i}(\log n)}$. We regard the functions in these classes as growth rates. In particular, $G_{0}$ contains the linearly bounded growth rates and $G_{1}$ contains the polynomially bounded growth rates. It is easy to show that each $G_{i}$ is closed under composition, that each $f \in G_{i}$ is $o\left(\hat{g}_{i+1}\right)$, and that each $\hat{g}_{i}$ is $o\left(2^{n}\right)$. Thus $G_{i}$ contains superpolynomial growth rates for all $i>1$, but all growth rates in the $G_{i}$-hierarchy are subexponential.

Within the class REC of all decidable languages, we are interested in the uniform complexity classes $\mathrm{E}_{i}=\operatorname{DTIME}\left(2^{G_{i-1}}\right)$ and $\mathrm{E}_{i} \operatorname{SPACE}=\operatorname{DSPACE}\left(2^{G_{i-1}}\right)$ for $i \geq 1$. The well-known exponential complexity classes $\mathrm{E}=\mathrm{E}_{1}=\mathrm{DTIME}\left(2^{\text {linear }}\right), \mathrm{E}_{2}=\mathrm{DTIME}\left(2^{\text {polynomial }}\right)$, ES PACE $=\mathrm{E}_{1} \mathrm{SPACE}=\operatorname{DSPACE}\left(2^{\text {linear }}\right)$, and $\mathrm{E}_{2} \mathrm{SPACE}=\operatorname{DSPACE}\left(2^{\text {polynomial }}\right)$ are of particular interest.

In this extended abstract, our discussion of functions, functionals, and their complexities is somewhat informal and is presented in terms of examples.

Many of the functions in this paper are real-valued functions on discrete domains. A typical example might have the form

$$
f: \mathbb{N} \times\{0,1\}^{*} \rightarrow \mathbb{R}
$$

and we often write $f_{k}(w)$ for $f(k, w)$. A computation of such a function $f$ is a function

$$
\widehat{f}: \mathbb{N} \times \mathbb{N} \times\{0,1\}^{*} \rightarrow \mathbb{Q}
$$

such that, for all $r, k \in \mathbb{N}$ and $w \in\{0,1\}^{*}$,

$$
\left|\widehat{f}_{r, k}(w)-f_{k}(w)\right|<2^{-r}
$$

The canonical computation of such a function $f$ is the unique computation $\widehat{f}$ of $f$ such that, for all $r, k \in \mathbb{N}$ and $w \in\{0,1\}^{*}, \widehat{f}_{r, k}(w)$ is of the form $a \cdot 2^{-k}$, where $a$ is an integer.

Given a set $\Delta$, we say that $f$ is $\Delta$-computable if there exists $\widehat{f} \in \Delta$ such that $\widehat{f}$ is a computation of $f$. For the sets $\Delta$ that we consider in this paper, this is equivalent to saying that the canonical computation of $f$ is an element of $\Delta$.

We also consider functionals, whose arguments and values may themselves be real-valued functions. A typical example might have the form

$$
\Phi: \mathbb{N} \times\left(\{0,1\}^{*} \longrightarrow \mathbb{R}\right) \rightarrow\left(\{0,1\}^{*} \longrightarrow \mathbb{R}\right)
$$

in which case for each $k \in \mathbb{N}$ and $f:\{0,1\}^{*} \rightarrow \mathbb{R}$, the value of $\Phi$ is a function $\Phi_{k}(f)$ : $\{0,1\}^{*} \rightarrow \mathbb{R}$. Formally, we regard such a functional as operating not on the real-valued functions themselves, but rather on canonical computations of these functions, as defined in the preceding paragraph. Thus we identify the example functional $\Phi$ above with (any) functional

$$
\Phi^{\prime}: \mathbb{N} \times\left(\mathbb{N} \times\{0,1\}^{*} \longrightarrow \mathbb{Q}\right) \rightarrow\left(\mathbb{N} \times\{0,1\}^{*} \longrightarrow \mathbb{Q}\right)
$$

such that, for all $k \in \mathbb{N}$ and $f:\{0,1\}^{*} \rightarrow \mathbb{R}$, if $\widehat{f}$ is the canonical computation of $f$, then $\Phi_{k}^{\prime}(\widehat{f})$ is a computation of $\Phi_{k}(f)$. Note that $\Phi^{\prime}$ is a type-2 functional; it is in this sense that we regard $\Phi$ as a type-2 functional.

If we wish to discuss the computability or complexity of the functional $\Phi$ above, then we curry the functional $\Phi^{\prime}$, obtaining a functional

$$
\Phi^{\prime \prime}: \mathbb{N} \times\left(\mathbb{N} \times\{0,1\}^{*} \longrightarrow \mathbb{Q}\right) \times \mathbb{N} \times\{0,1\}^{*} \rightarrow \mathbb{Q}
$$

We then say that $\Phi^{\prime}$ is computable if there is a function oracle Turing machine $M$ such that, for all $k \in \mathbb{N}, g: \mathbb{N} \times\{0,1\}^{*} \rightarrow \mathbb{Q}, r \in \mathbb{N}$, and $w \in\{0,1\}^{*}, M$ on input $(k, r, w)$ with oracle $g$ computes the value $\Phi^{\prime \prime}(k, g, r, w)=\Phi_{k}^{\prime}(g)_{r}(w)$ We say that $\Phi$ is computable if $\Phi$ is, in the above manner, identified with some computable functional $\Phi^{\prime}$. Similarly, we say that $\Phi$ is computable in time $t(n)$ if there is an oracle Turing machine $M$ as above that runs in at most $t(k+r+|w|)$ steps, and we say that $\Phi$ is computable in space $s(n)$ if there is an oracle Turing machine $M$ as above that uses at most $s(k+r+|w|)$ tape cells, including cells used for oracle queries and output.

We use the following classes of functionals.

1. The class "all," consisting of all functionals of type $\leq 2$ in the above sense.
2. The class

$$
\operatorname{rec}=\{\Phi \in \text { all } \mid \Phi \text { is computable }\} .
$$

3. For each $i \geq 1$, the class

$$
\mathrm{p}_{i}=\left\{\Phi \in \text { all } \mid \Phi \text { is computable in } G_{i} \text { time }\right\} .
$$

4. For each $i \geq 1$, the class

$$
\mathrm{p}_{i} \text { space }=\left\{\Phi \in \text { all } \mid \Phi \text { is computable in } G_{i} \text { space }\right\} .
$$

We write p for $\mathrm{p}_{1}$ and pspace for $\mathrm{p}_{1}$ space. Throughout this paper, a resource bound (generically denoted by $\Delta$ or $\Delta^{\prime}$ ) is one of the above classes of functionals.

As in [6], a constructor is a function $\delta:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ such that $x \not \equiv \delta(x)$ for all $x \in\{0,1\}^{*}$. The result of a constructor $\delta$ is the unique language $R(\delta)$ such that $\delta^{k}(\lambda) \sqsubseteq R(\delta)$ for all $k \in \mathbb{N}$, where $\delta^{k}$ is the $k$-fold composition of $\delta$ with itself. The result class of a resource bound $\Delta$ is the set of all languages $R(\delta)$ such that $\delta$ is a constructor and $\delta \in \Delta$. As noted in [6], we have

$$
\begin{aligned}
& R(\text { all })=\mathbf{C}, \\
& R(\mathrm{rec})=\mathrm{REC} \\
& R\left(\mathrm{p}_{i}\right)=\mathrm{E}_{i}, \\
& R\left(\mathrm{p}_{i}(\text { space })=\mathrm{E}_{i} \mathrm{SPACE}\right.
\end{aligned}
$$

for all $i \geq 1$.

## 3 Martingales

We work in the Cantor space $\mathbf{C}$, consisting of all languages (i.e., decision problems) $A \subseteq$ $\{0,1\}^{*}$. We identify each language $A$ with its characteristic sequence, which is the infinite binary sequence $A$ whose $n^{\text {th }}$ bit is $A[n]=\llbracket s_{n} \in A \rrbracket$ for each $n \in \mathbb{N}$. Relying on this identification, we also consider $\mathbf{C}$ to be the set of all infinite binary sequences.

For each string $w \in\{0,1\}^{*}$, the cylinder generated by $w$ is the set

$$
\mathbf{C}_{w}=\{A \in \mathbf{C} \mid w \sqsubseteq A\}
$$

Note that $\mathbf{C}_{\lambda}=\mathbf{C}$.
A probability measure on $\mathbf{C}$ is a function

$$
\nu:\{0,1\}^{*} \rightarrow[0,1]
$$

such that $\nu(\lambda)=1$ and, for all $w \in\{0,1\}^{*}$,

$$
\nu(w)=\nu(w 0)+\nu(w 1)
$$

Intuitively, $\nu(w)$ is the probability that $A \in \mathbf{C}_{w}$ when we "choose a language $A \in \mathbf{C}$ according to the probability measure $\nu$." We sometimes write $\nu\left(\mathbf{C}_{w}\right)$ for $\nu(w)$.

If $v, w \in\{0,1\}^{*}$ and $\nu(w)>0$, then we write

$$
\nu(v \mid w)= \begin{cases}1 & \text { if } v \sqsubseteq w \\ \frac{\nu(v)}{\nu(w)} & \text { if } w \sqsubseteq v \\ 0 & \text { otherwise }\end{cases}
$$

for the conditional $\nu$-measure of $v$ given $w$.
The uniform probability measure $\mu$ is defined by

$$
\mu(w)=2^{-|w|}
$$

for all $w \in\{0,1\}^{*}$.
A bias sequence is a sequence $\vec{\beta}=\left(\beta_{k} \mid k \in \mathbb{N}\right)$, where each $\beta_{k} \in[0,1]$. Given a bias sequence $\vec{\beta}$, the $\vec{\beta}$-coin-toss probability measure (also called the $\vec{\beta}$-product probability measure) is the probability measure $\mu^{\vec{\beta}}$ defined by

$$
\mu^{\vec{\beta}}(w)=\prod_{k=0}^{|w|-1}\left[\left(1-\beta_{k}\right) \cdot(1-w[k])+\beta_{k} \cdot w[k]\right]
$$

for all $w \in\{0,1\}^{*}$. Intuitively, $\mu^{\vec{\beta}}(w)$ is the probability that $w \sqsubseteq A$ when the language $A \subseteq\{0,1\}^{*}$ is chosen according to the following random experiment. For each string $s_{k}$ in the standard enumeration $\left(s_{k} \mid k \in \mathbb{N}\right)$ of $\{0,1\}^{*}$, we (independently of all other strings) toss a special coin, whose probability is $\beta_{k}$ of coming up heads, in which case $s_{k} \in A$, and $1-\beta_{k}$ of coming up tails, in which case $s_{k} \notin A$. Note that, in the special case where $\beta_{k}=\frac{1}{2}$ for all $k \in \mathbb{N}, \mu^{\vec{\beta}}$ is the uniform probability measure $\mu$.

If $\Delta$ is a resource bound, as specified in section 2, then a $\Delta$-probability measure on $\mathbf{C}$ is a probability measure $\nu$ on $\mathbf{C}$ with the following two properties.
(i) $\nu$ is $\Delta$-computable.
(ii) There is a $\Delta$-computable function $l: \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $w \in\{0,1\}^{*}, \nu(w)=0$ or $\nu(w) \geq 2^{-l(|w|)}$.

Note that, if $\nu$ is a $\Delta$-probability measure on $\mathbf{C}$, then the Boolean value $\llbracket \nu(w)=0 \rrbracket$ is $\Delta$-computable.

We now recall the well-known notion of a martingale over a probability measure $\nu$. Computable martingales were used by Schnorr [9, 10, 11, 12] in his investigations of randomness, and have more recently been used by Lutz [6] and Breutzmann and Lutz [2] in the development of resource-bounded measure.

If $\nu$ is a probability measure on $\mathbf{C}$, then a $\nu$-martingale is a function $d:\{0,1\}^{*} \rightarrow[0, \infty)$ such that, for all $w \in\{0,1\}^{*}$,

$$
d(w) \nu(w)=d(w 0) \nu(w 0)+d(w 1) \nu(w 1)
$$

The initial value of a $\nu$-martingale $d$ is $d(\lambda)$. A $\Delta$ - $\nu$-martingale is a $\nu$-martingale that is $\Delta$-computable. We reserve the symbol $\mathbf{1}$ for the unit martingale define by $\mathbf{1}(w)=w$ for all $w \in\{0,1\}^{*}$. Note that $\mathbf{1}$ is a $\nu$-martingale for every probability measure $\nu$ on $\mathbf{C}$.

Let $d$ be a $\nu$-martingale, and let $A \subseteq\{0,1\}^{*}$. We say that $d$ covers $A$ if there exists $n \in \mathbb{N}$ such that $d(A[0 . . n-1]) \geq 1$. We say that $d$ succeeds on $A$ if

$$
\limsup _{n \rightarrow \infty} d(A[0 . . n-1])=\infty
$$

We say that $d$ succeeds strongly on $A$ if

$$
\lim _{n \rightarrow \infty} d(A[0 . . n-1])=\infty
$$

The set covered by $d$ is the set

$$
S^{1}[d]=\{A \mid d \text { covers } A\} .
$$

The success set of $d$ is

$$
S^{\infty}[d]=\{A \mid d \text { succeeds on } A\}
$$

The strong success set of $d$ is

$$
S_{\mathrm{str}}^{\infty}[d]=\{A \mid d \text { succeeds strongly on } A\} .
$$

The set $S^{1}[d]$ is also called the unitary success set of $d$.
A prefix set is a language $A \subseteq\{0,1\}^{*}$ with the property that no element of $A$ is a prefix of another element of $A$. A routine induction on the definition of martingales yields the following.

Lemma 3.1. If $d$ is a $\nu$-martingale, then for every prefix set $A \subseteq\{0,1\}^{*}$,

$$
\sum_{w \in A} d(w) \nu(w) \leq d(\lambda)
$$

An open set in $\mathbf{C}$ is any set of the form

$$
\mathbf{C}_{A}=\bigcup_{w \in A} \mathbf{C}_{w}
$$

for $A \subseteq\{0,1\}^{*}$. Every open set can in fact be written in the form $\mathbf{C}_{A}$, where $A$ is a prefix set. This representation is not unique, but if $A$ and $B$ are prefix sets such that $\mathbf{C}_{A}=\mathbf{C}_{B}$, then it is easy to see that

$$
\sum_{w \in A} \nu(w)=\sum_{w \in B} \nu(w)
$$

for every probability measure $\nu$ on $\mathbf{C}$. The $\nu$-measure of an open set $X \subseteq \mathbf{C}$, given by

$$
\nu(X)=\sum_{w \in A} \nu(w)
$$

where $A$ is any prefix set such that $X=\mathbf{C}_{A}$, is thus well-defined.
If $d$ is any $\nu$-martingale, then the set $S^{1}[d]$ is open because $S^{1}[d]=\mathrm{C}_{A}$, where $A=$ $\{w \mid d(w) \geq 1\}$. Thus the $\nu$-measure $\nu\left(S^{1}[d]\right)$ is a well-defined real number. Before proceeding, however, we note that $\nu\left(S^{1}[d]\right)$ may fail to be computable, even when $d$ is p-computable.

