# Derandomizing from Random Strings 

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#### Abstract

In this paper we show that BPP is truth-table reducible to the set of Kolmogorov random strings $R_{K}$. It was previously known that PSPACE, and hence BPP is Turing-reducible to $R_{K}$. The earlier proof relied on the adaptivity of the Turingreduction to find a Kolmogorov-random string of polynomial length using the set $R_{K}$ as oracle. Our new non-adaptive result relies on a new fundamental fact about the set $R_{K}$, namely each initial segment of the characteristic sequence of $R_{K}$ is not compressible by recursive means. As a partial converse to our claim we show that strings of high Kolmogorov-complexity when used as advice are not much more useful than randomly chosen strings.


## 1 Introduction

Kolmogorov complexity studies the amount of randomness in a string by the smallest program that can generate it. The most random strings are those we cannot compress at all making the set $R_{K}=\{x|K(x) \geq|x|\}$ of Kolmogorov random strings worthy of close analysis.

Allender et al. $\mathrm{ABK}^{+} 02$ showed the surprising computational power of $R_{K}$ including that polynomial time adaptive (Turing) access to $R_{K}$ enables one to do PSPACEcomputations: PSPACE $\subseteq P^{R_{K}}$. One of the ingredients in the proof shows how on input $0^{n}$ one can in polynomial time with adaptive access to $R_{K}$ generate a polynomially long Kolmogorov random string. With non-adaptive access it is only possible to generate in polynomial time a random string of length at most $O(\log n)$.

[^0]In an attempt to characterize PSPACE as the class of sets reducible to $R_{K}$, Allender, Buhrman and Koucký ABK06 noticed that this question depends on the choice of universal machine used in the definition of the notion of Kolmogorov complexity. They also started a systematic study of weaker and non-adaptive access to $R_{K}$. They showed for example that

$$
\mathrm{P}=\operatorname{REC} \cap \bigcap_{U}\left\{A \mid A \leq_{\mathrm{dtt}}^{\mathrm{p}} R_{K_{U}}\right\} .
$$

This result and the fact that with non-adaptive access to $R_{K}$ in general only logarithmically small strings can be found seems to suggest that adaptive access to $R_{K}$ is needed in order to be useful.

Our first result proves this intuition false: We show that polynomial time nonadaptive access to $R_{K}$ can be used to derandomize any BPP computation. In order to derandomize a BPP computation one needs a (pseudo)random string of polynomial size. As mentioned before one can only obtain short, $O(\log n)$ sized, random strings from $R_{K}$. Instead we show that the characteristic sequence formed by the strings of length $c \log n$, $R_{K}^{=c \log n}$, itself a strings of length $n^{c}$, is complex enough to figure as a hard function in the hardness versus randomness framework of Impagliazzo and Wigderson [IW97. This way we construct a pseudorandom generator that is strong enough to derandomize BPP.

In particular we show that for every time bound $t$, there is a constant $c$ such that $R_{K} \notin$ i.o.-DTIME $(t) / 2^{n-c}$. This is in stark contrast with the time-unbounded case where only $n$ bits of advice are necessary Bar68. As a consequence we give an alternative proof of the existence of an r.e. set $A$, due to Barzdin [Bar68], such that for all time bounds $t$, there exists $c_{t}$ such that $K^{t(n)}\left(A_{1: n} \mid n\right) \geq n / c_{t}$. We simply take for $A$ the complement of $R_{K}$. Barzdin also showed that this lower bound is optimal for r.e. sets. Hence the constant depending on the time-bound in our Theorem 4 is optimal.

Next we try to establish whether we can characterize BPP as the class of sets that non-adaptively reduce to $R_{K}$. One can view the truth-table reduction to $R_{K}$ as a computation with advice of $K^{t(n)}$ complexity $\Omega(n)$. We can show that for sets in EXP and $t(n) \in 2^{n^{\Omega(1)}}$, polynomial-time computation with polynomial (exponential, resp.) size advice of $K^{t(n)}$ complexity $n-O(\log n)(n-O(\log \log n)$, resp.) can be simulated by bounded error probabilistic machine with almost linear size advice. For paddable sets that are complete for $\mathrm{NP}, \mathrm{P}^{\# \mathrm{P}}, \mathrm{PSPACE}$, or EXP we do not even need the linear size advice. Hence, advice of high $K^{t(n)}$ complexity is no better than a truly random string.

