Hardness of Finding Independent Sets in 2-Colorable Hypergraphs and of Satisfiable CSPs

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September 15, 2018

Abstract

This work revisits the PCP Verifiers used in the works of Håstad [Hås01], Guruswami et al. [GHS02], Holmerin [Hol02] and Guruswami [Gur00] for *satisfiable* MAX-E3-SAT and MAX-Ek-SET-SPLITTING, and independent set in 2-colorable 4-uniform hypergraphs. We provide simpler and more efficient PCP Verifiers to prove the following improved hardness results:

Assuming that NP $\not\subseteq$ DTIME $(N^{O(\log \log N)})$,

- There is no polynomial time algorithm that, given an *n*-vertex 2-colorable 4-uniform hypergraph, finds an independent set of $\frac{n}{(\log n)^c}$ vertices, for some constant c > 0.
- There is no polynomial time algorithm that satisfies $\frac{7}{8} + \frac{1}{(\log n)^c}$ fraction of the clauses of a *satis-fiable* MAX-E3-SAT instance of size n, for some constant c > 0.
- For any fixed $k \ge 4$, there is no polynomial time algorithm that finds a partition splitting $(1 2^{-k+1}) + \frac{1}{(\log n)^c}$ fraction of the *k*-sets of a *satisfiable* MAX-E*k*-SET-SPLITTING instance of size *n*, for some constant c > 0.

Our hardness factor for independent set in 2-colorable 4-uniform hypergraphs is an exponential improvement over the previous results of Guruswami et al. [GHS02] and Holmerin [Hol02]. Similarly, our inapproximability of $(\log n)^{-c}$ beyond the random assignment threshold for MAX-E3-SAT and MAX-E*k*-SET-SPLITTING is an exponential improvement over the previous bounds proved in [Hås01], [Hol02] and [Gur00].

The PCP Verifiers used in our results avoid the use of a *variable bias* parameter used in previous works, which leads to the improved hardness thresholds in addition to simplifying the analysis substantially. Apart from standard techniques from Fourier Analysis, for the first mentioned result we use a mixing estimate of Markov Chains based on *uniform* reverse hypercontractivity over general product spaces from the work of Mossel et al. [MOS13].

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1 Introduction

A k-uniform hypergraph consists of a set of vertices and a collection of hyperedges where each hyperedge is a subset of exactly k vertices. A hypergraph is said to be q-colorable if its vertices can be colored with q distinct colors such that no hyperedge contains all vertices of the same color. A related notion is that of an independent set, which is a subset of vertices that does not completely contain any hyperedge. It is easy to see that a q-colorable hypergraph has at least one independent set of q^{-1} fraction of vertices, i.e. relative size.

Computing the minimum number q – the *chromatic number* – of colors required to color a hypergraph is a very well studied optimization problem. There is a simple polynomial time algorithm to decide whether a given graph (k = 2) can be colored using q = 2 colors, i.e. is bipartite. However, for $k \ge 3$ or $q \ge 3$, this problem is NP-hard. A natural question in this context is how well can the chromatic number be approximated.

The first strong inapproximability for hypergraph coloring was given by Guruswami, Håstad and Sudan [GHS02] who showed that it is NP-hard to color an *n*-vertex 2-colorable 4-uniform hypergraph using constantly many colors, and quasi-NP-hard¹ to color it with $O((\log \log \log n)^{-1} \log \log n)$ colors. They used a notion of *covering complexity* combined with techniques developed in the seminal work of Håstad [Hås01]. In particular, the Probabilistically Checkable Proof (PCP) verifier of [GHS02] is identical to the one used in [Hås01] for the satisfiable MAX-E4-SET-SPLITTING problem. Subsequently, Holmerin [Hol02] used a more direct approach – with the same PCP verifier – to obtain a qualitatively stronger result. Holmerin showed that, given a 2-colorable 4-uniform it is NP-hard to compute an independent set of relative size δ , for any constant $\delta > 0$, and it is quasi-NP-hard to do so for $\delta = \Omega((\log \log n)^{-1} \log \log \log n)$.

In this work we prove the following quantitatively stronger result with an exponential improvement in the hardness factor.

Theorem 1.1. Given an n-vertex 2-colorable 4-uniform hypergraph it is quasi-NP-hard to find an independent set of relative size $\frac{1}{(\log n)^c}$ for some constant c > 0.

As mentioned above, the results of [GHS02] and [Hol02] are based on the PCP verifier used by Håstad [Hås01] for satisfiable MAX-E4-SET-SPLITTING. In this problem the input is a ground set and a collection of its 4-sets, and the goal is to partition the ground set into two subsets to maximize the number of *split* 4-sets. Another fundamental constraint satisfaction problem studied by Håstad [Hås01] is MAX-E3-SAT, where the goal is to satisfy the maximum number of a collection of 3-literal clauses. Håstad showed that approximating both these problems – on satisfiable instances – within δ of their random assignment threshold of $\frac{7}{8}$ is NP-hard for any constant $\delta > 0$ and quasi-NP-hard for $\delta = \Omega$ ($(\log \log n)^{-1} \log \log \log n$). Using a strengthened analysis of Holmerin [Hol02], Guruswami [Gur00] extended the inapproximability to MAX-Ek-SET-SPLITTING, for any constant $k \ge 4$, with the corresponding threshold of $(1 - 2^{-k+1})$.

In this work we prove the following hardness thresholds for these problems improving exponentially the non-constant parameter δ .

Theorem 1.2. Given an instance of MAX-E3-SAT of size n, it is quasi-NP-hard to decide whether it is satisfiable or at most $\frac{7}{8} + \delta$ fraction of the clauses can be satisfied, where $\delta = \frac{1}{(\log n)^c}$ for some positive constant c > 0.

¹For ease of presentation, in this paper we exclusively use a *stronger* notion of *quasi-NP-hardness*, i.e. a problem is quasi-NP-hard if it admits a DTIME($N^{O(\log \log N)}$) reduction from 3SAT. This differs from the weaker requirement of DTIME($N^{\operatorname{poly}(\log N)}$) reductions.

Theorem 1.3. For any fixed $k \ge 4$, given an instance of MAX-Ek-SET-SPLITTING of size n, it is quasi-NP-hard to decide whether there is a partition of the ground set into two subsets splitting all the k-sets in the collection or at most $(1 - 2^{-k+1}) + \delta$ fraction of the k-sets are split by any such partition, where $\delta = \frac{1}{(\log n)^c}$ for some positive constant c > 0.

The results of this paper are obtained using simpler PCP verifiers for the above problems, compared to the ones used by Håstad [Hås01], Guruswami et al. [GHS02] and Holmerin [Hol02]. In particular, we avoid the use of a *variable bias* parameter, which yields an exponential improvement in the inapproximability thresholds. This also considerably simplifies our analysis compared to previous works. In addition, for proving Theorem 1.1, we are able to use an estimate of the mixing probability of Markov Chains over general product spaces shown – using *uniform* reverse hypercontractivity – by Mossel et al. [MOS13]. The proofs of Theorems 1.2 and 1.3 use well known techniques from Fourier Analysis, while avoiding some of the complications in previous results. We remark that the starting point of our hardness reductions is the standard Label Cover problem instead of the so-called *Smooth* Label Cover which can also be used to avoid the variable bias but incurs the same loss in the hardness factors [Kho]. Section 1.4 elaborates more on the techniques used in this paper.

Our results also yield similar improvements in the hardness for satisfiable instances for other predicates whose inapproximability in [Hås01] is shown to follow from the PCP verifiers used for satisfiable MAX-E3-SAT and MAX-E4-SET-SPLITTING. The reader is referred to Theorems 6.15, 6.18, 7.17 and 7.18 and Section 9 of [Hås01] for more details on these predicates.

1.1 Problem Definition

For a hypergraph G, let IS(G) be the size of its maximum independent set and let $\chi(G)$ be its chromatic number. The following is the problem of finding independent sets in q-colorable hypergraphs.

Definition 1.4. ISCOLOR(k, q, Q): Given a k-uniform hypergraph G(V, E), decide between,

(i) YES Case:
$$\chi(G) \le q$$
. (ii) NO Case: $\mathsf{IS}(G) < \frac{|V|}{Q}$

The problem defined above is a generalization of hypergraph coloring: if ISCOLOR(k, q, Q) is NP-hard for some parameters $q, Q \in \mathbb{Z}^+$ then it is NP-hard to color a q-colorable k-uniform hypergraph with Q colors.

The following constraint satisfaction problems are studied in this paper.

Definition 1.5. An Ek-CNF formula is a conjunction of clauses (disjunctions), where each clause has exactly k literals. It is said to be satisfiable is there is an assignment to the variables such that each clause has at least one true literal, i.e. is satisfied.

Definition 1.6. An instance of MAX-Ek-SAT is an Ek-CNF formula, and the goal is to find an assignment to satisfy the maximum number of clauses. In satisfiable MAX-Ek-SAT, the input is a satisfiable Ek-CNF formula.

In this paper we study the above for k = 3, i.e. MAX-E3-SAT.

Definition 1.7. An instance of MAX-Ek-SET-SPLITTING is a ground set and a collection of its subsets, each of size exactly k. The goal is to find a partition of the ground set into two subsets to maximize the number of split k-sets in the collection, i.e. which are not contained in one of the subsets of the partition. In satisfiable MAX-Ek-SET-SPLITTING, the input admits a partition that splits all k-sets in the collection.