Example 3.2. Let ( $M_{i} \mid i \in \mathbb{N}$ ) be a standard enumeration of Turing machines, and let

$$
K=\left\{i \in \mathbb{N} \mid M_{i}\left(0^{i}\right) \text { halts }\right\}
$$

be the diagonal halting problem. For each $i \in K$, let $t(i)$ be the number of steps executed by $M_{i}\left(0^{i}\right)$, and let $T_{i}=\left\{0^{i} 1 u \mid u \in\{0,1\}^{t(i)}\right\}$. Let $T=\bigcup_{i \in K} T_{i}$, and note that $T$ is a prefix set. Define $d:\{0,1\}^{*} \rightarrow[0, \infty)$ by

$$
d(w)= \begin{cases}1 & \text { if } v 1 \sqsubseteq w \text { for some } v \in T \\ 0 & \text { if } v 0 \sqsubseteq w \text { for some } v \in T \\ \frac{1}{2} & \text { otherwise. }\end{cases}
$$

It is easy to check that $d$ is a p - $\mu$-martingale, where $\mu$ is the uniform probability measure on C. It is also easy to see that

$$
\mu\left(S^{1}[d]\right)=\frac{1}{4} \sum_{i \in K} 2^{-i}
$$

It follows that $\mu\left(S^{1}[d]\right)$ is Turing-equivalent to $K$, and hence not computable. (In fact, $\mu\left(S^{1}[d]\right)$ is a version of Chaitin's random real number $\Omega$ [3].)

Even though $\nu\left(S^{1}[d]\right)$ may not be computable, it is always bounded above by the initial value $d(\lambda)$.

Lemma 3.3. For every $\nu$-martingale $d$,

$$
\nu\left(S^{1}[d]\right) \leq d(\lambda)
$$

Proof of Lemma 3.3 Fix a prefix set $A \subseteq\{w \mid d(w) \geq 1\}$ such that $S^{1}[d]=\mathbf{C}_{A}$. Then

$$
\nu\left(S^{1}[d]\right)=\sum_{w \in A} d(w) \nu(w) \leq d(\lambda)
$$

by Lemma 3.1
If $d$ is $\Delta$-computable and $X \subseteq S^{1}[d]$, then Lemma 3.3 says that we can regard $d$ as an explicit, $\Delta$-computable certification that $X$ does not have $\nu$-measure greater than the initial value $d(\lambda)$, which is itself $\Delta$-computable.

By Lemma 3.3, no $\nu$-martingale $d$ can cover a cylinder $\mathbf{C}_{w}$ unless $d(\lambda)$ is at least $\nu(w)$. The following theorem, which is central to resource-bounded measure, says that, for $\Delta-\nu$ martingales $d$, this remains true even if we intersect the cylinder with $R(\Delta)$.

Theorem 3.4 (Measure Conservation Theorem). If $w \in\{0,1\}^{*}$ and $d$ is a $\Delta$ - $\nu$-martingale such that $\mathbf{C}_{w} \cap R(\Delta) \subseteq S^{1}[d]$, then $d(\lambda) \geq \nu(w)$.

Proof of Theorem 3.4 Assume that $w \in\{0,1\}^{*}$ and $d$ is a $\Delta$ - $\nu$-martingale such that $d(\lambda)<\nu(w)$. It suffices to exhibit a constructor $\delta \in \Delta$ such that $R(\delta) \in \mathbf{C}_{w}-S^{1}[d]$.

First note, for every prefix $w^{\prime} \sqsubseteq w$, the definition of $\nu$-martingales(applied inductively) tells us that

$$
\begin{aligned}
d\left(w^{\prime}\right) \nu\left(w^{\prime}\right) & \leq \sum_{|u|=\left|w^{\prime}\right|} d(u) \nu(u)=d(\lambda) \nu(\lambda) \\
& =d(\lambda)<\nu(w) \leq \nu\left(w^{\prime}\right)
\end{aligned}
$$

so $d\left(w^{\prime}\right)<1$. In particular, then fix a constant $m \in \mathbb{N}$ such that $d(w) \leq 1-2^{1-m}$.
Let $\widehat{d}$ be a $\Delta$-computation of $d$. Using $\widehat{d}$ and the constants $w$ and $m$, define $\delta:\{0,1\}^{*} \rightarrow$ $\{0,1\}^{*}$ by

$$
\delta(x)= \begin{cases}w & \text { if } x \nsucceq w \\ x 0 & \text { if } \widehat{d}_{a(x)}(x 0) \leq \widehat{d}_{a(x)}(x 1) \text { and note } x \not \approx w \\ x 1 & \text { otherwise },\end{cases}
$$

where $a(x)=|x|+m+2$. It is clear that $\delta$ is a constructor, $\delta \in \Delta$, and $R(\delta) \in \mathbf{C}_{w}$.
For any string $x$ such that $w \sqsubseteq x$, the definition of $\delta$ and the fact that $d$ is a $\nu$-martingale ensure that

$$
\begin{aligned}
d(\delta(x)) & \leq \widehat{d}_{a(x)}(\delta(x))+2^{-a(x)} \\
& =\min _{b \in\{0,1\}} \widehat{d}_{a(x)}(x b)+2^{-a(x)} \\
& \leq \min _{b \in\{0,1\}} d(x b)+2^{1-a(x)} \\
& \leq d(x)+2^{1-a(x)} \\
& =d(x)+2^{-(|x|+m+1)} .
\end{aligned}
$$

It follows that, for all $k \in \mathbb{N}$,

$$
\begin{aligned}
d\left(\delta^{k}(w)\right) & \leq d(w)+\sum_{j=0}^{k-1} 2^{-(j+m+1)} \\
& <d(w)+2^{-m} \\
& \leq 1-2^{-m}
\end{aligned}
$$

Since every prefix of $R(\delta)$ is either a prefix of $w$ or of the form $\delta^{k}(w)$ for some $k \in \mathbb{N}$, we have now shown that $R(\delta) \notin S^{1}[d]$, completing the proof.

A $\nu$-martingale is regular if, for all $v, w \in\{0,1\}^{*}$, if $\nu(v) \geq 1$ and $v \sqsubseteq w$, then $\nu(w) \geq 1$. It is often technically convenient to have a uniform means of ensuring that martingales are regular. The following lemma provides such a mechanism. Let $\Delta$ be a resource bound, as specified in section 2, and let $\nu$ be a probability measure on $\mathbf{C}$.

Lemma 3.5 (Regularity Lemma). There is a functional

$$
\Lambda: \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu}
$$

with the following properties.

1. For all $d \in \mathcal{D}_{\nu}, \Lambda(d)$ is a regular $\nu$-martingale such that $\Lambda(d)(\lambda)=d(\lambda)$ and $S^{1}[d] \subseteq$ $S^{1}[\Lambda(d)]$.
2. $\Lambda(1)=1$.
3. If $\nu$ is a $\Delta$-probability measure on $\mathbf{C}$, then $\Lambda$ is $\Delta$-computable.

Proof of Lemma 3.5 (sketch) Given $\alpha \in(0,1)$, it is convenient to have notations for the $\alpha$-weighted averaging function

$$
\begin{gathered}
m_{\alpha}: \mathbb{R}^{2} \rightarrow \mathbb{R} \\
m_{\alpha}(s, t)=\alpha s+(1-\alpha) t
\end{gathered}
$$

the half-plane

$$
H_{\alpha}=\left\{(s, t) \in \mathbb{R}^{2} \mid m_{\alpha}(s, t) \geq 1\right\}
$$

and the region

$$
D_{\alpha}=H_{\alpha} \cup[0, \infty)^{2}
$$

The "Robin Hood function"

$$
r h_{\alpha}: D_{\alpha} \rightarrow[0, \infty)^{2}
$$

is then defined as follows.
(i) If $(s, t) \in[0,1]^{2}$, then

$$
r h_{\alpha}(s, t)=(s, t)
$$

(ii) If $(s, t) \in H_{\alpha}$, then

$$
r h_{\alpha}(s, t)=\left(m_{\alpha}(s, t), m_{\alpha}(s, t)\right)
$$

(iii) If $(s, t) \in D_{\alpha}-H_{\alpha}$ and $s \geq 1$, then

$$
r h_{\alpha}(s, t)=\left(1, \frac{m_{\alpha}(s-1, t)}{1-\alpha}\right) .
$$

(iv) If $(s, t) \in D_{\alpha}-H_{\alpha}$ and $t \geq 1$, then

$$
r h_{\alpha}(s, t)=\left(\frac{m_{\alpha}(s, t-1)}{\alpha}, 1\right) .
$$

We also use the notation $r h_{\alpha}(s, t)=\left(r h_{\alpha}^{(0)}(s, t), r h_{\alpha}^{(1)}(s, t)\right)$.
The following essential properties of the Robin Hood function $r h_{\alpha}$ are routine to verify.

1. The transformation $r h_{\alpha}$ is a continuous, piecewise linear mapping from $D_{\alpha}$ into $[0, \infty)^{2}$.
2. The transformation $r h_{\alpha}$ preserves $\alpha$-weighted averages, i.e., $m_{\alpha}\left(r h_{\alpha}(s, t)\right)=m_{\alpha}(s, t)$ for all $(s, t) \in D_{\alpha}$.
3. The transformation $r h_{\alpha}$ maps $H_{\alpha}$ into $[1, \infty)^{2}$. That is, if the average $m_{\alpha}(s, t)$ is at least 1 , then $r h_{\alpha}$ "steals from the richer and gives to the poorer" of $s$ and $t$ so that both $r h_{\alpha}^{(0)}(s, t)$ and $r h_{\alpha}^{(1)}(s, t)$ are at least 1.
4. For all $(s, t) \in D_{\alpha}, r h_{\alpha}^{(0)}(s, t) \geq \min \{1, s\}$ and $r h_{\alpha}^{(1)}(s, t) \geq \min \{1, t\}$. That is, the transformation $r h_{\alpha}$ never "steals" more than the excess above 1 .
5. The transformation $r h_{\alpha}$ leaves points of $[0,1]^{2}$ unchanged.

Using the Robin Hood function, we define the functional $\Lambda: \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu}$ as follows. For $d \in \mathcal{D}_{\nu}$, we define the $\nu$-martingale $\Lambda(d)$ by the following recursion. (In all clauses, $w \in\{0,1\}^{*}$ and $b \in\{0,1\}$.)
(i) $\Lambda(d)(\lambda)=d(\lambda)$.
(ii) If $\nu(w)=0$ or $\nu(w b \mid w) \in\{0,1\}$, then $\Lambda(d)(w b)=\Lambda(d)(w)$.
(iii) If $\nu(w)>0$ and $0<\nu(w b \mid w)<1$, then

$$
\Lambda(d)(w b)=r h_{\nu(w 0 \mid w)}^{(b)}\left(g_{0}(w), g_{1}(w)\right)
$$

where $g_{b}(w)=\Lambda(d)(w)-d(w)+d(w b)$.

It is now routine (if tedious) to verify that $\Lambda$ has the desired properties.

## 4 Measure

In this, the main section of the paper, we develop the general theory of resource-bounded measurability and measure in complexity classes. Throughout this paper, $\Delta$ and $\Delta^{\prime}$ are resource bounds, as specified in section 2, and $\nu$ is a probability measure on $\mathbf{C}$, as defined in section 3 ,

The fundamental objects in resource-bounded measure are martingale splitting operators. If $X^{+}$and $X^{-}$are disjoint subsets of $\mathbf{C}$, then a martingale splitting operator for $X^{+}$and $X^{-}$is a type-2 functional that, given a martingale $d$ and error tolerance $2^{-r}$, "splits" $d$ into two martingales $d_{r}^{+}$and $d_{r}^{-}$such that $d_{r}^{+}$covers every element of $X^{+}$covered by $d, d_{r}^{-}$covers every element of $X^{-}$covered by $d$, and the sum of initial values $d_{r}^{+}(\lambda)+d_{r}^{-}(\lambda)$ does not exceed the initial value $d(\lambda)$ by more than $2^{-r}$. The formal definition follows.

Definition. If $X^{+}$and $X^{-}$are disjoint subsets of $\mathbf{C}$, then a $\nu$-splitting operator for $\left(X^{+}, X^{-}\right)$ is a functional

$$
\Phi: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

where we write

$$
\Phi(r, d)=\left(\Phi_{r}^{+}(d), \Phi_{r}^{-}(d)\right),
$$

such that the following conditions hold for all $r \in \mathbb{N}$ and $d \in \mathcal{D}_{\nu}$.
(i) $X^{+} \cap S^{1}[d] \subseteq S^{1}\left[\Phi_{r}^{+}(d)\right]$,
(ii) $X^{-} \cap S^{1}[d] \subseteq S^{1}\left[\Phi_{r}^{-}(d)\right]$,
(iii) $\Phi_{r}^{+}(d)(\lambda)+\Phi_{r}^{-}(d)(\lambda) \leq d(\lambda)+2^{-r}$.

A $\Delta$ - $\nu$-splitting operator for $\left(X^{+}, X^{-}\right)$is a $\nu$-splitting operator for $\left(X^{+}, X^{-}\right)$that is $\Delta$ computable.

The resource-bounded measurement of sets, both globally and in $R(\Delta)$, is now defined in terms of splitting operators.

Definition. Let $X \subseteq \mathbf{C}$.

1. A $\Delta-\nu$-measurement of $X$ is a $\Delta-\nu$-splitting operator for $\left(X, X^{\mathrm{c}}\right)$.
2. A $\nu$-measurement of $X$ in $R(\Delta)$ is a $\Delta$ - $\nu$-splitting operator for $(R(\Delta) \cap X, R(\Delta)-X)$.

If $Y^{+} \subseteq X^{+}$and $Y^{-} \subseteq X^{-}$, then every $\nu$-splitting operator for $\left(X^{+}, X^{-}\right)$is clearly a $\nu$-splitting operator for $\left(Y^{+}, Y^{-}\right)$. In particular, then, every $\Delta$ - $\nu$-measurement of $X$ is a $\nu$-measurement of $X$ in $R(\Delta)$. As we shall see, the converse is (fortunately) not true.

We now develop some basic properties of $\nu$-measurements. Recall the unit martingale 1 from section 3 .

Lemma 4.1. Let $X \subseteq \mathbf{C}$. If $\Phi$ and $\Psi$ are $\nu$-measurements of $X$ in $R(\Delta)$, then for all $j, k \in \mathbb{N}$,

$$
\Phi_{j}^{+}(\mathbf{1})(\lambda)+\Psi_{k}^{-}(\mathbf{1})(\lambda) \geq 1
$$

Proof of Lemma 4.1 Assume the hypothesis, let $j, k \in \mathbb{N}$, and let

$$
d=\Phi_{j}^{+}(\mathbf{1})+\Psi_{k}^{-}(\mathbf{1}) .
$$

Then $d$ is a $\Delta-\nu$-martingale and

$$
\begin{aligned}
R(\Delta) & =(R(\Delta) \cap X) \cup(R(\Delta)-X) \\
& \subseteq S^{1}\left[\Phi_{j}^{+}(\mathbf{1})\right] \cup S^{1}\left[\Psi_{k}^{-}(\mathbf{1})\right] \\
& \subseteq S^{1}[d]
\end{aligned}
$$

so $d(\lambda) \geq 1$ by the Measure Conservation Theorem.
Notation. If $\Phi$ is a $\nu$-splitting operator, then we write

$$
\begin{aligned}
& \Phi_{\infty}^{+}=\inf _{r \in \mathbb{N}} \Phi_{r}^{+}(\mathbf{1})(\lambda) \\
& \Phi_{\infty}^{-}=\inf _{r \in \mathbb{N}} \Phi_{r}^{-}(\mathbf{1})(\lambda)
\end{aligned}
$$

Lemma 4.2. Let $X \subseteq \mathbf{C}$. If $\Phi$ is a $\nu$-measurement of $x$ in $R(\Delta)$, then $\Phi_{\infty}^{+}+\Phi_{\infty}^{-}=1$ and, for all $r \in \mathbb{N}$,

$$
\Phi_{\infty}^{+} \leq \Phi_{r}^{+}(\mathbf{1})(\lambda) \leq \Phi_{\infty}^{+}+2^{-r}
$$

and

$$
\Phi_{\infty}^{-} \leq \Phi_{r}^{-}(\mathbf{1})(\lambda) \leq \Phi_{\infty}^{-}+2^{-r} .
$$

Thus $\Phi_{r}^{+}(\mathbf{1})(\lambda) \longrightarrow \Phi_{\infty}^{+}$and $\Phi_{r}^{-}(\mathbf{1})(\lambda) \longrightarrow \Phi_{\infty}^{-}$as $r \longrightarrow \infty$.