Summarizing our results:

- For every computable time bound $t$ there is a constant c (depending on $t$ ) such that $R_{K} \notin$ i.o.-DTIME $(t) / 2^{n-c}$.
- The complement of $R_{K}$ is a natural example of an computably enumerable set whose characteristic sequence has high time bounded Kolmorogov complexity for every $n$.
- BPP is truth-table reducible to $R_{K}$.
- A poly- up-to exponential-size advice that has very large $K^{t(n)}$ complexity can be replaced by $O(n \log n)$ bit advice and true randomness.


## 2 Preliminaries

We remind the reader of some of the definitions we use. Let $M$ be a Turing machine. For any string $x \in\{0,1\}^{*}$, the Kolmogorov complexity of $x$ relative to $M$ is $K_{M}(x)=$ $\min \left\{|p| \mid p \in\{0,1\}^{*} \& M(p)=x\right\}$, where $|p|$ denotes the length of string $p$. It is well known that for a universal Turing machine $U$ and any other machine $M$ there is a constant $c_{M}$ such that for all strings $x, K_{U}(x) \leq K_{M}(x)+c_{M}$. For the rest of the paper we will fix some universal Turing machine $U$ and we will measure Kolmogrov complexity relative to that $U$. Thus, we will not write the subscript $U$ explicitly.

We define $K^{t}(x)=\min \{|p| \mid U(p)=x$ and $U(p)$ uses at most $t(|x|)$ steps $\}$. Unlike traditional computational complexity the time bound is a function of the length of the output of $U$.

A string $x$ is said to be Kolmogorov-random if $K(x) \geq|x|$. The set of Kolmogorovrandom strings is denoted by $R_{K}=\left\{x \in\{0,1\}^{*}|K(x) \geq|x|\}\right.$. For an integer $n$ and set $A \subseteq\{0,1\}^{*}, A^{=n}=A \cap\{0,1\}^{n}$. The following well known claim can be proven by considering the Kolmogorov complexity of $\left|R_{K}^{=n}\right|$ (see [LV08]).

Proposition 1 There is a constant $d$ such that for all $n,\left|R_{K}^{=n}\right| \geq 2^{n} / d$.
We also use computation with advice. We deviate slightly from the usual definition of computation with advice in the way how we express and measure the running time. For an advice function $\alpha: \mathbf{N} \rightarrow\{0,1\}^{*}$, we say that $L \in \mathrm{P} / \alpha$ if there is a Turing machine $M$ such that for every $x \in\{0,1\}^{*}, M(x, \alpha(|x|))$ runs in time polynomial in the length of $x$ and $M(x, \alpha(|x|))$ accepts iff $x \in L$. We assume that $M$ has random access to its input so the length of $\alpha(n)$ can grow faster than any polynomial in $n$. Similarly, we define EXP/ $\alpha$ where we allow the machine $M$ to run in exponential time in length of $x$ on the input $(x, \alpha(|x|))$. Furthermore, we are interested not only in Boolean languages (decision problems) but also in functions, so we naturally extend both definitions also to computation with advice of functions. Typically we are interested in the amount of advice that we need for inputs of length $n$ so for $f: \mathbf{N} \rightarrow \mathbf{N}, C / f$ is the union of all $C / \alpha$ for $\alpha$ satisfying $|\alpha(n)| \leq f(n)$.

Let $L$ be a language and $C$ be a language class. We say that $L \in$ i.o. $-C$ if there exists a language $L^{\prime} \in C$ such that for infinitely many $n, L^{=n}=L^{\prime=n}$. For a Turing machine $M$, we say $L \in$ i.o.-M/f if there is some advice function $\alpha$ with $|\alpha(n)| \leq f(n)$ such that for infinitely many $n, L^{=n}=\left\{x \in \Sigma^{n} \mid M(x, \alpha(|x|))\right.$ accepts $\}$.