1.2 Previous Work

The problem of finding large independent sets in q-colorable graphs and hypergraphs (for small values of q) is very well studied algorithmically. On 2-colorable, i.e. bipartite, graphs, the maximum independent set can be computed in polynomial time. A long line of research – [Wig83], [Blu94], [KMS98], [BK97], [ACC06], and [KT12] – has shown that a 3-colorable graph can be efficiently colored with n^{α} colors thus solving ISCOLOR(2, 3, n^{α}). The current best value of $\alpha \approx 0.2038$ is due to [KT12]. For 2-colorable 3-uniform hypergraphs Krivelevich et al. [KNS01] gave a coloring algorithm using $O(n^{1/5})$ colors. An upper bound of $O(n^{3/4})$ was shown for coloring 2-colorable 4-uniform hypergraphs by Chen and Frize [CF96] and Kelsen, Mahajan and Ramesh [KMH96].

On the complexity side, the work of Guruswami, Håstad and Sudan [GHS02] and Holmerin [Hol02] showed that ISCOLOR $(4, 2, O((\log \log \log n)^{-1}(\log \log n)))$ is quasi-NP-hard. Khot [Kho02a, Kho02b] showed the inapproximability of ISCOLOR $(4, 5, (\log n)^c)$ and ISCOLOR $(3, 3, (\log \log n)^c)$. Assuming the so called *Alpha Conjecture*, Dinur et al. [DMR09] showed that ISCOLOR(2, 3, C) is NP-hard for arbitrarily large constant C > 0. Recently, assuming the *d-to-1 Games Conjecture*, Khot and Saket [KS14] showed that ISCOLOR(3, 2, C) is similarly NP-hard.

In another recent work, Dinur and Guruswami [DG13] showed a hardness factor of $\exp\left(2^{\sqrt{\log \log n}}\right)$ for a variant of coloring 2-colorable 6-uniform hypergraphs. They also showed that ISCOLOR(6, 2, $(\log n)^c$) is quasi-NP-hard. The former result is obtained via a novel use of the recently introduced *Short* Code, while the latter result uses a more standard PCP verifier based on the *Long* Code. Building upon [DG13] and concurrent to our work, Guruswami, Harsha, Håstad, Srinivasan and Varma [GHH⁺13] proved the first super-polylogarithmic hardness for hypergraph coloring showing, in particular, the hardness of ISCOLOR $\left(8, 2, \exp\left(2^{\sqrt{\log \log n}}\right)\right)$, ISCOLOR $\left(4, 4, \exp\left(2^{\sqrt{\log \log n}}\right)\right)$ and ISCOLOR $\left(3, 3, (\log n)^{O(1/\log \log \log n)}\right)$. However, previous to our work the best inapproximability for case of 2-colorable 4-uniform hypergraphs remained the result of [GHS02].

For satisfiable MAX-E3-SAT studied in this paper, the random assignment gives a $\frac{7}{8}$ approximation. Karloff and Zwick [KZ97] showed a semi-definite programming (SDP) relaxation based algorithm yields the same factor on instances where each clause has *at most* 3 literals. Their algorithm can be used to obtain a (folklore) $\frac{7}{8} + \delta$ approximation in time poly $(n)2^{O(\delta n)}$. Håstad [Hås01] showed the inapproximability of satisfiable MAX-E3-SAT beyond the random assignment threshold. In particular an approximation of $\frac{7}{8} + \delta$ is NP-hard for any constant $\delta > 0$ and quasi-NP-hard for $\delta = \Omega$ ($(\log \log n)^{-1}(\log \log \log n)$). On the other hand, Max-E3-Set-Splitting is known to admit an approximation factor of 0.912 in the satisfiable case, while the best inapproximability is $\frac{19}{20} + \delta$ by Guruswami [Gur00]. However, satisfiable MAX-E4-SET-SPLITTING was shown by Håstad [Hås01] to be hard to approximate beyond its random assignment threshold, i.e. an approximation factor of $\frac{7}{8} + \delta$ is NP-hard for for any constant $\delta > 0$ and quasi-NPhard for $\delta = \Omega$ ($(\log \log n)^{-1}(\log \log \log n)$). Guruswami [Gur00] extended this to satisfiable MAX-E*k*-SET-SPLITTING for $k \ge 5$ with a corresponding inapproximability of $(1 - 2^{-k+1}) + \delta$. The techniques used in the above results can also be combined with the subconstant error Label Cover of Moshkovitz and Raz [MR10] to obtain NP-hardness for $\delta = \Omega$ ($(\log \log n)^{-O(1)}$).

1.3 Our Results

This paper shows the following quasi-NP-hardness results obtained via $DTIME(N^{O(\log \log N)})$ reductions from 3SAT.

Theorem. [Theorem 1.1] ISCOLOR $(4, 2, (\log n)^c)$ is quasi-NP-hard for some constant c > 0.

Theorem. [Theorem 1.2] Satisfiable MAX-E3-SAT on *n* variables is quasi-NP-hard to approximate within $\frac{7}{8} + \frac{1}{(\log n)^c}$ for some constant c > 0.

Theorem. [Theorem 1.3] For any $k \ge 4$, satisfiable MAX-Ek-SET-SPLITTING on a ground set of n elements is quasi-NP-hard to approximate within $(1 - 2^{-k+1}) + \frac{1}{(\log n)^c}$ for some constant c > 0.

The proofs of Theorems 1.1 and 1.2 are given in Sections 3 and 4 respectively. Theorem 1.3 follows from the following inapproximability of MAX-E4-SET-SPLITTING.

Theorem 1.8. There is a $DTIME(N^{O(\log \log N)})$ reduction from 3SAT to an instance of MAX-E4-SET-SPLITTING over a ground set of size n such that:

YES Case: There is a partition of the ground set which splits every 4-set of the instance.

NO Case: Any fraction $\rho > 0$ of the ground set completely contains at least $\rho^4 - \frac{1}{(\log n)^c}$ fraction of the 4-sets of the instance, for some constant c > 0.

Theorem 1.8 is proved in Section 5. For the reduction from Theorem 1.8 to Theorem 1.3 we refer the reader to Theorem 8 of [Gur00].

1.4 Techniques

The results of this paper, as well as those of [Hås01], [GHS02] and [Hol02] are obtained by constructing PCPs based on Long Codes, i.e. the verifier accepts or rejects based on a 3 or 4 query test of the supposed Long Code encodings. The main technical difference from previous works is our construction of the these tests. Let us for now focus on PCP verifier used to prove hardness of independent set in 2-colorable 4-uniform hypergraphs, and abstract out the essence of the verifier's test. The main ingredient is a joint distribution over $x, x', y, y' \in \{-1, 1\}^d$ for some parameter d, to satisfy (among others) the following property: for each $i, j \in [d]$, $(x_i, x'_i, y_j, y'_j) \notin \{(1, 1, 1, 1), (-1, -1, -1, -1)\}$. The tuples x and x' constitute the building blocks of two queries made by the verifier to one purported Long Code encoding, whereas y and y' correspond to those made to a second encoding.

The difference of our PCP test in comparison to that of previous works is illustrated in the distribution of the pair (x, x') (which is identical to that of (y, y')). In previous works this is induced by the following randomized process. Select x uniformly at random and with probability $\frac{1}{2}$ set x' = -x, otherwise: for each $i \in [d]$ independently, set $x'_i = x_i$ with probability $1 - \varepsilon$ and $x'_i = -x_i$ with probability ε . Here ε is a bias parameter which itself is chosen from an appropriate distribution. This variable bias leads to an exponential loss in the hardness factor.

In our case, the distribution on (x, x') is simpler and obtained as follows. Select x u.a.r. and with probability $\frac{1}{2}$ set x' = -x, otherwise select x' u.a.r. This test does not use any variable bias. Additionally, -x' is a $\frac{1}{2}$ -correlated copy of x which allows us to use results of Mossel, Oleszkiewicz and Sen [MOS13] to estimate the mixing probability of the corresponding Markov Chain. This estimate is based on the uniform reverse hypercontractivity proved in the same work, wherein the parameters do not depend on the measure of the smallest atom in the probability space. This property – unlike the usual hypercontractivity inequality – is crucial for us, as the smallest atom in our application has measure exponential in d, which one cannot afford.

The PCP verifiers for satisfiable MAX-E3-SAT and MAX-E4-SET-SPLITTING also use similar distributions as above. While their analysis does not require any mixing probability estimate, the avoidance of the variable bias improves the inapproximability threshold and simplifies the analysis substantially.

2 **Preliminaries**

Let us define the following notion of ρ -correlated spaces used by Mossel et al. [MOS13].

Definition 2.1. Consider a product space $(\Omega, \mu) = (\prod_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mu_i)$ where (Ω_i, μ_i) are finite probability spaces. We say that $(X, Y) \in \Omega^2$ are ρ -correlated if X is distributed according to μ and the conditional distribution of Y given X is as follows: for each i independently, with probability ρ , $Y_i = X_i$ and with probability $1 - \rho$, Y_i is sampled independently from μ_i .