Proof of Lemma 4.2 Assume the hypothesis. It follows immediately from Lemma4.1 that $\Phi_{\infty}^{+}+\Phi_{\infty}^{-} \geq 1$. Also, for any $r \in \mathbb{N}$,

$$
\begin{aligned}
\Phi_{\infty}^{+}+\Phi_{\infty}^{-} & \leq \Phi_{r}^{+}(\mathbf{1})(\lambda)+\Phi_{r}^{-}(\mathbf{1})(\lambda) \\
& \leq \mathbf{1}(\lambda)+2^{-r} \\
& =1+2^{-r},
\end{aligned}
$$

so $\Phi_{\infty}^{+}+\Phi_{\infty}^{-} \leq 1$. Thus $\Phi_{\infty}^{+}+\Phi_{\infty}^{-}=1$.
Now fix $r \in \mathbb{N}$. Then, by the definitions of $\Phi_{\infty}^{+}$and $\Phi_{\infty}^{-}, \Phi_{\infty}^{+} \leq \Phi_{r}^{+}(\mathbf{1})(\lambda)$ and $\Phi_{\infty}^{-} \leq$ $\Phi_{r}^{-}(\mathbf{1})(\lambda)$. Also,

$$
\begin{aligned}
& \left(\Phi_{r}^{+}(\mathbf{1})(\lambda)-\Phi_{\infty}^{+}\right)+\left(\Phi_{r}^{-}(\mathbf{1})(\lambda)-\Phi_{\infty}^{-}\right) \\
& =\Phi_{r}^{+}(\mathbf{1})(\lambda)+\Phi_{r}^{-}(\mathbf{1})(\lambda)-1 \\
& \leq \mathbf{1}(\lambda)+2^{-r}-1 \\
& =2^{-r}
\end{aligned}
$$

so $\Phi_{r}^{+}(\mathbf{1})(\lambda) \leq \Phi_{\infty}^{+}+2^{-r}$ and $\Phi_{r}^{-}(\mathbf{1})(\lambda) \leq \Phi_{\infty}^{-}+2^{-r}$.

Lemma 4.3. Let $X \subseteq \mathbf{C}$. If $\Phi$ and $\Psi$ are $\nu$-measurements of $X$ in $R(\Delta)$, then for all $r \in \mathbb{N}$,

$$
\left|\Phi_{r}^{+}(\mathbf{1})(\lambda)-\Psi_{r}^{+}(\mathbf{1})(\lambda)\right| \leq 2^{-r}
$$

and

$$
\left|\Phi_{r}^{-}(\mathbf{1})(\lambda)-\Psi_{r}^{-}(\mathbf{1})(\lambda)\right| \leq 2^{-r}
$$

Thus $\Phi_{\infty}^{+}=\Psi_{\infty}^{+}$and $\Phi_{\infty}^{-}=\Psi_{\infty}^{-}$.

Proof of Lemma 4.3 Assume the hypothesis, and let $r \in \mathbb{N}$. Then, by Lemma 4.1,

$$
\begin{aligned}
1 & \leq \Phi_{r}^{+}(\mathbf{1})(\lambda)+\Psi_{r}^{-}(\mathbf{1})(\lambda) \\
& \leq \Phi_{r}^{+}(\mathbf{1})(\lambda)+\left(\mathbf{1}(\lambda)+2^{-r}-\Psi_{r}^{+}(\mathbf{1})(\lambda)\right)
\end{aligned}
$$

so $\Psi_{r}^{+}(\mathbf{1})(\lambda) \leq \Phi_{r}^{+}(\mathbf{1})(\lambda)+2^{-r}$. Similarly, $\Phi_{r}^{+}(\mathbf{1})(\lambda) \leq \Psi_{r}^{+}(\mathbf{1})(\lambda)+2^{-r}$, so

$$
\left|\Phi_{r}^{+}(\mathbf{1})(\lambda)-\Psi_{r}^{+}(\mathbf{1})(\lambda)\right| \leq 2^{-r}
$$

The proof that

$$
\left|\Phi_{r}^{-}(\mathbf{1})(\lambda)-\Psi_{r}^{-}(\mathbf{1})(\lambda)\right| \leq 2^{-r}
$$

is similar.
We now have the machinery we need to give unambiguous definitions of resource bounded measure.

Definition. A set $X \subseteq \mathbf{C}$ is $\nu$-measurable in $R(\Delta)$, and we write $X \in \mathcal{F}_{R(\Delta)}^{\nu}$, if there exists a $\nu$-measurement $\Phi$ of $X$ in $R(\Delta)$. In this case, the $\nu$-measure of $X$ in $R(\Delta)$ is the real number $\nu(X \mid R(\Delta))=\Phi_{\infty}^{+}$. (By Lemma 4.3, $\nu(X \mid R(\Delta))$ does not depend on the choice of $\Phi$.)

Definition. A set $X \subseteq \mathbf{C}$ is $\Delta$ - $\nu$-measurable, and we write $X \in \mathcal{F}_{\Delta}^{\nu}$, if there exists a $\Delta-\nu$ measurement $\Phi$ of $X$. In this case, the $\Delta$ - $\nu$-measure of $X$ is the real number $\nu_{\Delta}(X)=\Phi_{\infty}^{+}$. (By Lemma 4.3, $\nu_{\Delta}(X)$ does not depend on the choice of $\Phi$.)

An intuitive remark on these definitions is in order here. As noted in section 3, a martingale $d$ that covers a set $X$ can be regarded as an explicit certification that $X$ does not have measure greater than the total value $d(\lambda)$. Thus, if we apply a measurement $\Phi$ of $X$ to the unit martingale $\mathbf{1}$, then the resulting martingales $\Phi_{r}^{+}(\mathbf{1})$ and $\Phi_{r}^{-}(\mathbf{1})$, for $r \in \mathbb{N}$, can be regarded collectively as an explicit certification that $X$ does not have measure greater than $\Phi_{\infty}^{+}$or less than $1-\Phi_{\infty}^{-}=\Phi_{\infty}^{+}$. This is clearly a necessary condition for measurability but we require further that the measurement $\Phi$ split arbitrary martingales (which may be more exotic and of lesser total measure than 1) in analogous fashion. The utility of this requirement is evident in the proof of Theorem 4.12, where we establish that the measurable sets form an algebra over which measure is additive.

In general, when we make an assertion of the form " $\nu(X \mid R(\Delta))=\alpha$," it is implicit that $X$ is $\nu$-measurable in $R(\Delta)$. On the other hand, an assertion of the form " $\nu(X \mid R(\Delta) \neq \alpha$ " is the negation of " $\nu(X \mid R(\Delta))=\alpha$," and thus means that either $X$ is not $\nu$-measurable in $R(\Delta)$ or $\nu(X \mid R(\Delta))=\beta$ for some $\beta \neq \alpha$. Similar remarks hold for the assertions " $\nu_{\Delta}(X)=\alpha$ " and " $\nu_{\Delta}(X) \neq \alpha$."

The following two lemmas are now very obvious, but also very useful.

Lemma 4.4. Let $X \subseteq \mathbf{C}$.

1. If $X$ is $\nu$-computable in $R(\Delta)$, then for every $\nu$-measurement $\Phi$ of $X$ in $R(\Delta)$ and every $r \in \mathbb{N}$,

$$
\nu(X \mid \mathbb{R}(\Delta)) \leq \Phi_{r}^{+}(\mathbf{1})(\lambda) \leq \nu(X \mid R(\Delta))+2^{-r}
$$

2. If $X$ is a $\Delta$ - $\nu$-computable, then for every $\Delta$ - $\nu$-measurement $\Phi$ of $X$ and every $r \in \mathbb{N}$,

$$
\nu_{\Delta}(X) \leq \Phi_{r}^{+}(\mathbf{1})(\lambda) \leq \nu_{\Delta}(X)+2^{-r}
$$

Lemma 4.5. Let $X \subseteq \mathbf{C}$.

1. If $X$ is $\Delta-\nu$-measurable and $\Delta \subseteq \Delta^{\prime}$, then $X$ is $\Delta^{\prime}-\nu$-measurable and $\nu_{\Delta^{\prime}}(X)=\nu_{\Delta}(X)$.
2. If $X$ is $\Delta$ - $\nu$-measurable, then $X$ is $\nu$-measurable in $R(\Delta)$ and $\nu(X \mid R(\Delta))=\nu_{\Delta}(X)$.
3. $X$ is $\nu$-measurable in $R(\Delta)$ if and only if $X \cap R(\Delta)$ is $\nu$-measurable in $R(\Delta)$, in which case $\nu(X \mid R(\Delta))=\nu(X \cap R(\Delta) \mid R(\Delta))$.

Before proceeding further, we show that resource-bounded measure is a generalization of classical measure theory. More precisely, classical measure theory is the special case $\Delta=$ all of resource-bounded measure. We first recall the classical definitions (specialized to the Cantor space C.)

Recall the definitions of open sets and their measures in section 3,
Definition. The outer $\nu$-measure of a set $X \subseteq \mathbf{C}$ is

$$
\nu^{*}(X)=\inf \{\nu(Y) \mid Y \text { is open and } X \subseteq Y\}
$$

Note that the outer $\nu$-measure $\nu^{*}(X)$ is defined for every set $X \subseteq \mathbf{C}$, and that $0 \leq$ $\nu^{*}(X) \leq 1$. It is well known that outer measure is subadditive, i.e., that

$$
\nu^{*}(X \cup Y) \leq \nu^{*}(X)+\nu^{*}(Y)
$$

for all $X, Y \subseteq \mathbf{C}$.
Definition. A set $X \subseteq \mathbf{C}$ is $\nu$-measurable if, for every set $Y \subseteq \mathbf{C}$,

$$
\nu^{*}(Y)=\nu^{*}(Y \cap X)+\nu^{*}(Y-X) .
$$

In this case, the $\nu$-measure of $X$ is

$$
\nu(X)=\nu^{*}(X)
$$

Since the outer measure is subadditive, it is evident that a set $X \subseteq \mathbf{C}$ is $\nu$-measurable if, for every set $Y \subseteq \mathbf{C}$,

$$
\nu^{*}(Y) \geq \nu^{*}(Y \cap X)+\nu^{*}(Y-X)
$$

We now show that classical measurability and measure are the special case $\Delta=$ all of resource-bounded measure.

Theorem 4.6. A set $X \subseteq \mathbf{C}$ is all- $\nu$-measurable if and only if it is $\nu$-measurable, in which case $\nu_{\text {all }}(X)=\nu(X)$.

Proof of Theorem 4.6 Assume first that $X$ is all- $\nu$-measurable, and let $\Phi$ be an all- $\nu$ measurement of $X$. To see that $X$ is $\nu$-measurable, let $Y \subseteq \mathbf{C}$ be arbitrary. As noted above, to show that $X$ is $\nu$-measurable, it suffices to prove that

$$
\nu^{*}(Y) \geq \nu^{*}(Y \cap X)+\nu^{*}(Y-X)
$$

For this purpose, let $r \in \mathbb{N}$ be arbitrary. Then there is a prefix set $A \subseteq\{0,1\}^{*}$ such that $Y \subseteq \mathbf{C}_{A}$ and $\nu\left(\mathbf{C}_{A}\right) \leq \nu^{*}(Y)+2^{-(r+1)}$. Define $d:\{0,1\}^{*} \rightarrow[0, \infty)$ by

$$
d(w)= \begin{cases}\sum_{u \in A} \nu(u \mid w) & \text { if } \nu(w)>0 \\ 1 & \text { if } \nu(w)=0\end{cases}
$$

Then $d$ is a $\nu$-martingale and $Y \subseteq S^{1}[d]$, so

$$
Y \cap X \subseteq S^{1}[d] \cap X \subseteq S^{1}\left[\Phi_{r+1}^{+}(d)\right]
$$

Since $S^{1}\left[\Phi_{r+1}^{+}(d)\right]$ is open, it follows by Lemma 3.3 that

$$
\nu^{*}(Y \cap X) \leq \nu\left(S^{1}\left[\Phi_{r+1}^{+}(d)\right]\right) \leq \Phi_{r+1}^{+}(d)(\lambda)
$$

A similar argument shows that

$$
\nu^{*}(Y-X) \leq \Phi_{r+1}^{-}(d)(\lambda)
$$

We thus have

$$
\begin{aligned}
\nu^{*}(Y \cap X)+\nu^{*}(Y-X) & \leq \Phi_{r+1}^{+}(d)(\lambda)+\Phi_{r+1}^{-}(d)(\lambda) \\
& \leq d(\lambda)+2^{-(r+1)} \\
& =\nu\left(\mathbf{C}_{A}\right)+2^{-(r+1)} \\
& \leq \nu^{*}(Y)+2^{-r}
\end{aligned}
$$

Since $r \in \mathbb{N}$ is arbitrary here, this shows that

$$
\nu^{*}(Y) \geq \nu^{*}(Y \cap X)+\nu^{*}(Y-X)
$$

confirming that $X$ is $\nu$-measurable.
If we let $Y=\mathbf{C}$ in the above argument, then $d=\mathbf{1}$ and we have

$$
\nu(X)=\nu^{*}(\mathbf{C} \cap X) \leq \Phi_{r+1}^{+}(\mathbf{1})(\lambda)
$$

for all $r \in \mathbb{N}$, so

$$
\nu(X) \leq \Phi_{\infty}^{+}=\nu_{\mathrm{all}}(X)
$$

Similarly,

$$
1-\nu(X)=\nu\left(X^{\mathrm{c}}\right)=\nu^{*}(\mathbf{C}-X) \leq \Phi_{r+1}^{-}(\mathbf{1})(\lambda)
$$

for all $r \in \mathbb{N}$, so by Lemma 4.2,

$$
1-\nu(X) \leq \Phi_{\infty}^{-}=1-\Phi_{\infty}^{+}=1-\nu_{\mathrm{all}}(X)
$$

so $\nu_{\mathrm{all}}(X)=\nu(X)$.
Conversely, assume that $X$ is $\nu$-measurable. Define

$$
\Phi: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

as follows. Let $r \in \mathbb{N}$ and $d \in \mathcal{D}_{\nu}$. Since $S^{1}[d]$ is open, we know that $S^{1}[d], S^{1}[d] \cap X$ and $S^{1}[d]-X$ are $\nu$-measurable, with

$$
\nu(S[d])=\nu(S[d] \cap X)+\nu(S[d]-X)
$$

Fix prefix sets $A^{+}, A^{-} \subseteq\{0,1\}^{*}$ such that

$$
\begin{array}{ll}
S^{1}[d] \cap X \subseteq \mathbf{C}_{A^{+}}, & \nu\left(\mathbf{C}_{A^{+}}\right) \leq \nu(S[d] \cap X)+2^{-(r+1)}, \\
S^{1}[d]-X \subseteq \mathbf{C}_{A^{-}}, & \nu\left(\mathbf{C}_{A^{-}}\right) \leq \nu(S[d]-X)+2^{-(r+1)}
\end{array}
$$

For $a \in\{+,-\}$, define $\Phi_{r}^{a}(d)$ by

$$
\Phi_{r}^{a}(d)(w)= \begin{cases}\sum_{u \in A^{a}} \nu(u \mid w) & \text { if } \nu(w)>0 \\ 1 & \text { if } \nu(w)=0\end{cases}
$$

It is clear that the functional $\Phi$ so defined is all-computable (a vacuous condition) and maps $\mathbb{N} \times \mathcal{D}_{\nu}$ into $\mathcal{D}_{\nu} \times \mathcal{D}_{\nu}$. To see that $\Phi$ is an all- $\nu$-measurement of $X$, let $r \in \mathbb{N}$ and $d \in \mathcal{D}_{\nu}$. Three things are to be verified.