We say that a set $A$ polynomial-time Turing reduces to a set $B$, if there is an oracle machine $M$ that on input $x$ runs in polynomial time and with oracle $B$ decides whether $x \in A$. If $M$ asks its questions non-adaptively, i.e., each oracle question does not depend
on the answers to the previous oracle questions, we say that $A$ polynomial-time truthtable reduces to $B\left(A \leq_{\mathrm{tt}}^{\mathrm{p}} B\right)$. Moreover, $A \leq_{\mathrm{dtt}}^{\mathrm{p}} B$ if machine $M$ outputs as its answer the disjunction of the oracle answers. Similarly, $A \leq_{\mathrm{ctt}}^{\mathrm{p}} B$ for the conjunction of the answers.

## 3 High circuit complexity of $R_{K}$

In this section we prove that the characteristic sequence of $R_{K}$ has high circuit complexity almost everywhere. We will first prove the following lemma.

Lemma 2 For every total Turing machine $M$ there is a constant $c_{M}$ such that $R_{K}$ is not in i.o.-M/2 $2^{n-c_{M}}$.

There is a (non-total) Turing machine $M$ such that $R_{K}$ is in $M / n+1$ where the advice is the number of strings in $R_{K}^{=n}$. Simply find all the non-random strings of length $n$. This machine will fail to halt if the advice underestimates the number of random strings.
Proof of Lemma圆. Suppose the theorem is false. Fix a total machine $M$. We have that, $(x, \alpha) \in L(M)$ if and only if $x \in R_{K}$, for some advice $\alpha$ of length $k \leq 2^{n-c_{M}}$ and every $x$ of some large enough length $n$. By padding the advice we can assume $k=2^{n-c_{M}}$. We will set $c_{M}$ later in order to get a contradiction.

Let $R_{\beta}=\left\{x \in \Sigma^{n} \mid(x, \beta) \in L(M)\right\}$. By Proposition 1 for some constant $d$, $\left|R_{\alpha}\right| \geq 2^{n} / d$ so we know that if $\left|R_{\beta}\right|<2^{n} / d$ then $\beta \neq \alpha$. We call $\beta$ good if $\left|R_{\beta}\right| \geq 2^{n} / d$.

Fix a good $\beta$ and choose $x_{1}, \ldots, x_{m}$ at random. The probability that all the $x_{i}$ are not in $R_{\beta}$ is at most $(1-1 / d)^{m}<2^{-m / d}$. There are $2^{k}$ advice strings $\beta$ of length $k$ so if $2^{-m / d} \leq 2^{-k}$ then there is a sequence $x_{1}, \ldots, x_{m}$ such that for every good $\beta$ of length $k$ there is an $i$ such that $x_{i} \in R_{\beta}$.

We can computably search all such sequences so let $x_{1}, \ldots, x_{m}$ be the lexicographically least sequence such that for each good $\beta$ of length $k$, there is some $x_{i} \in R_{\beta}$. This also means $x_{i} \in R_{\alpha}$ for some $i$ so for one of the $x_{i}$ we have $K\left(x_{i}\right) \geq n$.

Fix $m=2^{n-a}$ for a constant $a$ to be chosen later.
We can describe $x_{i}$ by $n-a+b \log a$ bits for some constant $b: n-a$ bits to describe $i, O(\log a)$ bits to recover $n$ and a constant number of additional bits to describe $k, M$, $d$ and the algorithm above for finding $x_{1}, \ldots, x_{m}$. If we pick $a$ such that $a>b \log a$ we contradict the fact that $K\left(x_{i}\right) \geq n$.

If we pick $c_{M} \geq a+\log d$ we then have $2^{n-a} \geq 2^{n-c_{M}} d, m>k d$ and $2^{-m / d} \leq 2^{-k}$ completing our contradiction.

In order to get our statement about time bounded advice classes we instantiate Lemma 2 with universal machines $U_{t}$ that run in time $t$, use the first part of their advice, in prefix free form, as a code for a machine that runs in time $t$ and has the second part of the advice for $U_{t}$ as its advice. The following is a direct consequence of Lemma 2

Lemma 3 For every computable time bound $t$ and universal advice machine $U_{t}$ there is a constant $c_{t}$ such that $R_{K}$ is not in i.o.- $\mathrm{U}_{t} / 2^{n-c_{t}}$.