For our analysis in Section 3 we require an estimate of the mixing probability of Markov Chains over general product spaces. The corresponding bound for the case of the boolean hypercube was proved by Mossel, O'Donnell, Regev, Steif and Sudakov [MOR⁺06] using reverse hypercontractivity over the boolean domain. The generalization below was subsequently shown by Mossel et al. [MOS13], using uniform reverse hypercontractivity over general product spaces proved in the same work. We refer the reader to [MOS13] for more details.

Theorem 2.2. Let (Ω, μ) be the product probability space in Definition 2.1. Let $A, B \subseteq \Omega$ be two sets such that $\mu\{A\}, \mu\{B\} \ge \delta \ge 0$. Let X be distributed according to the product measure μ and Y be a ρ -correlated copy of X for some $0 \le \rho \le 1$. Then,

$$\Pr\left[X \in A, Y \in B\right] \ge \delta^{\frac{2-\sqrt{\rho}}{1-\sqrt{\rho}}}.$$

The starting point of the reductions in this paper is the LABELCOVER problem which is defined as follows.

Definition 2.3. An instance \mathcal{L} of LABELCOVER consists of a bipartite graph G(U, V, E) along with label sets [k] and [m]. For each edge e between $u \in U$ and $v \in V$, there is a projection $\pi_{vu} : [m] \mapsto [k]$. A labeling $l_u \in [k]$ to u and $l_v \in [m]$ to v satisfies the edge if $\pi_{vu}(l_v) = l_u$. The goal is to find a labeling of Uand V to satisfy the maximum number of edges.

The inapproximability of LABELCOVER stated below follows from the PCP Theorem [AS98, ALM⁺98], Raz's Parallel Repetition Theorem [Raz98] and a structural property proved by Håstad [Hås01].

Theorem 2.4. For every positive integer r, there is a deterministic $N^{O(r)}$ time reduction from a 3SAT instance of size N to an instance $\mathcal{L}(G(U, V, E), \{\pi_{vu}\}_{\{v,u\}\in E}, [k], [m])$ of LABELCOVER with the following properties:

- a. $|U|, |V| \leq N^{O(r)}$. $k, m \leq 2^{3r}$. G is bi-regular with left and right degrees bounded by $2^{O(r)}$.
- b. There is a universal constant $c_0 > 0$ such that for any $v \in V$ and $S \subseteq [m]$, taking an expectation over a random neighbor u of v,

$$\mathbb{E}\left[|\pi_{vu}(S)|^{-1}\right] \le |S|^{-2c_0}.$$

The above implies that with probability over a random neighbor u of v,

$$\Pr\left[|\pi_{vu}(S)| < |S|^{c_0}\right] \le |S|^{-c_0}.$$

c. There is a universal constant $\gamma_0 > 0$ such that,

YES Case: If the 3SAT instance is satisfiable then there is a labeling to U and V that satisfies all edges of \mathcal{L} .

NO Case: If the 3SAT instance is unsatisfiable then any labeling to U and V satisfies at most $2^{-\gamma_0 r}$ fraction of the edges.

3 Independent Set in 2-Colorable 4-Uniform Hypergraphs

In this section we give a hardness reduction from an instance of LABELCOVER to a 4-uniform hypergraph proving Theorem 1.1.

The input is an instance \mathcal{L} of LABELCOVER from Theorem 2.4 consisting of a bipartite graph G(U, V, E), label sets [k] and [m] and projections $\{\pi_{vu} : [m] \mapsto [k] \mid \{u, v\} \in E, u \in U, v \in V\}$. The following is the construction of the 4-uniform hypergraph $\mathcal{G}(\mathcal{H}, \mathcal{E})$.

Vertices. For each vertex $v \in V$, we have a copy of the binary Long Code over domain [m], viz. $\mathcal{H}^v := \{-1, 1\}^m$. Clearly the number of vertices in each \mathcal{H}^v is the same : 2^m . The set of vertices \mathcal{H} is the union of all the copies, i.e. $\mathcal{H} = \bigcup_{v \in V} \mathcal{H}^v$.

Hyperedges. The hyperedges \mathcal{E} are added via the following procedure.

- 1. Choose a vertex $u \in U$ u.a.r and two of its neighbors $v, w \in V$ independently and u.a.r.
- 2. Let $x, x' \in \mathcal{H}^v$ and $y, y' \in \mathcal{H}^w$ be chosen as follows. For each $i \in [k]$, with probability $\frac{1}{2}$ do Step 2a and with probability $\frac{1}{2}$ do Step 2b.
 - 2a. Independently for each $j \in \pi_{vu}^{-1}(i)$ choose x_j u.a.r. from $\{-1, 1\}$ and set x'_j to be $-x_j$. Independently for each $j \in \pi_{wu}^{-1}(i)$ choose y_j and y'_j independently and u.a.r. from $\{-1, 1\}$.
 - 2b. Independently for each $j \in \pi_{wu}^{-1}(i)$ choose y_j u.a.r. from $\{-1, 1\}$ and set y'_j to be $-y_j$. Independently for each $j \in \pi_{vu}^{-1}(i)$ choose x_j and x'_j independently and u.a.r. from $\{-1, 1\}$.
- 3. For all possible choices of $u \in U$, $v, w \in V$, $x, x' \in \mathcal{H}^v$ and $y, y' \in \mathcal{H}^w$ in the above steps, add a hyperedge between x, x', y, y'.

3.1 Analysis: YES Case

In the YES case there is a labeling $\sigma : V \mapsto [m]$ such that for any $u \in U$ and neighbors v, w of u, $\pi_{vu}(\sigma(v)) = \pi_{wu}(\sigma(w))$. We partition the vertex set \mathcal{H} into two disjoint subsets \mathcal{H}_{-1} and \mathcal{H}_1 where, $\mathcal{H}_{\ell} \cap \mathcal{H}^v = \{z \in \mathcal{H}^v \mid z_{\sigma(v)} = \ell\}$, for $\ell \in \{-1, 1\}$.

Consider a choice of $u \in U$ and two of its neighbors v and w in Step 1 of the hyperedges construction. Steps 2a and 2b ensure that either $x_{\sigma(v)} = -x'_{\sigma(v)}$ or $y_{\sigma(w)} = -y'_{\sigma(w)}$, as $\pi_{vu}(\sigma(u)) = \pi_{wu}(\sigma(w))$. Thus, no hyperedge lies completely in either \mathcal{H}_{-1} or \mathcal{H}_{1} and hypergraph \mathcal{G} is 2-colorable.

3.2 Analysis: NO Case

Suppose for a contradiction that there is an independent set $\mathcal{I} \subseteq \mathcal{H}$ such that $|\mathcal{I}| \ge \delta |\mathcal{H}|$. Our analysis shall show that this implies a labeling to the LABELCOVER instance \mathcal{L} that satisfies $\delta^{O(1)}$ fraction of its edges. This is in contrast to the bound of $\delta^{O(\delta^{-1})}$ obtained in [GHS02], [Hol02].

By averaging, for at least $\delta/2$ fraction of the vertices $v \in V$, $|\mathcal{I} \cap \mathcal{H}^v| \ge (\delta/2)|\mathcal{H}^v|$. Call such vertices as "good" vertices. We use \mathcal{I}^v to denote $\mathcal{I} \cap \mathcal{H}^v$ for any $v \in V$.

For now fix a choice of "good" vertices v and w that share a neighbor $u \in U$. Let $A : \mathcal{H}^v \mapsto \{0, 1\}$ be the indicator of the subset \mathcal{I}^v . Similarly, let $B : \mathcal{H}^w \mapsto \{0, 1\}$ be the indicator for \mathcal{I}^w . Thus we have,

$$\mathbb{E}_{x \in \mathcal{H}^v}[A(x)] \ge \delta/2 \quad , \quad \mathbb{E}_{y \in \mathcal{H}^w}[B(y)] \ge \delta/2. \tag{1}$$

Furthermore, since \mathcal{I} is an independent set, we have,

$$\mathbb{E}_{x,x',y,y'}\left[A(x)A(x')B(y)B(y')\right] = 0,$$
(2)

where the expectation is according to the distribution induced by Steps 2, 2a and 2b of the hyperedges construction. Expanding out the Fourier expansion of the above product we obtain,

$$\mathbb{E}_{x,x',y,y'}\left[\sum_{\substack{\alpha,\alpha',\\\beta,\beta'\subseteq[m]}}\widehat{A}_{\alpha}\widehat{A}_{\alpha'}\widehat{B}_{\beta}\widehat{B}_{\beta'}\chi_{\alpha}(x)\chi_{\alpha'}(x')\chi_{\beta}(y)\chi_{\beta'}(y')\right]=0.$$

Dropping the subscripts from the expectation and taking it inside summation,

$$\sum_{\substack{\alpha,\alpha',\\\beta,\beta'\subseteq[m]}} \widehat{A}_{\alpha}\widehat{A}_{\alpha'}\widehat{B}_{\beta}\widehat{B}_{\beta'}\mathbb{E}\left[\chi_{\alpha}(x)\chi_{\alpha'}(x')\chi_{\beta}(y)\chi_{\beta'}(y')\right] = 0.$$
(3)

Using the properties of the distribution induced in Steps 2-2b of the construction, we have the following lemma.

