# Extracting Quantum Entanglement (General Entanglement Purification Protocols) 4 

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#### Abstract

We study the problem of general entanglement purification protocols. Suppose Alice and Bob share a bipartite state $\rho$ which is "reasonably close" to perfect EPR pairs. The only information Alice and Bob possess is a lower bound on the fidelity of $\rho$ and a maximally entangled state. They wish to "purify" $\rho$ using local operations and classical communication and create a state that is arbitrarily close to EPR pairs. We prove that on average, Alice and Bob cannot increase the fidelity of the input state significantly. We also construct protocols that may fail with a small probability, and otherwise will output states arbitrarily close to EPR pairs with very high probability. Our constructions are efficient, i.e., they can be implemented by polynomial-size quantum circuits.


## 1 Introduction

Random bits are an important computational resource in the randomized computation. There has been a lot of work on extracting good random bits from imperfect sources of randomness. The beginning of this study goes back to at least von Neumann [VN51] who showed that a linear number of perfect random bits can be extracted from independent tosses of a biased coin. More recent research has constructed extractors NT99, T99 which can extract almost perfect random bits from any source with a certain min-entropy without any other assumptions. The best constructions of extractors allow to extract a number of random bits close to the min-entropy of the random source with min-entropy if we can use a polylogarithmic number of perfect random bits TUZ01.

Quantum entanglement is an important resource in quantum computation, similar to random bits in probabilistic computation. It comes in the form of Einstein-Podolsky-Rosen pairs. An EPR pair is the state of two quantum bits $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ shared by two parties, with one party (Alice) holding one quantum bit and the other party (Bob) holding the second bit. This is the quantum counterpart of a random bit shared by two parties.

Einstein-Podolsky-Rosen EPR35 (EPR) pairs are among the most interesting objects of study in quantum mechanics and quantum information theory. They behave very differently from classical random bits shared by two parties. The phenomenon of having entangled states separated by space is one of the quintessential features in quantum mechanics and it has no analogue in classical physics.

Besides being conceptually interesting in quantum mechanics, EPR pairs are also very useful in quantum information theory. Using an EPR pair, Alice and Bob can perform quantum teleportation. Using only local operations and classical communication (LOCC), Alice can "transport" a qubit to Bob,

[^0]who could be miles away from Alice BBC+93]. So EPR pairs, along with a classical communication channel, effectively constitute a quantum channel. Conversely, "superdense coding" is possible with EPR pairs: if Alice and Bob share an EPR pair, then Alice can transport 2 classical bits to Bob by just sending one qubit BW92.

For the teleportation and dense coding to work perfectly, perfect EPR pairs are needed. Nevertheless, individual qubits are prone to errors, which may end up creating imperfect EPR pairs. These imperfect EPR pairs behave like a noisy channel - qubits teleported with these EPR pairs can become distorted.

This creates the need for generating perfect (or almost perfect) EPR pairs from imperfect ones. This is known as "entanglement purification". Bennett et. al. BBP+96a gave a protocol for the case that Alice and Bob share identical copies of the pure state $|\phi\rangle=(\cos \theta|01\rangle+\sin \theta|10\rangle)$. This was extended to the case when Alice and Bob share identical copies of a mixed state [BBP+96b, BDS+96, HHH96]. Vidal V99], and subsequently, Jonathan and Plenio [JP99], Hardy H99], and Vidal, Jonathan, and Nielsen [VJN0]] considered extracting entanglement from a single copy of an arbitrary pure state, assuming that we know a complete description of the state.

All of this work uses relatively simple models for imperfect EPR pairs. The model where Alice and Bob share identical copies of the same state corresponds to generating perfect random bits from the sequence of i.i.d. biased coin flips. Extracting entanglement from a single copy of a known state corresponds to constructing uniform/almost uniform random bits from a biased distribution if we know a complete description of the distribution. Both of those are very easy tasks classically but dealing with quantum states makes them much harder.

Can we extract entanglement if we do not have such a detailed knowledge of the quantum state (like extractors in the classical setting which work for any probability distribution satisfying certain constraints)? This is the question that we consider in this paper.

### 1.1 Our error model

We no longer assume that there is a single "distortion" operator that acts independently on each qubit pair, neither do we assume that Alice and Bob have complete information about the distortion. The only assumption that we have is that the distortion is not very large. More precisely, we assume that Alice and Bob share a state $\rho$ with fidelity at least $(1-\epsilon) \downarrow$. We call this model of imperfect EPR pairs the "General Error" model. We call the protocols for this model General Entanglement Purification Protocols (GEPPs)

In the General Error model, the techniques used in previous literature don't appear to work. Some of the techniques rely on the Law of Large Number heavily. For example, both the "Schmidt projection" method BBP+96a] and the "hashing" method BDS+96] try to reduce the state to a "typical sequence", and then do purification over the typical sequences. In the General Error model, it is not clear what a "typical sequence" would be. Some techniques, like the "Procrustean" method BBP+96a, are designed to work with individual states that Alice and Bob have complete knowledge of. Apparently they don't work in the General Error model, where Alice and Bob only have very limited information about the state they share. In fact it is not obvious if Alice and Bob can do anything at all to extract EPR pairs in this model.

### 1.2 Our Contribution

Some features about GEPPs are:

## 1. Arbitrary Maximally Entangled States

Instead of working only with EPR pairs, a GEPP works with arbitrary maximally entangled states. In every symmetric bipartite system of dimension $T \times T$, there exist a maximally entangled state $\Psi_{T}=\frac{1}{\sqrt{T}} \sum_{i=0}^{T-1}|i\rangle^{A}|i\rangle^{B}$. In the case that $T$ is a power of $2, \Psi_{T}$ is the state of $\log _{2} T$ EPR pairs.

[^1]Thus, extracting EPR pairs is a particular case of our setting. We note that, by a result of Nielsen [N99], Alice and Bob can transform $\Psi_{N}$ into $\Psi_{M}$ for any $M<N$. Therefore, if we want the final state to be a set of EPR pairs, we can just add a step at the end of protocol that maps $\Psi_{N}$ to $\Psi_{2\lfloor\log N\rfloor}$ and obtain a state of $\left\lfloor\log _{2} N\right\rfloor$ EPR pairs.

## 2. Auxiliary Input

Besides an input state $\rho$, a GEPP also has a maximally entangled state $\Psi_{K}$ as auxiliary input. This assumption is similar to having extra perfect random bits in randomness extractors.

## 3. Possibility to Fail

We allow a GEPP to fail with a reasonably low probability. As we will prove later, a GEPP that never fails won't be able to increase the fidelity of the input state significantly, even if it has an extra input $\Psi_{K}$. However, if we allow a GEPP to fail, then in the case that it doesn't fail, it will be able to output states of very high fidelity with very high probability. In general, a GEPP will either output a special symbol FAIL or output a state, which has (hopefully) very high fidelity.

We consider 3 types of GEPPs. Roughly speaking, we say a GEPP is absolutely successful, if it never fails, and always outputs a state of very high fidelity. We say a GEPP is deterministically conditional successful, if the probability it fails is small, and when it doesn't fail, it will output a state of very high fidelity with certainty. We say a GEPP is probabilistically conditionally successful, if the probability it fails is small, and in the case it doesn't fail, it outputs a state of very high fidelity with high probability. Each definition is a generalization of the previous one: an absolutely successful GEPP is deterministically conditionally successful, and a deterministically successful GEPP is probabilistically conditionally successful. We prove the following results:

1. There don't exist absolutely successful GEPPs with "interesting" parameters. To be more precise, we prove the following result: Suppose Alice and Bob share a state of fidelity $1-\epsilon$, and they have an auxiliary input $\Psi_{K}$. They then perform LOCC to create a state $\sigma$ in a subspace of dimension $M \times M$. Then the maximal fidelity Alice and Bob can guarantee about the state $\sigma$ is at most $1-\frac{N}{N-1}\left(1-\frac{K}{M}\right) \epsilon$. If $K$ is significantly smaller than $M$ improvement of fidelity is very small. In other words, Alice and Bob cannot arbitrarily increase the average fidelity of the input state.
2. There exist deterministically conditionally successful GEPPs for states in the diagonal subspace. A diagonal subspace is spanned by states of the form $\sum_{i} \alpha_{i}|i\rangle^{A}|i\rangle^{B}$. As we will show later, a state in the diagonal subspace is "easy" to work with and there exists an efficient protocol that is deterministically conditionally successful. For an input state of fidelity $1-\epsilon$, the protocol will fail with probability at most $\epsilon$, and when it doesn't fail, it always outputs a state of fidelity $1-\frac{\epsilon}{L}$, where $L$ is a parameter that can be made very large. We call our protocol the "Simple Scrambling protocol". The Simple Scrambling protocol is optimal in the sense that the average fidelity of its output matches the upper bound asymptotically.
3. There exist probabilistically conditionally successful GEPPs for arbitrary states. We present a protocol, namely the "Hash and Compare protocol". The Hash and Compare protocol converts an arbitrary state $|\phi\rangle$ of fidelity $1-\epsilon$ to another state $|\phi\rangle^{\prime}$ of fidelity at least $1-\epsilon$, such that $\left|\phi^{\prime}\right\rangle$ is "almost" in the diagonal subspace. Then the Simple Scrambling protocol can be used on state $\left|\phi^{\prime}\right\rangle$ to create a state with very high fidelity.

Both the Simple Scrambling protocol and the Hash and Compare protocol are efficient, i.e., they can be implemented by polynomial-size quantum circuits. We present the precise definitions and results in the next section.

## 2 Notations and Definitions

### 2.1 General Notations

All logarithms are base-2. We use $[N]$ to denote the set $\{0,1, \ldots, N-1\}$. We identify an integer with its binary representation, and view its binary representation as a bit vector. The XOR of two integers $x$ and $y$, denoted by $x \oplus y$, is the XOR of the two bit vectors $x$ and $y$ represent. The inner product of $x$ and $y$, denoted by $x \bullet y$, is defined as the inner product in $G F_{2}$ of the two bit vectors $x$ and $y$ represent.

We study quantum systems of finite dimension. We identify a pure state (written in the "braket" notation as $|\phi\rangle$ ) with a (column) vector of unit length. We identify a mixed state with the density matrix of this state. For a quantum system whose states lie in the Hilbert space $\mathcal{H}$ of dimension $N$, we always assume that it has a canonical computational basis and we denote it by $\{|0\rangle,|1\rangle, \ldots,|N-1\rangle\}$. Furthermore, we often denote $|0\rangle \in \mathcal{H}$ by $\left|Z_{N}\right\rangle$ to specify the dimension of this state.

We are mostly interested in symmetric, bipartite quantum systems, namely, systems shared between Alice and Bob, whose states lie in a Hilbert space $\mathcal{H}=\mathcal{H}^{A} \otimes \mathcal{H}^{B}$ and $\mathcal{H}^{A} \equiv \mathcal{H}^{B}$. Alice can access $\mathcal{H}^{A}$ and Bob can access $\mathcal{H}^{B}$. We always superscript subspaces and states to distinguish states accessible by Alice and Bob. For example, a general bipartite state $|\varphi\rangle$ can written in the following way:

$$
|\varphi\rangle=\sum_{i, j} \alpha_{i j}|i\rangle^{A}|j\rangle^{B}
$$

where $|i\rangle^{A}$ denotes the state of Alice and $|j\rangle^{B}$ denotes the state of of Bob. We sometimes subscript a space by its dimension. For example, $\mathcal{H}_{N}$ means a space of dimension $N$.

A quantum state is unentangled if it is of the form $|\psi\rangle^{A} \otimes\left|\psi^{\prime}\right\rangle^{B}$. Any other pure state in $\mathcal{H}^{A} \otimes \mathcal{H}^{B}$ is entangled. For a pure state $|\varphi\rangle$ in a bipartite system, we define its entanglement to be the von Neumann entropy of the reduced sub-system of Bob when we trace out Alice:

$$
\begin{equation*}
E(|\varphi\rangle)=S\left(\operatorname{Tr}_{A}(|\varphi\rangle\langle\varphi|)\right) \tag{1}
\end{equation*}
$$

where $S(\rho)=-\operatorname{Tr}(\rho \log \rho)$ is the von Neumann entropy. We have $S=0$ if and only the state is unentangled. For mixed states, a mixed state $\rho$ is unentangled if and only if it is equivalent to a state that is a mixture of pure states $\left|\varphi_{i}\right\rangle$ with probabilities $p_{i}$. Any other mixed state is entangled. However, there is no universally agreed definition for the amount of entanglement in a mixed state.

If we denote the dimension of $\mathcal{H}^{A}$ by $N$, then the maximum amount of entanglement in this system is $\log N$. We define the state $\Psi_{N}$ to be

$$
\begin{equation*}
\Psi_{N}=\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1}|i\rangle^{A}|i\rangle^{B} \tag{2}
\end{equation*}
$$

It is a maximally entangled state in $\mathcal{H}^{A} \otimes \mathcal{H}^{B}$. Notice it is a state in a space of dimension $N^{2}$. In particular, if $N$ is a power of 2: $N=2^{n}$, then the state $\Psi_{N}$ is the state of $n$ EPR pairs. We call this special kind of states $E P R$ states.

### 2.2 Diagonal Subspaces

For a symmetric, bipartite system $\mathcal{H}=\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$, we denote by $\mathcal{H}^{\mathcal{D}}$ the $N$-dimensional subspace spanned by

$$
\left\{\sum_{i=0}^{N-1} \alpha_{i} \cdot|i\rangle^{A}|i\rangle^{B}\right\}
$$

and we call it the diagonal subspace of $\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$. The reason for the name is: for a general state

$$
|\varphi\rangle=\sum_{i, j} \alpha_{i j}|i\rangle^{A}|j\rangle^{B}
$$

we can write its coefficients (totally $N^{2}$ of them) in a matrix form, where the $(i, j)$-th entry is $\alpha_{i, j}$, then the elements in $\mathcal{H}^{\mathcal{D}}$ correspond to the diagonal matrices. Notice that this definition is also consistent with the "Bell-diagonal" BDS+96] states for $N=2$. A mixed state $\rho$ is in the diagonal subspace, if there exists a decomposition of $\rho$ :

$$
\rho=\sum_{i} p_{i} \cdot\left|\phi_{i}\right\rangle\left\langle\phi_{i}\right|
$$

such that all pure states $\left|\phi_{i}\right\rangle$ are in the diagonal subspace.

### 2.3 Fidelity

For two (mixed) states $\rho$ and $\sigma$ in the same quantum system, their fidelity is defined as

$$
\begin{equation*}
F(\rho, \sigma)=\operatorname{Tr}\left(\rho^{1 / 2} \sigma \rho^{1 / 2}\right) \tag{3}
\end{equation*}
$$

This definition simplifies if one $\sigma=|\varphi\rangle\langle\varphi|$ is a pure state. Then the fidelity of $\rho$ and $\sigma$ is

$$
\begin{equation*}
F(\rho,|\varphi\rangle\langle\varphi|)=\langle\varphi| \rho|\varphi\rangle \tag{4}
\end{equation*}
$$

In the special case that $|\varphi\rangle=\Psi_{N}$ is the maximally entangled state, we call the fidelity of $\rho$ and $|\varphi\rangle$ the fidelity of state $\rho$, and the definition simplifies to:

$$
\begin{equation*}
F(\rho)=\left\langle\Psi_{N}\right| \rho\left|\Psi_{N}\right\rangle \tag{5}
\end{equation*}
$$

When the state $\rho=|\phi\rangle\langle\phi|$ is also a pure state, we have

$$
F(\rho)=F(|\phi\rangle\langle\phi|)=\left|\left\langle\phi \mid \Psi_{N}\right\rangle\right|^{2}
$$

and the fidelity is just the square of the inner product with $\Psi_{N}$.
One property for the fidelity is: it is linear with respect to ensembles.
Claim 1 Let $\rho$ be the density matrix for a mixed state that is an ensemble $\left\{p_{i},\left|\phi_{i}\right\rangle\right\}$. The fidelity of $\rho$ is the weighted averages of the qualities of the pure states:

$$
F(\rho)=\sum_{i} p_{i} \cdot F\left(\left|\phi_{i}\right\rangle\left\langle\phi_{i}\right|\right)
$$

This linearity is particularly convenient in some of the proofs in this paper.

## 3 General Entanglement Purification Protocols

### 3.1 The general setting

Alice and Bob are given some entangled state in $\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$. They are also given an auxiliary input $\Psi_{K}$ in $\mathcal{H}_{K}^{A} \otimes \mathcal{H}_{K}^{B}$. Alice can perform unitary transformations on her part of the state $\left(\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{K}^{A}\right)$ and Bob can perform unitary transformations on his part $\left(\mathcal{H}_{N}^{B} \otimes \mathcal{H}_{K}^{B}\right)$. Since those transformations only affect one part of the state, they are called local operations. Alice and Bob are also allowed to communicate classical bits but not quantum bits. This model is called LOCC (local operations and classical communication) BBP+96a, N99.