We are now ready to prove the main theorem from this section.
Theorem 4 For every computable time bound $t$ there is a constant $d_{t}$ such that $R_{K}$ is not in i.o.-DTIME $(t) / 2^{n-d_{t}}$.

Proof. Suppose the theorem is false, that is there is a time bound $t$ such that for every $d$ there is a machine $M_{d}$ that runs in time $t$ such that $R_{k} \in$ i.o.- $\mathrm{M}_{d} / 2^{n-d}$. Set $t^{\prime}=t \log t$ and let $c_{t^{\prime}}$ be the constant that comes out of Lemma 3 when instantiated with time bound $t^{\prime}$. Set $d=c_{t^{\prime}}+1$ and let the code of machine $M_{d}$ from the (false) assumption have size $e$. So we have that $R_{k} \in$ i.o. $-\mathrm{M}_{d} / 2^{n-d}$. This in turn implies that $R_{K} \in$ i.o.- $\mathrm{U}_{t^{\prime}} / 2^{n-d}+e+2 \log e$, which implies that $R_{K} \in$ i.o. $\mathrm{U}_{t^{\prime}} / 2^{n-c_{t^{\prime}}}$ a contradiction with Lemma 3. The last step is true because the universal machine running for at most time $t^{\prime}=t \log t$, can simulate $M_{d}$, who runs in time $t$.

As an immediate corollary we get an alternative, more natural candidate for Barzdin's computably enumerable set that has high resource bounded Kolomorov complexity, namely the set of compressible strings.

Corollary 5 For every computable time bound there is a constant $c$ such that $K^{t}\left(\overline{R_{k}}(1\right.$ : $n) \mid n) \geq n / c$

Barzdin [Bar68] also showed that this lower bound is optimal. That is the dependence of $c$ on the time bound $t$ is needed for the characteristic sequence of every r.e. set. Hence this depence is also necessary in our Theorem 4 .

## 4 BPP truth-table reduces to $R_{k}$

In this section we investigate what languages are reducible to $R_{k}$. We start with the following theorem which one can prove using nowadays standard derandomization techniques.

Theorem 6 Let $\alpha:\{0\}^{*} \rightarrow\{0,1\}^{*}$ be a length preserving function and $\delta>0$ be a constant. If $\alpha\left(0^{n}\right) \notin$ i.o.-EXP $/ n^{\delta}$ then for every $A \in \operatorname{BPP}$ there exists $d>0$ such that $A \in \mathrm{P} / \alpha\left(0^{n^{d}}\right)$.

Proof. $\alpha\left(0^{n}\right) \notin$ i.o.-EXP $/ n^{\delta}$ implies that when $\alpha\left(0^{n}\right)$ is interpreted as a truth-table of a function $f_{\alpha\left(0^{n}\right)}:\{0,1\}^{\log n} \rightarrow\{0,1\}, f_{\alpha\left(0^{n}\right)}$ does not have boolean circuits of size $n^{\delta / 3}$ for all $n$ large enough. It is known that such a function can be used to build the ImpagliazzoWigderson pseudorandom generator [IW97 which can be used to derandomize boolean
circuits of size $n^{\delta^{\prime}}$ for some $\delta^{\prime}>0$ (see [IW97, KvM99, $\mathrm{ABK}^{+} 02$ ). Hence, bounded-error probabilistic computation running in time $n^{\ell}$ can be derandomized in polynomial time given access to $\alpha\left(0^{n^{2 \ell / \delta^{\prime}}}\right)$.