Lemma 3.1. Unless $\alpha = \alpha'$, $\beta = \beta'$ and $\pi_{vu}(\alpha) \cap \pi_{wu}(\beta) = \emptyset$,

$$\mathbb{E}\left[\chi_{\alpha}(x)\chi_{\alpha'}(x')\chi_{\beta}(y)\chi_{\beta'}(y')\right] = 0.$$

Proof. It can be seen that x_j and $x'_{j'}$ are independent for $j' \neq j$, and either one is independent of y and y'. Thus the expectation vanishes if $\alpha \neq \alpha'$. An identical argument handles the case when $\beta \neq \beta'$.

We may assume that $\alpha = \alpha'$ and $\beta = \beta'$. Consider the case when $i \in \pi_{vu}(\alpha) \cap \pi_{wu}(\beta)$. From the construction, in Step 2a, the variables $\{y_j, y'_j | j \in \pi_{wu}^{-1}(i) \cap \beta\}$ are chosen independently and u.a.r. from $\{-1, 1\}$. Otherwise, in Step 2b, the variables $\{x_j, x'_j | j \in \pi_{vu}^{-1}(i) \cap \alpha\}$ are chosen independently and u.a.r. from $\{-1, 1\}$. In both cases the expectation vanishes.

Observe that $\pi_{vu}(\alpha) \cap \pi_{wu}(\beta) = \emptyset$ implies that the variable $\chi_{\alpha}(x)\chi_{\alpha}(x') = \chi_{\alpha}(xx')$ is independent of $\chi_{\beta}(y)\chi_{\beta}(y') = \chi_{\beta}(yy')$. For convenience we use the following notation:

$$\Gamma_{\alpha}^{vu} := \mathbb{E}\left[\chi_{\alpha}(xx')\right] \quad \text{and} \quad \Gamma_{\beta}^{wu} := \mathbb{E}\left[\chi_{\beta}(yy')\right]. \tag{4}$$

Note that Γ_{α}^{vu} and Γ_{β}^{wu} depend on the projections π_{vu} and π_{wu} respectively. Using Lemma 3.1 and Equation (3) we obtain,

$$\sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)=\emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\Gamma_{\alpha}^{vu}\Gamma_{\beta}^{wu} = 0.$$
(5)

Using standard arguments along with the fact that x_i is independent of x'_j for $i \neq j$ we obtain,

$$\mathbb{E}_{x,x'} \left[A(x)A(x') \right] = \sum_{\alpha} \widehat{A}_{\alpha}^{2} \mathbb{E} \left[\chi_{\alpha}(xx') \right]$$
$$= \sum_{\alpha} \widehat{A}_{\alpha}^{2} \Gamma_{\alpha}^{vu}.$$
(6)

Similarly,

$$\mathbb{E}_{y,y'}\left[B(y)B(y')\right] = \sum_{\beta} \widehat{B}_{\beta}^2 \Gamma_{\beta}^{wu}.$$
(7)

To use the above equalities, the goal of the next lemma is to lower bound $\mathbb{E}[A(x)A(x')]$ and $\mathbb{E}[B(y)B(y')]$.

Lemma 3.2. For x, x', y and y' as distributed in Steps 2-2b of the construction of the hyperedges,

$$\mathbb{E}[A(x)A(x')] \ge (\delta/2)^{c_1} \quad , \quad \mathbb{E}[B(y)B(y')] \ge (\delta/2)^{c_1}$$

where $c_1 = \frac{2\sqrt{2}-1}{\sqrt{2}-1}$ is an absolute constant.

Proof. Let us consider $\mathbb{E}[A(x)A(x')]$. The proof for $\mathbb{E}[B(y)B(y')]$ is analogous. Recall that A is the indicator for $\mathcal{I}^v \subseteq \mathcal{H}^v$. Let $-\mathcal{I}^v := \{-x \mid x \in \mathcal{I}^v\}$. It is easy to see that,

$$\mathbb{E}\left[A(x)A(x')\right] = \Pr\left[x \in \mathcal{I}^{v}, x' \in \mathcal{I}^{v}\right] = \Pr\left[x \in \mathcal{I}^{v}, -x' \in -\mathcal{I}^{v}\right] = \Pr\left[x \in \mathcal{I}^{v}, x'' \in -\mathcal{I}^{v}\right], \quad (8)$$

where we use x'' to denote -x'.

Consider the product probability space $(\Omega, \mu) = (\prod_{i=1}^{k} \Omega_i, \otimes_{i=1}^{k} \mu_i)$, where for each $i \in [k]$, $\Omega_i = \{-1, 1\}^{\pi_{vu}^{-1}(i)}$ and μ_i is the uniform measure. Thus, a uniformly random $x \in \mathcal{H}^v$ (as chosen in Steps 2-2b of the construction) can be thought of as belonging to (Ω, μ) with $x|_{\pi_{vu}^{-1}(i)}$ being drawn from (Ω_i, μ_i) independently for each $i \in [k]$. In Equation (8), both x and x'' have uniform marginals distributions. Furthermore, given x, independently for each $i \in [k]$, with probability $\frac{1}{2}$, $x''|_{\pi_{vu}^{-1}(i)} = x|_{\pi_{vu}^{-1}(i)}$ and with probability $\frac{1}{2}$, $x''|_{\pi_{vu}^{-1}(i)}$ is chosen uniformly from (Ω_i, μ_i) . Thus, x and x'' are ρ -correlated elements of (Ω, μ) with $\rho = \frac{1}{2}$, according to Definition 2.1. Since $\mu(\mathcal{I}^v) = \mu(-\mathcal{I}^v) \geq \delta$, applying Theorem 2.2 to Equation 8 we obtain,

$$\mathbb{E}\left[A(x)A(x')\right] = \Pr\left[x \in \mathcal{I}^{v}, x'' \in -\mathcal{I}^{v}\right] \ge (\delta/2)^{c_{1}},\tag{9}$$

which completes the proof of the lemma.

Using the above lemma along with Equations (6) and (7) we have,

$$\left(\frac{\delta}{2}\right)^{2c_1} \le \left(\sum_{\alpha} \widehat{A}^2_{\alpha} \Gamma^{vu}_{\alpha}\right) \left(\sum_{\beta} \widehat{B}^2_{\beta} \Gamma^{wu}_{\beta}\right) = \sum_{\alpha,\beta} \widehat{A}^2_{\alpha} \widehat{B}^2_{\beta} \Gamma^{vu}_{\alpha} \Gamma^{wu}_{\beta}.$$
(10)

Subtracting Equation (5) from Equation (10), we obtain,

$$\sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\Gamma_{\alpha}^{vu}\Gamma_{\beta}^{wu} \ge \left(\frac{\delta}{2}\right)^{2c_{1}}.$$
(11)

To continue with the analysis we calculate Γ^{vu}_{α} and Γ^{wu}_{β} in the following lemma.

Lemma 3.3. Let,

$$\pi_{vu}^{\mathsf{odd}}(\alpha) := \left\{ i \in \pi_{vu}(\alpha) | \left| \pi_{vu}^{-1}(i) \cap \alpha \right| \text{ is odd} \right\}, \text{ and, } \pi_{wu}^{\mathsf{odd}}(\beta) := \left\{ i \in \pi_{wu}(\beta) | \left| \pi_{wu}^{-1}(i) \cap \beta \right| \text{ is odd} \right\}.$$

Then,

$$\Gamma_{\alpha}^{vu} = \frac{(-1)^{\left|\pi_{vu}^{\mathsf{odd}}(\alpha)\right|}}{2^{\left|\pi_{vu}(\alpha)\right|}}, \quad \text{and,} \quad \Gamma_{\beta}^{wu} = \frac{(-1)^{\left|\pi_{wu}^{\mathsf{odd}}(\beta)\right|}}{2^{\left|\pi_{wu}(\beta)\right|}}.$$

Proof. From the definition of Γ_{α}^{vu} we can rewrite it as,

$$\Gamma_{\alpha}^{vu} = \mathbb{E}\left[\prod_{i \in [k]} \left(\prod_{j \in \pi_{vu}^{-1}(i) \cap \alpha} x_j x_j'\right)\right] = \prod_{i \in [k]} \mathbb{E}\left[\prod_{j \in \pi_{vu}^{-1}(i) \cap \alpha} x_j x_j'\right].$$
(12)

For a given $i \in [k]$, with probability $\frac{1}{2}$ all the variables x_j, x'_j $(j \in \pi_{vu}^{-1}(i))$ are uniformly random and independent. Otherwise, x_j $(j \in \pi_{vu}^{-1}(i))$ are chosen independently u.a.r and each x'_j is set to $-x_j$. Thus,

$$\mathbb{E}\left[\prod_{j\in\pi_{vu}^{-1}(i)\cap\alpha} x_j x_j'\right] = \begin{cases} \frac{1}{2} & \text{if } |\pi_{vu}^{-1}(i)\cap\alpha| \text{ is even.} \\ -\frac{1}{2} & \text{otherwise.} \end{cases}$$
(13)

Substituting the above in Equation (12) proves the lemma for Γ_{α}^{vu} . The proof for Γ_{β}^{wu} is analogous.