If the starting state is unentangled, applying LOCC operations keeps the state unentangled [BBP+96a]. Thus, LOCC operations cannot create entanglement but they can be used to extract the entanglement that already exists in the state.

We use the letter $\mathcal{P}$ to denote protocols for extracting entanglement by LOCC operations. At the end of a protocol $\mathcal{P}$, Alice and Bob have two options:

1. They can abort and claim failure by outputting a special symbol FAIL. We denote this by $\mathcal{P}(\rho)=$ FAIL.
2. They can output a (possibly mixed) state $\sigma$ in $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$. We denote this by $\mathcal{P}(\rho)=\sigma$.

We now define the error model. We first give an unsuccessful definition to illuminate some of difficulties that we face and to explain the reasons behind our final definition.

### 3.2 Extracting entanglement from an arbitrary state

Ideally, we would like to have a protocol that takes any entangled state in $\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$ with at least a certain amount of entanglement and extracts a state close to $\Psi_{M}$ for some $M<N$. This would correspond the definition of extractors where extractor can transform any probability distribution with min-entropy at least $m$ into a probability distribution that is close to uniform.

Unfortunately, this is not possible, even if we restrict ourselves to starting states with the maximum possible entanglement. Unlike in the classical world where there is just one probability distribution over $N$ elements with entropy $\log N$ (the uniform distribution), there are infinitely many quantum states with entanglement $\log N$. Namely, any quantum state of the form

$$
\begin{equation*}
|\phi\rangle=\sum_{i=0}^{N-1} \alpha_{i}|i\rangle|i\rangle \tag{6}
\end{equation*}
$$

with $\left|\alpha_{i}\right|^{2}=1 / N$ for all $i \in\{0, \ldots, N-1\}$ has entanglement $\log N$. In particular, this includes

$$
\left|\phi_{j}\right\rangle=\sum_{i=0}^{N-1} \frac{1}{\sqrt{N}} e^{2 i j \pi / N}|i\rangle|i\rangle
$$

for $j \in\{1, \ldots, N\}$. Assume that we have a protocol that extracts $\Psi_{M}$ from any $\left|\phi_{j}\right\rangle$. This means that, given $\left|\phi_{j}\right\rangle$, the protocol ends with the final state of the from $\Psi_{M} \otimes\left|\phi_{j}^{\prime}\right\rangle$. We consider running this protocol on the mixed state $\rho$ that is $\left|\phi_{0}\right\rangle$ with probability $1 / \mathrm{N},\left|\phi_{2}\right\rangle$ with probability $1 / \mathrm{N}, \ldots,\left|\phi_{N-1}\right\rangle$ with probability $1 / \mathrm{N}$. Then, the final state is of the form $\Psi_{M} \otimes \rho^{\prime}$ where $\rho^{\prime}$ is some mixed state.

The problem is that $\rho$ is equivalent to the mixed state that is $|0\rangle|0\rangle$ with probability $1 / N,|1\rangle|1\rangle$ with probability $1 / N, \ldots,|N-1\rangle|N-1\rangle$ with probability $1 / N$. (This equivalence can be verified by writing out the density matrices of both states.) Neither of states $|i\rangle|i\rangle$ is entangled, so the mixed state obtained by combining them is not entangled as well. Yet, since this mixed state is equivalent to $\rho$, it gets transformed into $\Psi_{M} \otimes \rho^{\prime}$ which is entangled because $\Psi_{M}$ is entangled.

We have constructed a protocol that transforms an unentangled starting state into entangled end state without quantum communication. Since this is impossible [BBP+96a], our assumption is wrong and there is no protocol that extracts any $\Psi_{M}$ from an arbitrary $\left|\phi_{j}\right\rangle$.

The argument described above is still valid if we relax the requirement to extracting a state close to $\Psi_{M}$ and if we allow to use a perfect auxiliary state $\Psi_{K}$. In the second case, we can get the perfect $\Psi_{K}$ back but cannot get an entangled state of higher dimension.

### 3.3 Extracting from a state close to $\Psi_{M}$

The reason for the problem in the previous section is that there are multiple maximally entangled states and combining them into a mixed state can cancel the entanglement and create a state with no entanglement. The consequence is that if we want to be able to extract entanglement we have to restrict ourselves to states that are close to one particular highly entangled state (rather than some highly entangled state). Therefore, we assume that the starting state is close to $\Psi_{M}{ }^{\text {P }}$.

A common way to measure the closeness to $\Psi_{M}$ is the fidelity (section 2.3). This gives the following definitions.

[^2]Definition 1 (Absolutely Successful GEPP) A General Entanglement Purification Protocol $\mathcal{P}$ is absolutely successful with parameter $\langle N, K, M, \epsilon, \delta\rangle$, if for all states $\rho$ such that $F(\rho) \geq 1-\epsilon$,

$$
\operatorname{Prob}[\mathcal{P}(\rho)=\mathrm{FAIL}]=0
$$

and

$$
\operatorname{Prob}[F(\mathcal{P}(\rho)) \geq 1-\delta]=1
$$

Definition 2 (Deterministically Successful GEPP) A General Entanglement Purification Proto$\operatorname{col} \mathcal{P}$ is deterministically conditionally successful with parameter $\langle N, K, M, \epsilon, \delta, p\rangle$, if for all input states $\rho$ such that $F(\rho)=1-\epsilon$,

$$
\operatorname{Prob}[\mathcal{P}(\rho)=\mathrm{FAIL}] \leq p
$$

and

$$
\operatorname{Prob}[F(\mathcal{P}(\rho)) \geq 1-\delta \mid \mathcal{P}(\rho) \neq \mathrm{FAIL}]=1
$$

Definition 3 (Probabilistically Successful GEPP) A General Entanglement Purification Protocol $\mathcal{P}$ is probabilistically conditionally successful with parameter $\langle N, K, M, \epsilon, \delta, p, q\rangle$, if for all input states $\rho$ such that $F(\rho)=1-\epsilon$,

$$
\operatorname{Prob}[\mathcal{P}(\rho)=\mathrm{FAIL}] \leq p
$$

and

$$
\operatorname{Prob}[F(\mathcal{P}(\rho)) \geq 1-\delta \mid \mathcal{P}(\rho) \neq \text { FAIL }] \geq 1-q
$$

Definition 4 (Efficient GEPP) A General Entanglement Purification Protocol $\mathcal{P}$ is efficient, if there exists a constant $c$ such that $\mathcal{P}$ can be implemented by quantum circuits of size $O\left((\log N+\log K)^{c}\right)$.

## 4 Results

### 4.1 Impossibility result for absolutely successful protocols

Theorem 1 (a) For all absolutely successful General Entanglement Purification Protocols with parameter $\langle N, K, M, \epsilon, \delta\rangle$, we have the following inequality:

$$
\delta \geq \frac{M-K}{M} \frac{N}{N-1} \epsilon
$$

(b) The bound in (a) is tight. For any integers $N, M, K$ such that $N K / M$ and $M / K$ are both integers, there exists an absolutely successful GEPP with parameter $\left\langle N, K, M, \epsilon, \frac{M-K}{M} \frac{N}{N-1} \epsilon\right\rangle$.

This shows that absolutely succesful protocols are quite weak. If we just want to extract the auxiliarly state $\Psi_{K}$ and $c$ more EPR pairs, then $M=2^{c} K$ and we can achieve the fidelity of at most $1-\frac{2^{c}-1}{2^{c}} \frac{N}{N-1} \epsilon<1-\left(1-\frac{1}{2^{c}}\right) \epsilon$ which is less than $\frac{1}{2^{c}} \epsilon$ better than $1-\epsilon$ that we had at the beginning. If we want to get $\Psi_{K}$ plus a linear number of EPR pairs, the improvement in fidelity is an exponentially small fraction of $\epsilon$.

We prove theorem 11 in appendix A.

### 4.2 Constructions of conditionally successful protocols

On the other hand, there are good conditionally successful protocols.
Theorem 2 For all integers $n, t, d$ such that $n>t$ and any real $\epsilon<1 / 2$, there exist efficient deterministic conditionally successful general entanglement purification protocols of following parameters

- $\left\langle 2^{n},\left(2^{n}-1\right), 2^{n-t}\left(2^{n}-1\right), \epsilon, \frac{\epsilon}{2^{2}}, \epsilon\right\rangle$
- $\left\langle 2^{2 n}, 2^{n}+1,2^{n}\left(2^{n}+1\right), \epsilon, \frac{\epsilon}{2^{n}}, \epsilon\right\rangle$
- $\left\langle 2^{d n}, \frac{2^{d n}-1}{2^{n}-1}, 2^{(d-1) n}\left(\frac{2^{d n}-1}{2^{n}-1}\right), \epsilon, \frac{\epsilon}{2^{n}}, \epsilon\right\rangle$
for mixed states in the diagonal subspace.
The first protocol achieves the smallest loss of EPR pairs, increasing the fidelity from $1-\epsilon$ to $1-\frac{\epsilon}{2^{t}}$ at the cost of losing just $t$ EPR pairs. Is starts with $n$ imperfect EPR pairs and an auxiliary state of dimension $2^{n}-1$ and outputs a state of dimension $2^{n-t}\left(2^{n}-1\right)$. The disadvantage is that we have to use an auxiliary state of almost the same dimension $\left(2^{n}-1\right)$ as the state that we try to purify $\left(2^{n}\right)$. The second and the third construction use smaller auxiliary states but lose more EPR pairs. All 3 results are achieved by Simple Scrambling Protocol (appendix C) using 3 different constructions of scrambling permutations (appendix E).

The constructions fail with probability at most $\epsilon$. We can extend Theorem 2 to show that the trade-off between the probability of failure and increase in fidelity achieved by Theorem 2 is optimal. It might be possible to improve the theorem with respect to other parameters (the dimensionality of the extra state $\Psi_{K}$ and the amount of entanglement that is lost if the prototocol does not fails).

For states not in the diagonal subspace, we can construct a probabilistically succesful protocol with almost the same parameters.

Theorem 3 For all integers $n, t, l, d$ such that $n>t, n>l$ and all real $\epsilon<1 / 2$, there exist efficient probabilistic conditionally successful general entanglement purification protocols of following parameters

- $\left\langle 2^{n},\left(2^{n}-1\right) 2^{2 t}, 2^{n-t}\left(2^{n}-1\right), \epsilon, \frac{\epsilon}{2^{t-3}}, 2 \epsilon+\sqrt{\frac{2 \epsilon}{2^{t}}}, \frac{1}{2^{t}}\right\rangle$
- $\left\langle 2^{2 n},\left(2^{n}+1\right) 2^{2 t}, 2^{n}\left(2^{n}+1\right), \epsilon, \frac{\epsilon}{2^{t-3}}, 2 \epsilon+\sqrt{\frac{2 \epsilon}{2^{t}}}, \frac{1}{2^{t}}\right\rangle$
- $\left\langle 2^{d n}, \frac{2^{d n}-1}{2^{n}-1} \cdot 2^{2 t}, 2^{(d-1) n}\left(\frac{2^{d n}-1}{2^{n}-1}\right), \epsilon, \frac{\epsilon}{2^{t-3}}, 2 \epsilon+\sqrt{\frac{2 \epsilon}{2^{t}}}, \frac{1}{2^{t}}\right\rangle$

We note that the extra probability of failure $\left(\sqrt{\frac{2 \epsilon}{2^{t}}}\right)$ can be made arbitrarily small by increasing $t$. This theorem is shown by Complete Scrambling protocol which combines the Simple Scrambling protocol with another protocol, Hash-and-Compare (appendix D).

## 5 Conclusions and Open Problems

We investigated the problem of entanglement purification by Alice and Bob via LOCC. We used a very general model of the input state, where the only information Alice and Bob have is a lower bound on the fidelity of the input state. This contrasts with the previous models which assumed that the "noise" is identical and independent, or Alice and Bob have complete knowledge of the input state they share. Because of the generality of the General Error model, the techniques used in previous works don't appear viable.

We defined three types of General Entanglement Purification Protocols. Absolutely successful GEPPs never fail, and they always output states that have very high fidelity. Deterministically conditionally successful GEPPs fail with small probability, and otherwise, they output states of high fidelity
with certainty. Probabilistically conditionally successful GEPPs fail with small probability, and otherwise output states of high fidelity with very high probability.

We proved a negative result that there don't exist absolutely successful GEPPs of interesting parameters, i.e., on average, the ability of Alice and Bob to purify the entanglement is very limited.

We constructed efficient GEPPs that are deterministically conditionally successful for mixed states in the diagonal subspace (Simple Scrambling protocol) and probabilistically conditionally successful for an arbitrary state of sufficiently high fidelity (Complete Scrambling protocol).

In our construction of the protocols, Scrambling Permutations play a very important role. We give 3 different constructions of efficient Scrambling Permutations in Appendix E. Each construction has its own advantage. By plugging them into the construction of Complete Scrambling protocol, we obtain different protocols with different parameters. We notice that the notion of Scrambling Permutations are closely related to universal hash functions. By being more lax on the "scrambling" property, they can have more efficient constructions than universal hash functions.

There are several open problems:

1. Remove the auxiliary input or reduce its size. In our paper, both the Simple Scrambling protocol and the Hash and Compare protocol need maximally entangled states as auxiliary input. The Simple Scrambling protocol needs them to "scramble" the coefficients of the input state, and the Hash and Compare protocol needs them to perform teleportation. In the final construction of the Complete Scrambling protocols, if Alice and Bob share an input of $n_{0}$ qubit pairs, they need to invest $O\left(n_{0}\right)$ perfect EPR pairs in order to perform the purification. It would be very desirable to reduce the number of auxiliary perfect EPR pairs as much as possible: the ideal case would be removing them completely, but even reducing them to $o\left(n_{0}\right)$ would be interesting.
2. Optimal GEPPs The Simple Scrambling protocol is optimal in the sense that the average fidelity of its output matches the upper bound of Theorem 1 asymptotically (in case Alice and Bob fail, they can output $\left|Z_{M}\right\rangle^{A} \otimes\left|Z_{M}\right\rangle^{B}$ instead). However it is not true for Hash and Compare and Complete Scrambling protocols. Do there exist optimal GEPPs that work for all states?
3. Relationship to classical randomness extraction. As we described in the introduction, we view the problem of extracting entanglement as a quantum counterpart of extracting randomness from a weak random source. There is also a similarity between the techniques used in those two problems. One of the main techniques used in the classical randomness extraction is the universal hash function, and we used scrambling permutations in our construction of GEPP. Are there deeper relationships between the two problems? Also, can some of the techniques from classical randomness extractors be used in the entanglement purification? Notice that the state of art in randomness extraction is that only logarithmic number of truly random bits need to be invested and almost all the entropy can be extracted NT99], whereas in the case of entanglement extraction, our constructions call for linear number of perfect EPR pairs to be invested and considerable amount of entanglement is wasted. Can we make the entanglement purification protocols more efficient, or are these inefficiencies inherent?

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## References

[B64] J. S. Bell, Physics 1, 195 (1964) [reprinted in J. S. Bell, Speakable and Unspeakable in Quantum Mechanics, Cambridge University Press, Cambridge, 1987].
[BBC+93] C. H. Bennett, G. Brassard, C. Crépeau, R. Josza, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. 70, 1895 (1993).
[BBP $+96 \mathrm{a}]$ C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, In Physical Review A, vol. 53, No. 4, April 1996.
[BBP+96b] C. H. Bennett, H. J. Bernstein, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels, In Physics Review Letters, vol. 76, page 722, 1996.
[BDS+96] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction. In Physical Review A, vol. 54, No. 5, November 1996.
[BW92] C. H. Bennett and S. J. Wiesner, Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, Phyc. Rev. Lett. 69, 2881 (1992).
[CW79] J. Carter, and M. Wegman, Universal Classes of Hash Functions, in Journal of Computer and System Sciences, vol. 18, 1979, pp. 143-154.
[EPR35] A. Einstein, B. Podolsky, and N. Rosen, Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? Phys. Rev. 47, 777 (1935) [reprinted in Quantum Theory and Measurement, edited by J. A. Wheeler and W. Z. Zurek, Princeton University Press, 1983].
[H99] L. Hardy, Method of areas for manipulating the entanglement properties of one copy of a twoparticle pure entangled state, Phys. Rev. A, 60, 1912 (1999). also available at quant-ph/9903001.
[HHH96] M. Horodecki, P. Horodecki, and R. Horodecki, Distillability of Inseparable Quantum Systems. In quant-ph/9607009.
[JP99] D. Jonathan and M. Plenio, Minimal conditions for local pure-state entanglement manipulation, Phys. Rev. Lett. 83, 1455 (1999), also available at quant-ph/9903054.
[L96] M. Luby, Pseudorandomness and Cryptographic Applications, Princeton University Press, 1996.
[L01] J. C. Langford, Generic Quantum Block Compression, available at quant-ph/0109074.
[MR95] R. Motwani and P. Raghavan, Randomized Algorithms, Cambridge University Press, 1995.
[vN51] J. von Neumann, Various techniques used in connection with random digits. Notes by G. E. Forsythe, National Bureau of Standards, 1952, vol. 12, pages 36-38.
[N99] M. Nielsen, Conditions for a class of entanglement transformations, Phys. Rev. Lett, 83(2):436:439, 1999.
[NC00] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2000.
[NT99] N. Nisan and A. Ta-Shma, Extracting Randomness: A Survey and New Constructions. JCSS 58(1): 148-173 (1999).
[TUZ01] A. Ta-Shma, C. Umans, D. Zuckerman. Loss-less condensers, unbalanced expanders and extractors. Proceedings of STOC'01, pp. 143-152.
[T99] L. Trevisan. Construction of extractors using pseudo-random generators. Proceedings of STOC'99, pp. 141-148.
[V99] G. Vidal, Entanglement of pure states for a single copy, Phys. Rev. Lett. 83 (1999) 1046-1049, quant-ph/9902033
[VJN00] G. Vidal, D. Jonathan, and M. Nielsen Approximation Transformations and Robust Manipulation of Bipartite Pure State Entanglement Phys. Rev. A 62, 012304 (2000) Also available at quant-ph/9910099.
[WC81] M. Wegman and J. Carter. New Hash Functions and Their Use in Authentication and Set Equality, In Journal of Computer and System Sciences, vol. 22, pp 265-279, 1981.