From Theorem 4 and the above Theorem we obtain the following corollary.
Corollary $7 \mathrm{BPP} \leq_{t t}^{p} R_{K}$.
Proof. Let $\alpha\left(0^{n}\right)$ be the truth-table of $R_{K}$ on strings of length $\lfloor\log n\rfloor$ padded by zeros to the length of $n$. By Theorem 4, $\alpha\left(0^{n}\right) \notin$ i.o.-EXP $/(n / c)$ for some $c>0$. Consider any $A \in \mathrm{BPP}$. By Theorem 6 for some $d, A \in P / \alpha\left(0^{n^{d}}\right)$. The claim follows by noting that a truth-table reduction to $R_{k}$ may query the membership of all the strings of length $\left\lfloor\log n^{d}\right\rfloor$ to construct $\alpha\left(0^{n^{d}}\right)$ and then run the $P / \alpha\left(0^{n^{d}}\right)$ algorithm for $A$.

Our goal would be to show that using $R_{K}$ as a source of randomness is the only way to make use of it. Ideally we would like to show that any recursive set that is truth-table reducible to $R_{K}$ must be in BPP. We fall short of such a goal. However we can show the following claim.

Theorem 8 Let $\alpha:\{0\}^{*} \rightarrow\{0,1\}^{*}$ be a length preserving function and $c>0$ be a constant. If $\alpha\left(0^{n}\right) \notin$ i.o.-EXP $/ n-c \log n$ then for every $A \in \operatorname{EXP}$ if $A \in \mathrm{P} / \alpha\left(0^{n^{d}}\right)$ for some $d>0$ then $A \in \mathrm{BPP} / O(n \log n)$.

This theorem says that Kolmogorov random advice of polynomial size can be replaced by almost linear size advice and true randomness. We come short of proving a converse of the above corollary in two respects. First, the advice is supposed to model the initial segment of the characteristic sequence of $R_{K}$ which the truth-table can access. However, by providing only polynomial size advice we restrict the hypothetical truthtable reduction to query strings of only logarithmic length. Second, the randomness that we require from the initial segment is much stronger than what one can prove and what is in fact true for the initial segment of the characteristic sequence of $R_{K}$. One can deal with the first issue as is shown by Theorem 9 but we do not know how to deal with the second one.
Proof. Let $M$ be a polynomial time Turing machine and $A \in$ EXP be a set such that $A(x)=M\left(x, \alpha\left(|x|^{d}\right)\right)$. We claim that for all $n$ large enough there is a non-negligible fraction of advice strings $r$ of size $n^{d}$ that could be used in place of $\alpha\left(n^{d}\right)$ more precisely:

$$
\operatorname{Pr}_{r \in\{0,1\}^{n^{d}}}[\forall x, x \in A \Longleftrightarrow M(x, r)=1]>\frac{1}{n^{c d}} .
$$

To prove the claim consider the set $G=\left\{r \in\{0,1\}^{n^{d}} ; \forall x \in\{0,1\}^{n}, x \in A \Longleftrightarrow\right.$ $M(x, r)=1\}$. Clearly, $G \in \operatorname{EXP}$ and $\alpha\left(0^{n^{d}}\right) \in G$. If $\left|G^{=n^{d}}\right| \leq 2^{n^{d}} / n^{c d}$ then $\alpha\left(0^{n^{d}}\right)$ can be computed in exponential time from its index in the set $G^{=n^{d}}$ of length $n^{d}-c d \log n$. Since $\alpha\left(0^{n^{d}}\right) \notin$ i.o.-EXP $/ n^{d}-c d \log n$ this cannot happen infinitely often.

Now we present an algorithm that on input $x$ samples from $G$ using only $O(n \log n)$ bits of advice (in fact $O(\log n)$ entries from the truth table of $A$ ) and outputs $A(x)$ with high probability. Consider the following algorithm:

1. Given an input $x$ of length $n$, and an advice string $x_{1}, A\left(x_{1}\right), \ldots, x_{k}, A\left(x_{k}\right)$,
2. sample at most $2 n^{c d}$ strings of length $n^{d}$ until the first string $r$ is found such that $M\left(x_{i}, r\right)=A\left(x_{i}\right)$ for all $i \in\{1, \ldots, k\}$.
3. If we find $r$ consistent with the advice then output $M(x, r)$ otherwise output 0 .

For all $n$ large enough the probability that the second step does not find $r$ compatible with the advice is upper-bounded by the probability that we do not sample any string from $G$ which is at most $\left(1-\frac{1}{n^{c d}}\right)^{2 n^{c d}}<e^{-2}<1 / 6$.