1. $S^{1}[d] \cap X \subseteq S^{1}\left[\Phi_{r}^{+}(d)\right]$. To see this, let $B \in S^{1}[d] \cap X$. Then $B \in \mathbf{C}_{A^{+}}$, so there exists $w \in A^{+}$such that $w \sqsubseteq B$. If $\nu(w)>0$, then

$$
\Phi_{r}^{+}(d)(w)=\sum_{u \in A^{+}} \nu(u \mid w) \geq \nu(w \mid w)=1
$$

so $\Phi_{r}^{+}(d)(w) \geq 1$ in any case, whence $B \in S^{1}\left[\Phi_{r}^{+}(d)\right]$.
2. $S^{1}[d]-X \subseteq S^{1}\left[\Phi_{r}^{-}(d)\right]$. The verification is analogous to that of 1 .
3. $\Phi_{r}^{+}(d)(\lambda)+\Phi_{r}^{-}(d)(\lambda) \leq d(\lambda)+2^{-r}$. To see this note that

$$
\begin{aligned}
\Phi_{r}^{+}(d)(\lambda) & +\Phi_{r}^{-}(d)(\lambda) \\
& =\nu\left(\mathbf{C}_{A^{+}}\right)+\nu\left(\mathbf{C}_{A^{-}}\right) \\
& \leq \nu\left(S^{1}[d] \cap X\right)+\nu\left(S^{1}[d]-X\right)+2^{-r} \\
& =\nu\left(S^{1}[d]\right)+2^{-r} \\
& \leq d(\lambda)+2^{-r}
\end{aligned}
$$

by Lemma 3.3.

We have now shown that $\Phi$ is an all- $\nu$-measurement of $X$, whence $X$ is all- $\nu$-measurable.

In the classical case, two definitions of $\nu$-measurability are frequently used. (See Royden [8] for example.) The definition we have used above is known as the Carathéodory definition. The other definition frequently used is the Lebesgue definition, which is stated in terms of equality of inner and outer measures. These two definitions are known to be equivalent in the classical case. Our definition of $\Delta-\nu$-measurability in terms of $\Delta$-computable splitting operators is motivated by the classical Carathéodory definition, and the proof of Theorem 4.6 articulates this motivation more precisely. Corollary 5.5 below suggests that resource-bounded measurability in complexity classes does not have an equivalent definition in the style of Lebesgue.

We now note that the $\Delta$-measures of $\Delta$-measurable sets must themselves be $\Delta$-computable.

Theorem 4.7. Let $X \subseteq \mathbf{C}$.

1. If $X$ is $\nu$-measurable in $R(\Delta)$, then $\nu(X \mid R(\Delta))$ is $\Delta$-computable.
2. If $X$ is $\Delta-\nu$-measurable, then $\nu_{\Delta}(X)$ is $\Delta$-computable.

Proof of Theorem 4.7 It suffices to prove 1 , since 2 then follows by part 2 of Lemma 4.5,
Assume that $X$ is $\nu$-measurable in $R(\Delta)$ with the $\nu$-measurement $\Phi$ as witness. Let $\widehat{\Phi}^{+}$ be a $\Delta$-computation of $\Phi^{+}$, and define $f: \mathbb{N} \rightarrow \mathbb{Q}$ by

$$
f(r)=\widehat{\Phi}_{r+1, r+1}^{+}(\mathbf{1})(\lambda)
$$

Then $f$ is $\Delta$-computable and, for all $r \in \mathbb{N}$, Lemma 4.4 tells us that

$$
\begin{aligned}
& |f(r)-\nu(X \mid R(\Delta))| \\
& \leq\left|f(r)-\Phi_{r+1}^{+}(\mathbf{1})(\lambda)\right|+\left|\Phi_{r+1}^{+}(\mathbf{1})(\lambda)-\nu(X \mid R(\Delta))\right| \\
& \leq 2^{-(r+1)}+2^{-(r+1)} \\
& =2^{-r}
\end{aligned}
$$

so $f$ is a $\Delta$-computation of $\nu(X \mid R(\Delta))$.
Our next step is to show that cylinders are $\Delta$ - $\nu$-measurable, provided that $\nu$ itself is a $\Delta$-probability measure, as defined in section 3.

Lemma 4.8. If $\nu$ is a $\Delta$-probability measure on $\mathbf{C}$, then for each $w \in\{0,1\}^{*}$, the cylinder $\mathbf{C}_{w}$ is $\Delta-\nu$-measurable, with $\nu_{\Delta}\left(\mathbf{C}_{w}\right)=\nu(w)$.

Proof of Lemma 4.8 Assume the hypothesis, and let $w \in\{0,1\}^{*}$. We have two cases.

I: $\nu(w)=0$. (Note that $w \neq \lambda$ in this case because $\nu(\lambda)=1$.) Then define

$$
\Phi: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

as follows. For each $r \in \mathbb{N}, d \in \mathcal{D}_{\nu}$, and $v \in\{0,1\}^{*}$, set

$$
\begin{aligned}
& \Phi_{r}^{+}(d)(v)= \begin{cases}1 & \text { if } w \sqsubseteq v \\
0 & \text { otherwise },\end{cases} \\
& \Phi_{r}^{-}(d)(v)=d(v)
\end{aligned}
$$

It is clear that $\Phi$ is $\Delta$-computable and that $\Phi_{r}^{+}(d), \Phi_{r}^{-}(d) \in \mathcal{D}_{\nu}$ for all $r \in \mathbb{N}$ and $d \in \mathcal{D}_{\nu}$. Also, for all $r \in \mathbb{N}$ and $d \in \mathcal{D}_{\nu}$, the following conditions hold.
(i) $S^{1}[d] \cap \mathbf{C}_{w} \subseteq \mathbf{C}_{w}=S^{1}\left[\Phi_{r}^{+}(d)\right]$.
(ii) $S^{1}[d]-\mathbf{C}_{w} \subseteq S^{1}[d]=S^{1}\left[\Phi_{r}^{-}(d)\right]$.
(iii) $\Phi_{r}^{+}(d)(\lambda)+\Phi_{r}^{-}(d)(\lambda)=d(\lambda)$.

Thus $\Phi$ is a $\Delta-\nu$-measurement of $\mathbf{C}_{w}$, so $\mathbf{C}_{w}$ is $\Delta-\nu$-measurable, with

$$
\begin{aligned}
\nu_{\Delta}\left(\mathbf{C}_{w}\right) & =\Phi_{\infty}^{+} \\
& =\inf _{r \in \mathbb{N}} \Phi_{r}^{+}(\mathbf{1})(\lambda) \\
& =\Phi_{0}^{+}(\mathbf{1})(\lambda) \\
& =0 \\
& =\nu(w) .
\end{aligned}
$$

II: $\nu(w)>0$. In this case, define

$$
\Phi: \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

as follows. For each $d \in \mathcal{D}_{\nu}$ and $v \in\{0,1\}^{*}$, set

$$
\begin{aligned}
& \Phi^{+}(d)(v)= \begin{cases}d(w) \nu(w \mid v) & \text { if } v \sqsubseteq w \\
d(v) & \text { if } w \sqsubseteq v \\
0 & \text { otherwise }\end{cases} \\
& \Phi^{-}(d)(v)=d(v)-\Phi^{+}(d)(v) .
\end{aligned}
$$

Then define

$$
\Psi: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

by

$$
\Psi(r, d)=\Phi(\Lambda(d))
$$

where $\Lambda$ is the functional given by the Regularity Lemma. It is clear that $\Psi$ is $\Delta$ computable.
Let $d \in \mathcal{D}_{\nu}$. It is routine to check that $\Phi^{+}(d) \in \mathcal{D}_{\nu}$. Also, for $v \sqsubseteq w$,

$$
d(w) \nu(w) \leq \sum_{|u|=|w|} d(u) \nu(u)=d(v) \nu(v)
$$

so $d(w) \nu(w \mid v) \leq d(v)$. It follows readily that $\Phi^{+}(d)(v) \leq d(v)$ for all $v \in\{0,1\}^{*}$, whence $\Phi^{-}(d) \in \mathcal{D}_{\nu}$. This confirms that $\Psi$ maps $\mathbb{N} \times \mathcal{D}_{\nu}$ into $\mathcal{D}_{\nu} \times \mathcal{D}_{\nu}$.
Now let $r \in \mathbb{N}$ and $d \in \mathcal{D}_{\nu}$.

To see that $S^{1}[d] \cap \mathbf{C}_{w} \subseteq S^{1}\left[\Psi_{r}^{+}(d)\right]$, let $A \in S^{1}[d] \cap \mathbf{C}_{w}$. Then $A \in S^{1}[\Lambda(d)] \cap \mathbf{C}_{w}$, so there exists $n \geq|w|$ such that $w \sqsubseteq A[0 . . n-1]$ and $\Lambda(d)(A[0 . . n-1]) \geq 1$. Then

$$
\begin{aligned}
\Psi_{r}^{+}(d)(A[0 . . n-1]) & =\Phi^{+}(\Lambda(d))(A[0 . . n-1]) \\
& =\Lambda(d)(A[0 . . n-1]) \\
& \geq 1
\end{aligned}
$$

so $A \in S^{1}\left[\Psi_{r}^{+}(d)\right]$.
To see that $S^{1}[d]-\mathbf{C}_{w} \subseteq S^{1}\left[\Psi_{r}^{-}(d)\right]$, let $A \in S^{1}[d]-\mathbf{C}_{w}$. Then $A \in S^{1}[\Lambda(d)]-\mathbf{C}_{w}$, so there exists $n \geq|w|$ such that $\Lambda(d)(A[0 . . n-1]) \geq 1$ and neither of $w, A[0 . . n-1]$ is a prefix of the other. Then $\Phi^{+}(\Lambda(d))(A[0 . . n-1])=0$, so

$$
\begin{aligned}
\Psi_{r}^{-}(d)(A[0 . . n-1]) & =\Phi^{-}(\Lambda(d))(A[0 . . n-1]) \\
& =\Lambda(d)(A[0 . . n-1]) \\
& \geq 1
\end{aligned}
$$

so $A \in S^{1}\left[\Psi_{r}^{-}(d)\right]$.
Finally, note that

$$
\begin{aligned}
\Psi_{r}^{+}(d)(\lambda)+\Psi_{r}^{-}(d)(\lambda) & =\Phi^{+}(\Lambda(d))(\lambda)+\Phi^{-}(\Lambda(d))(\lambda) \\
& =\Lambda(d)(\lambda) \\
& =d(\lambda)
\end{aligned}
$$

We have now shown that $\Psi$ is a $\Delta-\nu$-measurement of $\mathbf{C}_{w}$. The $\Delta-\nu$-measure of $\mathbf{C}_{w}$ is then

$$
\begin{aligned}
\nu_{\Delta}\left(\mathbf{C}_{w}\right) & =\Psi_{\infty}^{+} \\
& =\Phi^{+}(\Lambda(\mathbf{1}))(\lambda) \\
& =\Phi^{+}(\mathbf{1})(\lambda) \\
& =\mathbf{1}(w) \nu(w \mid \lambda) \\
& =\nu(w)
\end{aligned}
$$

Lemma 4.8 has important consequences for measure in $R(\Delta)$, as shown by the following corollaries and example.

Corollary 4.9. If $\nu$ is a $\Delta$-probability measure on $\mathbf{C}$, then for each $w \in\{0,1\}^{*}$, the cylinder $\mathbf{C}_{w}$ is $\nu$-measurable in $R(\Delta)$, with $\nu\left(\mathbf{C}_{w} \mid R(\Delta)\right)=\nu(w)$.

Proof of Corollary 4.9 This follows immediately from Lemma 4.8 and part 2 of Lemma 4.5

Corollary 4.10. If $\nu$ is a $\Delta$-probability measure on $\mathbf{C}$, then $R(\Delta)$ is $\nu$-measurable in $R(\Delta)$, with $\nu(R(\Delta) \mid R(\Delta))=1$.

Proof of Corollary 4.10 This follows immediately from Corollary 4.9 with $w=\lambda$.

Example 4.11. Consider the uniform probability measure $\mu$ on $\mathbf{C}$ and the set REC, consisting of all decidable languages. By Corollary 4.10 (with $\nu=\mu$ and $\Delta=$ REC), REC is $\mu$-measurable in REC, with $\mu(\mathrm{REC} \mid \mathrm{REC})=1$. On the other hand, REC is countable, so REC is (classically) $\mu$-measurable, with $\mu(\mathrm{REC})=0$. Since $\mu(\mathrm{REC} \mid \mathrm{REC}) \neq \mu(\mathrm{REC})$, it follows by Lemma 4.7 and parts 1 and 2 of Lemma 4.5 that REC is not rec- $\mu$-measurable. This example shows that the converses of parts 1 and 2 of Lemma 4.5 do not hold.

The rest of this section is devoted to showing that resource-bounded measurability and measure are well-behaved with respect to set-theoretic operations.

Definition. Let $R \subseteq \mathbf{C}$. An algebra on $R$ is a collection $\mathcal{F}$ of subsets of $\mathbf{C}$ with the following three properties.
(i) $R \in \mathcal{F}$.
(ii) If $X \in \mathcal{F}$, then $X^{\mathrm{c}} \in \mathcal{F}$.
(iii) If $X, Y \in \mathcal{F}$, then $X \cup Y \in \mathcal{F}$.

If $\mathcal{F}$ is an algebra on $R$, then a subalgebra of $\mathcal{F}$ on $R$ is a set $\mathcal{E} \subseteq \mathcal{F}$ that is also an algebra on $R$.

In this paper, we only use the above definition in cases where $R=\mathbf{C}$ or $R=R(\Delta)$. In the latter case, our terminology is somewhat nonstandard, in that we allow an algebra on $R(\Delta)$ to have elements that are not themselves subsets of $R(\Delta)$. Our terminology is in accordance with past usage in resource-bounded measure, where it is often convenient to speak of "the measure of $X$ in $R(\Delta)$ " (e.g., the measure of P/Poly in ESPACE) in cases where $X$ is not a subset of $R(\Delta)$. In any case, if one so prefers, it is a straightforward matter to define the notion of an algebra in $R(\Delta)$, and to adapt the results here to this notion.