Let R and T (R > T) be positive integers to be determined later. Using the above lemma and Equation (11) we obtain,

$$\sum_{\substack{|\alpha|< R, |\beta|< R\\ \pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}} \widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2} + \sum_{\substack{[(|\alpha|\geq R, |\pi_{vu}(\alpha)|< T)\\ \vee(|\beta|\geq R, |\pi_{wu}(\beta)|< T)],\\ \pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}} \widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}$$

$$+ \sum_{\substack{[(|\alpha|\geq R, |\pi_{vu}(\alpha)|\geq T)\\ \vee(|\beta|\geq R, |\pi_{wu}(\beta)|\geq T)\\ \vee(|\beta|\geq R, |\pi_{wu}(\beta)|\geq T)],\\ \pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}} \widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}2^{-(|\pi_{vu}(\alpha)|+|\pi_{wu}(\beta)|)} \geq \left(\frac{\delta}{2}\right)^{2c_{1}}.$$
(14)

The third term on the LHS of the above inequality is at most $2^{-T} \left(\sum_{\alpha} \widehat{A}_{\alpha}^2 \right) \left(\sum_{\beta} \widehat{B}_{\beta}^2 \right) \leq 2^{-T}$, using Parseval's identity and the fact that A and B are indicator functions. Similarly, the second term in the LHS of Equation (14) is upper bounded by,

$$\left(\sum_{\substack{|\alpha|\geq R,\\ |\pi_{vu}(\alpha)|< T}} \widehat{A}^2_{\alpha}\right) \left(\sum_{\beta} \widehat{B}^2_{\beta}\right) + \left(\sum_{\alpha} \widehat{A}^2_{\alpha}\right) \left(\sum_{\substack{|\beta|\geq R,\\ |\pi_{wu}(\beta)|< T}} \widehat{B}^2_{\beta}\right) \leq \sum_{\substack{|\alpha|\geq R,\\ \pi_{vu}(\alpha)< T}} \widehat{A}^2_{\alpha} + \sum_{\substack{|\beta|\geq R,\\ \pi_{wu}(\beta)< T}} \widehat{B}^2_{\beta}.$$

Substituting the above in Equation (14) we obtain that for any two good vertices $v, w \in V$ which share a neighbor $u \in U$,

$$\sum_{\substack{|\alpha|< R, |\beta|< R\\ \pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}} \widehat{A}_{\alpha}^2 \widehat{B}_{\beta}^2 + \sum_{\substack{|\alpha|\geq R, \\ |\pi_{vu}(\alpha)|< T}} \widehat{A}_{\alpha}^2 + \sum_{\substack{|\beta|\geq R, \\ |\pi_{wu}(\beta)|< T}} \widehat{B}_{\beta}^2 \geq \left(\frac{\delta}{2}\right)^{2c_1} - 2^{-T}.$$
(15)

Consider the following process of selecting u, v and w. Choose u u.a.r from U and v and w be two neighbors of u chosen independently and u.a.r from its neighborhood. Let p_u be the fraction of the neighbors of u that are "good". Since, $\delta/2$ fraction of the vertices in V are good and the graph G(U, V, E) is bi-regular, $\mathbb{E}_{u \in U}[p_u] \ge (\delta/2)$. Thus, the probability that both v and w are "good" is $\mathbb{E}_u[p_u^2] \ge (\mathbb{E}_u[p_u])^2 \ge (\delta/2)^2$.

Taking an expectation over the choice of u, v and w, and noting that the LHS of Equation (15) is always positive, we obtain,

$$\mathbb{E}_{u,v,w}\left[\sum_{\substack{|\alpha|< R, |\beta|< R\\ \pi_{vu}(\alpha)\cap \pi_{wu}(\beta)\neq \emptyset}} \widehat{A}_{\alpha}^{2} \widehat{B}_{\beta}^{2} + \sum_{\substack{|\alpha|\geq R, \\ \pi_{vu}(\alpha)< T}} \widehat{A}_{\alpha}^{2} + \sum_{\substack{|\beta|\geq R, \\ \pi_{wu}(\beta)< T}} \widehat{B}_{\beta}^{2}\right] \geq \left(\frac{\delta}{2}\right)^{2} \left[\left(\frac{\delta}{2}\right)^{2c_{1}} - 2^{-T}\right].$$
(16)

In order to bound the second and third terms in the above expectation we use property (b) in Theorem 2.4. For a fixed vertex $v \in V$ and subset $\alpha \subseteq [m]$ such that $|\alpha| \ge R$,

$$\Pr_{u}[|\pi_{vu}(\alpha)| < R^{c_0}] \leq \frac{1}{R^{c_0}},\tag{17}$$

where the probability is over a random neighbor u of v. Thus,

$$\mathbb{E}_{u}\left[\sum_{\substack{|\alpha|\geq R,\\|\pi_{vu}(\alpha)|< R^{c_{0}}}}\widehat{A}_{\alpha}^{2}\right] = \sum_{|\alpha|\geq R}\left[\widehat{A}_{\alpha}^{2}\cdot\Pr_{u}\left[|\pi_{vu}(\alpha)|< R^{c_{0}}\right]\right] \leq \sum_{|\alpha|\geq R}\widehat{A}_{\alpha}^{2}\cdot\frac{1}{R^{c_{0}}} \leq \frac{1}{R^{c_{0}}}.$$
 (18)

Setting $T = R^{c_0}$ and substituting the above into Equation (16) we obtain,

$$\mathbb{E}_{u,v,w}\left[\sum_{\substack{|\alpha|< R, |\beta|< R\\ \pi_{vu}(\alpha)\cap \pi_{wu}(\beta)\neq \emptyset}} \widehat{A}_{\alpha}^2 \widehat{B}_{\beta}^2\right] \ge \left(\frac{\delta}{2}\right)^2 \left[\left(\frac{\delta}{2}\right)^{2c_1} - 2^{-R^{c_0}}\right] - \frac{2}{R^{c_0}}.$$
(19)

Let $c' = 2 + 2c_1$. Setting $R = 8/(\delta/2)^{c'/c_0}$ and using $2^{-R^{c_0}} \leq R^{-c_0}$ in the above inequality yields,

$$\mathbb{E}_{u,v,w}\left[\sum_{\substack{|\alpha|,|\beta|<8/(\delta/2)^{c'/c_0}\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\right] \geq \frac{1}{4}\left(\frac{\delta}{2}\right)^{c'}.$$
(20)

Labeling. The above analysis yields the following randomized labeling σ of the vertices of \mathcal{L} . For a vertex $v \in V$, choose a subset α probability \widehat{A}^2_{α} and assign as label $\sigma(v)$ a random $i \in \alpha$. For any vertex $u \in U$, randomly choose a neighbor w and assign $\pi_{wu}(\sigma(w))$ as the label to u. Equation (20) implies that the expected fraction of constraints satisfied is at least,

$$\frac{1}{256} \left(\frac{\delta}{2}\right)^{c'+2c'/c_0}.$$
(21)

3.2.1 Choice of parameters

In Theorem 2.4 we can choose $r = (\log \log N)/4$. This ensures that the instance \mathcal{G} is of size $n = N^{O(r)}2^{2^{3r}} \leq N^{O(\log \log N)}$. The soundness of \mathcal{L} is $2^{-\Omega(\log \log N)} = 2^{-\Omega(\log \log n)}$. Combining this with the above analysis in the NO Case, choosing $\delta = \frac{1}{(\log n)^c}$ for some positive constant c (depending on c_0, c_1, γ_0) we obtain a contradiction to our assumption on the size of the independent set.

we obtain a contradiction to our assumption on the size of the independent set. Thus, in the NO Case, \mathcal{G} does not contain independent set of $\frac{1}{(\log n)^c}$ relative size. This completes the proof of Theorem 1.1.

4 Satisfiable MAX-E3-SAT

As before, the input is an instance \mathcal{L} of LABELCOVER from Theorem 2.4 consisting of a bipartite graph G(U, V, E), label sets [m] and [k] and projections $\{\pi_{vu} : [m] \mapsto [k] \mid \{u, v\} \in E, u \in U, v \in V\}$.

The PCP proof is the same as in [Hås01]. For each vertex $u \in U$ there is a Long Code $\mathcal{H}^u = \{-1, 1\}^k$. Similarly, for each $v \in V$, there is $\mathcal{H}^v = \{-1, 1\}^m$. The assignments to these Long Codes are $A^u : \mathcal{H}^u \mapsto \{-1, 1\}$ and $B^v : \mathcal{H}^v \mapsto \{-1, 1\}$. We can assume that these assignments are *folded* over -1, i.e. $A^u(x) = -A^u(-x)$ and $B^v(y) = -B^v(-y)$.

The instance of MAX-E3-SAT is given by the following PCP verifier whose acceptance predicate corresponds to a 3-literal clause. Let $\varepsilon > 0$ be a parameter which we shall set later.