It suffices to show that we can find an advice sequence such that for at least $5 / 6$ fraction of the $r$ 's compatible with the advice $M(x, r)=A(x)$. For given $n$, we will find the advice by prnning iteratively the set of bad random strings $B=\{0,1\}^{n^{d}} \backslash G$. Let $i=0,1, \ldots, 2 c d \log _{6 / 5} n$. Set $B_{0}=B$. If there is a string $x \in\{0,1\}^{n}$ such that for at least $1 / 6$ of $r \in B_{i}, M(x, r) \neq A(x)$, then set $x_{i+1}=x$ and $B_{i+1}=B_{i} \cap\left\{r \in\{0,1\}^{n^{d}} \mid\right.$ $\left.M\left(x_{i+1}, r\right)=A\left(x_{i+1}\right)\right\}$. If there is no such string $x$ then stop and the $x_{i}$ 's obtained so far will form our advice. Notice, if we stop for some $i<2 c d \log _{6 / 5} n$ then for all $x \in\{0,1\}^{n}, \operatorname{Pr}_{r \in \mathcal{B}_{i}}[M(x, r) \neq A(x)]<1 / 6$. Hence, any $r$ found by the algorithm to be compatible with the advice will give the correct answer for a given input with probability at least $5 / 6$. On the other hand, if we stop building the advice at $i=2 c d \log _{6 / 5} n$ then $\left|B_{3 c d \log _{6 / 5} n}\right| \leq 2^{n^{d}} \cdot(5 / 6)^{2 c d \log _{6 / 5} n} \leq\left|G^{=n^{d}}\right| / n^{c d}$. Hence, any string $r$ found by the algorithm to be compatible with the advice $x_{1}, A\left(x_{1}\right), \ldots, x_{i}, A\left(x_{i}\right)$ will come from $G$ with good probability, i.e., with probability $>5 / 6$ for $n$ large enough.

The following theorem can be established by a similar argument. It again relies on the fact that a polynomially large fraction of all advice strings of length $2^{n^{d}}$ must work well as an advice. By a pruning procedure similar to the proof of Theorem 8 we can avoid bad advice. In the BPP algorithm one does not have to explicitly guess the whole advice but only the part relevant to the pruning advice and to the current input.

Theorem 9 Let $\alpha:\{0\}^{*} \rightarrow\{0,1\}^{*}$ be a length preserving function and $c>0$ be a constant. If $\alpha\left(0^{n}\right) \notin$ i.o.-EXP $/ n-c \log \log n$ then for every $A \in \operatorname{EXP}$ if $A \in \mathrm{P} / \alpha\left(0^{2^{n^{d}}}\right)$ for some $d>0$ then $A \in \mathrm{BPP} / O(n \log n)$.

We show next that if the set $A$ has some suitable properties we can dispense with the linear advice all together and replace it with only random bits. Thus for example if SAT $\in \mathrm{P} / \alpha\left(0^{n}\right)$ for some computationally hard advice $\alpha\left(0^{n}\right)$ then SAT $\in$ BPP.

Theorem 10 Let $\alpha:\{0\}^{*} \rightarrow\{0,1\}^{*}$ be a length preserving function and $c>0$ be $a$ constant such that $\alpha\left(0^{n}\right) \notin$ i.o.-EXP $/ n-c \log n$. Let $A$ be paddable and polynomial-time
many-one-complete for a class $\mathcal{C} \in\left\{\mathrm{NP}, \mathrm{P}^{\# \mathrm{P}}, \mathrm{PSPACE}, \mathrm{EXP}\right\}$. If $A \in \mathrm{P} / \alpha\left(0^{n^{d}}\right)$ for some $d>0$ then $A \in \mathrm{BPP}$ (and hence $C \subseteq \mathrm{BPP}$ ).

To prove the theorem we will need the notion of instance checkers. We use the definition of Trevisan and Vadhan [TV02].