## A Tight bounds on Absolutely Successful GEPPs

We prove Theorem 1 in this section.

## A. 1 A Negative Result

We show that on average, Alice and Bob cannot increase the fidelity of their input state significantly, even if they have an auxiliary input $\Psi_{K}$.

We first study a simpler problem. Suppose Alice and Bob share a maximally entangled state $\Psi_{K}$ and some private ancillary bits, initialized to $|0\rangle$. We describe this shared state by

$$
|\phi\rangle=\left(\left|Z_{N}\right\rangle^{A} \otimes\left|Z_{N}\right\rangle^{B}\right) \otimes \Psi_{K}
$$

The fidelity of this state is $K / M$ by a simple computation.
Alice and Bob try to convert state $|\phi\rangle$ as close to $\Psi_{M}$ as possible by LOCC. The problem is: how close can they get? If $M / K$ is an integer, Alice and Bob just trace out a subsystem of their ancillary bits to bring the dimension of each their subsystem to $M$, then they obtain a state

$$
\left|\psi_{0}\right\rangle=\left(\left|Z_{M / K}\right\rangle^{A} \otimes\left|Z_{M / K}\right\rangle^{B}\right) \otimes \Psi_{K}
$$

which has fidelity $K / M$ by a straightforward computation. In fact, this is actually the best Alice and Bob can do:

Lemma 1 Let $|\phi\rangle=\left(\left|Z_{N}\right\rangle^{A} \otimes\left|Z_{N}\right\rangle^{B}\right) \otimes \Psi_{K}$ be a state in a bipartite system $\mathcal{H}_{N K}^{A} \otimes \mathcal{H}_{N K}^{B}$ shared between Alice and Bob. Let $\sigma$ be the state Alice and Bob output after performing LOCC operations. Suppose that $\sigma$ is in the subspace $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$. We have $F(\sigma) \leq \frac{K}{M}$.

This lemma is a direct corollary of a result by Vidal, Jonathan, and Nielsen VJN00. There is also a simple direct proof in Appendix B.

Proof: [Proof to Theorem 1, part (a)]
We prove the theorem by demonstrating a particular mixed state $\rho$ such that $\rho$ has a fidelity $1-\epsilon$, and no LOCC can increase its fidelity to more than $1-\frac{M-K}{M} \frac{N}{N-1} \epsilon$.

Let $\epsilon^{\prime}=\frac{N}{N-1} \epsilon$. We define the state $\rho$ to be

$$
\rho=\left(1-\epsilon^{\prime}\right) \cdot\left|\Psi_{N}\right\rangle\left\langle\Psi_{N}\right|+\epsilon^{\prime} \cdot\left|Z_{N}^{A} \otimes Z_{N}^{B}\right\rangle\left\langle Z_{N}^{A} \otimes Z_{N}^{B}\right|
$$

In fact, $\rho$ is the maximally entangled state $\Psi_{M}$ with probability $\left(1-\epsilon^{\prime}\right)$ and the totally disentangled state $Z_{N}^{A} \otimes Z_{N}^{B}$ with probability $\epsilon^{\prime}$.

It is easy to verify that $F(\rho)=1-\epsilon$, since $\left\langle\Psi_{N} \mid Z_{N}^{A} \otimes Z_{N}^{B}\right\rangle=1 / \sqrt{N}$ and, therefore,

$$
F(\rho)=\left(1-\epsilon^{\prime}\right) F\left(\left|\Psi_{N}\right\rangle\left\langle\Psi_{N}\right|\right)+\epsilon^{\prime} F\left(\left|Z_{N}^{A} \otimes Z_{N}^{B}\right\rangle\left\langle Z_{N}^{A} \otimes Z_{N}^{B}\right|\right)=\left(1-\epsilon^{\prime}\right)+\frac{1}{N} \epsilon^{\prime}=1-\left(1-\frac{1}{N}\right) \epsilon^{\prime}=1-\epsilon
$$

For an arbitrary GEPP $\mathcal{P}$ that never fails, we define

$$
f_{1}=F\left(\mathcal{P}\left(\left|\Psi_{N}\right\rangle\left\langle\Psi_{N}\right|\right)\right)
$$

and

$$
f_{2}=F\left(\mathcal{P}\left(\left|Z_{N}^{A} \otimes Z_{N}^{B}\right\rangle\left\langle Z_{N}^{A} \otimes Z_{N}^{B}\right|\right)\right)
$$

Then we have $f_{1} \leq 1$ and by Lemma 1,,$f_{2} \leq K / M$.
By the linearity of fidelity of quantum operations, we know that

$$
F(\mathcal{P}(\rho))=\left(1-\epsilon^{\prime}\right) f_{1}+\epsilon^{\prime} f_{2} \leq 1-\frac{M-K}{M} \epsilon^{\prime}=1-\frac{M-K}{M} \frac{N}{N-1} \epsilon .
$$

We will prove the part (b) of the theorem in the next subsection.
Therefore, there don't exist absolutely successful GEPPs with very interesting parameters - we hope that our protocol is able to "boost" the fidelity of the input state to arbitrarily close to 1 , but clearly this is impossible for absolutely successful protocols.

## A. 2 A Protocol That Matches the Bound

We now prove the second part Theorem 11, that the bound is tight. We do so by showing that there is a protocol that achieves this slight increase in fidelity.

The input to the protocol is a state $\rho$ in $\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$. The protocol outputs a state in $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$ where $M<N$ and $M$ divides $N$. There is no auxiliary state used, i.e., $K=1$.

Construction 1 (Random Permutation Protocol) The input to this protocol is a state in $\mathcal{H}_{N}^{A} \otimes$ $\mathcal{H}_{N}^{B}$. The steps are:

1. Alice generates a uniformly random permutation $\pi$ on $N$ elements using classical randomness and transmits the permutation to Bob.
2. Alice applies permutation $\pi$ on $\mathcal{H}_{N}^{A}$, mapping $|i\rangle$ to $|\pi(i)\rangle$, Bob does the same on $\mathcal{H}_{N}^{B}$.
3. Alice and Bob decompose $\mathcal{H}_{N}$ as $\mathcal{H}_{M} \otimes \mathcal{H}_{L}, L=N / M$ and measure the $\mathcal{H}_{L}$ part.
4. Alice sends the result of her measurement to Bob, Bob sends his result to Alice.
5. They compare the results. If the results are the same, they output the state that they have in $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$. If the results are different, they output $\left|Z_{M}\right\rangle \otimes\left|Z_{M}\right\rangle$.

We start with the case when the state of Alice and Bob is in the diagonal subspace.
Lemma 2 If the input state to the Random Permutation Protocol is in the diagonal subspace, the protocol is absolutely successful with parameters $\left\langle N, 1, M, \epsilon, \frac{M-1}{M} \frac{N}{N-1} \epsilon\right\rangle$.

Proof: Without loss of generality, we assume that the starting state is pure. Let $|\phi\rangle=\sum_{i=1}^{N} \alpha_{i}|i\rangle^{A}|i\rangle^{B}$ be the starting state. For a permutation $\pi$, let $U_{\pi}$ be the unitary transformation defined by $U_{\pi}\left(|i\rangle^{A} \otimes|j\rangle^{B}\right)=$ $|\pi(i)\rangle^{A}|\pi(j)\rangle^{B}$. Then, if Alice and Bob use a permutation $\pi$, the resulting state is

$$
\left|\phi_{\pi}\right\rangle=U_{\pi}|\phi\rangle=\sum_{i=1}^{N} \alpha_{i}|\pi(i)\rangle^{A}|\pi(i)\rangle^{B}=\sum_{i=1}^{N} \alpha_{\pi^{-1}(i)}|i\rangle^{A}|i\rangle^{B} .
$$

There are $N$ ! permutations $\pi$ on a set of $N$ elements. Therefore, each of them gets applied with probability $1 / N$ !. This means that the final state is a mixed state of $\left|\phi_{\pi}\right\rangle$ with probabilities $1 / N$ ! each. We calculate the density matrix $\rho$ of this state. It is equal to

$$
\sum_{\pi} \frac{1}{N!}\left|\phi_{\pi}\right\rangle\left\langle\phi_{\pi}\right|=\sum_{\pi} \frac{1}{N!}\left(\begin{array}{llll}
\alpha_{\pi^{-1}(1)} \alpha_{\pi \pi^{-1}(1)}^{*} & \alpha_{\pi^{-1}(1)} \alpha_{\pi^{-1}(2)}^{*} & \ldots & \alpha_{\pi^{-1}(1)} \alpha_{\pi \pi^{-1}(N)}^{*} \\
\alpha_{\pi^{-1}(2)} \alpha_{\pi^{-1}(1)}^{*} & \alpha_{\pi^{-1}(2)} \alpha_{\pi^{-1}(2)}^{*} & \ldots & \alpha_{\pi^{-1}(2)} \alpha_{\pi^{-1}(N)}^{*} \\
\ldots & \ldots & \ldots & \ldots \\
\alpha_{\pi^{-1}(N)} \alpha_{\pi^{-1}(1)}^{*} & \alpha_{\pi^{-1}(N)} \alpha_{\pi^{-1}(2)}^{*} & \ldots & \alpha_{\pi^{-1}(N)} \alpha_{\pi^{-1}(N)}^{*}
\end{array}\right)
$$

We claim that all diagonal entries $\rho_{i i}$ are equal to $1 / N$ and all off-diagonal entries $\rho_{i j}, i \neq j$ are equal to some value $a$ which is real. This follows from the symmetries created by summing over all permutations.

Consider a diagonal entry $\rho_{i i}$. For each $j \in\{1, \ldots, N\}$, there are ( $N-1$ )! permutations that map $j$ to $i$. Therefore,

$$
\rho_{i i}=\sum_{j=1}^{N}(N-1)!\frac{1}{N!} \alpha_{j} \alpha_{j}^{*}=\frac{1}{N} \sum_{j=1}^{N}\left|\alpha_{j}\right|^{2} .
$$

$\sum_{j=1}^{N}\left|\alpha_{j}\right|^{2}$ is the same as $\|\phi\|^{2}$ which is equal to 1 . Therefore, $\rho_{i i}=\frac{1}{N}$.

Next, consider an off-diagonal entry $\rho_{i j}$. For each $k, l, k \neq l$, there are $(N-2)$ ! permutations that $\operatorname{map} k$ to $i$ and $l$ to $j$. Therefore,

$$
\rho_{i j}=\sum_{k=1}^{N} \sum_{l=1, l \neq k}^{N}(N-2)!\frac{1}{N!} \alpha_{k} \alpha_{l}^{*}=\sum_{k=1}^{N} \sum_{l=1, l \neq k}^{N} \frac{1}{N(N-1)} \alpha_{k} \alpha_{l}^{*} .
$$

This immediately implies that $\rho_{i j}$ is the same for all $i \neq j$. Also, notice that $\left(\alpha_{k} \alpha_{l}^{*}\right)^{*}=\alpha_{k}^{*} \alpha_{l}$. Therefore, $\alpha_{k} \alpha_{l}^{*}+\alpha_{l} \alpha_{k}^{*}$ is real and $\rho_{i j}$ (which is a sum of terms of this form) is real as well. Let $a=\rho_{i j}$. We have shown that

$$
\rho=\left(\begin{array}{llll}
\frac{1}{N} & a & \ldots & a \\
a & \frac{1}{N} & \ldots & a \\
\ldots & \ldots & \ldots & \ldots \\
a & a & \ldots & \frac{1}{N}
\end{array}\right)
$$

Notice that the density matrix $\rho$ can be also obtained from a mixed state that is $\Psi_{N}$ with probability $N a$ and each of basis states $|i\rangle^{A}|i\rangle^{B}$ with probability $\frac{1}{N}-a$.

We now consider applying steps $3-5$ to those states. Measuring $\mathcal{H}_{L}^{A} \otimes \mathcal{H}_{L}^{B}$ for $\left|\Psi_{N}\right\rangle$ always gives the same results and leaves Alice and Bob with the state $\left|\Psi_{M}\right\rangle$ in $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$. The fidelity of this state with $\left|\Psi_{M}\right\rangle$ is, of course, 1. Measuring $\mathcal{H}_{L}^{A}$ and $\mathcal{H}_{L}^{B}$ for $|i\rangle^{A}|i\rangle^{B}$ also gives the same results and leaves Alice and Bob with some basis state $\left|i^{\prime}\right\rangle^{A}\left|i^{\prime}\right\rangle^{B}$ in the diagonal subspace of $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$. The fidelity of this state and $\left|\Psi_{M}\right\rangle$ is $\frac{1}{M}$. By Claim 目, if we apply those steps to the state $\rho$, we get that the final fidelity

$$
\begin{equation*}
N a+N\left(\frac{1}{N}-a\right) \frac{1}{M}=\frac{1}{M}+N a\left(1-\frac{1}{M}\right) . \tag{7}
\end{equation*}
$$

We now lower-bound $a$. By Claim [1], $F(\rho)=\frac{1}{N!} \sum_{\pi} F\left(\left|\phi_{\pi}\right\rangle\right)$. Since permuting the basis states $|i\rangle^{A}|i\rangle^{B}$ preserves the maximally entangled state $\Psi_{N}=\frac{1}{\sqrt{N}} \sum_{i=1}^{N}|i\rangle^{A}|i\rangle^{B}$, the fidelity of any $\left|\phi_{\pi}\right\rangle$ is the same as the fidelity of $|\phi\rangle$. Therefore, $F(\rho)=F(|\phi\rangle) \geq 1-\epsilon$. By applying the definition of fidelity,
$F(\rho)=\left(\begin{array}{c}\frac{1}{\sqrt{N}} \\ \frac{1}{\sqrt{N}} \\ \cdots \\ \frac{1}{\sqrt{N}}\end{array}\right)\left(\begin{array}{cccc}\frac{1}{N} & a & \ldots & a \\ a & \frac{1}{N} & \ldots & a \\ \ldots & \ldots & \ldots & \ldots \\ a & a & \ldots & \frac{1}{N}\end{array}\right)\left(\begin{array}{llll}\frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \ldots & \frac{1}{\sqrt{N}}\end{array}\right)=N \frac{1}{N^{2}}+N(N-1) \frac{1}{N} a=\frac{1}{N}+(N-1) a$.
Since $F(\rho) \geq 1-\epsilon$, it must be the case that $a \geq \frac{1}{N}-\frac{\epsilon}{N-1}$. By substituting that into (7), the fidelity of the final state with $\left|\Psi_{K}\right\rangle$ is at least

$$
\frac{1}{M}+N\left(\frac{1}{N}-\frac{\epsilon}{N-1}\right)\left(1-\frac{1}{M}\right)=1-\frac{N}{N-1}\left(1-\frac{1}{M}\right) \epsilon .
$$

To prove the second part of Theorem 1, it remains to show that the protocol also succeeds for states not in the diagonal subspace. Let $|\phi\rangle$ be a state such that $F(|\phi\rangle) \geq 1-\epsilon$. We decompose

$$
|\phi\rangle=\sqrt{1-\delta}\left|\phi_{1}\right\rangle+\sqrt{\delta}\left|\phi_{2}\right\rangle,
$$

with $\left|\phi_{1}\right\rangle \in \mathcal{H}^{D}$ and $\left|\phi_{2}\right\rangle \in\left(\mathcal{H}^{D}\right)^{\perp}$. Let $F\left(\left|\phi_{1}\right\rangle\right)=1-\delta^{\prime}$. Since $\Psi_{N}$ is in $\mathcal{H}^{D}$ and $\left|\phi_{2}\right\rangle$ is orthogonal to $\mathcal{H}^{D}$, we have $F\left(\left|\phi_{2}\right\rangle\right)=0$ and $F(|\phi\rangle)=(1-\delta)\left(1-\delta^{\prime}\right)$. Notice that $(1-\delta)\left(1-\delta^{\prime}\right) \geq 1-\epsilon$ because $F(|\psi\rangle) \geq 1-\epsilon$.