PCP Verifier

- 1. Choose a vertex $u \in U$ u.a.r and one of its neighbors $v \in V$ u.a.r.
- 2. Choose $x \in \mathcal{H}^u$ u.a.r.
- 3. Let $y, y' \in \mathcal{H}^v$ be chosen as follows. For each $i \in [k]$, if $x_i = 1$ do Step 3 otherwise do Step 4.
- 4. $x_i = 1$: Independently for each $j \in \pi_{vu}^{-1}(i)$ choose y_j u.a.r from $\{-1, 1\}$ and set $y'_j = -y_j$.
- 5. $x_i = -1$: Do Step 5a with probability 1ε , or Step 5b with probability ε .
 - 5a. Independently for each $j \in \pi_{vu}^{-1}(i)$ choose y_j u.a.r. from $\{-1, 1\}$ and set y'_j to be y_j .
 - 5b. Independently for each $j \in \pi_{vu}^{-1}(i)$, choose y_j and y'_j independently and u.a.r. from $\{-1, 1\}$.
- 6. Accept if $(A^u(x), B^v(y), B^v(y')) \neq (1, 1, 1)$.

The above PCP predicate (after folding) is equivalent – in terms of its completeness and soundness – to a gap instance of MAX-E3-SAT.

4.1 Analysis: YES Case

In the YES case there is a labeling σ to the vertices of \mathcal{L} that satisfies all the constraints. Consider the assignment $A^u(x) = x_{\sigma(u)}$ and similarly $B^v(y) = y_{\sigma(v)}$ for all $u \in U$ and $v \in V$. Clearly, these assignments are folded over -1. Furthermore, in the choice of x, y, y' in the PCP test, it is easy to see that $(x_{\sigma(u)}, y_{\sigma(v)}, y'_{\sigma(v)}) \neq (1, 1, 1)$ since $\pi_{vu}(\sigma(v)) = \sigma(u)$. Thus, the PCP test is always satisfied and there is an assignment that satisfies all the clauses of the corresponding MAX-E3-SAT instance.

For notational simplicity in the rest of the analysis we shall drop the superscripts to denote A^u by A and B^v by B.

4.2 Analysis: NO Case

Suppose for a contradiction that,

$$\mathbb{E}_{\substack{u,v\\x,y,y'}}\left[1 - \frac{(1+A(x))(1+B(y))(1+B(y'))}{8}\right] \ge \frac{7}{8} + \delta,$$
(22)

where the expectation is over the choices of the verifier and thus the LHS denotes the probability that the verifier accepts. We shall show that (for an appropriate setting of ε) there is a labeling to the vertices of \mathcal{L}

that satisfies $\delta^{O(1)}$ fraction of edges. This is in contrast to the PCP test in [Hås01] which yields a bound of $\delta^{O(\delta^{-1})}$.

In the following analysis we fix the choice of u and v for the time being.

Since the Long Codes are folded, we have $\mathbb{E}[A(x)] = \mathbb{E}[B(y)] = \mathbb{E}[B(y')] = 0$, as the distributions of $x \in \mathcal{H}^u$ and $y, y' \in \mathcal{H}^v$ are respectively uniform. Further, x is independent of y and independent of y', and thus $\mathbb{E}[A(x)B(y)] = \mathbb{E}[A(x)B(y')] = \mathbb{E}[A(x)]\mathbb{E}[B(y)] = 0$. The rest of the terms are analyzed as follows.

Lemma 4.1. $|\mathbb{E}[B(y)B(y')]| \leq \varepsilon/2.$

Proof. Using the Fourier expansion of *B*, and since *B* is folded,

$$\mathbb{E}[B(y)B(y')] = \sum_{|\beta| \text{ odd}} \widehat{B}_{\beta}^{2} \mathbb{E}\left[\chi_{\beta}(yy')\right]$$
$$= \sum_{|\beta| \text{ odd}} \widehat{B}_{\beta}^{2} \prod_{i \in [k]} \mathbb{E}\left[\chi_{\beta \cap \pi_{vu}^{-1}(i)}(yy')\right].$$
(23)

For an odd sized β , there is a $i \in [k]$ such that $|\beta \cap \pi_{vu}^{-1}(i)|$ is odd. It is easy to check that for such a $i, \mathbb{E}\left[\chi_{\beta \cap \pi_{vu}^{-1}(j)}(yy')\right] = -\varepsilon/2$. Also note that for any $i, \left|\mathbb{E}\left[\chi_{\beta \cap \pi_{vu}^{-1}(i)}(yy')\right]\right| \leq 1$. Thus, Equation (23) yields,

$$\left|\mathbb{E}[B(y)B(y')]\right| \leq (\varepsilon/2) \sum_{|\beta| \text{ odd}} \widehat{B}_{\beta}^2 = \varepsilon/2.$$

Lemma 4.2. For any positive integers R, T such that $R \ge T$,

$$\left| \mathbb{E}[A(x)B(y)B(y')] \right| \leq \left(\sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta) \\ |\beta| < R}} \widehat{A}_{\alpha}^2 \widehat{B}_{\beta}^2 \right)^{\frac{1}{2}} + \sum_{\substack{|\beta| \ge R \\ |\pi_{vu}(\beta)| < T}} \widehat{B}_{\beta}^2 + \left(1 - \frac{\varepsilon}{2}\right)^{\frac{T}{2}}.$$

Proof. Using the fact $y_j(y'_j)$ is independent of x_i for any i, the term $\mathbb{E}\left[\chi_{\alpha}(x)\chi_{\beta}(y)\chi_{\beta'}(y')\right]$ is zero unless $\beta = \beta'$ and $\alpha \subseteq \pi_{vu}(\beta)$. Thus,

$$\mathbb{E}[A(x)B(y)B(y')] = \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta)}} \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2}\mathbb{E}\left[\chi_{\alpha}(x)\chi_{\beta}(yy')\right]$$
$$= \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta)}} \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2}\left(\prod_{i \in \alpha} \mathbb{E}\left[x_{i}\chi_{\beta \cap \pi_{vu}^{-1}(i)}(yy')\right]\right)$$
$$\left(\prod_{i \in \pi_{vu}(\beta) \setminus \alpha} \mathbb{E}\left[\chi_{\beta \cap \pi_{vu}^{-1}(i)}(yy')\right]\right).$$
(24)

To simplify the above equation we require the following lemma.

Lemma 4.3. Fix $\beta \neq \emptyset$ and let $r = |\pi_{vu}(\beta)|$. For any $\alpha \subseteq \pi_{vu}(\beta)$ let,

$$p_{\beta}(\alpha) := \left| \prod_{i \in \alpha} \mathbb{E} \left[x_i \chi_{\beta \cap \pi_{vu}^{-1}(i)}(yy') \right] \right| \cdot \left| \prod_{i \in \pi_{vu}(\beta) \setminus \alpha} \mathbb{E} \left[\chi_{\beta \cap \pi_{vu}^{-1}(i)}(yy') \right] \right|.$$

Then,

$$p_{\beta}(\alpha) = \left(\frac{\varepsilon}{2}\right)^{r'} \left(1 - \frac{\varepsilon}{2}\right)^{r-r'},$$

where $r' = |\alpha \Delta \pi_{vu}^{\mathsf{odd}}(\beta)|$ and π_{vu}^{odd} is as defined in Lemma 3.3. Thus, $p_{\beta}(\alpha)$ is a probability measure over $\alpha \subseteq \pi_{vu}(\beta)$.

Proof. It is easy to verify that for any $i \in [k]$ and $J \subseteq \pi_{vu}^{-1}(i), J \neq \emptyset, |J|$ even,

$$\left|\mathbb{E}\left[x_i\chi_J(yy')\right]\right| = \frac{\varepsilon}{2}, \quad \left|\mathbb{E}\left[\chi_J(yy')\right]\right| = 1 - \frac{\varepsilon}{2}.$$
(25)

Similarly, for $i \in [k]$ and $J \subseteq \pi_{vu}^{-1}(i)$, |J| odd,

$$\left|\mathbb{E}\left[x_i\chi_J(yy')\right]\right| = 1 - \frac{\varepsilon}{2}, \quad \left|\mathbb{E}\left[\chi_J(yy')\right]\right| = \frac{\varepsilon}{2}.$$
(26)

The above equations imply,

$$\left|\prod_{i\in\alpha} \mathbb{E}\left[x_i \chi_{\beta\cap\pi_{vu}^{-1}(i)}(yy')\right]\right| = \left(1 - \frac{\varepsilon}{2}\right)^{\left|\alpha\cap\pi_{vu}^{\mathsf{odd}}(\beta)\right|} \left(\frac{\varepsilon}{2}\right)^{\left|\alpha\cap\overline{\pi_{vu}^{\mathsf{odd}}(\beta)}\right|},\tag{27}$$

and,

$$\prod_{i \in \pi_{vu}(\beta) \setminus \alpha} \mathbb{E}\left[\chi_{\beta \cap \pi_{vu}^{-1}(i)}(yy')\right] = \left(1 - \frac{\varepsilon}{2}\right)^{\left|\overline{\alpha} \cap \overline{\pi_{vu}^{\text{odd}}(\beta)}\right|} \left(\frac{\varepsilon}{2}\right)^{\left|\overline{\alpha} \cap \pi_{vu}^{\text{odd}}(\beta)\right|},$$
(28)

where, in the above two equations $\overline{}$ denotes the $\pi_{vu}(\beta) \setminus .$ operation. Combining them we obtain the lemma.