Definition $11 A n$ instance checker $C$ for a boolean function $f$ is a polynomial-time probabilistic oracle machine whose output is in $\{0,1$, fail $\}$ such that

- for all inputs $x, \operatorname{Pr}\left[C^{f}(x)=f(x)\right]=1$, and
- for all inputs $x$, and all oracles $f^{\prime}, \operatorname{Pr}\left[C^{f^{\prime}}(x) \notin\{f(x)\right.$, fail $\left.\}\right] \leq 1 / 4$.

It is immediate that by linearly many repetitions and taking the majority answer one can reduce the error of an instance checker to $2^{-n}$. Vadhan and Trevisan also state the following claim:

Theorem 12 ([BFL91],[LFKN92, Sha92]) Every problem that is complete for EXP, PSPACE or $\mathrm{P}^{\# \mathrm{P}}$ has an instance checker. Moreover, there are EXP-complete problems, PSPACE-complete problems, and $\mathrm{P}^{\#} \mathrm{P}_{\text {-complete problems for which the instance checker }}$ $C$ only makes oracle queries of length exactly $\ell(n)$ on inputs of length $n$ for some polynomial $\ell(n)$.

However, it is not known whether NP has instance checkers.
Proof of Theorem 10. To prove the claim for $\mathrm{P}^{\# \mathrm{P}_{-}}$, PSPACE- and EXP-complete problems we use the instance checkers. We use the same notation as in the proof of Theorem 8, i.e., $M$ is a Turing machine such that $A(x)=M\left(x, \alpha\left(|x|^{d}\right)\right)$ and the set of good advice is $G=\left\{r \in\{0,1\}^{n^{d}} ; \forall x \in\{0,1\}^{n}, x \in A \Longleftrightarrow M(x, r)=1\right\}$. We know from the previous proof that $\left|G^{=n^{d}}\right| \geq 2^{n^{d}} / n^{c d}$ because $\alpha\left(0^{n}\right) \notin$ i.o.-EXP $/ n-c \log n$.

Let $C$ be the instance checker for $A$ which on input of length $n$ asks oracle queries of length only $\ell(n)$ and makes error on a wrong oracle at most $2^{-n}$. The following algorithm is a bounded error polynomial time algorithm for $A$ :

1. On input $x$ of length $n$, repeat $2 n^{c d}$ times
(a) Pick a random string $r$ of length $(\ell(n))^{d}$.
(b) Run the instance checker $C$ on input $x$ and answer each of his oracle queries $y$ by $M(y, r)$.
(c) If $C$ outputs fail continue with another iteration otherwise output the output of $C$.
2. Output 0 .

Clearly, if we sample $r \in G$ then the instance checker will provide a correct answer and we stop. The algorithm can produce a wrong answer either if the instance checker always fails (so we never sample $r \in G$ during the iterations) or if the instance checker gives a wrong answer. Probability of not sampling good $r$ is at most $1 / 6$. The probability of getting a wrong answer from the instance checker in any of the iterations is at most $2 n^{c d} / 2^{n}$. Thus the algorithm provides the correct answer with probability at least $2 / 3$.

To prove the claim for NP-complete languages we show it for the canonical example of SAT. The following algorithm solves SAT correctly with probability at least $5 / 6$ :

1. On input $\phi$ of length $n$, repeat $2 n^{c d}$ times
(a) Pick a random string $r$ of length $n^{d}$.
(b) If $M(\phi, r)=1$ then use the self-reducibility of SAT to find a presumably satisfying assignment $a$ of $\phi$ while asking queries $\psi$ of size $n$ and answering them according to $M(\psi, r)$. If the assignment $a$ indeed satisfies $\phi$ then output 1 otherwise continue with another iteration.
2. Output 0 .

Clearly, if $\phi$ is satisfiable we will answer 1 with probability at least $5 / 6$. If $\phi$ is not satisfiable we will always answer 0 .

## 5 Open Problems

We have shown that the set $R_{K}$ cannot be compressed using a computable algorithm and used this fact to reduce BPP non-adaptively to $R_{K}$. We conjecture that every computable set that non-adaptively reduces in polynomial-time to $R_{K}$ sits in BPP and have shown a number of partial results in that directions.

The classification of languages that polynomial-time adaptively reduce to $R_{K}$ also remains open. Can we characterize PSPACE this way?

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