Applying $U_{\pi}$ maps $|\phi\rangle$ to $\left|\phi_{\pi}\right\rangle=\sqrt{1-\delta}\left|\phi_{\pi, 1}\right\rangle+\sqrt{\delta}\left|\phi_{\pi, 2}\right\rangle$ where $\left|\phi_{\pi, 1}\right\rangle=U_{\pi}\left|\phi_{1}\right\rangle,\left|\phi_{\pi, 2}\right\rangle=U_{\pi}\left|\phi_{2}\right\rangle$. Since $U_{\pi}$ preserves the diagonal subspace, $\left|\phi_{\pi, 1}\right\rangle \in \mathcal{H}^{\mathcal{D}}$ and $\left|\phi_{\pi, 2}\right\rangle \in\left(\mathcal{H}^{\mathcal{D}}\right)^{\perp}$. Measuring $\mathcal{H}_{L}^{A}$ and $\mathcal{H}_{L}^{B}$ for
a state in $\mathcal{H}^{\mathcal{D}}$ always gives the same results and produces a state in the diagonal subspace of $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$. Measuring $\mathcal{H}_{L}^{A}$ and $\mathcal{H}_{L}^{B}$ for a state in $\left(\mathcal{H}^{\mathcal{D}}\right)^{\perp}$ either gives the different results for Alice and Bob or gives the same results but produces a state orthogonal to the diagonal subspace of $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$.

The fidelity of the final state consists of two parts: the fidelity of the final state if Alice's and Bob's measurements of $\mathcal{H}_{L}$ give the same answer and the fidelity if measurements give the different answer. The first part is just $(1-\delta)$ times the fidelity of the final state if the starting state was $\left|\phi_{1}\right\rangle$ (instead of $|\phi\rangle$ ). Since $\left|\phi_{1}\right\rangle$ is in the diagonal subspace, Lemma 2 implies that the final state of the protocol $\left|\phi_{1}\right\rangle$ has the fidelity at least $1-D \delta^{\prime}$ where $D=\frac{M-1}{M} \frac{N}{N-1}$. Therefore, the first part is at least

$$
\begin{equation*}
(1-\delta)\left(1-D \delta^{\prime}\right)=(1-\delta)\left(1-\delta^{\prime}\right)+(1-D) \delta^{\prime}(1-\delta) \tag{8}
\end{equation*}
$$

The second part is the probability of measurements giving different answers times the fidelity of the state $|0\rangle \otimes|0\rangle$ which Alice and Bob output in this case. The fidelity of this state is $\frac{1}{M}$ and the probability of this case is given by

Lemma 3 The probability that Alice's and Bob's measurements give different answers is $\frac{N-M}{N-1} \delta$.
Proof: First, we look at the state $\left|\phi_{2}\right\rangle$. Since this state is in $\left(\mathcal{H}^{\mathcal{D}}\right)^{\perp}$, it is of the form

$$
\left|\phi_{2}\right\rangle=\sum_{i, j=1, i \neq j}^{N} \alpha_{i, j}|i\rangle^{A}|j\rangle^{B} .
$$

Applying $U_{\pi}$ maps it to

$$
\left|\phi_{\pi, 2}\right\rangle=\sum_{i \neq j} \alpha_{i, j}|\pi(i)\rangle^{A}|\pi(j)\rangle^{B}=\sum_{i \neq j} \alpha_{\pi^{-1}(i), \pi^{-1}(j)}|i\rangle^{A}|j\rangle^{B} .
$$

The probability of Alice and Bob getting different results is equal to the sum of $\left|\alpha_{\pi^{-1}(i), \pi^{-1}(j)}\right|^{2}$ over all basis $|i\rangle^{A},|j\rangle^{B}$ that differ in the $\mathcal{H}_{L}$ part. If this sum is averaged over all permutations $\pi$, it becomes the same for all $i, j, i \neq j$. Therefore, the probability of Alice and Bob getting different results is just the fraction of pairs $(i, j)$ that differ in the $\mathcal{H}_{L}$ part. It is $\frac{N-M}{N-1}$ because for each $i$, there are $(N-1)$ $j \in\{1, \ldots, N\}, j \neq i$ and $M-1$ of them differ only in the $\mathcal{H}_{K}$ but the remaining $N-M$ differ in the $\mathcal{H}_{L}$ part.

If the starting state is $|\phi\rangle$, the probability of Alice and Bob getting different results is $\delta$ times the probability for $\left|\phi_{2}\right\rangle$ because $|\phi\rangle=\sqrt{1-\delta}\left|\phi_{1}\right\rangle+\sqrt{\delta}\left|\phi_{2}\right\rangle$ and the measurements always give the same answer on $\left|\phi_{1}\right\rangle$.

Therefore, the second part of the fidelity is $\frac{1}{M} \frac{N-M}{N-1} \delta$. Notice that $1-D=1-\frac{(M-1) N}{M(N-1)}=$ $\frac{(M-1) N-M(N-1)}{M(N-1)}=\frac{N-M}{M(N-1)}$. Thus, the second part is $(1-D) \delta$ and the overall fidelity is at least

$$
(1-\delta)\left(1-\delta^{\prime}\right)+(1-D)(1-\delta) \delta^{\prime}+(1-D) \delta=1-D\left(\delta\left(1-\delta^{\prime}\right)+\delta^{\prime}\right) .
$$

Since $(1-\delta)\left(1-\delta^{\prime}\right) \geq 1-\epsilon, \delta\left(1-\delta^{\prime}\right)+\delta^{\prime} \leq \epsilon$. Therefore, the overall fidelity is at least $1-D \epsilon$. This completes the proof of the second part of Theorem 1 for $K=1$.

For $K>1$, we can just produce an entangled state of dimension $M^{\prime}=M / K$ without the use of $\left|\Psi_{K}\right\rangle$ by the protocol above and then output this state and the original $\left|\Psi_{K}\right\rangle$. This achieves the fidelity of at least $1-D \epsilon$ for $D=\frac{M^{\prime}-1}{M^{\prime}} \frac{N}{N-1}=\frac{M / K-1}{M / K} \frac{N}{N-1}=\frac{M-K}{M} \frac{N}{N-1}$, proving that the bound of Theorem 1 (a) is tight for $K>1$ as well.

## B Proof to Lemma 1$]$

We give a (somewhat) simpler proof to Lemma 1 than the proof by Vidal, Jonathan, and Nielsen VJN00].
For a self-adjoint matrix $M$, we define its spectrum written as $\mathcal{S}(M)$, to be a vector formed by the eigenvalues of $M$, and whose entries are sorted in a decreasing order. In other words, if the eigenvalues of $M$ are $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{d}$, where $\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{d}$, then $\mathcal{S}(M)=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{d}\right)$.

For a mixed state $\rho$, if we write $\rho$ as

$$
\rho=\sum_{i=1}^{d} p_{i} \cdot\left|\phi_{i}\right\rangle\left\langle\phi_{i}\right|
$$

where $p_{1} \geq p_{2} \geq \cdots \geq p_{d}$, and $\left\{\left|\phi_{i}\right\rangle\right\}$ is an orthonormal basis, then

$$
\mathcal{S}(\rho)=\left(p_{1}, p_{2}, \ldots, p_{d}\right)
$$

A useful Fact about the spectrum of a tensor product of two matrices is the following:
Fact 1 Let $A$ and $B$ be square matrices such that the eigenvalues for $A$ are $\left\{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m}\right\}$ and the eigenvalues for $B$ are $\left\{\mu_{1}, \mu_{2}, \ldots, \mu_{m}\right\}$. Then the eigenvalues for the matrix $A \otimes B$ are $\left\{\lambda_{i} \cdot \mu_{j}\right\}_{i=1,2, \ldots, m, j=1,2, . ., n}$.

Proof: It is easy to verify that if $A \cdot \vec{v}=\lambda \cdot \vec{v}$ and $B \cdot \vec{u}=\mu \cdot \vec{u}$, then $(A \otimes B) \cdot(v \otimes \vec{\otimes} u)=(\lambda \cdot \mu)(v \vec{\otimes} u)$
and a corollary the above fact is:
Corollary 1 Let $\rho^{A}, \rho^{B}$ be the density matrices for quantum systems $\mathcal{H}^{A}$ and $\mathcal{H}^{B}$. Then we have

$$
\begin{equation*}
\operatorname{rank}\left(\rho^{A} \otimes \rho^{B}\right) \geq \operatorname{rank}\left(\rho^{A}\right) \tag{9}
\end{equation*}
$$

Proof: Notice that the rank of a matrix equals the number of non-zero eigenvalues of this matrix. Since $\rho^{B}$ is a density matrix, it has trace 1 , and thus it has at least one non-zero eigenvalue - assume it is $\mu_{1}$. We denote the eigenvalues of $\rho^{A}$ by $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m}$, then by Fact $1, \lambda_{1} \cdot \mu_{1}, \lambda_{2} \cdot \mu_{1}, \ldots, \lambda_{m} \cdot \mu_{1}$ are all eigenvalues of $\rho^{A} \cdot \rho^{B}$, and they contain at many non-zero numbers as the eigenvalues of $\rho^{A}$.

Proof: [Proof to Lemma 1]
We consider an arbitrary protocol $\mathcal{P}$ between Alice and Bob involving only LOCC. We assume that $\mathcal{P}$ consists of steps, where each step could be one of the following operations 阬:

1. Unitary Operation:

Alice (or Bob) applies a unitary operation to her (or his) subsystem.
2. Measurement:

Alice (or Bob) performs a measurement to her (or his) subsystem.

## 3. Tracing Out:

Alice (or Bob) discards part of her (or his) subsystem, or equivalently, traces out part of the subsystem.
4. Classical Operation:

Alice (or Bob) sends a (classical) message to the other party.

[^3]We first convert this protocol $\mathcal{P}$ into another protocol $\mathcal{P}^{\prime}$ in the following way: for each tracing-out operation Alice (or Bob) performs, we insert a measurement operation right before the tracing-out, and the measurement is a full measurement of the subsystem to be traced out. Notice that $\mathcal{P}^{\prime}$ will have exactly the same output as $\mathcal{P}$, since the subsystem that was traced out isn't part of the output. However, $\mathcal{P}^{\prime}$ has the property that for each subsystem traced out in the protocol, that subsystem is disentangled from the rest, since it is already completely measured.

Now we analyze the new protocol $\mathcal{P}^{\prime}$. We denote the partial density matrix of Alice for the state $|\phi\rangle$ by $\rho^{A}$ :

$$
\begin{equation*}
\rho^{A}=\operatorname{Tr}_{B}(|\phi\rangle\langle\phi|) \tag{10}
\end{equation*}
$$

Since we know $|\phi\rangle$ precisely, we can compute $\rho^{A}$ precisely, and in particular, its spectrum. It is easy to verify that the spectrum of $\rho^{A}$ is

$$
\mathcal{S}\left(\rho^{A}\right)=(\underbrace{1 / K, 1 / K, \ldots 1 / K}_{K}, \underbrace{0,0, \ldots, 0}_{(N-1) K})
$$

So the rank of $\rho^{A}$ (which is also the Schmidt Number of $|\phi\rangle$ ) is $K$.
We focus on how $\rho^{A}$ changes with the local operations Alice performs (apparently it doesn't change with Bob's local operations): we shall prove that the rank of $\rho^{A}$ never increases. There are 3 types of operations Alice can perform: unitary operations, local measurements, and tracing out a subsystem, we analyze them one by one:

## - Unitary Operations

This operation changes a mixed state $\rho^{A}$ to $U \rho^{A} U^{\dagger}$, where $U$ is a unitary operation. Obviously the rank doesn't change.

## - Local Measurements

Suppose measurement operator is $\left\{M_{m}\right\}$ satisfying $\sum_{m} M_{m}^{\dagger} M_{m}=I$, and the measurement yields result $m$. Then Alice ends in state

$$
\rho_{m}=\frac{M_{m} \rho^{A} M_{m}^{\dagger}}{\operatorname{Tr}\left(M_{m}^{\dagger} M_{m} \rho^{A}\right)}
$$

Again, we have $\operatorname{rank}\left(\rho_{m}\right) \leq \operatorname{rank}\left(\rho^{A}\right)$.

## - Tracing Out a Subsystem

We write $\mathcal{H}^{A}=\mathcal{H}^{A_{0}} \otimes \mathcal{H}^{A_{1}}$, and we suppose that the subsystem $\mathcal{H}^{A_{1}}$ is traced out. We write the partial density matrix for $\mathcal{H}^{A_{0}}$ as $\rho^{A_{0}}$, and we have $\rho^{A_{0}}=\operatorname{Tr}_{A_{1}}\left(\rho^{A}\right)$.
We know that in protocol $\mathcal{P}^{\prime}$, the subsystem $\mathcal{H}^{A_{0}}$ is disentangled from the subsystem $\mathcal{H}^{A_{1}}$. Thus we have

$$
\rho^{A}=\rho^{A_{0}} \otimes \rho^{A_{1}}
$$

for some density matrix $\rho^{A_{1}}$. and by Corollary [1] we have rank $\left(\rho^{A_{0}}\right) \leq \operatorname{rank}\left(\rho^{A}\right)$.
So, as Alice and Bob perform local operations, the rank of the partial density matrix for Alice never increases. This fact remains true even if Alice and Bob perform classical communications (this just means that Alice has the ability to perform different local operations according to Bob's measurement result, but no local operation Alice performs can increase the rank).

We denote the density matrix for the final state after the protocol $\mathcal{P}$ to be $\rho_{E}$, and we define $\rho_{E}^{A}=\operatorname{Tr}_{B}\left(\rho_{E}\right)$ to be the partial density matrix for Alice. Then we have $\operatorname{rank}\left(\rho_{E}^{A}\right) \leq K$. Notice $\rho_{E}^{A}$ should be an $M \times M$ matrix since Alice and Bob are supposed to arrive at a state in $\mathcal{H}_{M}^{A} \otimes \mathcal{H}_{M}^{B}$. We use $\rho_{0}^{A}$ to denote the partial density matrix for Alice if we trace out the system $\mathcal{H}_{M}^{B}$ from the target state $\Psi_{M}$. It is easy to verify that $\rho_{0}^{A}=\frac{1}{M} I$, where $I$ is the identity matrix.

By monotonicity of fidelity, we have

$$
F\left(\rho_{E},\left|\Psi_{M}\right\rangle\left\langle\Psi_{M}\right|\right) \leq F\left(\rho_{E}^{A}, \rho_{0}^{A}\right)
$$

However, we have

$$
\begin{aligned}
F\left(\rho_{E}^{A}, \rho_{0}^{A}\right) & =\operatorname{Tr} \sqrt{\left(\rho_{E}^{A}\right)^{1 / 2} \rho_{0}^{A}\left(\rho_{E}^{A}\right)^{1 / 2}} \\
& =\sqrt{\frac{1}{M}} \cdot \operatorname{Tr} \sqrt{\rho_{E}^{A}}
\end{aligned}
$$

We write the spectrum of $\rho_{E}^{A}$ as

$$
\mathcal{S}\left(\rho_{E}^{A}\right)=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{M}\right)
$$

and we know that $\lambda_{K+1}=\lambda_{K+2}=\cdots=\lambda_{M}=0 \operatorname{since} \operatorname{rank}\left(\rho_{E}^{A}\right) \leq K$. Therefore, we have

$$
\operatorname{Tr} \sqrt{\rho_{E}^{A}}=\sum_{l=1}^{M} \sqrt{\lambda_{l}}=\sum_{l=1}^{K} \sqrt{\lambda_{l}} \leq \sqrt{K} \cdot\left(\sum_{l=1}^{K} \lambda_{l}\right)=\sqrt{K}
$$

and thus

$$
F\left(\rho_{E}^{A}, \rho_{0}^{A}\right)=\sqrt{\frac{1}{M}} \cdot \operatorname{Tr} \sqrt{\rho_{E}^{A}} \leq \sqrt{\frac{K}{M}}
$$

Therefore we have

$$
F\left(\rho_{E}\right)=F\left(\rho_{E},\left|\Psi_{M}\right\rangle\left\langle\Psi_{M}\right|\right) \leq F\left(\rho_{E}^{A}, \rho_{0}^{A}\right) \leq \frac{K}{M}
$$

## C Constructions of Deterministically Conditionally Successful GEPPs

We describe the construction of the "Simple Scrambling" protocol. This protocol is deterministically conditionally successful GEPP for input states in the diagonal subspace.

The construction relies on a special family of permutations, namely the Scrambling Permutations. We give a definition of Scrambling Permutations first, and postpone the actual construction of these permutations to the Appendix $\mathbb{E}$.