Using the above in Equation (24) we obtain,

$$\left|\mathbb{E}[A(x)B(y)B(y')]\right| \leq \sum_{\substack{|\alpha|,|\beta| \text{ odd}\\\alpha \subseteq \pi_{vu}(\beta)}} \left|\widehat{A}_{\alpha}\widehat{B}_{\beta}^{2}\right| p_{\beta}(\alpha).$$
⁽²⁹⁾

A categorization of the terms of the above inequality based on the parameters R and T yields,

$$\begin{aligned} \left| \mathbb{E}[A(x)B(y)B(y')] \right| &\leq \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta) \\ |\beta| < R}} \left| \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) + \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta) \\ |\beta| \geq R \\ |\pi_{vu}(\beta)| < T}} \left| \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) + \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta) \\ |\beta| \geq R \\ |\pi_{vu}(\beta)| \geq T}} \left| \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) + \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta) \\ |\beta| \geq R \\ |\pi_{vu}(\beta)| \geq T}} \left| \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) + \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ |\beta| \geq R \\ |\pi_{vu}(\beta)| \geq T}} \left| \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) + \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ |\beta| \geq R \\ |\pi_{vu}(\beta)| \geq T}} \left| \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) + \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ |\beta| \geq R \\ |\pi_{vu}(\beta)| \geq T}} \left| \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) + \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ |\beta| \geq R \\ |\beta| \geq R \\ |\pi_{vu}(\beta)| \geq T}} \left| \widehat{A}_{\alpha}\widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) + \sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ |\beta| \geq R \\ |\beta$$

For each of the three terms in the RHS above, we apply Cauchy-Schwarz in the following manner. Each \hat{B}_{β}^2 is multiplied by,

$$\sum_{\substack{\alpha \subseteq \pi_{vu}(\beta) \\ |\alpha| \text{ odd}}} \left| \widehat{A}_{\alpha} \right| p_{\beta}(\alpha) \leq \sum_{\substack{\alpha \subseteq \pi_{vu}(\beta) \\ |\alpha| \text{ odd}}} \left| \widehat{A}_{\alpha} \right| \sqrt{p_{\beta}(\alpha)} \sqrt{p_{\beta}(\alpha)}$$
$$\leq \left(\sum_{\substack{\alpha \subseteq \pi_{vu}(\beta) \\ |\alpha| \text{ odd}}} \widehat{A}_{\alpha}^{2} \right)^{\frac{1}{2}} \left(\sum_{\substack{\alpha \subseteq \pi_{vu}(\beta) \\ |\alpha| \text{ odd}}} p_{\beta}(\alpha) \right)^{\frac{1}{2}} \left(\max_{\substack{\alpha \subseteq \pi_{vu}(\beta) \\ \alpha \subseteq \pi_{vu}(\beta)}} \sqrt{p_{\beta}(\alpha)} \right) \tag{31}$$

By Parseval's and since $p_{\beta}(\alpha)$ is a probability measure over α we obtain that the RHS of Equation (31) is bounded by 1. Thus, the second term in the RHS of Equation (30) is bounded by,

$$\sum_{\substack{|\beta| \ge R \\ |\pi_{vu}(\beta)| < T}} \widehat{B}_{\beta}^2.$$
(32)

Further, observe that for β such that $\pi_{vu}(\beta) \geq T$,

$$p_{\beta}(\alpha) \leq \left(1 - \frac{\varepsilon}{2}\right)^T$$
,

for any $\alpha \subseteq \pi_{vu}(\beta)$. Thus, the third term in the RHS of Equation (30) is bounded by,

$$\left(1 - \frac{\varepsilon}{2}\right)^{\frac{T}{2}} \sum_{\beta} \widehat{B}_{\beta}^{2} \le \left(1 - \frac{\varepsilon}{2}\right)^{\frac{T}{2}}.$$
(33)

For the first term in the RHS of Equation (30), we use Equation (31) to obtain the following upper bound.

$$\sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta) \\ |\beta| < R}} \left| \widehat{A}_{\alpha} \widehat{B}_{\beta}^{2} \right| p_{\beta}(\alpha) \leq \sum_{\substack{|\beta| \text{ odd}, |\beta| < R}} \left(\sum_{\substack{\alpha \subseteq \pi_{vu}(\beta) \\ |\alpha| \text{ odd}}} \widehat{A}_{\alpha}^{2} \right)^{\frac{1}{2}} \widehat{B}_{\beta}^{2}$$

$$\leq \left(\sum_{\substack{|\beta| \text{ odd}, |\beta| < R}} \left(\sum_{\substack{\alpha \subseteq \pi_{vu}(\beta) \\ |\alpha| \text{ odd}}} \widehat{A}_{\alpha}^{2} \right) \widehat{B}_{\beta}^{2} \right)^{\frac{1}{2}} \left(\sum_{\substack{|\beta| \text{ odd}, |\beta| < R}} \widehat{B}_{\beta}^{2} \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{\substack{|\alpha|,|\beta| \text{ odd} \\ \alpha \subseteq \pi_{vu}(\beta) \\ |\beta| < R}} \widehat{A}_{\alpha}^{2} \widehat{B}_{\beta}^{2} \right)^{\frac{1}{2}}, \qquad (34)$$

where the second last inequality above is obtained by an application of Cauchy-Schwarz and the last by Parseval's. Substituting Equations (32), (33) and (34) into (30) completes the proof of the Lemma 4.2. \Box

Setting $T = R^{c_0}$, taking the expectation over a random neighbor u of a fixed v in Lemma 4.2 and using the analysis of Equation (18), we obtain,

$$\mathbb{E}_{u}\left[\left|\mathbb{E}_{x,y,y'}[A(x)B(y)B(y')]\right|\right] \leq \mathbb{E}_{u}\left[\left(\sum_{\substack{|\alpha|,|\beta| \text{ odd}\\\alpha \subseteq \pi_{vu}(\beta)\\|\beta| < R}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\right)^{\frac{1}{2}}\right] + \frac{1}{R^{c_{0}}} + \left(1 - \frac{\varepsilon}{2}\right)^{\frac{R^{c_{0}}}{2}}.$$
 (35)

Using the above, Lemma 4.1 and Equation (22) we obtain,

$$\mathbb{E}_{u,v}\left[\left(\sum_{\substack{|\alpha|,|\beta| \text{ odd}\\\alpha \subseteq \pi_{vu}(\beta)\\|\beta| < R}} \widehat{A}_{\alpha}^{2} \widehat{B}_{\beta}^{2}\right)^{\frac{1}{2}}\right] \ge 8\delta - \frac{\varepsilon}{2} - \frac{1}{R^{c_{0}}} - \left(1 - \frac{\varepsilon}{2}\right)^{\frac{R^{c_{0}}}{2}}.$$
(36)

Applying Cauchy-Schwarz and setting $R = \left(\frac{4}{\varepsilon} \log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{c_0}}$, simplifies the above to,

$$\left(\mathbb{E}_{u,v}\left[\sum_{\substack{|\alpha|,|\beta| \text{ odd}\\\alpha \subseteq \pi_{vu}(\beta)\\|\beta| < R}} \widehat{A}_{\alpha}^2 \widehat{B}_{\beta}^2\right]\right)^{\frac{1}{2}} \ge 8\delta - 2\varepsilon.$$
(37)

Finally, we set $\varepsilon = \delta$ to make the RHS of the above at least 6δ . This yields a labeling for the LABELCOVER instance \mathcal{L} : for every vertex $u \in U$ uniformly choose a subset α of labels with probability $\widehat{A^u}_{\alpha}^2$ and assign it a random label from α . Similarly, for every vertex $v \in V$ uniformly select a set of labels β with probability $\widehat{B^v}_{\beta}^2$ and assign it a random label from β . The above analysis shows that the expected fraction of edges satisfied is,

$$\frac{36\delta^2}{R} = \Omega\left(\delta^{c'}\right),\,$$

for some positive constant c' depending on c_0 .

4.2.1 Choice of parameters

In Theorem 2.4 we can choose $r = (\log \log N)/4$. This ensures that the reduction to MAX-E3-SAT is of size $n = N^{O(r)} 2^{2^{3r}} \leq N^{O(\log \log N)}$. The soundness of \mathcal{L} is $2^{-\Omega(\log \log N)} = 2^{-\Omega(\log \log n)}$. Combining this with the above analysis in the NO Case, choosing $\delta = \frac{1}{(\log n)^c}$ for some positive constant c (depending on c_0 and γ_0) we obtain a contradiction to our assumption on the probability of acceptance of the verifier.

Thus, in the NO Case, the verifier accepts with probability at most $\frac{7}{8} + \frac{1}{(\log n)^c}$. This completes the proof of Theorem 1.2.

5 Satisfiable MAX-E4-SET-SPLITTING

As in the previous sections, the input is an instance \mathcal{L} of LABELCOVER from Theorem 2.4 consisting of a bipartite graph G(U, V, E), label sets [m] and [k] and projections $\{\pi_{vu} : [m] \mapsto [k] \mid \{u, v\} \in E, u \in U, v \in V\}$.

The PCP proof is similar to the previous sections. For each vertex $v \in V$, there is a Long Code $\mathcal{H}^v = \{-1, 1\}^m$. The assignments to these Long Codes are $A^v : \mathcal{H}^v \mapsto \{-1, 1\}$. In the case of MAX-E4-SET-SPLITTING we *do not* have folding of the Long Codes.