Theorem 4.12. The set $\mathcal{F}_{R(\Delta)}^{\nu}$, consisting of all sets that are $\nu$-measurable in $R(\Delta)$, is an algebra on $R(\Delta)$. For $X, Y \in \mathcal{F}_{R(\Delta)}^{\nu}$, we have

$$
\nu\left(X^{\mathrm{c}} \mid R(\Delta)\right)=1-\nu(X \mid R(\Delta))
$$

and

$$
\nu(X \cup Y \mid R(\Delta))=\nu(X \mid R(\Delta))+\nu(Y \mid R(\Delta))-\nu(X \cap Y \mid R(\Delta))
$$

Proof of Theorem 4.12 By Corollary 4.10, $R(\Delta) \in \mathcal{F}_{R(\Delta)}^{\nu}$.
Assume that $X \in \mathcal{F}_{R(\Delta)}^{\nu}$. Then there is a $\nu$-measurement $\Phi$ of $X$ in $R(\Delta)$. Define

$$
\begin{gathered}
\Psi: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu} \\
\Psi_{r}^{+}(d)=\Phi_{r}^{-}(d) \\
\Psi_{r}^{-}(d)=\Phi_{r}^{+}(d)
\end{gathered}
$$

It is easily verified that $\Psi$ is a $\nu$-measurement of $X^{\mathrm{c}}$ in $R(\Delta)$, so $X^{\mathrm{c}} \in \mathcal{F}_{R(\Delta)}^{\nu}$ and, by Lemma 4.2,

$$
\begin{aligned}
\nu\left(X^{\mathrm{c}} \mid R(\Delta)\right) & =\Psi_{\infty}^{+}=\left\{\Phi_{\infty}^{-}\right. \\
& =1-\Phi_{\infty}^{+} \\
& =1-\nu(X \mid R(\Delta))
\end{aligned}
$$

Now assume that $X, Y \in \mathcal{F}_{R(\Delta)}^{\nu}$. Let $\Phi$ and $\Psi$ be $\nu$-measurements of $X$ and $Y$, respectively, in $R(\Delta)$. For each $a, b \in\{+,-\}$, define the functional

$$
\Theta[a b]: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu}
$$

by

$$
\Theta[a b]_{r}(d)=\Psi_{r+2}^{b}\left(\Phi_{r+1}^{a}(d)\right)
$$

It is clear that each of the four functionals $\Theta[a b]$ so defined is $\Delta$-computable. For all $r \in \mathbb{N}$ and $d \in \mathcal{D}_{\nu}$, our choice of $\Phi$ and $\Psi$ gives us the inclusions

$$
\begin{array}{r}
R(\Delta) \cap(X \cap Y) \cap S^{1}[d] \subseteq S^{1}\left[\Theta[++]_{r}(d)\right], \\
R(\Delta) \cap(X-Y) \cap S^{1}[d] \subseteq S^{1}\left[\Theta[+-]_{r}(d)\right], \\
R(\Delta) \cap(Y-X) \cap S^{1}[d] \subseteq S^{1}\left[\Theta[-+]_{r}(d)\right], \\
R(\Delta) \cap(X \cup Y)^{\mathrm{c}} \cap S^{1}[d] \subseteq S^{1}\left[\Theta[--]_{r}(d)\right]
\end{array}
$$

and the bound

$$
\begin{aligned}
\Theta[++]_{r}(d)(\lambda)+\Theta[+-]_{r}(d)(\lambda) & +\Theta[-+]_{r}(d)(\lambda)+\Theta[--]_{r}(d)(\lambda) \\
& \leq \Phi_{r+1}^{+}(d)(\lambda)+\Phi_{r+1}^{-}(d)(\lambda)+2^{-(r+1)} \\
& \leq d(\lambda)+2^{-r} .
\end{aligned}
$$

It follows that the functionals

$$
\begin{aligned}
\Theta[X] & =(\Theta[++]+\Theta[+-], \Theta[-+]+\Theta[--]), \\
\Theta[Y] & =(\Theta[++]+\Theta[-+], \Theta[+-]+\Theta[--]), \\
\Theta[X \cap Y] & =(\Theta[++], \Theta[+-]+\Theta[-+]+\Theta[--]), \\
\Theta[X \cup Y] & =(\Theta[++]+\Theta[+-]+\Theta[-+], \Theta[--])
\end{aligned}
$$

are $\nu$-measurements of $X, Y, X \cap Y, X \cup Y$, respectively, in $R(\Delta)$. Thus $X \cap Y$ and $X \cup Y$ are $\nu$-measurable in $R(\Delta)$, with

$$
\begin{aligned}
\nu(X \cup Y \mid R(\Delta)) & +\nu(X \cap Y \mid R(\Delta)) \\
& =\lim _{r \rightarrow \infty} \Theta[X \cup Y]_{r}^{+}(\mathbf{1})(\lambda)+\lim _{r \rightarrow \infty} \Theta[X \cap Y]_{r}^{+}(\mathbf{1})(\lambda) \\
& =\sum_{a, b \in\{+,-\}} \lim _{r \rightarrow \infty} \Theta[a b]_{r}(\mathbf{1})(\lambda) \\
& =\lim _{r \rightarrow \infty} \Theta[X]_{r}^{+}(\mathbf{1})(\lambda)+\lim _{r \rightarrow \infty} \Theta[Y]_{r}^{+}(\mathbf{1})(\lambda) \\
& =\nu(X \mid R(\Delta))+\nu(Y \mid R(\Delta))
\end{aligned}
$$

The following immediate consequence of Theorem 4.12 says that measure in $R(\Delta)$ is additive and monotone.

Corollary 4.13. Let $X, Y \in \mathcal{F}_{R(\Delta)}^{\nu}$.

1. If $X \cap Y \cap R(\Delta)=\emptyset$, then

$$
\nu(X \cup Y \mid R(\Delta))=\nu(X \mid R(\Delta))+\nu(Y \mid R(\Delta))
$$

2. If $X \cap R(\Delta) \subseteq Y$, then $\nu(X \mid R(\Delta)) \leq \nu(Y \mid R(\Delta))$.

The following analogs of Theorem 4.12 and its corollary are proven in similar fashion.

Theorem 4.14. The set $\mathcal{F}_{\Delta}^{\nu}$, consisting of all sets that are $\Delta$ - $\nu$-measurable, is an algebra over C. For $X, Y \in \mathcal{F}_{\Delta}^{\nu}$, we have

$$
\nu_{\Delta}\left(X^{\mathrm{c}}\right)=1-\nu_{\Delta}(X)
$$

and

$$
\nu_{\Delta}(X \cup Y)=\nu_{\Delta}(X)+\nu_{\Delta}(Y)-\nu_{\Delta}(X \cap Y)
$$

Corollary 4.15. Let $X, Y \in \mathcal{F}_{\Delta}^{\nu}$.

1. If $X \cap Y=\emptyset$, then

$$
\nu_{\Delta}(X \cup Y)=\nu_{\Delta}(X)=\nu_{\Delta}(Y)
$$

2. If $X \subseteq Y$, then $\nu_{\Delta}(X) \leq \nu_{\Delta}(Y)$.

Note that by Example 4.11, $\mathcal{F}_{\text {rec }}^{\mu} \varsubsetneqq \mathcal{F}_{\text {REC }}^{\mu}$, and $\mathcal{F}_{\text {rec }}^{\mu}$ is not an algebra over REC.
An important property of classical measurability is its completeness, which means that all subsets of measure-0 sets are measurable. We now show that resource-bounded measurability also enjoys this property.

Definition. Let $\mathcal{F}$ be a subalgebra of $\mathcal{F}_{R(\Delta)}^{\nu}$ on $R(\Delta)$. Then $\mathcal{F}$ is $\nu$-complete on $R(\Delta)$ if, for all $X, Y \subseteq \mathbf{C}$, if $X \subseteq Y \in \mathcal{F}$ and $\nu(Y \mid R(\Delta))=0$, then $X \in \mathcal{F}$.

Definition. Let $\mathcal{F}$ be a subalgebra of $\mathcal{F}_{\Delta}^{\nu}$ on $\mathbf{C}$. Then $\mathcal{F}$ is $\Delta$ - $\nu$-complete if, for all $X, Y \subseteq \mathbf{C}$, if $X \subseteq Y \in \mathcal{F}$ and $\nu_{\Delta}(Y)=0$, then $X \in \mathcal{F}$.

## Theorem 4.16.

1. The algebra $\mathcal{F}_{R(\Delta)}^{\nu}$ is $\nu$-complete on $R(\Delta)$.
2. The algebra $\mathcal{F}_{\Delta}^{\nu}$ is $\Delta-\nu$-complete.

Proof of Theorem 4.16 To prove part 1, assume that $X \subseteq Y \in \mathcal{F}$ and $\nu(Y \mid R(\Delta))=0$. Let $\Psi$ be a $\nu$-measurement of $Y$ in $R(\Delta)$. Define

$$
\begin{gathered}
\Phi: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu} \\
\Phi_{r}(d)=\left(\Psi_{r}^{+}(\mathbf{1}), d\right)
\end{gathered}
$$

It is easy to check that $\Phi$ is a $\nu$-measurement of $X$ in $R(\Delta)$, whence $X \in \mathcal{F}_{R(\Delta)}^{\nu}$.

The proof of part 2 is identical.
The rest of this section concerns infinitary unions and intersections of resource-bounded measurable sets. Classical measure is countably additive, which means that the union of a countable collection of disjoint measurable sets is measurable, and the measure of the union is the sum of the measures of the disjoint measurable sets. However, this property does not hold for arbitrary resource bounds $\Delta$. For example, if $\Delta=\mathrm{p}$ or $\Delta=\mathrm{rec}$, then $R(\Delta)$ can be written as a countable union of singleton sets, each of which has $\Delta$ - $\mu$-measure 0 [6], hence $\mu$-measure 0 in $R(\Delta)$. The union of these singletons, $R(\Delta)$, has $\mu$-measure 1 in $R(\Delta)-$ exceeding the sum of the $\mu$-measures of the singletons - and is not $\Delta$ - $\mu$-measurable at all. Thus any resource-bounded analog of countable additivity must involve a restricted class of countable unions.

The obvious approach is to restrict attention to countable sequences of sets that are not only individually measurable, but for which there is a uniform witness of their measurability. This leads to the following definitions.

Definition. Let $\mathcal{F}$ be a subalgebra of $\mathcal{F}_{R(\Delta)}^{\nu}$ on $R(\Delta)$. A $\Delta$-sequence in $\mathcal{F}$ is a sequence $\left(X_{k} \mid k \in \mathbb{N}\right)$ of sets $X_{k} \in \mathcal{F}$ for which there exists a $\Delta$-computable functional

$$
\Phi: \mathbb{N} \times \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

such that, for each $k \in \mathbb{N}, \Phi_{k}$ is a $\nu$-measurement of $X_{k}$ in $R(\Delta)$.
Definition. Let $\mathcal{F}$ be a subalgebra of $\mathcal{F}_{\Delta}^{\nu}$ on $\mathbf{C}$. A $\Delta$-sequence in $\mathcal{F}$ is a sequence $\left(X_{k} \mid k \in\right.$ $\mathbb{N}$ ) of sets $X_{k} \in \mathcal{F}$ for which there exists a $\Delta$-computable functional

$$
\Phi: \mathbb{N} \times \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

such that, for each $k \in \mathbb{N}, \Phi_{k}$ is a $\Delta$ - $\nu$-measurement of $X_{k}$.
Although these definitions are useful for our purposes here, the following example shows that they are not sufficient.

Example 4.17. Let ( $M_{i} \mid i \in \mathbb{N}$ ) be a standard enumeration of Turing machines. Using the standard enumeration $\left(s_{j} \mid j \in \mathbb{N}\right)$ of $\{0,1\}^{*}$ and a standard pairing bijection $<,>$ : $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, define a sequence $\left(X_{k} \mid k \in \mathbb{N}\right)$ as follows. For $k=<i, j>\in \mathbb{N}$, if $M_{i}\left(0^{i}\right)$ halts in exactly $\left|s_{j}\right|$ steps, then let $X_{k}=\mathbf{C}_{0^{i} 1 s_{j}}$; otherwise, let $X_{k}=\emptyset$. Since the construction in the proof of Lemma 4.8 is easily uniformized, it is not hard to show that $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a p-sequence in $\mathcal{F}_{\mathrm{p}}^{\mu}$. Let $X=\bigcup_{k=0}^{\infty} X_{k}$. Then the sets $X_{0}, X_{1}, \ldots$ are disjoint, so

$$
\mu(X)=\sum_{k=0}^{\infty} \mu\left(X_{k}\right)=\frac{1}{2} \sum_{i \in K} 2^{-i}
$$

where

$$
K=\left\{i \in \mathbb{N} \mid M_{i}\left(0^{i}\right) \text { halts }\right\}
$$

is the diagonal halting problem. It is thus clear that $\mu(X)$ is Turing-equivalent to $K$, hence not computable. (In fact, $\mu(X)$ is a version of Chaitin's random real number $\Omega$ [3].) It follows by Lemmas 4.5 and 4.7 that $X \notin \mathcal{F}_{\text {rec }}^{\mu}$. We thus have a p-sequence of disjoint elements of $\mathcal{F}_{\mathrm{p}}^{\mu}$ whose union is not an element of $\mathcal{F}_{\mathrm{p}}^{\mu}$.

The set $X$ of Example 4.17 fails to be rec- $\mu$-measurable because the sum $\sum_{k=0}^{\infty} \mu\left(X_{k}\right)$ converges too slowly. The following technical definition prohibits such phenomena.

Definition. A functional

$$
\Phi: \mathbb{N} \times \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

is $\Delta$-modulated if the sequences $\left(\Phi_{k, r}^{a}(d)(w) \mid k \in \mathbb{N}\right)$, for $a \in\{+,-\}, r \in \mathbb{N}, d \in \mathcal{D}_{\nu}$, and $w \in\{0,1\}^{*}$, are uniformly $\Delta$-convergent. Equivalently, $\Phi$ is $\Delta$-modulated if there is a $\Delta$ computable functional

$$
\Gamma: \mathbb{N} \times \mathbb{N} \times \mathcal{D}_{\nu} \times\{0,1\}^{*} \rightarrow \mathbb{N}
$$

such that, for all $a \in\{+,-\}, t, r \in \mathbb{N}, d \in \mathcal{D}_{\nu}, w \in\{0,1\}^{*}$, and $k \geq \Gamma(t, r, d, w)$,

$$
\left|\Phi_{k, r}^{a}(d)(w)-\Phi_{\infty, r}^{a}(d)(w)\right| \leq 2^{-t}
$$

where the limit $\Phi_{\infty, r}^{a}(d)(w)=\lim _{k \rightarrow \infty} \Phi_{k, r}^{a}(d)(w)$ is implicitly assumed to exist.
We now define the restricted infinitary unions and intersections with which we deal.
Definition. Let $\mathcal{F}$ be a subalgebra of $\mathcal{F}_{R(\Delta)}^{\nu}$ on $R(\Delta)$.

1. A union $\Delta$-sequence in $\mathcal{F}$ is a sequence $\left(X_{k} \mid k \in \mathbb{N}\right)$ of sets $X_{k} \in \mathcal{F}$ for which there exists a $\Delta$-modulated functional $\Phi$ such that $\Phi_{k, r}^{+}(d)(w)$ is nondecreasing in $k$, $\Phi_{k, r}^{-}(d)(w)$ is nonincreasing in $k$, and $\Phi$ testifies that $\left(\bigcup_{j=0}^{k-1} X_{j} \mid k \in \mathbb{N}\right)$ is a $\Delta$-sequence in $\mathcal{F}$.
2. An intersection $\Delta$-sequence in $\mathcal{F}$ is a sequence ( $X_{k} \mid k \in \mathbb{N}$ ) of sets $X_{k} \in \mathcal{F}$ for which there exists a $\Delta$-modulated functional $\Phi$ such that $\Phi_{k, r}^{+}(d)(w)$ is nonincreasing in $k, \Phi_{k, r}^{-}(d)(w)$ is nondecreasing in $k$, and $\Phi$ testifies that $\left(\bigcap_{j=0}^{k-1} X_{j} \mid k \in \mathbb{N}\right)$ is a $\Delta$-sequence in $\mathcal{F}$.
3. $\mathcal{F}$ is closed under $\Delta$-unions if $\bigcup_{k=0}^{\infty} X_{k} \in \mathcal{F}$ for every union $\Delta$-sequence ( $X_{k} \mid k \in \mathbb{N}$ ) in $\mathcal{F}$.
4. $\mathcal{F}$ is closed under $\Delta$-intersections if $\bigcap_{k=0}^{\infty} X_{k} \in \mathcal{F}$ for every intersection $\Delta$-sequence $\left(X_{k} \mid k \in \mathbb{N}\right)$ in $\mathcal{F}$.