## C. 1 Scrambling Permutations

We define a class of permutations that would be useful in constructing the Simple Scrambling protocol. We work on functions over binary strings, and we use $x \circ y$ to denote string $x$ concatenated with string $y$. For finite sets $A$ and $B$ of binary strings, we define the concatenation of $A$ and $B$ to be set

$$
A \circ B=\{a \circ b \mid a \in A, b \in B\}
$$

We will be working with 4 finite sets of binary strings, and we call them $X, Y, G$, and $H$. These sets have the property that $X=G \circ H$. We define $N=|X|, K=|Y|, W=|H|, L=|G|$, and we will be using this size convention for the rest of this paper. Obviously we have $N=W L$.

Definition 5 (Scrambling Permutation) A class of parameterized function pairs $\left\langle g_{y}(x), h_{y}(x)\right\rangle$ of types $g_{y}: X \mapsto G$ and $h_{y}: X \mapsto H$ is called $a$ scrambling permutation pair of parameter ( $N, K, W, L$ ), or simply scrambling permutation, if the following 2 conditions are satisfied:

1. (Permutation) For all $y \in Y, x \longmapsto g_{y}(x) \circ h_{y}(x)$ is a permutation in $X$.
2. (Scrambling) There exists a positive number $p$, such that or any pair of elements $x_{1} \neq x_{2}$ in $X$,

$$
\operatorname{Prob}_{y}\left[h_{y}\left(x_{1}\right)=h_{y}\left(x_{2}\right)\right]=p
$$

where the probability is taken over the $y$ uniformly chosen from $Y$. We call this $p$ the "collision probability".
Furthermore, the pair will be called efficient scrambling permutation pair if both the function $g_{y}(x) \circ h_{y}(x)$ and its inverse can be efficiently computed (i.e., has polynomial-size circuits).

It is interesting to compare the definition of scrambling permutations to that of universal hash functions CW79, WC81. On one hand, the scrambling permutations are permutations, while the universal hash functions don't have to be. On the other hand, the "scrambling" property in the scrambling permutation is weaker than that of the universal hash functions. For scrambling permutations, the function $h_{y}(x)$ only need to have a constant collision probability for all pairs $\left(x_{1}, x_{2}\right)$. However, for universal hash functions, the setting is that $\operatorname{Prob}_{y}\left[h_{y}\left(x_{1}\right)=a \wedge h_{y}\left(x_{2}\right)=b\right]$ is the same for all $\left(x_{1}, x_{2}, a, b\right)$ tuples. Obviously, any universal hash function that can be extended to a permutation will induce a scrambling permutation. An example is the linear map construction $\left(f_{a, b}(x)=a \cdot x+b\right.$, see MR95, page 219, or L96], page 85). However, there exist more efficient constructions of scrambling permutations - We postpone the detailed construction and discussion to Appendix $\mathbb{E}$, and we just state the results here:

Theorem 4 There exist efficient scrambling permutations with the following parameters:

- $\left\langle 2^{n},\left(2^{n}-1\right), 2^{n-t}, 2^{t}\right\rangle$
- $\left\langle 2^{2 n},\left(2^{n}+1\right) 2^{2 t}, 2^{n}, 2^{n}\right\rangle$
- $\left\langle 2^{d n}, \frac{2^{d n}-1}{2^{n}-1} \cdot 2^{2 t}, 2^{(d-1) n}, 2^{n}\right\rangle$
where $n, t, d$ are integers such that $n>t$.
A useful fact about Scrambling Permutation is that the collision probability $p$ can be computed.
Theorem 5 Let $\left\langle g_{y}(x), h_{y}(x)\right\rangle$ be a scrambling permutation pairs of parameter $\langle N, K, W, L\rangle$. The collision probability $p$ equals $(L-1) /(N-1)$.

Proof: We call a triple $\left\langle x_{1}, x_{2}, y\right\rangle$ a "collision instance", if $h_{y}\left(x_{1}\right)=h_{y}\left(x_{2}\right)$. Now we count how many such collision instances there are. There are two ways to count them.

- For each $\left(x_{1}, x_{2}\right)$ pair, there are $K \cdot p y$ 's such that $h_{y}\left(x_{1}\right)=h_{y}\left(x_{2}\right)$. So the total number of collision instances is

$$
K p \cdot N(N-1) / 2
$$

- For each fixed $h_{y}(\cdot)$, it is a function that maps $X$ of size $N$ to $H$ of size $W$. Since $g_{y} \cdot h_{y}$ is a permutation, the mapping $h_{y}(x)$ has to be an "even" one: for each $u \in H$, there must be precisely $L$ elements in $X$ that are mapped to $u$. So the $N$ elements in $X$ are partitioned into $W$ subsets, each of size $L$. The number of pairs that are in the same subset is therefore $W \cdot L(L-1) / 2$. So the number of collision instances is

$$
K \cdot W \cdot L(L-1) / 2
$$

The two ways should give the same result. Thus we have

$$
K p \cdot N(N-1) / 2=K \cdot W \cdot L(L-1) / 2
$$

or

$$
p=\frac{L-1}{N-1}
$$

## C. 2 The Construction of the Simple Scrambling Protocol

We describe the construction of the Simple Scrambling protocol, which is deterministically conditional successful for input states in the diagonal subspace.

First, we recall the definitions of the Fourier Operator and the Hadamard Operator.
For a Hilbert space of dimension $N$, the Fourier Operator is defined by the matrix $F$ where the $(x, y)$-th entry of $F$ is $\frac{1}{\sqrt{N}} \omega^{-x \cdot y}$, for $x, y=0,1, \ldots, N-1$, where $\omega=e^{i \frac{2 \pi}{N}}$ is a root of the unity, and $x \cdot y$ denotes the integer multiplication. $F$ is a unitary operator. We call its inverse, $F^{\dagger}$, the Inverse Fourier Operator.

For a Hilbert space of dimension $N=2^{n}$, the Hadamard Operator is defined by the matrix $H$ where the $(x, y)$-th entry of $H$ is $\frac{1}{\sqrt{N}}(-1)^{x \bullet y}$, where $x \bullet y$ is the inner product of $x$ and $y$, for $x, y=0,1, \ldots, N-1$.

This protocol is parameterized by 4 integers: $N, K, W, L$, such that there exists a scrambling permutation pair $\left\langle g_{y}(x), h_{y}(x)\right\rangle$ of parameter $(N, K, W, L)$.

Construction 2 (Simple Scrambling Protocol) The input to the protocol is a state $\rho$ in $\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$. The protocol also has an auxiliary input of $\Psi_{K}$. The steps are:

1. Alice and Bob both apply the scrambling permutation to their qubits: using the qubits from $\rho$ as $x$ and the qubits from $\Phi_{K}$ as $y$, and outputs the values of both functions and $y$ :

$$
\begin{equation*}
|x\rangle|y\rangle \longrightarrow\left|g_{y}(x)\right\rangle\left|h_{y}(x)\right\rangle|y\rangle \tag{11}
\end{equation*}
$$

Here we identify the Hilbert space $\mathcal{H}_{N} \otimes \mathcal{H}_{K}$ with $\mathcal{H}_{L} \otimes \mathcal{H}_{W} \otimes \mathcal{H}_{K}$.
2. Alice applies the Fourier Operator to the state $\left|g_{y}(x)\right\rangle^{A}$, and Bob applies the Inverse Fourier Operator to the state $\left|g_{y}(x)\right\rangle^{B}$. Then both measure these qubits in the computational basis.
In the case that $L$ is a power of 2, Alice and Bob can, alternatively, both apply a Hadamard operator to their states of $\left|g_{y}(x)\right\rangle$, instead of the Fourier and Inverse Fourier operators.
3. Alice and Bob compare their results via classical communication.
4. If the results are the same, they discard the measured state (or equivalently, trace out the subspace $\left.\mathcal{H}_{L}\right)$, and output the remaining state, which is in Hilbert space $\mathcal{H}_{W K}^{A} \otimes \mathcal{H}_{W K}^{B}$.
5. If the results are different, they discard everything and output FAIL.

We point out that this protocol can be implemented by LOCC. In step 1, both Alice and Bob apply a scrambling permutation to their state. It is easy to verify that the mapping in Equation 11 is a permutation and thus is possible to realize quantum-mechanically. Next, if the scrambling permutation is efficient, there exists a polynomial-size quantum circuit that implements it [L01]. In step 2, Fourier Operators and Inverse Fourier Operators are applied by Alice and Bob, respectively. Fourier Operators exist for every $N$ and when $N$ is a power of 2, there exists an efficient implementation of both the Fourier Operators and Inverse Fourier Operators. Also, in the case $N$ is a power of 2, there exists a very efficient algorithm for performing Hadamard operators. Therefore we have:

Claim 2 The simple scrambling protocol can be implemented by LOCC. Furthermore, if the scrambling permutation used in the protocol is efficient and $L$ is power of 2, the protocol can be efficiently implemented.

## C. 3 The Analysis of the Simple Scrambling Protocol

Now we prove a lemma that the Simple Scrambling protocol is deterministically conditionally successful for input states that are pure states in the diagonal subspace.

Lemma 4 If the input state to a Simple Scrambling protocol is a pure state in the diagonal subspace, then this protocol is deterministically conditionally successful with parameter $\left\langle N, K, W K, \epsilon, \frac{2 W}{N} \epsilon, \epsilon\right\rangle$ for $\epsilon<1 / 2$.

Intuitively, this lemma is true because the Scrambling Permutation "shuffles" the coefficients of $|\phi\rangle$ very "evenly". Then the Fourier operator (or the Hadamard operator) "mixes" all the coefficients together. Therefore when the protocol doesn't fail, the coefficients of the output state are much more "smooth" than that of $|\phi\rangle$.

Proof: We write the input state $|\phi\rangle$ as

$$
\begin{equation*}
|\phi\rangle=\sum_{x \in X} \alpha_{x}|x\rangle^{A}|x\rangle^{B} \tag{12}
\end{equation*}
$$

and we have that

$$
\sum_{x \in X}\left|\alpha_{x}\right|^{2}=1
$$

We denote $\sum_{x \in X} \alpha_{x}$ by $D$. Then we have

$$
\begin{equation*}
1-\epsilon=\left|\left\langle\phi \mid \Psi_{N}\right\rangle\right|^{2}=\frac{1}{N} \cdot\left|\sum_{x \in X} \alpha_{x}\right|^{2}=\frac{|D|^{2}}{N} \tag{13}
\end{equation*}
$$

We will go through the protocol and keep track of the state.

1. The initial state for Alice and Bob is

$$
\begin{equation*}
\left|\psi_{1}\right\rangle=|\phi\rangle \otimes \Psi_{K}=\frac{1}{\sqrt{K}} \sum_{x \in X} \sum_{y \in Y} \alpha_{x}|x \circ y\rangle^{A}|x \circ y\rangle^{B} \tag{14}
\end{equation*}
$$

2. After applying the scrambling permutation, the state becomes

$$
\begin{equation*}
\left|\psi_{2}\right\rangle=\frac{1}{\sqrt{K}} \sum_{x \in X} \sum_{y \in Y} \alpha_{x} \cdot\left|g_{y}(x) \circ h_{y}(x) \circ y\right\rangle^{A} \cdot\left|g_{y}(x) \circ h_{y}(x) \circ y\right\rangle^{B} \tag{15}
\end{equation*}
$$

3. After the Fourier and Inverse Fourier operators, the state is

$$
\begin{equation*}
\left|\psi_{3}\right\rangle=\frac{1}{L \sqrt{K}} \sum_{x \in X} \sum_{y \in Y} \sum_{g_{A} \in G} \sum_{g_{B} \in G} \omega^{g_{y}(x) \cdot\left(g_{B}-g_{A}\right)} \cdot \alpha_{x} \cdot\left|g_{A} \circ h_{y}(x) \circ y\right\rangle^{A} \cdot\left|g_{B} \circ h_{y}(x) \circ y\right\rangle^{B} \tag{16}
\end{equation*}
$$

Alternatively, if $L$ is a power of 2 , and Hadamard operators are used instead of Fourier and Inverse Fourier operators, the state is

$$
\begin{equation*}
\left|\psi_{3}^{\prime}\right\rangle=\frac{1}{L \sqrt{K}} \sum_{x \in X} \sum_{y \in Y} \sum_{g_{A} \in G} \sum_{g_{B} \in G}(-1)^{g_{y}(x) \bullet\left(g_{A} \oplus g_{B}\right)} \cdot \alpha_{x} \cdot\left|g_{A} \circ h_{y}(x) \circ y\right\rangle^{A} \cdot\left|g_{B} \circ h_{y}(x) \circ y\right\rangle^{B} \tag{17}
\end{equation*}
$$

In either cases, if Alice and Bob both measure their qubits and they both obtain the result $g$, the state becomes

$$
\begin{equation*}
\left|\psi_{4, g}\right\rangle=\frac{\Delta}{L \sqrt{K}} \sum_{x \in X} \sum_{y \in Y} \alpha_{x} \cdot\left|g \circ h_{y}(x) \circ y\right\rangle^{A} \cdot\left|g \circ h_{y}(x) \circ y\right\rangle^{B} \tag{18}
\end{equation*}
$$

where $\Delta$ is a normalization factor. Notice that if Alice and Bob both discard the qubits $|g\rangle^{A}$ and $|g\rangle^{B}$ (which are disentangled from the rest), the resultant state is the same for different $|g\rangle^{\prime}$ 's:

$$
\begin{equation*}
\left|\psi_{5}\right\rangle=\frac{\Delta}{L \sqrt{K}} \sum_{x \in X} \sum_{y \in Y} \alpha_{x} \cdot\left|h_{y}(x) \circ y\right\rangle^{A} \cdot\left|h_{y}(x) \circ y\right\rangle^{B} \tag{19}
\end{equation*}
$$

Now let's compute $\Delta$.
To normalize $\left|\psi_{5}\right\rangle$, we can re-group the terms and re-write it as

$$
\begin{equation*}
\left|\psi_{5}\right\rangle=\frac{\Delta}{L \sqrt{K}} \sum_{u \in H} \sum_{y \in Y}\left(\sum_{h_{y}(x)=u} \alpha_{x}\right) \cdot|u \circ y\rangle^{A} \cdot|u \circ y\rangle^{B} \tag{20}
\end{equation*}
$$

Therefore we should have

$$
\frac{\Delta^{2}}{L^{2} K} \sum_{u \in H} \sum_{y \in Y}\left|\sum_{h_{y}(x)=u} \alpha_{x}\right|^{2}=1
$$

Furthermore, we have

$$
\begin{aligned}
\sum_{u \in H} \sum_{y \in Y}\left|\sum_{h_{y}(x)=u} \alpha_{x}\right|^{2} & =\sum_{y \in Y} \sum_{u \in H}\left|\sum_{h_{y}(x)=u} \alpha_{x}\right|^{2} \\
& =\sum_{y \in Y} \sum_{x \in X}\left|\alpha_{x}\right|^{2}+\sum_{y \in Y} \sum_{x_{1} \neq x_{2}, h_{y}\left(x_{1}\right)=h_{y}\left(x_{2}\right)}\left(\alpha_{x_{1}} \bar{\alpha}_{x_{2}}+\bar{\alpha}_{x_{1}} \alpha_{x_{2}}\right) \\
& =\sum_{y \in Y} 1+\sum_{x_{1} \neq x_{2}} \sum_{y \in Y: h_{y}\left(x_{1}\right)=h_{y}\left(x_{2}\right)}\left(\alpha_{x_{1}} \bar{\alpha}_{x_{2}}+\bar{\alpha}_{x_{1}} \alpha_{x_{2}}\right) \\
& =K+\sum_{x_{1} \neq x_{2}} p K\left(\alpha_{x_{1}} \bar{\alpha}_{x_{2}}+\bar{\alpha}_{x_{1}} \alpha_{x_{2}}\right) \\
& =K+p K\left(\left|\sum_{x \in X} \alpha_{x}\right|^{2}-\sum_{x \in X}\left|\alpha_{x}\right|^{2}\right) \\
& =K\left[1+p\left(|D|^{2}-1\right)\right]
\end{aligned}
$$

and thus

$$
\begin{aligned}
\Delta^{2} & =\frac{L^{2} K}{K\left[1+p\left(|D|^{2}-1\right)\right]} \\
& =\frac{L^{2}}{1+p\left(|D|^{2}-1\right)} \\
& =\frac{L^{2}}{1+\frac{L-1}{N-1}(N-1-N \epsilon)} \\
& =\frac{L}{1-\epsilon \cdot \frac{N(L-1)}{L(N-1)}}
\end{aligned}
$$

Notice $\Delta^{2}$ is the probability that Alice and Bob both obtain $|g\rangle$ for their measurement. There are $L$ possible $|g\rangle$ 's that Alice and Bob can obtain. So the probability that Alice and Bob obtain the same result is

Prob [Alice and Bob obtain the same result] $=\frac{L}{\Delta^{2}}=1-\epsilon \cdot \frac{N(L-1)}{L(N-1)} \geq 1-\epsilon$

And the fidelity of $\left|\psi_{5}\right\rangle$ is

$$
\begin{aligned}
F\left(\left|\psi_{5}\right\rangle\left\langle\psi_{5}\right|\right) & =\left|\left\langle\psi_{5} \mid \Psi_{W K}\right\rangle\right|^{2} \\
& =\left|\frac{1}{\sqrt{W K}} \cdot \frac{\Delta}{L \sqrt{K}} \sum_{u \in H} \sum_{y \in Y}\left(\sum_{h_{y}(x)=u} \alpha_{x}\right)\right|^{2} \\
& =\frac{\Delta^{2}}{L^{2} K^{2} W}\left|\sum_{u \in H} \sum_{y \in Y} \sum_{h_{y}(x)=u} \alpha_{x}\right|^{2} \\
& =\frac{\Delta^{2}}{L^{2} K^{2} W}\left|\sum_{x \in X} \sum_{y \in Y} \alpha_{x}\right|^{2} \\
& =\frac{\Delta^{2}}{L^{2} K^{2} W} \cdot|K D|^{2} \\
& =\frac{(1-\epsilon) \cdot \Delta^{2}}{L} \\
& =\frac{1-\epsilon}{1-\epsilon \cdot \frac{N(L-1)}{L(N-1)}} \\
& =1-\epsilon \cdot \frac{N-L}{L(N-1)} \cdot \frac{1}{1-\epsilon \cdot \frac{N(L-1)}{L(N-1)}} \\
& \geq 1-2 \epsilon \cdot \frac{N-L}{L(N-1)} \\
& =1-2 \epsilon \cdot \frac{W-1}{N-1} \\
& \geq 1-\frac{2 W}{N} \epsilon
\end{aligned}
$$

Therefore, if the input state $|\phi\rangle$ has a sufficiently high fidelity, then with high probability the Simple Scrambling protocol will succeed, and the resultant state, which is also a pure state, can have a much higher fidelity.

Next, we prove that the Simple Scrambling protocol works without modification for mixed states in Diagonal Subspaces, with exactly the same parameters.

Lemma 5 If the input state to a Simple Scrambling protocol is a mixed state in the diagonal subspace, then this protocol is deterministically conditionally successful with parameter $\left\langle N, K, W K, \epsilon, \frac{2 W}{N} \epsilon, \epsilon\right\rangle$ for $\epsilon<1 / 2$.

Proof: We write the mixed state as an ensemble: $\left\{p_{i},\left|\phi_{i}\right\rangle\right\}_{i=1,2, \ldots s}$. We use $1-\epsilon_{i}$ to denote the fidelity of pure state $\phi_{i}$. Then we have $\epsilon=\sum_{i} p_{i} \cdot \epsilon_{i}$ by the linearity of fidelity.

As in the proof to Lemma 团, the protocol will succeed with probability $1-\epsilon_{i} \cdot \frac{N(L-1)}{L(N-1)}$ for state $\left|\phi_{i}\right\rangle$. So the overall probability that the protocol doesn't fail is

$$
\begin{aligned}
\operatorname{Prob}\left[\mathcal{P}\left(\left|\phi_{i}\right\rangle\left\langle\phi_{i}\right|\right)=\text { FAIL }\right] & =\sum_{i} p_{i} \cdot \epsilon_{i} \cdot \frac{N(L-1)}{L(N-1)} \\
& =\frac{N(L-1)}{L(N-1)} \cdot \sum_{i} p_{i} \cdot \epsilon_{i}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{N(L-1)}{L(N-1)} \cdot \epsilon \\
& \leq \epsilon
\end{aligned}
$$

If we use $Q_{i}$ to denote the fidelity of the output of the protocol on state $\left|\phi_{i}\right\rangle$ conditioned on that it doesn't fail, then the overall fidelity of the output if the protocol doesn't fail is:

$$
\begin{aligned}
Q & =\frac{\sum_{i} Q_{i} \cdot p_{i} \cdot \operatorname{Prob}\left[\mathcal{P}\left(\left|\phi_{i}\right\rangle\left\langle\phi_{i}\right|\right)=\text { FAIL }\right]}{\sum_{i} p_{i} \cdot \operatorname{Prob}\left[\mathcal{P}\left(\left|\phi_{i}\right\rangle\left\langle\phi_{i}\right|\right)=\text { FAIL }\right]} \\
& =\frac{\sum_{i} \frac{1-\epsilon_{i}}{1-\epsilon_{i} \cdot \frac{N(L-1)}{L(N-1)}} \cdot p_{i} \cdot\left[1-\epsilon_{i} \cdot \frac{N(L-1)}{L(N-1)}\right]}{\sum_{i} p_{i} \cdot\left[1-\epsilon_{i} \cdot \frac{N(L-1)}{L(N-1)}\right]} \\
& =\frac{\sum_{i}\left(1-\epsilon_{i}\right) \cdot p_{i}}{\sum_{i}\left(1-\frac{N(L-1)}{L(N-1)} \cdot \epsilon_{i}\right) \cdot p_{i}} \\
& =\frac{1-\epsilon}{1-\epsilon \cdot \frac{N(L-1)}{L(N-1)}} \\
& \geq 1-\frac{2 W}{N} \epsilon
\end{aligned}
$$

So the Simple Scrambling protocol is deterministically conditionally successful even for mixed states in the diagonal subspace.

Now we are ready to prove Theorem 2 .
Proof: [Proof to Theorem 2] We simply combine Theorem 1 and Lemma 5.

## D Toward the Diagonal Subspace: The Hash and Compare Protocol

In this section, we present the construction of the Hash and Compare protocol. With high probability, this protocol converts any state of reasonably high fidelity into a state that is "almost completely" in the diagonal subspace. Therefore, if we combine this protocol with the Simple Scrambling protocol, we obtain a GEPP that is probabilistically conditionally successful for arbitrary states.

Before describing the actual protocol, we give some motivations and intuitions behind it. Suppose Alice and Bob share a state of fidelity at least $1-\epsilon$. For simplicity, we assume that the state is a pure state, and we will show how to extend our result to mixed states later. We write the input state as $|\phi\rangle$. In this situation, the Simple Scrambling protocol doesn't work anymore. Essentially what this protocol does is to "shuffle" and "mix" the coefficients in the diagonal subspace in a very "even" way to increase the fidelity. The Scrambling Permutation guarantees that coefficients in the diagonal subspace will be mixed "evenly". However it gives no guarantee for coefficients outside this subspace. Nevertheless, it is worth noting that the maximally entangled state, $\Psi_{N}$, is completely in the diagonal subspace. So if $|\phi\rangle$ is close to $\Psi_{N}$, then a large "fraction" of $|\phi\rangle$ must lie in the diagonal subspace.

We write

$$
\begin{equation*}
|\phi\rangle=\alpha \cdot\left|\phi_{\|}\right\rangle+\beta \cdot\left|\phi_{\perp}\right\rangle \tag{21}
\end{equation*}
$$

where $\left|\phi_{\|}\right\rangle$is a vector in the diagonal subspace $\mathcal{H}^{\mathcal{D}}$ and $\left|\phi_{\perp}\right\rangle$ is a vector orthogonal to $\mathcal{H}^{\mathcal{D}}$ subspace. Both vectors are normalized, and thus we have $|\alpha|^{2}+|\beta|^{2}=1$. Obviously we have $\left\langle\phi_{\perp} \mid \Psi_{N}\right\rangle=0$ and thus $|\alpha|^{2} \geq 1-\epsilon$.

The Simple Scrambling protocol works well for state $\left|\phi_{\|}\right\rangle$, but does not work for state $\left|\phi_{\perp}\right\rangle$. So if we can first "eliminate" $\left|\phi_{\perp}\right\rangle$, or at least decrease its coefficient from $\beta$ to a much smaller one, we can use the Simple Scrambling protocol to obtain a state with high fidelity. The Hash and Compare protocol does exactly this.

Construction 3 (Hash and Compare) The input to the protocol is a state $\rho$ in the subspace $\mathcal{H}_{N}^{A} \otimes$ $\mathcal{H}_{N}^{B}$. The protocol also has an auxiliary input $\Psi_{S}$, where $S=2^{s}$ is a power of 2. The output of the protocol is a state $\sigma$ in $\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$. The steps are:

1. Alice randomly generates $s$ numbers $r_{0}, r_{1}, \ldots, r_{s-1} \in[N]$ and introduces $s$ ancillary qubits, $\left|b_{0}\right\rangle,\left|b_{1}\right\rangle, \ldots,\left|b_{s-1}\right\rangle$ all initialized to $|0\rangle$.
2. Alice performs s unitary operations:

$$
|x\rangle\left|y_{j}\right\rangle \longrightarrow|x\rangle\left|y_{j} \oplus\left(x \bullet r_{j}\right)\right\rangle
$$

She uses the qubits from state $\rho$ as $x$, and the ancillary qubit $\left|b_{j}\right\rangle$ as $y_{j}$, for $j=0,1, \ldots, s-1$.
3. Alice send $r_{0}, r_{1}, \ldots, r_{s-1}$ to Bob.
4. Alice and Bob engage in s teleportation protocols. They use the shared state $\Psi_{S}$ as $s$ EPR pairs, to teleport the s ancillary qubits $\left|b_{0}\right\rangle^{A},\left|b_{1}\right\rangle^{A}, \ldots,\left|b_{s-1}\right\rangle^{A}$ from Alice to Bob. Then Alice discards all her ancillary qubits. Bob obtains the qubits $\left|b_{0}\right\rangle^{B},\left|b_{1}\right\rangle^{B}, \ldots,\left|b_{s-1}\right\rangle^{B}$.
5. Bob performs s unitary operations (the same operations as Alice did):

$$
|x\rangle\left|y_{j}\right\rangle \longrightarrow|x\rangle\left|y_{j} \oplus\left(x \bullet r_{j}\right)\right\rangle
$$

He uses the qubits from state $|\phi\rangle$ as $x$, and qubit $\left|b_{j}\right\rangle^{B}$ as $y_{j}$, for $j=0,1, \ldots, s-1$.
6. Bob measures all his ancillary bits $\left|b_{0}\right\rangle^{B},\left|b_{1}\right\rangle^{B}, \ldots,\left|b_{s-1}\right\rangle^{B}$.
7. If all the results of the measurements are 0, Bob discards all the ancillary qubits. Then Alice and Bob output the remaining state, which is in Hilbert space $\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$.
8. If not all the results of the measurements are 0, Alice and Bob discard everything and output FAIL.

We point out that the Hash and Compare protocol can be efficiently implemented.
Now we prove that the Hash and Compare protocol will bring the input state $|\phi\rangle$ to another state that is "almost" in the diagonal subspace.

First, we extend the definition of fidelity. We define the fidelity between a pure state $|\varphi\rangle$ and a linear subspace $L$ to be the square of the length of the projection of $|\varphi\rangle$ on $L$. Alternatively, we have

$$
\begin{equation*}
F(|\varphi\rangle, L)=\max _{|\psi\rangle \in L}|\langle\varphi \mid \psi\rangle|^{2} \tag{22}
\end{equation*}
$$

Now we state and prove our lemma about the Hash and Compare protocol.
Lemma 6 Let state $|\phi\rangle$ be a pure state of fidelity at least $1-\epsilon$, where $\epsilon<1 / 2$. If $|\phi\rangle$ is the input state to the Hash and Compare protocols, then the probability this protocol outputs FAIL is at most $\epsilon$. Given that the protocol doesn't fail, we use $|\psi\rangle$ to denote the output state, which is a pure state. We have $F(|\psi\rangle\langle\psi|) \geq 1-\epsilon$, and

$$
\operatorname{Prob}\left[F\left(|\psi\rangle, \mathcal{H}^{\mathcal{D}}\right) \geq 1-\frac{2}{\sqrt{S}} \epsilon\right] \geq 1-\frac{1}{\sqrt{S}}
$$

Proof: We write the state $|\phi\rangle$ as

$$
\begin{equation*}
|\phi\rangle=\sum_{x_{A} \in X} \sum_{x_{B} \in X} \alpha_{x_{A}, x_{B}}\left|x_{A}\right\rangle^{A}\left|x_{B}\right\rangle^{B} \tag{23}
\end{equation*}
$$

and we have

$$
\sum_{x_{A} \in X} \sum_{x_{B} \in X}\left|\alpha_{x_{A}, x_{B}}\right|^{2}=1
$$

Comparing this to Equation 21, we conclude that

$$
\begin{aligned}
|\alpha|^{2} & =\sum_{x \in X}\left|\alpha_{x, x}\right|^{2} \\
|\beta|^{2} & =\sum_{x_{A} \neq x_{B}}\left|\alpha_{x_{A}, x_{B}}\right|^{2}
\end{aligned}
$$

We go through the protocol:

1. The initial state for Alice and Bob, excluding the auxiliary input $\Psi_{T}$ is:

$$
\left|\phi_{1}\right\rangle=\sum_{x_{A} \in X} \sum_{x_{B} \in X} \alpha_{x_{A}, x_{B}}\left|x_{A}\right\rangle^{A}\left|x_{B}\right\rangle^{B}
$$

2. After Alice introduces her ancillary qubits and done with the $t$ unitary operations, the state is:

$$
\begin{equation*}
\left|\phi_{2}\right\rangle=\sum_{x_{A} \in X} \sum_{x_{B} \in X} \alpha_{x_{A}, x_{B}}\left|x_{A}\right\rangle^{A}\left|x_{A} \bullet r_{0}\right\rangle^{A}\left|x_{A} \bullet r_{1}\right\rangle^{A} \cdots\left|x_{A} \bullet r_{s-1}\right\rangle^{A}\left|x_{B}\right\rangle^{B} \tag{24}
\end{equation*}
$$

as we can see, the ancillary qubits are entangled with the qubits from $|\phi\rangle$.
3. After the teleportation, Alice's ancillary qubits becomes disentangled from the qubits of $|\phi\rangle$, and after discarding all the ancillary qubits of Alice, the state becomes

$$
\begin{equation*}
\left|\phi_{3}\right\rangle=\sum_{x_{A} \in X} \sum_{x_{B} \in X} \alpha_{x_{A}, x_{B}}\left|x_{A}\right\rangle^{A}\left|x_{B}\right\rangle^{B}\left|x_{A} \bullet r_{0}\right\rangle^{B}\left|x_{A} \bullet r_{1}\right\rangle^{B} \cdots\left|x_{A} \bullet r_{s-1}\right\rangle^{B} \tag{25}
\end{equation*}
$$

4. After Bob has done with his unitary operations, the state becomes

$$
\begin{equation*}
\left|\phi_{4}\right\rangle=\sum_{x_{A} \in X} \sum_{x_{B} \in X} \alpha_{x_{A}, x_{B}}\left|x_{A}\right\rangle^{A}\left|x_{B}\right\rangle^{B}\left|\left(x_{A} \oplus x_{B}\right) \bullet r_{0}\right\rangle^{B}\left|\left(x_{A} \oplus x_{B}\right) \bullet r_{1}\right\rangle^{B} \cdots\left|\left(x_{A} \oplus x_{B}\right) \bullet r_{s-1}\right\rangle^{B} \tag{26}
\end{equation*}
$$

5. Next, Bob measures all his ancillary qubits. Now it should be clear that if the state Alice and Bob start with, $|\phi\rangle$, is indeed in the diagonal subspace, then all the measurements will yield 0 with probability one, since we have $x_{A}=x_{B}$ for all non-zero $\alpha_{x_{A}, x_{B}}$ 's.
Now that $|\phi\rangle$ is not in the diagonal subspace, but it is close. Thus intuitively, Bob should have a high probability getting all 0 's in his measurement.
We do a more formal analysis: we denote by $Z$ the subset of $[N]$ whose elements have inner product 0 with all $r_{0}, r_{1}, \ldots, r_{s-1}$ :

$$
Z=\left\{x \mid x \in[N], x \bullet r_{j}=0, j=0,1, \ldots, s-1\right\}
$$

We group all the terms in Equation 26 into 3 parts:

$$
\left|\phi_{4}\right\rangle=\lambda_{0} \cdot\left|\psi_{0}\right\rangle+\lambda_{1} \cdot\left|\psi_{1}\right\rangle+\lambda_{2} \cdot\left|\psi_{2}\right\rangle
$$

where

$$
\begin{aligned}
& \lambda_{0} \cdot\left|\psi_{0}\right\rangle=\sum_{x \in X} \alpha_{x, x} \cdot|x\rangle^{A}|x\rangle^{B}|0\rangle^{B} \cdots|0\rangle^{B} \\
& \lambda_{1} \cdot\left|\psi_{1}\right\rangle=\sum_{x_{A} \neq x_{B}, x_{A} \oplus x_{B} \in Z} \alpha_{x_{A}, x_{B}} \cdot\left|x_{A}\right\rangle^{A}\left|x_{B}\right\rangle^{B}|0\rangle^{B} \cdots|0\rangle^{B} \\
& \lambda_{2} \cdot\left|\psi_{2}\right\rangle=\sum_{x_{A} \neq x_{B}, x_{A} \oplus x_{B} \notin Z} \alpha_{x_{A}, x_{B}} \cdot\left|x_{A}\right\rangle^{A}\left|x_{B}\right\rangle^{B}\left|\left(x_{A} \oplus x_{B}\right) \bullet r_{0}\right\rangle^{B}\left|\left(x_{A} \oplus x_{B}\right) \bullet r_{1}\right\rangle^{B} \cdots\left|\left(x_{A} \oplus x_{B}\right) \bullet r_{s-1}\right\rangle
\end{aligned}
$$