The motivation for these definitions will be discussed further in the final version of this paper. We note in the next lemma that, when the sets $X_{k}$ have resource-bounded measure 0 , every $\Delta$-sequence is already a union $\Delta$-sequence.

Lemma 4.18. Let $\mathcal{F}$ be a subalgebra of $\mathcal{F}_{R(\Delta)}^{\nu}$ on $R(\Delta)$. If $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a $\Delta$-sequence in $\mathcal{F}$ and $\nu\left(X_{k} \mid R(\Delta)\right)=0$ for all $k \in \mathbb{N}$, then $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a union $\Delta$-sequence in $\mathcal{F}$.

Proof of Lemma 4.18 Assume the hypothesis, and let $\Phi$ be a witness that $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a $\Delta$-sequence in $\mathcal{F}$. Then the functional

$$
\begin{gathered}
\Psi: \mathbb{N} \times \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu} \\
\Psi_{k, r}(d)=\left(\sum_{j=0}^{k} \Phi_{j, j+r+1}^{+}(\mathbf{1}), d\right)
\end{gathered}
$$

is easily seen to testify that $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a union $\Delta$-sequence in $\mathcal{F}$.
The following easy de Morgan law is useful.

Lemma 4.19. Let $\mathcal{F}$ be a subalgebra of $\mathcal{F}_{R(\Delta)}^{\nu}$ on $R(\Delta)$. Then a sequence ( $X_{k} \mid k \in \mathbb{N}$ ) is a union $\Delta$-sequence in $\mathcal{F}$ if and only if the complemented sequence ( $X_{k}^{\mathrm{c}} \mid k \in \mathbb{N}$ ) is an intersection $\Delta$-sequence in $\mathcal{F}$. Thus $\mathcal{F}$ is closed under $\Delta$-unions if and only if $\mathcal{F}$ is closed under $\Delta$-intersections.

Proof of Lemma 4.19 It is clear that $\left(\Phi^{+}, \Phi^{-}\right)$testifies that $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a union $\Delta$ sequence in $\mathcal{F}$ if and only if $\left(\Phi^{-}, \Phi^{+}\right)$testifies that $\left(X_{k}^{\mathrm{c}} \mid k \in \mathbb{N}\right)$ is an intersection $\Delta$-sequence in $\mathcal{F}$.

We now show that measure in $R(\Delta)$ is well-behaved with respect to $\Delta$-unions and $\Delta$ intersections.

## Theorem 4.20.

1. $\mathcal{F}_{R(\Delta)}^{\nu}$ is closed under $\Delta$-unions and $\Delta$-intersections.
2. If $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a union $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$, then

$$
\nu\left(\cup_{k=0}^{\infty} X_{k} \mid R(\Delta)\right) \leq \sum_{k=0}^{\infty} \nu\left(X_{k} \mid R(\Delta)\right)
$$

with equality if the sets $X_{0}, X_{1}, \ldots$ are disjoint.
3. If ( $X_{k} \mid k \in \mathbb{N}$ ) is a union $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$ with each $X_{k} \subseteq X_{k+1}$, then

$$
\nu\left(\cup_{k=0}^{\infty} X_{k} \mid R(\Delta)\right)=\lim _{k \rightarrow \infty} \nu\left(X_{k} \mid R(\Delta)\right)
$$

4. If $\left(X_{k} \mid k \in \mathbb{N}\right)$ is an intersection $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$ with each $X_{k+1} \subseteq X_{k}$, then

$$
\nu\left(\cap_{k=0}^{\infty} X_{k} \mid R(\Delta)\right)=\lim _{k \rightarrow \infty} \nu\left(X_{k} \mid R(\Delta)\right) .
$$

Proof of Theorem 4.20 Let $\left(X_{k} \mid k \in \mathbb{N}\right)$ be a union $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$, with the functional $\Phi$ as witness, and let $X \bigcup_{k=0}^{\infty} X_{k}$. Fix a functional $\Gamma$ testifying that $\Phi$ is $\Delta$ modulated, and define a functional

$$
\Theta: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

by

$$
\Theta_{r}^{+}(d)(w)=\Phi_{\infty, r+1}^{+}(d)(w)
$$

and

$$
\Theta_{r}^{-}(d)(w)=\Phi_{m, r+1}^{-}(d)(w),
$$

where $m=\Gamma(r+1, r+1, d, \lambda)$. By our choice of $\Phi$ and $\Gamma, \Theta$ is a well-defined, $\Delta$-computable functional mapping $\mathbb{N} \times \mathcal{D}_{\nu}$ into $\mathcal{D}_{\nu} \times \mathcal{D}_{\nu}$. To see that $\Theta$ is a $\nu$-measurement of $X$ in $R(\Delta)$, let $r \in \mathbb{N}$ and $d \in \mathcal{D}_{\nu}$, and note the following three things.

1. $R(\Delta) \cap X \cap S^{1}[d] \subseteq S^{1}\left[\Theta_{r}^{+}(d)\right]$. To see this, let $A \in R(\Delta) \cap X \cap S^{1}[d]$. Then there exists $k \in \mathbb{N}$ such that

$$
\begin{aligned}
A \in R(\Delta) \cap\left(\cup_{j=0}^{k-1} X_{j}\right) \cap S^{1}[d] & \subseteq S^{1}\left[\Phi_{k, r+1}^{+}(d)\right] \\
& \subseteq S^{1}\left[\Phi_{\infty, r+1}^{+}(d)\right] \\
& =S^{1}\left[\Theta_{r}^{+}(d)\right]
\end{aligned}
$$

2. $R(\Delta) \cap X^{\mathrm{c}} \cap S^{1}[d] \subseteq S^{1}\left[\Theta_{r}^{-}(d)\right]$. To see this, let $A \in R(\Delta) \cap X^{\mathrm{c}} \cap S^{1}[d]$. Then, writing $m=\Gamma(r+1, r+1, d, \lambda)$, we have

$$
\begin{aligned}
& A \in R(\Delta) \cap\left(\cup_{j=0}^{m-1} X_{j}\right)^{\mathrm{c}} \cap S^{1}[d] \\
& \subseteq S^{1}\left[\Phi_{m, r+1}^{-}(d)\right] \\
&=S^{1}\left[\Theta_{r}^{-}(d)\right] .
\end{aligned}
$$

3. $\Theta_{r}^{+}(d)(\lambda)+\Theta_{r}^{-}(d)(\lambda) \leq d(\lambda)+2^{-r}$. To see this, let $m=\Gamma(r+1, r+1, d, \lambda)$. Then

$$
\begin{aligned}
\Theta_{r}^{+}(d)(\lambda) & =\Phi_{\infty, r+1}^{+}(d)(\lambda) \\
& \leq \Phi_{m, r+1}^{+}(d)(\lambda)+2^{-(r+1)},
\end{aligned}
$$

$$
\begin{aligned}
\Theta_{r}^{+}(d)(\lambda) & +\Theta_{r}^{-}(d)(\lambda) \\
& \leq \Phi_{m, r+1}^{+}(d)(\lambda)+\Phi_{m, r+1}^{-}(d)(\lambda)+2^{-(r+1)} \\
& \leq d(\lambda)+2^{-r} .
\end{aligned}
$$

We have now shown that $\Theta$ is a $\nu$-measurement of $X$ in $R(\Delta)$, whence $X \in \mathcal{F}_{R(\Delta)}^{\nu}$. This, together with Lemma 4.19, proves part 1 of the theorem.

Since $\Theta$ is a $\nu$-measurement of $X$ in $R(\Delta)$, Theorem 4.12 and its corollary (applied inductively) tell us that

$$
\begin{aligned}
\nu\left(\bigcup_{k=0}^{\infty} X_{k} \mid R(\Delta)\right) & =\lim _{r \rightarrow \infty} \Theta_{r}^{+}(\mathbf{1})(\lambda) \\
& =\lim _{r \rightarrow \infty} \lim _{k \rightarrow \infty} \Phi_{k, r+1}^{+}(\mathbf{1})(\lambda) \\
& =\lim _{k \rightarrow \infty} \lim _{r \rightarrow \infty} \Phi_{k, r+1}^{+}(\mathbf{1})(\lambda) \\
& =\lim _{k \rightarrow \infty} \nu\left(\bigcup_{j=0}^{k-1} X_{j} \mid R(\Delta)\right) \\
& \leq \lim _{k \rightarrow \infty} \sum_{j=0}^{k-1} \nu\left(X_{j} \mid R(\Delta)\right) \\
& =\sum_{k=0}^{\infty} \nu\left(X_{k} \mid R(\Delta)\right)
\end{aligned}
$$

with equality if the sets $X_{0}, X_{1}, \ldots$ are disjoint. This proves part 2 of the theorem.
To prove part 3 of the theorem, let $\left(X_{k} \mid k \in \mathbb{N}\right)$ be a union $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$ with each $X_{k} \subseteq X_{k+1}$. Let $Y_{0}=X_{0}$ and, for each $k \in \mathbb{N}$, let $Y_{k+1}=X_{k+1}-X_{k}$. By uniformly applying the technique of the proof of Theorem 4.12 to a functional testifying that $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a union $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$, it is routine to obtain a functional testifying that $\left(Y_{k} \mid k \in \mathbb{N}\right)$ is also a union $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$. Since the sets $Y_{0}, Y_{1}, \ldots$ are disjoint and $X_{k}=\bigcup_{j=0}^{k} Y_{j}$
for each $k \in \mathbb{N}$, it follows by part 2 of this theorem and Theorem 4.12 that

$$
\begin{aligned}
\nu\left(\bigcup_{k=0}^{\infty} X_{k} \mid R(\Delta)\right) & =\nu\left(\bigcup_{k=0}^{\infty} Y_{k} \mid R(\Delta)\right) \\
& =\sum_{k=0}^{\infty} \nu\left(Y_{k} \mid R(\Delta)\right) \\
& =\lim _{k \rightarrow \infty} \sum_{j=0}^{k} \nu\left(Y_{j} \mid R(\Delta)\right) \\
& =\lim _{k \rightarrow \infty} \nu\left(\bigcup_{j=0}^{k} Y_{j} \mid R(\Delta)\right) \\
& =\lim _{k \rightarrow \infty} \nu\left(X_{k} \mid R(\Delta)\right) .
\end{aligned}
$$

To prove part 4 of the theorem, let $\left(X_{k} \mid k \in \mathbb{N}\right)$ be an intersection $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$ with each $X_{k+1} \subseteq X_{k}$. By part 1 of this theorem, $\bigcap_{k=0}^{\infty} X_{k} \in \mathcal{F}_{R(\Delta)}^{\nu}$, so by Theorem 4.12,

$$
\begin{aligned}
\nu\left(\bigcap_{k=0}^{\infty} X_{k} \mid R(\Delta)\right) & =1-\nu\left(\left(\bigcap_{k=0}^{\infty} X_{k}\right)^{\mathrm{c}} \mid R(\Delta)\right) \\
& =1-\nu\left(\bigcup_{k=0}^{\infty} X_{k}^{\mathrm{c}} \mid R(\Delta)\right)
\end{aligned}
$$

By Lemma 4.19, the complemented sequence $\left(X^{\mathrm{c}} \mid k \in \mathbb{N}\right)$ is a union $\Delta$-sequence in $\mathcal{F}_{R(\Delta)}^{\nu}$. It is clear that each $X_{k}^{\mathrm{c}} \subseteq X_{k+1}^{\mathrm{c}}$, so by part 3 of this theorem and Theorem 4.12,

$$
\begin{aligned}
\nu\left(\bigcap_{k=0}^{\infty} X_{k} \mid R(\Delta)\right) & =1-\lim _{k \rightarrow \infty} \nu\left(X_{k}^{\mathrm{c}} \mid R(\Delta)\right) \\
& =1-\lim _{k \rightarrow \infty}\left(1-\nu\left(X_{k} \mid R(\Delta)\right)\right) \\
& =\lim _{k \rightarrow \infty} \nu\left(X_{k} \mid R(\Delta)\right)
\end{aligned}
$$

Corollary 4.21. If $\left(X_{k} \mid k \in \mathbb{N}\right)$ is a $\Delta$-sequence of sets, each of which has $\nu$-measure 0 in $R(\Delta)$, then $\nu\left(\bigcup_{k=0}^{\infty} X_{k} \mid R(\Delta)\right)=0$.

Proof of Corollary 4.21 This follows immediately from Lemma 4.19 and Theorem 4.20,

Our discussion of union $\Delta$-sequences and intersection $\Delta$-sequences has been confined to the algebra $\mathcal{F}_{R(\Delta)}^{\nu}$. It is a routine matter to develop the analogous definitions and results for the algebra $\mathcal{F}_{\Delta}^{\nu}$.

## 5 Measure Zero

In this section we prove a resource-bounded generalization of the classical Kolmogorov zeroone law that holds in complexity classes, and we develop some useful characterizations of sets of resource-bounded measure 0 . We begin our development with two simple consequences of resource-bounded measurability. Throughout this section, $\Delta$ is a resource bound, as specified in section 2, and $\nu$ is a probability measure on $\mathbf{C}$, as defined in section 3.

Definition. Let $X \in$ C. A $\nu$-bicover of $X$ in $R(\Delta)$ is an ordered pair $\left(d^{+}, d^{-}\right)$of $\Delta$ computable functions

$$
d^{+}, d^{-}: \mathbb{N} \times\{0,1\}^{*} \rightarrow[0, \infty)
$$

that satisfy the following conditions for all $r \in \mathbb{N}$.
(i) $d_{r}^{+}$and $d_{r}^{-}$are $\nu$-martingales.
(ii) $R(\Delta) \cap X \subseteq S^{1}\left[d_{r}^{+}\right]$.
(iii) $R(\Delta)-X \subseteq S^{1}\left[d_{r}^{-}\right]$.
(iv) $d_{r}^{+}(\lambda)+d_{r}^{-}(\lambda) \leq 1+2^{-r}$.

If $\left(d^{+}, d^{-}\right)$is a $\nu$-bicover of $X$ in $R(\Delta)$, then we use the notation

$$
d_{\infty}^{+}(\lambda)=\lim _{r \rightarrow \infty} d_{r}^{+}(\lambda)
$$

(This limit is shown to exist in Lemma 5.1)
Definition. Let $X \in \mathbf{C}$. The martingale outer $\nu$-measure of $X$ in $R(\Delta)$ is the infimum $\nu^{\circ}(X \mid R(\Delta))$ of all $d(\lambda)$ for which $d$ is a $\Delta$ - $\nu$-martingale such that $X \cap R(\Delta) \subseteq S^{1}[d]$.

Note that, for every set $X \subseteq \mathbf{C}, \nu^{\circ}(X \mid R(\Delta))$ is defined and

$$
0 \leq \nu^{\circ}(X \mid R(\Delta)) \leq 1
$$

Lemma 5.1. Let $X \subseteq \mathbf{C}$, and consider the following three conditions.
(A) $X$ is $\nu$-measurable in $R(\Delta)$.
(B) $X$ has a $\nu$-bicover in $R(\Delta)$.
(C) $\nu^{\circ}(X \mid R(\Delta))+\nu^{\circ}\left(X^{\mathrm{c}} \mid R(\Delta)\right)=1$.

Then $(\mathrm{A}) \Longrightarrow(\mathrm{B}) \Longrightarrow(\mathrm{C})$. Moreover, we have the following.

1. If (A) holds and $\left(d^{+}, d^{-}\right)$is any $\nu$-bicover of $X$ in $R(\Delta)$, then $d_{\infty}^{+}(\lambda)=\nu(X \mid R(\Delta))$.
2. If (B) holds and $\left(d^{+}, d^{-}\right)$is an $\nu$-bicover of $X$ in $R(\Delta)$, then $d_{\infty}^{+}(\lambda)=\nu^{\circ}(X \mid R(\Delta))$.