Both $\left|\psi_{0}\right\rangle$ and $\left|\psi_{1}\right\rangle$ have all 0 's in the ancillary qubits of Bob, while $\left|\psi_{2}\right\rangle$ doesn't. All these 3 states, $\left|\psi_{0}\right\rangle,\left|\psi_{1}\right\rangle$ and $\left|\psi_{2}\right\rangle$ are orthogonal to each other.
We again write

$$
|\phi\rangle=\alpha \cdot\left|\phi_{\|}\right\rangle+\beta \cdot\left|\phi_{\perp}\right\rangle
$$

and we notice that $\lambda_{0}=\alpha$, and $\left|\psi_{0}\right\rangle=\left|\phi_{\|}\right\rangle \otimes\left|Z_{T}\right\rangle^{B}$
Therefore the probability that Bob obtains all-zero in the measurement is at least $\left|\lambda_{0}\right|^{2}=|\alpha|^{2} \geq$ $1-\epsilon$.
After the measurement, and if is result is indeed all-zero, the state will become

$$
\begin{equation*}
|\psi\rangle=\frac{1}{\sqrt{\left|\lambda_{0}\right|^{2}+\left|\lambda_{1}\right|^{2}}} \cdot\left(\lambda_{0}\left|\psi_{0}\right\rangle+\lambda_{1}\left|\psi_{1}\right\rangle\right) \tag{27}
\end{equation*}
$$

where $\left|\psi_{0}\right\rangle$ is in the diagonal subspace $\mathcal{H}^{\mathcal{D}}$ and $\left|\psi_{1}\right\rangle$ is orthogonal to the $\mathcal{H}^{\mathcal{D}}$. The fidelity of $|\psi\rangle$ and $\mathcal{H}^{\mathcal{D}}$ is $\frac{\left|\lambda_{0}\right|}{\sqrt{\left|\lambda_{0}\right|^{2}+\left|\lambda_{1}\right|^{2}}}$.
Now we can prove that the fidelity of $|\psi\rangle$ is at least $1-\epsilon$ :

$$
\begin{aligned}
\left\langle\psi \mid \Psi_{N}\right\rangle & =\frac{\lambda_{0}}{\sqrt{\left|\lambda_{0}\right|^{2}+\left|\lambda_{1}\right|^{2}}}\left\langle\psi_{0} \mid \Psi_{N}\right\rangle+\frac{\lambda_{1}}{\sqrt{\left|\lambda_{0}\right|^{2}+\left|\lambda_{1}\right|^{2}}}\left\langle\psi_{1} \mid \Psi_{N}\right\rangle \\
& =\frac{\left\langle\lambda_{0} \psi_{0} \mid \Psi_{N}\right\rangle}{\sqrt{\left|\lambda_{0}\right|^{2}+\left|\lambda_{1}\right|^{2}}} \\
& =\frac{1}{\sqrt{\left|\lambda_{0}\right|^{2}+\left|\lambda_{1}\right|^{2}}} \cdot \frac{\left|\sum_{x \in X} \alpha_{x, x}\right|}{\sqrt{N}} \\
& =\frac{\left\langle\phi \mid \Psi_{N}\right\rangle}{\sqrt{\left|\lambda_{0}\right|^{2}+\left|\lambda_{1}\right|^{2}}} \\
& \geq\left\langle\phi \mid \Psi_{N}\right\rangle \geq 1-\epsilon
\end{aligned}
$$

Essentially, the hash and compare protocol leaves the coefficients in diagonal subspace untouched, and eliminates part of the "off-diagonal" coefficients. Therefore, after the re-normalization, the coefficients in the diagonal subspace will not decrease, and thus the fidelity of the output state is at least $1-\epsilon$.
Now we estimate the magnitude of $\lambda_{1}$ : we have

$$
\left|\lambda_{1}\right|^{2}=\sum_{x_{A} \neq x_{B}, x_{A} \oplus x_{B} \in Z}\left|\alpha_{x_{A}, x_{B}}\right|^{2}
$$

Notice that $\lambda_{1}$ is actually a random variable since the $r_{0}, r_{1}, \ldots, r_{t-1}$ are randomly chosen by Alice. Notice that each pair $x_{A} \neq x_{B}$, we have

$$
\operatorname{Prob}_{r}\left[\left(x_{A} \oplus x_{B}\right) \bullet r=0\right]=1 / 2
$$

and thus for random $r_{0}, r_{1}, \ldots, r_{s-1}$, the probability that all $\left(x_{A} \oplus x_{B}\right) \bullet r_{j}$ results in 0 for $j=0,1, \ldots, s-1$, is $1 / 2^{s}$.

In other words, the expected value of $\left|\lambda_{1}\right|^{2}$ is

$$
\begin{aligned}
E\left[\left|\lambda_{1}\right|^{2}\right] & =E\left[\sum_{x_{A} \neq x_{B}, x_{A} \oplus x_{B} \in Z}\left|\alpha_{x_{A}, x_{B}}\right|^{2}\right] \\
& =\sum_{x_{A} \neq x_{B}} \operatorname{Prob}_{r_{0}, r_{1}, \ldots, r_{s-1}}\left[x_{A} \oplus x_{B} \in Z\right] \cdot\left|\alpha_{x_{A}, x_{B}}\right|^{2} \\
& =\frac{1}{2^{s}} \sum_{x_{A} \neq x_{B}}\left|\alpha_{x_{A}, x_{B}}\right|^{2} \\
& \leq \frac{\epsilon}{S}
\end{aligned}
$$

and thus by Markov Inequality, we have

$$
\operatorname{Prob}\left[\left|\lambda_{1}\right|^{2} \leq \frac{\epsilon}{\sqrt{S}}\right] \leq \frac{1}{\sqrt{S}}
$$

Therefore, with probability at least $1-\frac{1}{\sqrt{S}}$, we have $\left|\lambda_{1}\right|^{2} \leq \frac{\epsilon}{\sqrt{S}}$. In that case, the fidelity of $|\psi\rangle$ and the diagonal subspace is

$$
\begin{aligned}
F\left(|\psi\rangle, \mathcal{H}^{\mathcal{D}}\right) & =\frac{\left|\lambda_{0}\right|^{2}}{\left|\lambda_{0}\right|^{2}+\left|\lambda_{1}\right|^{2}} \\
& \geq \frac{1-\epsilon}{1-\epsilon+\frac{\epsilon}{\sqrt{S}}} \\
& =1-\frac{\epsilon}{\sqrt{S} \cdot\left(1-\epsilon+\frac{\epsilon}{\sqrt{S}}\right)} \\
& \geq 1-\frac{2 \epsilon}{\sqrt{S}}
\end{aligned}
$$

when $\epsilon<1 / 2$.
Now, we can put everything together: for a general state $\rho$, we first apply the Hash and Compare protocol to $\rho$ to make it "almost completely in" the diagonal subspace $\mathcal{H}^{\mathcal{D}}$. Then we apply the Simple Scrambling protocol to enhance the fidelity. We describe the complete protocol in more details:

Construction 4 (Complete Scrambling Protocol) The Complete Scrambling protocol is parameterized by a quintuple: $\langle N, K, W, L, S\rangle$, such that there exists a Scrambling Permutation pair $\left\langle g_{y}(x), h_{y}(x)\right\rangle$ of parameter $\langle N, K, W, L\rangle$, where $S$ is a power of 2. The input to the protocol is a (mixed) state $\rho$ in space $\mathcal{H}_{N}^{A} \otimes \mathcal{H}_{N}^{B}$. The protocol also has an auxiliary input $\Psi_{T}$, where $T=S \cdot K$. We can also write the auxiliary input $\Psi_{T}$ as $\Psi_{S} \otimes \Psi_{K}$. The steps are:

1. Alice and Bob engage in the Hash and Compare protocol, using the input state $\rho$ as the input, and part of the auxiliary input, $\Psi_{S}$ as the auxiliary input.
2. If the Hash and Compare protocol fails, Alice and Bob output FAIL and terminate.
3. If the Hash and Compare protocol succeeds, it will output a state $\sigma$. Alice and Bob then engage in the Simple Scrambling protocol, using $\sigma$ as the input and the other part of the auxiliary input, $\Psi_{K}$ as the auxiliary input.
4. If the Simple Scrambling protocol fails, Alice and Bob output FAIL and terminate.
5. If the Simple Scrambling protocol succeeds, a state $\tau$ will be output, and Alice and Bob output $\tau$.

It is obvious that the complete scrambling protocol can be realized quantum-mechanically, and if the scrambling permutation used in the protocol is an efficient one, and $L$ is a power of 2 , the protocol can be realized efficiently.

Lemma 7 The Complete Scrambling protocol is a probabilistic conditional successful GEPP with parameter $\left\langle N, S K, W K, \epsilon,\left(\frac{4 M}{N}+\frac{4}{\sqrt{S}}\right) \epsilon, 2 \epsilon+\sqrt{\frac{2 \epsilon}{\sqrt{S}}}, \frac{1}{\sqrt{S}}\right\rangle$. If the Simple Scrambling protocol used inside the complete protocol is efficient, then so is the complete protocol.

Proof: To prove this lemma we need some claims about fidelity:

Claim 3 (Monotonicity) For any (mixed) states $\rho$ and $\sigma$ and any quantum operator $\mathcal{E}$ (not necessarily unitary), we have

$$
\begin{equation*}
F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma) \tag{28}
\end{equation*}
$$

It is a well-known result NC00.
Claim 4 (Triangle Inequality) For any 3 pure states $|A\rangle,|B\rangle$ and $|C\rangle$ in the same Hilbert space $\mathcal{H}$ such that $F(|A\rangle,|B\rangle)=1-\epsilon$ and $F(|A\rangle,|C\rangle)=1-\delta$, where both $\epsilon$ and $\delta$ are real numbers between 0 and $1 / 2$. then we have

$$
F(|B\rangle,|C\rangle) \geq 1-2(\epsilon+\delta)
$$

Claim 5 (Relationship to Statistical Distance) Let $\rho$ and $\sigma$ be 2 mixed states in the Hilbert space $\mathcal{H}$ such that that $F(\rho, \sigma)=1-\epsilon$. Let $\mathcal{E}$ be an arbitrary quantum operation over $\mathcal{H}$ that ends with a measurement. We use $M_{\rho}$ and $M_{\sigma}$ to denote the random variables describing the outcomes of the measurement of $\mathcal{E}$ on input $\rho$ and $\sigma$, respectively. Then the statistical distance between $M_{\rho}$ and $M_{\sigma}$ is at most $\sqrt{\epsilon}$.

We first consider the case that the input state is a pure state $|\phi\rangle$. By Lemma 6, with probability at least $1-\epsilon$, the Hash and Compare protocol will succeed. In the case it succeeds, the output state $|\psi\rangle$ will have a fidelity at least $\left(1-\frac{2 \epsilon}{\sqrt{S}}\right)$ with the diagonal subspace $\mathcal{H}^{\mathcal{D}}$ with probability $1-\frac{1}{\sqrt{S}}$. We define a "good event" to be the event that $|\psi\rangle$ has fidelity at least $\left(1-\frac{2 \epsilon}{\sqrt{S}}\right)$ with $\mathcal{H}^{\mathcal{D}}$. Then the probability a good event happens is at least $1-\frac{1}{\sqrt{S}}$. We focus on the good events. We write the normalized projection of $|\psi\rangle$ to $\mathcal{H}^{\mathcal{D}}$ as $\left|\psi_{\mathcal{D}}\right\rangle$. So we have

$$
F\left(|\psi\rangle,\left|\psi_{\mathcal{D}}\right\rangle\right) \geq 1-\frac{2 \epsilon}{\sqrt{S}}
$$

In other words, the fidelity of state $|\psi\rangle$ and the state $\left|\psi_{\mathcal{D}}\right\rangle$ is at least $1-\frac{2 \epsilon}{\sqrt{S}}$. If Alice and Bob, instead of feeding $|\psi\rangle$, had fed $\left|\psi_{\mathcal{D}}\right\rangle$ into the Simple Scrambling protocol, they would have succeeded with probability at least $1-\epsilon$, and output a pure state $\left|\psi_{\mathcal{D}}^{E}\right\rangle$ of fidelity at least $1-\frac{2 W}{N} \epsilon$. However, since Alice and Bob don't feed $\left|\psi_{\mathcal{D}}\right\rangle$ into the Simple Scrambling protocol, they don't get $\left|\psi_{\mathcal{D}}^{E}\right\rangle$ back: rather they get a state $\left|\psi^{E}\right\rangle$ if they don't fail ${ }^{\text {P }}$. By the monotonicity of fidelity, we have that

$$
\begin{equation*}
\left\langle\psi^{E} \mid \psi_{\mathcal{D}}^{E}\right\rangle \geq\left\langle\psi \mid \psi_{\mathcal{D}}\right\rangle \geq 1-\frac{2 \epsilon}{\sqrt{S}} \tag{29}
\end{equation*}
$$

Combining Equation 29 with the fact that $F\left(\left|\psi_{\mathcal{D}}^{E}\right\rangle,\left|\Psi_{W K}\right\rangle\right) \geq 1-\frac{2 W}{N} \epsilon$, we have, by Claim 0 ,

$$
F\left(\left|\psi^{E}\right\rangle,\left|\Psi_{W K}\right\rangle\right) \geq 1-\left(\frac{4 W}{N}+\frac{4}{\sqrt{S}}\right) \epsilon
$$

We denote by $p$ the failing probability of the Simple Scrambling protocol on input $|\psi\rangle$, and $p_{\mathcal{D}}$ the failing probability on input $\left|\psi_{\mathcal{D}}\right\rangle$. Then we have, by Claim 5 ,

$$
\left|p-p_{\mathcal{D}}\right| \leq \sqrt{\frac{2 \epsilon}{\sqrt{S}}}
$$

[^4]Putting things together, we have: with probability at least $1-2 \epsilon-\sqrt{\frac{2 \epsilon}{\sqrt{S}}}$, the Complete Scrambling protocol succeeds. In the case it succeeds, it outputs a state $\left|\psi^{E}\right\rangle$ of fidelity at least $1-\left(\frac{4 W}{N}+\frac{4}{\sqrt{S}}\right) \epsilon$ with probability at least $1-\frac{1}{\sqrt{S}}$.

Next we consider the case that the input state is a mixed state $\rho$. We have $F(\rho) \geq 1-\epsilon$. We write $\rho$ as an ensemble $\left\{p_{i},\left|\phi_{i}\right\rangle\right\}$. For each pure state $\left|\phi_{i}\right\rangle$, we assume that it has fidelity $1-\epsilon_{i}$, and then by the linearity of fidelity, we have $\sum_{i} p_{i} \epsilon=\epsilon$. The analysis above works for each pure state $\left|\psi_{i}\right\rangle$ : for each pure state $\left|\phi_{i}\right\rangle$, with probability at least $1-2 \epsilon_{i}-\sqrt{\frac{2 \epsilon_{i}}{\sqrt{S}}}$, the Complete Scrambling protocol succeeds. In the case it succeeds, it outputs a state $\left|\psi_{i}^{E}\right\rangle$ of fidelity at least $1-\left(\frac{4 W}{N}+\frac{4}{\sqrt{S}}\right) \epsilon_{i}$ with probability at least $1-\frac{1}{\sqrt{S}}$. The fidelity $1-\left(\frac{4 W}{N}+\frac{4}{\sqrt{S}}\right) \epsilon_{i}$ is a linear functions in $\epsilon_{i}$, and $1-2 \epsilon-\sqrt{\frac{2 \epsilon}{\sqrt{S}}}$ is a convex function. So overall, the Complete Scrambling protocol succeeds with probability at least $1-2 \epsilon-\sqrt{\frac{2 \epsilon}{\sqrt{S}}}$. In the case it succeeds, it outputs a state $\left|\psi^{E}\right\rangle$ of fidelity at least $1-\left(\frac{4 W}{N}+\frac{4}{\sqrt{S}}\right) \epsilon$ with probability at least $1-\frac{1}{\sqrt{S}}$.

Now we are ready to prove Theorem 3 .
Proof: [Proof to Theorem 3] We simply combine Theorem 0 and Lemma 7 and choose $S=2^{2 t}$.

## E Constructions of Scrambling Permutations

We discuss various constructions of Scrambling Permutations.
For a binary string $S=s_{1} s_{2} \ldots s_{n}$, we define the left sub-string and the right sub-string of the string $S$ as follows:

$$
\begin{aligned}
\operatorname{LEFT}(k, S) & =s_{1} s_{2} \ldots s_{k} \\
\operatorname{RIGHT}(k, S) & =s_{n-k+1} s_{n-k+2} \ldots s_{n}
\end{aligned}
$$

Obviously we have

$$
\operatorname{LEFT}(k, S) \circ \operatorname{RIGHT}(n-k, S)=S
$$

The first construction is a very simple one, and it is very closely related to a construction of universal hash functions.