Proof of Lemma 5.1 First, assume that (A) holds, and let $\Phi$ be a $\nu$-measurement of $X$ in $R(\Delta)$. Then it is easy to check that $\left(\Phi^{+}(\mathbf{1}), \Phi^{-}(\mathbf{1})\right)$ is a $\nu$-bicover of $X$ in $R(\Delta)$, whence (B) holds. Now let $\left(d^{+}, d^{-}\right)$be any $\nu$-bicover of $X$ in $R(\Delta)$. Then, by the Measure Conservation Theorem (arguing as in Lemma4.1), for all $r \in \mathbb{N}$,

$$
\begin{aligned}
1 & \leq \Phi_{r}^{+}(\mathbf{1})(\lambda)+d_{r}^{-}(\lambda) \\
& \leq \Phi_{r}^{+}(\mathbf{1})(\lambda)+\left(1+2^{-r}-d_{r}^{+}(\lambda)\right)
\end{aligned}
$$

so $d_{r}^{+}(\lambda) \leq \Phi_{r}^{+}(\mathbf{1})(\lambda)+2^{-r}$. Similary, $\Phi_{r}^{+}(\mathbf{1})(\lambda) \leq d_{r}^{+}(\lambda)+2^{-r}$, so

$$
\left|\Phi_{r}^{+}(\mathbf{1})(\lambda)-d_{r}^{+}(\lambda)\right| \leq 2^{-r} .
$$

It follows immediately that

$$
d_{\infty}^{+}(\lambda)=\Phi_{\infty}^{+}=\nu(X \mid R(\Delta)) .
$$

Next, assume that (B) holds, and let $\left(d^{+}, d^{-}\right)$be a $\nu$-bicover of $X$ in $R(\Delta)$. The Measure Conservation Theorem (arguing again as in Lemma 4.1) tells us that $\nu^{\circ}(X \mid R(\Delta))+\nu^{\circ}\left(X^{\mathrm{c}} \mid\right.$ $R(\Delta)) \geq 1$ in any case. To see that (C) holds, let $r \in \mathbb{N}$ be arbitrary. Then $R(\Delta) \cap X \subseteq$ $S^{1}\left[d_{r}^{+}\right]$and $R(\Delta) \cap X^{\mathrm{c}} \subseteq S^{1}\left[d_{r}^{-}\right]$, so

$$
\begin{aligned}
\nu^{\circ}(X \mid R(\Delta)) & +\nu^{\circ}\left(X^{\mathrm{c}} \mid R(\Delta)\right) \\
& \leq d_{r}^{+}(\lambda)+d_{r}^{-}(\lambda) \\
& \leq 1+2^{-r}
\end{aligned}
$$

Since $r$ is arbitrary here, this confirms (C). We have already noted that $\nu^{\circ}(X \mid R(\Delta)) \leq$ $d_{r}^{+}(\lambda)$ for all $r \in \mathbb{N}$; it follows immediately that $\nu^{\circ}(X \mid R(\Delta)) \leq d_{\infty}^{+}(\lambda)$. To see that $d_{\infty}^{+}(\lambda) \leq \nu^{\circ}(X \mid R(\Delta))$, let $d$ be an arbitrary $\Delta$ - $\nu$-martingale such that $X \cap R(\Delta) \subseteq S^{1}[d]$,
and let $r \in \mathbb{N}$ be arbitrary. Then, by the Measure Conservation Theorem (arguing yet again as in Lemma 4.1),

$$
\begin{aligned}
1 & \leq d(\lambda)+d_{r}^{-}(\lambda) \\
& \leq d(\lambda)+\left(1+2^{-r}-d_{r}^{+}(\lambda)\right),
\end{aligned}
$$

so $d(\lambda) \geq d_{r}^{+}(\lambda)-2^{-r}$. Since $r$ is arbitrary, it follows that $d(\lambda) \geq d_{\infty}^{+}(\lambda)$. Since $d$ is arbitrary, it follows in turn that $\nu^{\circ}(X \mid R(\Delta)) \geq d_{\infty}^{+}(\lambda)$, completing the proof.

Before proceeding, we mention two examples and a lemma that further illuminate Lemma 5.1

Example 5.2. Consider the uniform probability measure $\mu$ on $\mathbf{C}$ and the complexity class $R(\mathrm{p})=\mathrm{E}$. For each $k \in \mathbb{N}$, let

$$
X_{k}=\operatorname{DTIME}\left(2^{(k+1) n}\right)-\operatorname{DTIME}\left(2^{k n}\right),
$$

and let $X=\bigcup_{k=0}^{\infty} X_{2 k}$. The diagonalization technique used in the proof of the Measure Conservation Theorem can readily be adapted to show that

$$
\nu^{\circ}(X \mid \mathrm{E})=\mu^{\circ}\left(X^{\mathrm{c}} \mid \mathrm{E}\right)=1
$$

Thus condition (C) of Lemma 5.1 is not satisfied by all $X \subseteq \mathbf{C}$.

Example 5.3. Again, consider the uniform probability measure $\mu$ on $\mathbf{C}$ and the complexity $\operatorname{class} R(\mathrm{p})=\mathrm{E}$. For each $k \in \mathbb{N}$, let

$$
X_{k}=\operatorname{DTIME}\left(2^{k n}\right) \cap \mathbf{C}_{0^{k}},
$$

and let $X=\bigcup_{k=0}^{\infty} X_{k}$. We show in the full version of this paper that $X$ satisfies condition (C) of Lemma [5.1, with $\mu^{\circ}(X \mid \mathrm{E})=0$, but that $X$ does not satisfy condition (B). Thus, in Lemma 5.1, condition (C) does not imply condition (B) - or condition (A) - even in the measure 0 case.

## Proof of Example 5.3

To see that $X$ satisfies condition (C), let $r \in \mathbb{N}$ be arbitrary. It is well-known 6] that $\mu_{\mathrm{p}}\left(\operatorname{DTIME}\left(2^{r n}\right)\right)=0$, so there is a $\mathrm{p}-\mu$-martingale $d^{\prime}$ such that $d^{\prime}(\lambda) \leq 2^{-(r+1)}$ and $\operatorname{DTIME}\left(2^{r n}\right) \subseteq S^{1}\left[d^{\prime}\right]$. Define $d^{\prime \prime}:\{0,1\}^{*} \rightarrow[0, \infty)$ by

$$
d^{\prime \prime}(w)= \begin{cases}2^{|w|-(r+1)} & \text { if } w \sqsubseteq 0^{r+1} \\ 1 & \text { if } 0^{r+1} \sqsubseteq w \\ 0 & \text { otherwise }\end{cases}
$$

It is easy to check that $d^{\prime \prime}$ is a p - $\mu$-martingale with $d^{\prime \prime}(\lambda)=2^{-(r+1)}$ and $S^{1}\left[d^{\prime \prime}\right]=\mathbf{C}_{0^{r+1}}$. Thus, if we let

$$
d=d^{\prime}+d^{\prime \prime}
$$

then $d$ is a p - $\mu$-martingale,

$$
d(\lambda)=d^{\prime}(\lambda)+d^{\prime \prime}(\lambda) \leq 2^{-r}
$$

and

$$
\begin{aligned}
X & =\left(\bigcup_{k=0}^{r} X_{k}\right) \cup\left(\bigcup_{k=r+1}^{\infty} X_{k}\right) \\
& \subseteq \operatorname{DTIME}\left(2^{r n}\right) \cup \mathbf{C}_{0^{r+1}} \\
& \subseteq S^{1}\left[d^{\prime}\right] \cup S^{1}\left[d^{\prime \prime}\right] \\
& \subseteq S^{1}[d] .
\end{aligned}
$$

Since $X \subseteq E$, it follows that

$$
\nu^{\circ}(X \mid \mathrm{E}) \leq d(\lambda) \leq 2^{-r}
$$

Since $r$ is arbitrary here, this shows that $\nu^{\circ}(X \mid E)=0$. It follows directly from this that ( C ) holds.

To see that $X$ does not satisfy condition (B), let $d^{+}: \mathbb{N} \times\{0,1\}^{*} \rightarrow[0, \infty)$ be an arbitrary p-computable function such that, for each $r \in \mathbb{N}, d_{r}^{+}$is a p- $\mu$-martingale with $d_{r}^{+}(\lambda) \leq 2^{-r}$. Fix $k \in \mathbb{N}$ such that $d^{+}$is $n^{k}$-time computable. Using the diagonalization technique of the Measure Conservation Theorem, it is routine to show that $X_{k+1} \nsubseteq S^{1}\left[d_{k+2}^{+}\right]$. Since $X \subseteq \mathrm{E}$ and $d^{+}$is arbitrary, it follows that $X$ does not have a $\mu$-bicover in E.

Lemma 5.4. For every $A \in \operatorname{REC}$, there exist $X, Y \subseteq \mathbf{C}$ such that $X$ and $Y$ have $\mu$-bicovers in E and $A \leq_{\mathrm{T}}^{\mathrm{P}} \mu^{\circ}(X \cup Y)$.

Corollary 5.5. There exist $X, Y \subseteq \mathbf{C}$ such that $X$ and $Y$ have $\mu$-bicovers in E but $X \cup Y$ does not.

By Corollary 5.5 and Theorem4.12, condition (B) of Lemma 5.1]does not imply condition (A). This appears to suggest that, in contrast with the classical case, resource-bounded measurability in complexity classes does not admit a Lebesgue-style characterization in terms of type- 1 objects. More seriously, Corollary 5.5 says that the sets having $\mu$-bicovers in E do not even form an algebra on E . Note also that condition (C) is noneffective, in the sense that there need be no concrete witness to its being satisfied. Thus the sets that satisfy condition (C) cannot enjoy a closure property of the type established for $\mathcal{F}_{R(\Delta)}^{\nu}$ in Theorem 4.20,

We now turn to the zero-one law.
Definition. A set $X \subseteq \mathbf{C}$ is a tail set if, for all $A, B \in \mathbf{C}$, if $A \in X$ and the symmetric difference $(A-B) \cup(B-A)$ is finite, then $B \in X$.

Most subsets of $\mathbf{C}$ that are of interest in computational complexity are tail sets. If we are working with a coin-toss probability measure on $\mathbf{C}$, i.e., a probability measure $\mu^{\vec{\beta}}$ as defined in section 3, then the classical Kolmogorov zero-one law [5] says that every $\vec{\beta}$-measurable tail set has $\vec{\beta}$-measure 0 or $\vec{\beta}$-measure 1. (We simplify terminology by writing " $\vec{\beta}$-measurable" in place of " $\mu^{\vec{\beta}}$-measurable," etc.) That is, if $X \in \mathcal{F}^{\vec{\beta}}$ is a tail set, then $\mu^{\vec{\beta}}(X)=0$ or $\mu^{\vec{\beta}}(X)=1$.

By Lemma 4.5, the Kolmogorov zero-one law trivially implies its own generalization to the algebra $\mathcal{F}_{\Delta}^{\vec{\beta}}$. That is, if $X$ is a $\Delta$ - $\vec{\beta}$-measurable tail set, then $\mu_{\Delta}^{\vec{\beta}}(X)=0$ or $\mu_{\Delta}^{\vec{\beta}}(X)=1$. Our objective here is to generalize the Kolmogorov zero-one law to the algebra $\mathcal{F}_{R(\Delta)}^{\vec{\beta}}$. This generalization does not follow directly from the classical Kolmogorov zero-one law. The main part of our argument is the following lemma, which is a resource-bounded zero-one law for the martingale outer $\vec{\beta}$-measure $\mu^{\vec{\beta} \circ}(X \mid R(\Delta))$ of an arbitrary (not necessarily measurable) tail set $X$ in $R(\Delta)$.

Lemma 5.6. If $X$ is a tail set and $\vec{\beta}$ is a $\Delta$-computable bias sequence, then $\mu^{\vec{\beta} \circ}(X \mid$ $R(\Delta))=0$ or $\mu^{\vec{\beta} \circ}(X \mid R(\Delta))=1$.

Proof of Lemma 5.6 Assume the hypothesis, and let $\alpha=\mu^{\vec{\beta} \circ}(X \mid R(\Delta))$. It suffices to show that $\alpha \leq \alpha^{2}$.

Let $r \in \mathbb{N}$ be arbitrary, and fix a $\Delta$ - $\vec{\beta}$-martingale $d$ such that $X \cap R(\Delta) \subseteq S^{1}[d]$ and $d(\lambda) \leq \alpha+2^{-(r+2)}$. Without loss of generality, we can also assume that $d$ is regular (by the Regularity Lemma) and that $d(\lambda) \leq \mu^{\vec{\beta}}\left(S^{1}[d]\right)+2^{-(r+3)}$ (doing finite surgery on $d$ to acheive this, if necessary.)

For each $n \in \mathbb{N}$, let

$$
\begin{aligned}
& I_{n}=\left\{w \in\{0,1\}^{*} \mid d(w)<1\right\} \\
& J_{n}=\left\{w \in\{0,1\}^{n} \mid d(w) \geq 1\right\}
\end{aligned}
$$

and fix $m \in \mathbb{N}$ such that

$$
\sum_{w \in J_{m}} \mu^{\vec{\beta}}(w) \geq \mu^{\vec{\beta}}\left(S^{1}[d]\right)-2^{-(r+3)}
$$

Fix a string $u \in\{0,1\}^{m}$ such that $d(u) \leq d(\lambda)$. (Such a string $u$ must exist because $d$ is a martingale.) Note that

$$
\begin{aligned}
\sum_{w \in I_{m}} d(w) \mu^{\vec{\beta}}(w) & =\sum_{w \in\{0,1\}^{m}} d(w) \mu^{\vec{\beta}}(w)-\sum_{w \in J_{m}} d(w) \mu^{\vec{\beta}}(w) \\
& =d(\lambda)-\sum_{w \in J_{m}} d(w) \mu^{\vec{\beta}}(w) \\
& \leq d(\lambda)-\sum_{w \in J_{m}} \mu^{\vec{\beta}}(w) \\
& \leq d(\lambda)-\mu^{\vec{\beta}}\left(S^{1}[d]\right)-2^{-(r+3)} \\
& \leq 2^{-(r+2)}
\end{aligned}
$$

and

$$
\sum_{w \in J_{m}} \mu^{\vec{\beta}}(w) \leq \mu^{\vec{\beta}}\left(S^{1}[d]\right) \leq d(\lambda)
$$

Using the $\Delta$ - $\vec{\beta}$-martingale $d$, the natural number $m$, the sets $I_{m}$ and $J_{m}$, and the string $u$, define $d^{\prime}:\{0,1\}^{*} \rightarrow[0, \infty)$ as follows. Let $w \in\{0,1\}^{*}$.
(i) If $|w| \geq m$ and $w[0 . . m-1] \in J_{m}$, then

$$
d^{\prime}(w)=d(u * w)
$$

where $u * w$ is the string obtained by substituting $u$ for the first $m$ bits of $w$, i.e., $|u * w|=|w|$ and

$$
(u * w)[i]= \begin{cases}u[i] & \text { if } 0 \leq i<m \\ w[i] & \text { if } m \leq i<|w|\end{cases}
$$

(ii) If $|w| \geq m$ and $w[0 . . m-1] \in I_{m}$, then

$$
d^{\prime}(w)=d(w)
$$

(iii) If $|w|<m$, then $d^{\prime}(w)$ is defined from the values of $d^{\prime}$ on $\{0,1\}^{m}$ so that $d^{\prime}$ is a $\vec{\beta}$-martingale.

It is easy to check that $d^{\prime}$ is a $\Delta-\vec{\beta}$-martingale, (The fact that we have a coin-toss probability measure is crucial here in clause (i) of the definition of $d^{\prime}$.)