Construction 5 (Multiplication-table Scrambling Permutation) We work in $G F_{2^{n}}$, where each element is a polynomial of degree at most $n-1$, and can be written as

$$
a_{0}+a_{1} \cdot Z+\cdots+a_{n-1} \cdot Z^{n-1}
$$

We identify each element with an n-bit binary string in the most straight-forward way. We set $X=G F_{2^{n}}$ and $Y=G F_{2^{n}}^{*}=X \backslash\{\mathbf{0}\}$, where $\mathbf{0}$ is the additive identity in $G F_{2^{n}}$. We can pick an arbitrary l, such that $1 \leq l<n$. Then we let $G=\{0,1\}^{l}$ and $H=\{0,1\}^{n-l}$. The functions are:

$$
\begin{aligned}
g_{y}(x) & =\operatorname{LEFT}(l, x \cdot y) \\
h_{y}(x) & =\operatorname{RIGHT}(n-l, x \cdot y)
\end{aligned}
$$

and we have $N=2^{n}, K=2^{n}-1, L=2^{l}$, and $M=2^{n-l}$.
Notice that a very common construction for universal hash functions over $G F_{2^{n}}$ is $h_{y, z}(x)=x \cdot y+z$, and our construction can be viewed as a sub-family of this universal hash family, by setting $z=0$. Our construction here is not a universal hash function family, but is more efficient.

Lemma 8 The function pair given in Construction 5 is an efficient Scrambling Permutation pair.
Proof: It is obvious that $\left\langle g_{y}(\cdot), h_{y}(\cdot)\right\rangle$ is a permutation, since

$$
g_{y}(x) \circ h_{y}(x)=x \cdot y
$$

is a permutation for $y \neq \mathbf{0}$.
Now let's prove that for any $x_{1} \neq x_{2}, \operatorname{Prob}_{y}\left[h_{y}\left(x_{1}\right)=h_{y}\left(x_{2}\right)\right]$ is always the same. This is actually not hard: we have $h_{y}\left(x_{1}\right)=h_{y}\left(x_{2}\right)$, iff

$$
\left(x_{1}-x_{2}\right) \cdot y=0 \bmod \left(Z^{n-l}\right)
$$

There are exactly $2^{l}$ elements in $G F_{N}$ that are multiples of ( $Z^{n-l}$ ), and so there are exactly $2^{l} y$ 's that satisfy the equation. However, one such $y$ is $\mathbf{0}$ and has to to be excluded. So the probability is $p=\left(2^{l}-1\right) /\left(2^{n}-1\right)=(L-1) /(N-1)$. This is true for every pair $x_{1} \neq x_{2}$.

Finally, both the permutation and its inverse can be implemented efficiently (only field multiplication and inversion are involved). So this scrambling permutation is efficient.

A word about efficiency: it is desirable for us to construct families of scrambling permutations of relatively small $K$ and $L$, as compared $M$ : In the Simple Scrambling protocol, where the Scrambling Permutation is used, $N$ is the dimension of the input state that Alice and Bob try to purify, which is normally fixed; $K$ is the dimension of maximally entangled state Alice and Bob invests; $M$ is the "yield" of the protocol, or the dimension of the output; $L$ is the dimension of the subspace Alice and Bob discard. So the Simple Scrambling protocol invests about $\log K$ perfect EPR pairs and discard about $\log L$ amount of entanglement. For the Multiplication-table construction, $K$ is almost as large as $N$, which is a disadvantage since Alice and Bob has to invest as many perfect EPR pairs as the imperfect ones they try to purify. However, the $L$ in this construction is fully adjustable, and it provides a nice trade-off between the yield Alice and Bob wish to obtain and the fidelity of the output (the greater $L$ is, the less the yield is, and the higher fidelity the output has).

Below is another construction:
Construction 6 (Linear Function Scrambling Permutation) We work in $G F_{2^{n}}$, and let $X=$ $G F_{2^{n}} \times G F_{2^{n}}$. Therefore each element in $X$ is represented by $\left\langle x_{0}, x_{1}\right\rangle$. We let $Y=G F_{2^{n}} \cup\{\perp\}$, where $\perp$ is a special symbol.

Both functions $g_{y}\left(\left\langle x_{0}, x_{1}\right\rangle\right)$ and $h_{y}\left(\left\langle x_{0}, x_{1}\right\rangle\right)$ output elements in $G F_{2^{n}}$ and the actual functions are defined as follows:

$$
\begin{aligned}
& g_{y}\left(\left\langle x_{0}, x_{1}\right\rangle\right)= \begin{cases}x_{0} & \text { if } y \in G F_{2^{n}} \\
x_{1} & \text { if } y=\perp\end{cases} \\
& h_{y}\left(\left\langle x_{0}, x_{1}\right\rangle\right)= \begin{cases}x_{0} \cdot y+x_{1} & \text { if } y \in G F_{2^{n}} \\
x_{0} & \text { if } y=\perp\end{cases}
\end{aligned}
$$

and we have $N=2^{2 n}$, $K=2^{n}+1, M=2^{n}$, and $L=2^{n}$.

Lemma 9 The function pair given in Construction 6 is an efficient Scrambling Permutation pair.
Proof: It is easy to verify that for any $y, g_{y}\left(\left\langle x_{0}, x_{1}\right\rangle\right) \circ h_{y}\left(\left\langle x_{0}, x_{1}\right\rangle\right)$ is a permutation.
Next we prove the scrambling property: for any pair of inputs $x=\left\langle x_{0}, x_{1}\right\rangle$ and $x^{\prime}=\left\langle x_{0}^{\prime}, x_{1}^{\prime}\right\rangle$ :

- If $x_{0} \neq x_{0}^{\prime}$, then the unique $y=\left(x_{1}-x_{1}^{\prime}\right) \cdot\left(x_{0}-x_{0}^{\prime}\right)^{-1}$ makes $h_{y}\left(\left\langle x_{0}, x_{1}\right\rangle\right)=h_{y}\left(\left\langle x_{0}^{\prime}, x_{1}^{\prime}\right\rangle\right)$.
- If $x_{0}=x_{0}^{\prime}$, then the unique $y=\perp$ makes $h_{y}\left(\left\langle x_{0}, x_{1}\right\rangle\right)=h_{y}\left(\left\langle x_{0}^{\prime}, x_{1}^{\prime}\right\rangle\right)$.

Finally, it is easy both the permutation and its can be computed efficiently, and thus the linear function construction is an efficient Scrambling Permutation pair.

In this construction, $K$ is about the square root of $N$, which is much better than the Multiplicationtable construction. However, $L$ is fixed, and we don't have the flexibility as in the Multiplication-table construction. However, we can extend this construction to a class of Scrambling Permutations, and resolve the flexibility problem.

Construction 7 (Extended Linear Function Scrambling Permutation) We work in $G F_{2^{n}}$, and let $X=G F_{2^{n}}^{d}$, where $d$ is an integer. Therefore each element in $X$ is represented by a d-tuple $\left\langle x_{0}, x_{1}, \ldots, x_{d-1}\right\rangle$. We let $Y=\bigcup_{k=0}^{d-1} G F_{2^{n}}^{d}$, where we define $G F_{2^{n}}^{0}=\{\perp\}$.

The function $g_{y}\left(\left\langle x_{0}, x_{1}, \ldots, x_{d-1}\right\rangle\right)$ outputs an element in $G F_{2^{n}}$ and the function $h_{y}\left(\left\langle x_{0}, x_{1}, \ldots, x_{d-1}\right\rangle\right)$ output a (d-1)-tuple in $G F_{2^{n}}$ : For any $y \in Y$, we write $y=\left\langle y_{0}, y_{1}, \ldots, y_{k-1}\right\rangle$, where $0 \leq k<d$.

$$
\begin{aligned}
g_{y}\left(\left\langle x_{0}, x_{1}, \ldots, x_{d-1}\right\rangle\right) & =x_{k} \\
h_{y}\left(\left\langle x_{0}, x_{1}, \ldots, x_{d-1}\right\rangle\right) & =\left\langle x_{0}+x_{k} \cdot y_{0}, x_{1}+x_{k} \cdot y_{1}, \ldots, x_{k-1}+x_{k} \cdot y_{k-1}, x_{k+1}, x_{k+2}, \ldots, x_{d-1}\right\rangle
\end{aligned}
$$

and we have $N=2^{d n}, K=\frac{2^{d n}-1}{2^{n}-1}, M=2^{(d-1) n}$, and $L=2^{n}$.
Here is a concrete example for $d=4$ :

| $y$ | $g_{y}(x)$ | $h_{y}(x)$ |  |  |
| :--- | :---: | :---: | :---: | :---: |
| $y=\perp$ | $x_{0}$ | $x_{1} \quad, \quad x_{2}$ | ,$\quad x_{3}$ |  |
| $y=\left\langle y_{0}\right\rangle$ | $x_{1}$ | $x_{0}+x_{1} \cdot y_{0}$, | $x_{2}$ | $x_{3}$ |
| $y=\left\langle y_{0}, y_{1}\right\rangle$ | $x_{2}$ | $x_{0}+x_{2} \cdot y_{0}, x_{1}+x_{2} \cdot y_{1}$, | $x_{3}$ |  |
| $y=\left\langle y_{0}, y_{1}, y_{2}\right\rangle$ | $x_{3}$ | $x_{0}+x_{3} \cdot y_{0}, x_{1}+x_{3} \cdot y_{1}, x_{2}+x_{3} \cdot y_{2}$ |  |  |

Lemma 10 The function pair defined in Construction $\gamma^{7}$ is an efficient Scrambling Permutation pair.
Proof: The permutation property is obvious, and it is easy to see that both the permutation and its inverse can be computed efficiently.

Now the scrambling property: given any pair $x=\left\langle x_{0}, x_{1}, \ldots, x_{d-1}\right\rangle$ and $x^{\prime}=\left\langle x_{0}^{\prime}, x_{1}^{\prime}, \ldots, x_{d-1}^{\prime}\right\rangle$. we show that there is always a unique $y$ such that $h_{y}(x)=h_{y}\left(x^{\prime}\right)$. We define $k$ to be the largest index such that $x_{k} \neq x_{k}^{\prime}$. Then for $y \in G F_{2^{n}}^{l}$,

1. If $l<k$, then the $k$-th entry in $h_{y}(x)$ is $x_{k}$, and it is different from the $k$-th entry in $h_{y}\left(x^{\prime}\right)$, which is $x_{k}^{\prime}$;
2. If $l=k$, we are effectively solving a linear system:

$$
\begin{aligned}
x_{0}+x_{k} \cdot y_{0} & =x_{0}^{\prime}+x_{k}^{\prime} \cdot y_{0} \\
x_{1}+x_{k} \cdot y_{1} & =x_{1}^{\prime}+x_{k}^{\prime} \cdot y_{1} \\
& \cdots \\
x_{k-1}+x_{k} \cdot y_{k-1} & =x_{k-1}^{\prime}+x_{k}^{\prime} \cdot y_{k-1}
\end{aligned}
$$

and it has a unique solution

$$
\begin{aligned}
y_{0} & =\left(x_{0}-x_{0}^{\prime}\right) \cdot\left(x_{k}-x_{k}^{\prime}\right)^{-1} \\
y_{1} & =\left(x_{1}-x_{1}^{\prime}\right) \cdot\left(x_{k}-x_{k}^{\prime}\right)^{-1} \\
& \cdots \\
y_{k-1} & =\left(x_{k-1}-x_{k-1}^{\prime}\right) \cdot\left(x_{k}-x_{k}^{\prime}\right)^{-1}
\end{aligned}
$$

3. If $l>k$, the $k$-th entry of $x$ is $x_{k}+y_{k} \cdot x_{k+1}$, and it is different from the $k$-th entry of $x^{\prime}$, which is $x_{k}^{\prime}+y_{k} \cdot x_{k+1}$, since $x_{k} \neq x_{k}^{\prime}$, while $x_{k+1}=x_{k+1}^{\prime}$.

So there exists a unique $y \in Y$ such that $h_{y}(x)=h_{y}\left(x^{\prime}\right)$.
The extended linear function construction gives a class of Scrambling Permutations of different parameters: for a fixed $N$, we can pick a construction such that $K$ is about $N^{(d-1) / d}$ and $L$ is about $N^{1 / d}$ for any integer $d$. When $d=2$, the extended linear function construction becomes the linear function construction. So we get back some flexibility: not only in $K$, but also in $L$.

Of course, one question is: how good are our constructions in terms of the size of $K$ and $L$ as compared to $N$ ? We hope $K$ and $L$ are as small as possible, and how small can they be? We have the following theorem which essentially says that the Extended Linear Function construction is optimal in terms of the size of $K$ and $L$.

Theorem 6 Let $\left\langle g_{y}(x), h_{y}(x)\right\rangle$ be a scrambling permutation pairs of parameter $\langle N, K, M, L\rangle$. We have $N \leq K L$.

Proof: First, by Theorem 包, we know that the collision probability $p=(L-1) /(N-1)$.
Recall that $p$ is the probability that a random $y \in Y$ satisfies $h_{y}\left(x_{1}\right)=h_{y}\left(x_{2}\right)$, and thus it is at least $1 / K$. Therefore we have

$$
\frac{1}{K} \leq \frac{L-1}{N-1}
$$

or

$$
K \geq \frac{N-1}{L-1} \geq \frac{N}{L}
$$

It is easy to see that the Extended Linear Function construction achieves this bound asymptotically.
We summarize the 3 constructions in the following table, which essentially proves Theorem 4 .

| Construction | $N$ | $K$ | $M$ | $L$ | Comments |
| :--- | :---: | :---: | :---: | :---: | :--- |
| Multiplication-table | $2^{n}$ | $2^{n}-1$ | $2^{n-l}$ | $2^{l}$ | Fully adjustable $L$, not optimal |
| Linear Function | $2^{2 n}$ | $2^{n}+1$ | $2^{n}$ | $2^{n}$ | Minimal $K$ among all constructions, optimal, inflexibl |
| Extended Linear Function | $2^{d n}$ | $\frac{2^{d n}-1}{2^{n}-1}$ | $2^{(d-1) n}$ | $2^{n}$ | Optimal, flexible $K$ and $L$ |

## F Proofs to Two Claims About Fidelity

We give the proofs to 2 claims about fidelity that are used in this paper.
Proof: [Proof to Claim 4 Notice that $|A\rangle,|B\rangle$ and $|C\rangle$ are vectors in $\mathcal{H}$. We denote the angle between $|A\rangle$ and $|B\rangle$ by $\theta_{A B}$, and define $\theta_{B C}$, and $\theta_{C A}$ accordingly. Then it is easy to see (by the triangle inequality), that $\theta_{B C} \leq \theta_{A B}+\theta_{A C}$. It is also easy to see that $\cos \theta_{A B}=\langle A \mid B\rangle=\sqrt{1-\epsilon}$ and $\cos \theta_{A C}=\langle A \mid C\rangle=\sqrt{1-\delta}$ Therefore, we have

$$
\begin{aligned}
\langle B \mid C\rangle & =\cos \theta_{B C} \\
& \geq \cos \left(\theta_{A B}+\theta_{A C}\right) \\
& =\cos \theta_{A B} \cos \theta_{A C}-\sin \theta_{A B} \sin \theta_{A C} \\
& =\sqrt{(1-\epsilon)(1-\delta)}-\sqrt{\epsilon \delta} \\
& \geq \sqrt{1-2(\epsilon+\delta)}
\end{aligned}
$$

where the last step is a simple algebraic deduction.

Proof: [Proof to Claim 可] We use $D(\rho, \sigma)$ to denote the trace distance between $\rho$ and $\sigma$, and we have NC00

$$
D(\rho, \sigma) \leq \sqrt{1-F(\rho, \sigma)}=\sqrt{\epsilon}
$$

However the statistical distance between $M_{\rho}$ and $M_{\sigma}$ is bounded by $D(\rho, \sigma)$, which is bounded by $\sqrt{\epsilon}$.


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    ${ }^{\dagger}$ A preliminary version of this paper is submitted to STOC 2002

[^1]:    ${ }^{1}$ Most of the time in this paper, we are interested in the fidelity of a state $\rho$ and a (pre-defined) maximally entangled state (e.g., an EPR pair). In this case, we simply use the "fidelity of state $\rho$ " to denote the fidelity of $\rho$ and the pre-defined maximally entangled state of appropriate dimension.

[^2]:    ${ }^{2}$ The protocols can be modified to use any other fixed state of the form ( $\sigma^{6}$ ) instead of $\Psi_{M}$

[^3]:    ${ }^{3}$ We assume that Alice have enough ancillary qubit at the beginning of the protocol and not more new ancillary qubits need to be introduced during the protocol.

[^4]:    ${ }^{4}$ It is easy to check that the Simple Scrambling protocol always outputs a pure state if the input state is pure.