We now show that $X \cap R(\Delta) \subseteq S^{1}\left[d^{\prime}\right]$. To see this, let $A \in X \cap R(\Delta)$. If $A[0 . . m-1] \in I_{m}$, then $d^{\prime}(A[0 . . n-1])=d(A[0 . . n-1])$ for all $n \geq m$. Since $d$ is regular and $A \in S^{1}[d]$, this
implies that $A \in S^{1}\left[d^{\prime}\right]$. On the other hand, if $A[0 . . m-1] \in J_{m}$, then for all $n \geq m$,

$$
\begin{aligned}
d^{\prime}(A[0 . . n-1]) & =d(u *(A[0 . . n-1])) \\
& =d((u * A)[0 . . n-1]),
\end{aligned}
$$

where $u * A \in \mathbf{C}$ is defined by

$$
(u * A)[i]= \begin{cases}u[i] & \text { if } 0 \leq i<m \\ A[i] & \text { if } i \geq m\end{cases}
$$

Since $A \in X \cap R(\Delta)$ and $X$ is a tail set, we have

$$
u * A \in X \cap R(\Delta) \subseteq S^{1}[d]
$$

Since $d$ is regular, it now follows that $A \in S^{1}\left[d^{\prime}\right]$. This completes the demonstration that $X \cap R(\Delta) \subseteq S^{1}\left[d^{\prime}\right]$.

Since $X \cap R(\Delta) \subseteq S^{1}\left[d^{\prime}\right]$, we now have

$$
\begin{aligned}
\alpha & =\mu^{\vec{\beta} \circ}(X \mid R(\Delta)) \leq d^{\prime}(\lambda)=d^{\prime}(\lambda) \mu^{\vec{\beta}}(\lambda) \\
& =\sum_{w \in\{0,1\}^{m}} d^{\prime}(w) \mu^{\vec{\beta}}(w) \\
& \leq \sum_{w \in I_{m}} d^{\prime}(w) \mu^{\vec{\beta}}(w)+\sum_{w \in J_{m}} d^{\prime}(w) \mu^{\vec{\beta}}(w) \\
& =\sum_{w \in J_{m}} d(w) \mu^{\vec{\beta}}(w)+d(u) \sum_{w \in J-m} \mu^{\vec{\beta}}(w) \\
& \leq \sum_{w \in I_{m}} d(w) \mu^{\vec{\beta}}(w)+d(\lambda) \sum_{w \in J_{m}} \mu^{\vec{\beta}}(w) .
\end{aligned}
$$

Using the bounds we have already derived for the two sums in this last expression, it follows that

$$
\begin{aligned}
\alpha & \leq 2^{-(r+2)}+d(\lambda)^{2} \\
& \leq 2^{-(r+2)}+\left(\alpha+2^{-(r+2)}\right)^{2} \\
& \leq \alpha^{2}+2^{-r} .
\end{aligned}
$$

Since $r$ is arbitrary here, this establishes that $\alpha \leq \alpha^{2}$, completing the proof.
Our resource-bounded generalization of the Kolmogorov zero-one law now follows easily.

Theorem 5.7. If $\vec{\beta}$ is a $\Delta$-computable bias sequence and $X$ is a tail set that is $\vec{\beta}$ measurable in $R(\Delta)$, then $\mu^{\vec{\beta}}(X \mid R(\Delta))=0$ or $\mu^{\vec{\beta}}(X \mid R(\Delta))=1$.

Proof of Theorem 5.7 Assume the hypothesis. Then, by Lemma 5.1, $\mu^{\vec{\beta}}(X \mid R(\Delta))=$ $\mu^{\vec{\beta} \circ}(X \mid R(\Delta))$, so the conclusion follows immediately from Lemma 5.6

Theorem 5.7, like its classical counterpart, cannot be extended to arbitrary probability measures on C. For example, the set $X$ of all finite languages is a tail set, but if

$$
\nu(w)= \begin{cases}1 & \text { if } w=\lambda \\ \frac{1}{2} & \text { if } w \in\{0\}^{+} \cup\{1\}^{+} \\ 0 & \text { otherwise }\end{cases}
$$

then $\nu(X)=\nu(X \mid R(\Delta))=\frac{1}{2}$.
We conclude this paper with some useful characterizations of resource-bounded measure 0 sets.

The condition $\nu_{\Delta}(X \cap R(\Delta))=\alpha$ implies the condition $\nu(X \mid R(\Delta))=\alpha$, but the converse does not generally hold for $\alpha>0$. However, when $\alpha=0$, these two conditions are equivalent.

Lemma 5.8. For any set $X \subseteq \mathbf{C}$, the following conditions are equivalent.
(1) $\nu(X \mid R(\Delta))=0$.
(2) $\nu_{\Delta}(X \cap R(\Delta))=0$.

Proof of Lemma 5.8 The fact that (2) implies (1) follows immediately from Lemma 4.5 , To see that (1) implies (2), assume (1). Let $\Phi$ be a $\nu$-measurement of $X$ in $R(\Delta)$. Then the functional $\Psi$ defined by

$$
\Psi_{r}(d)=\left(\Phi_{r}^{+}(\mathbf{1}), d\right)
$$

is easily seen to be a $\Delta$ - $\nu$-measurement of $X \cap R(\Delta)$, so

$$
\nu_{\Delta}(X \cap R(\Delta))=\Psi_{\infty}^{+}=\Phi_{\infty}^{+}=\nu(X \mid R(\Delta))=0
$$

We now give several useful characterizations of resource-bounded measure 0 sets.
Definition. A $\Delta$ - $\nu$-null cover of a set $X \subseteq \mathbf{C}$ is a $\Delta$-computable function $d: \mathbb{N} \times\{0,1\}^{*} \rightarrow$ $[0, \infty)$ such that, for each $r \in \mathbb{N}, d_{r}$ is a $\nu$-martingale, $X \subseteq S^{1}\left[d_{r}\right]$, and $d_{r}(\lambda) \leq 2^{-r}$. A $\Delta$ - $\nu$-null cover $d$ is regular if the martingale $d_{r}$ is regular for each $r \in \mathbb{N}$.

Recall the martingale success sets $S^{\infty}[d]$ and $S_{\mathrm{str}}^{\infty}[d]$ defined in section 3.

Theorem 5.9. Let $X \in \mathbf{C}$, and let $\nu$ be a $\Delta$-probability measure on $\mathbf{C}$. The following conditions are equivalent.
(1) $\nu_{\Delta}(X)=0$.
(2) $X$ has a $\Delta$ - $\nu$-null cover.
(3) $X$ has a regular $\Delta$ - $\nu$-null cover.
(4) There is a $\Delta$ - $\nu$-martingale $d$ such that $X \subseteq S_{\mathrm{str}}^{\infty}[d]$.
(5) There is a $\Delta$ - $\nu$-martingale $d$ such that $X \subseteq S^{\infty}[d]$.

Proof of Theorem 5.9 To see that (1) implies (2), assume that $\nu_{\Delta}(X)=0$, and let $\Phi$ be a $\Delta$ - $\nu$-measurement of $X$. Then $\Phi^{+}(\mathbf{1})$ is easily seen to be a $\Delta$ - $\nu$-null cover of $X$, so (2) holds.

It is clear by the Regularity Lemma that (2) implies (3).
To see that (3) implies (4), let $d^{\prime}$ be a regular $\Delta$ - $\nu$-null cover of $X$. Define $d:\{0,1\}^{*} \rightarrow$ $[0, \infty)$ by

$$
d(w)= \begin{cases}\sum_{r=0}^{\infty} d_{r}^{\prime}(w) & \text { if } \nu(w)>0 \\ |w| & \text { if } \nu(w)=0\end{cases}
$$

For all $w \in\{0,1\}^{*}$ such that $\nu(w)>0$, the trivial martingale inequality $d_{r}^{\prime}(w) \nu(w) \leq d_{r}^{\prime}(\lambda)$ assures us that

$$
\begin{aligned}
d(w) & =\sum_{r=0}^{\infty} d_{r}^{\prime}(w) \leq \frac{1}{\nu(w)} \sum_{r=0}^{\infty} d^{\prime}(\lambda) \\
& \leq \frac{1}{\nu(w)} \sum_{r=0}^{\infty} 2^{-r}=\frac{2}{\nu(w)} \leq \infty
\end{aligned}
$$

so $d$ is well-defined. It is then easy to check that $d$ is a $\Delta$ - $\nu$-martingale. To see that $X \subseteq S_{\text {str }}^{\infty}[d]$, let $A \in X$, and let $m \in \mathbb{N}$ be arbitrary. We have two cases.

I: There exists $w \sqsubseteq A$ such that $\nu(w)=0$. Then, for all $n \geq \max \{|w|, m\}$,

$$
d(A[0 . . n-1])=n \geq m
$$

II: $\nu(w)>0$ for all $w \sqsubseteq A$. For each $0 \leq r<m$, fix $n_{r} \in \mathbb{N}$ such that $d_{r}^{\prime}\left(A\left[0 . . n_{r}-1\right]\right) \geq 1$. (Such $n_{r}$ exists because $A \in X \subseteq S^{1}\left[d_{r}^{\prime}\right]$.) Then, for all $n \geq \max \left\{n_{r} \mid 0 \leq r<m\right\}$, the fact that each $d_{r}^{\prime}$ is regular ensures that

$$
d(A[0 . . n-1]) \geq \sum_{r=0}^{m-1} d_{r}^{\prime}(A[0 . . n-1]) \geq m
$$

In either case, we have shown that $d(A[0 . . n-1]) \geq m$ for all sufficiently large $n$. Since $m$ is arbitrary here, it follows that $A \in S_{\mathrm{str}}^{\infty}[d]$, affirming (4).

It is trivial that (4) implies (5).
To see that (5) implies (1), let $d$ be a $\Delta$ - $\nu$-martingale such that $X \subseteq S^{\infty}[d]$. For each $r \in \mathbb{N}$ and $d^{\prime} \in \mathcal{D}_{\nu}$, let

$$
\begin{aligned}
& \Phi_{r}^{+}\left(d^{\prime}\right)=\frac{2^{-r}}{1+d(\lambda)} \cdot d, \\
& \Phi_{r}^{-}\left(d^{\prime}\right)=d^{\prime}
\end{aligned}
$$

It is easy to check that $\Phi=\left(\Phi^{+}, \Phi^{-}\right)$is $\Delta$-computable and that

$$
\Phi: \mathbb{N} \times \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu} \times \mathcal{D}_{\nu}
$$

To see that $\Phi$ is a $\Delta$ - $\nu$-measurement of $X$, let $r \in \mathbb{N}$ and $d^{\prime} \in \mathcal{D}_{\nu}$. Three things are to be checked.
(i) $S^{1}\left[d^{\prime}\right] \cap X \subseteq S^{1}\left[\Phi_{r}^{+}\left(d^{\prime}\right)\right]$. To see this, let $A \in S^{1}\left[d^{\prime}\right] \cap X$. Then $A \in X \subseteq S^{\infty}[d]$, so there exists $n \in \mathbb{N}$ such that $d(A[0 . . n-1]) \geq 2^{r} \cdot(1+d(\lambda))$. Then $\Phi_{r}^{+}\left(d^{\prime}\right)(A[0 . . n-1]) \geq 1$, so $A \in S^{1}\left[\Phi_{r}^{+}\left(d^{\prime}\right)\right]$.
(ii) $S^{1}\left[d^{\prime}\right]-X \subseteq S^{1}\left[\Phi_{r}^{-}\left(d^{\prime}\right)\right]$. This holds trivially.
(iii) $\Phi_{r}^{+}\left(d^{\prime}\right)(\lambda)+\Phi_{r}^{-}\left(d^{\prime}\right)(\lambda) \leq d^{\prime}(\lambda)+2^{-r}$. This holds because $\Phi_{r}^{+}\left(d^{\prime}\right)(\lambda)<2^{-r}$ and $\Phi_{r}^{-}\left(d^{\prime}\right)(\lambda)=d^{\prime}(\lambda)$.

We have now seen that $\Phi$ is a $\Delta-\nu$-measurement of $X$. Since $\Phi_{r}^{+}(\mathbf{1})(\lambda) \leq 2^{-r}$ for each $r \in \mathbb{N}$, it follows that $\nu_{\Delta}(X)=\Phi_{\infty}^{+}=0$, confirming (1).

Analogous characterizations of the sets of $\nu$-measure 0 in $R(\Delta)$ follow immediately from Theorem 5.9 via Lemma 5.8.

Note that conditions (2) through (5) in Theorem 5.9 do not involve type-2 functionals. It is these characterizations of measure 0 sets in terms of martingales (type- 1 objects) that have been used in resource-bounded measure to date.

## Acknowledgments

I thank Alekos Kechris and Yaser Abu-Mostafa for their hospitality during several visits at Caltech, where much of this research was performed. I thank Dexter Kozen and Juris

Hartmanis for their hospitality at Cornell, where much of the writing took place. I thank Klaus Weihrauch, Jim Royer, and Jack Dai for pointing out errors in an earlier draft of this paper. For helpful discussions over the past few years, I thank many colleagues, including Steve Fenner, David Juedes, Steve Kautz, Jim Lathrop, Elvira Mayordomo, Ken Regan, Jim Royer, Giora Slutzki, Martin Strauss, and Bas Terwijn. I am especially grateful to my late friend Ron Book for his constant encouragement of this project, and I dedicate this paper to his memory.

## References

[1] K. Ambos-Spies and E. Mayordomo. Resource-bounded measure and randomness. In A. Sorbi, editor, Complexity, Logic and Recursion Theory, Lecture Notes in Pure and Applied Mathematics, pages 1-47. Marcel Dekker, New York, N.Y., 1997.
[2] J. M. Breutzmann and J. H. Lutz. Equivalence of measures of complexity classes. SIAM Journal on Computing, 29(1):302-326, 2000.
[3] G. J. Chaitin. A theory of program size formally identical to information theory. Journal of the Association for Computing Machinery, 22:329-340, 1975.
[4] S. M. Kautz. Resource-bounded randomness and compressibility with respect to nonuniform measures. In Proceedings of the International Workshop on Randomization and Approximation Techniques in Computer Science, pages 197-211, 1997.
[5] A. N. Kolmogorov. Grundbegriffe der Wahrscheinlichkeitsrechnung. Berlin, 1933. English translation, Cheksea Publishing, New York, 1950.
[6] J. H. Lutz. Almost everywhere high nonuniform complexity. Journal of Computer and System Sciences, 44(2):220-258, 1992.
[7] J. H. Lutz. The quantitative structure of exponential time. In L. A. Hemaspaandra and A. L. Selman, editors, Complexity Theory Retrospective II, pages 225-254. SpringerVerlag, 1997.
[8] H. L. Royden. Real Analysis. Macmillan Publishing Company, third edition, 1988.
[9] C. P. Schnorr. Klassifikation der Zufallsgesetze nach Komplexität und Ordnung. Z. Wahrscheinlichkeitstheorie verw. Geb., 16:1-21, 1970.
[10] C. P. Schnorr. A unified approach to the definition of random sequences. Mathematical Systems Theory, 5:246-258, 1971.
[11] C. P. Schnorr. Zufälligkeit und Wahrscheinlichkeit. Springer-Verlag, Berlin, 1971.
[12] C. P. Schnorr. Process complexity and effective random tests. Journal of Computer and System Sciences, 7:376-388, 1973.


[^0]:    *This research was supported in part by National Science Foundation Grants 9157382 (with matching funds from Rockwell, Microware Systems Corporation, and Amoco Foundation) and 9610461. Much of the research was performed during several visits at the California Institute of Technology, and much of the writing took place during a sabbatical at Cornell University.