The instance of MAX-E4-SET-SPLITTING is given by the following PCP verifier whose 4-query tests correspond to the 4-sets of the instance. The rejection probability of the predicate estimates the fraction 4-query tests completely contained in the subset corresponding to the 1s of the proof locations. Let $\varepsilon > 0$ be a parameter which shall be set later.

PCP Verifier.

- 1. Choose a vertex $u \in U$ u.a.r and two of its neighbors $v, w \in V$ independently and u.a.r.
- 2. Choose $x \in \mathcal{H}^v$ and $y \in \mathcal{H}^w$ independently and u.a.r.
- 3. For each $i \in [k]$, either do Step 3a or Step 3b with probability $\frac{1}{2}$ each.
 - 3a. For each $j \in \pi_{vu}^{-1}(i)$ set $x'_j = -x_j$. Further, with probability 1ε do Step 3a.1, or Step 3a.2 with probability ε .
 - 3a.1 For each $j \in \pi_{wu}^{-1}(i)$ set $y'_j = y_j$.
 - 3a.2 For each $j \in \pi_{wu}^{-1}(i)$ independently, set y'_j u.a.r from $\{-1, 1\}$.
 - 3b. For each $j \in \pi_{wu}^{-1}(i)$ set $y'_j = -y_j$. Further, with probability 1ε do Step 3b.1, or Step 3b.2 with probability ε .
 - 3b.1 For each $j \in \pi_{vu}^{-1}(i)$ set $x'_j = x_j$.
 - 3b.2 For each $j \in \pi_{vu}^{-1}(i)$ independently, set x'_i u.a.r from $\{-1, 1\}$.
- 4. Reject if $(A^{v}(x), A^{v}(x'), A^{w}(y), A^{w}(y')) = (1, 1, 1, 1).$

The above PCP verifier is equivalent – in terms of its completeness and soundness – to a gap instance of MAX-E4-SET-SPLITTING.

5.1 Analysis: YES Case

In the YES case there is a labeling σ to the vertices \mathcal{L} that satisfies all its edges. Consider the assignment $A^v(x) = x_{\sigma(v)}$ for all $v \in V$. For the choice of u, v, and w in the above PCP we have $\pi_{vu}(\sigma(v)) = \pi_{wu}(\sigma(w))$. Thus, from the choice of x, x', y, y' in the PCP test, it is easy to see that $(x_{\sigma(v)}, x'_{\sigma(v)}, y_{\sigma(w)}, y'_{\sigma(w)}) \notin \{(1, 1, 1, 1), (-1, -1, -1, -1)\}$. Thus, the PCP test is always satisfied and the there is an assignment that splits all the 4-sets of the MAX-E4-SET-SPLITTING instance.

For notational simplicity in the rest of the analysis we shall drop the superscripts to denote A^v by A and A^w by B.

5.2 Analysis: NO Case

The probability that the PCP verifier rejects is given by,

$$\frac{1}{16} \cdot \mathbb{E}\left[(1+A(x))(1+A(x'))(1+B(y))(1+B(y')) \right],$$

where the expectation is above the random choice of u, v, w, x, x', y and y' by the PCP verifier. Expanding the above we obtain that the probability of rejection of the verifier is,

$$\frac{1}{16} \cdot \mathbb{E} \left[1 + A(x) + A(x') + B(x) + B(y') + A(x)A(x') + A(x)B(y) + A(x)B(y') + A(x')B(y) + A(x')B(y') + B(y)B(y') + A(x)A(x')B(y) + A(x')B(y)B(y') + A(x)B(y)B(y') + A(x)A(x')B(y') + A(x)A(x')B(y)B(y') \right].$$
(38)

Let the number of 1s in the proof be exactly ρ fraction, i.e.,

$$\mathbb{E}_{v,x}[A^{v}(x)] = 2\rho - 1.$$
(39)

where the expectation is over a random vertex $v \in V$ and a uniformly chosen $x \in \mathcal{H}^v$. Assume that the probability that the verifier rejects is at most $\rho^4 - \delta$ for some $\delta > 0$. We shall show that this implies a labeling to the vertices of \mathcal{L} that satisfies $\delta^{O(1)}$ fraction of edges (using an appropriate choice of $\varepsilon > 0$ depending only on δ). For the analysis we shall consider the terms in the expectation in Equation (38) one by one. Before proceeding, we fix the choice of u in the expectation for the time being. Let $p_u := \mathbb{E}_{v \sim u, x \in \mathcal{H}^v}[A^v(x)]$, where the expectation is over a random neighbor v of u.

Since v and w are u.a.r neighbors of u, and by the uniformity of x, x', y and y',

$$\mathbb{E}[A(x)] = \mathbb{E}[A(x')] = \mathbb{E}[B(y)] = \mathbb{E}[B(y')] = p_u.$$
(40)

Observe that x is independent of y and of y'. For a fixed choice of u, v and w are two independently chosen random neighbors of u. This implies that,

$$\mathbb{E}\left[A(x)B(y)\right] = \left(\mathbb{E}_{v,x\in\mathcal{H}^v}\left[A^v(x)\right]\right)^2 = p_u^2.$$

This also holds for the other cross terms and thus,

$$\mathbb{E}\left[A(x)B(y)\right] = \mathbb{E}\left[A(x)B(y')\right] = \mathbb{E}\left[A(x')B(y)\right] = \mathbb{E}\left[A(x')B(y')\right] = p_u^2.$$
(41)

Before analyzing the rest of the terms we require the following lemmas. Fix the choice of v and w for the next two lemmas.

Lemma 5.1. Let $i \in [k]$ and $J \subseteq \pi_{vu}^{-1}(i)$, be non-empty. Then,

$$\mathbb{E}\left[\chi_J(xx')\right] = \begin{cases} 1 - \frac{\varepsilon}{2} & \text{if } |J| \text{ even.} \\ -\frac{\varepsilon}{2} & \text{if } |J| \text{ odd.} \end{cases}$$
(42)

A similar property holds for π_{wu} with y and y'.

Proof. Note that in the choice of the verifier $x'|_{\pi_{vu}^{-1}(i)}$ is chosen to be $-x|_{\pi_{vu}^{-1}(i)}$ with probability $\frac{1}{2}$, $x|_{\pi_{vu}^{-1}(i)}$ with probability $\frac{1-\varepsilon}{2}$, and u.a.r with probability $\frac{\varepsilon}{2}$. The lemma follows, and holds analogously for π_{wu} with y and y'.

The above immediately implies the following lemma,

Lemma 5.2. Let $\alpha \subseteq [m]$, and $r = |\pi_{vu}(\alpha)|$ and $r' = |\pi_{vu}^{\mathsf{odd}}(\alpha)|$ (as per the definition in Lemma 3.3). Then,

$$\mathbb{E}\left[\chi_{\alpha}(xx')\right] = \left(1 - \frac{\varepsilon}{2}\right)^{r-r'} \left(-\frac{\varepsilon}{2}\right)^{r'}.$$
(43)

Similarly, for $\beta \subseteq [m]$, $r = |\pi_{wu}(\beta)|$ and $r' = |\pi_{wu}^{\mathsf{odd}}(\beta)|$,

$$\mathbb{E}\left[\chi_{\beta}(yy')\right] = \left(1 - \frac{\varepsilon}{2}\right)^{r-r'} \left(-\frac{\varepsilon}{2}\right)^{r'}.$$
(44)

We are now ready to bound the terms $\mathbb{E}[A(x)A(x')]$ and $\mathbb{E}[B(y)B(y')]$, where the choice of u is fixed. Lemma 5.3. $\mathbb{E}_{v,x,x'}[A(x)A(x')] = \mathbb{E}_{w,y,y'}[B(y)B(y')] \ge p_u^2 - \varepsilon/2.$

Proof. Using the Fourier expansion along with standard arguments we have,

$$\mathbb{E}_{v,x,x'}\left[A(x)A(x')\right] = \mathbb{E}_{v}\left[\widehat{A}_{\emptyset}^{2} + \sum_{\alpha \neq \emptyset}\widehat{A}_{\alpha}^{2}\mathbb{E}\left[\chi_{\alpha}(xx')\right]\right] \ge \left(\mathbb{E}_{v}\left[\widehat{A}_{\emptyset}\right]\right)^{2} + \mathbb{E}_{v}\left[\sum_{\alpha \neq \emptyset}\widehat{A}_{\alpha}^{2}\mathbb{E}\left[\chi_{\alpha}(xx')\right]\right].$$

Lemma 5.2 implies that $\mathbb{E}[\chi_{\alpha}(xx')] \ge -\varepsilon/2$. Using Parseval's we obtain the lemma. Also, by symmetry $\mathbb{E}[A(x)A(x')] = \mathbb{E}[B(y)B(y')]$.

Observe that x and x' individually are independent of the pair (y, y'). Similarly, y and y' individually are independent of the pair (x, x'). Thus, we obtain,

$$\mathbb{E}[A(x)B(y)B(y')] = \mathbb{E}[A(x')B(y)B(y')]$$

$$= \mathbb{E}[A(x)A(x')B(y)]$$

$$= \mathbb{E}[A(x)A(x')B(y')]$$

$$= \mathbb{E}[A(x)]\mathbb{E}[B(y)B(y')]$$

$$\geq p_u(p_u^2 - \varepsilon/2)$$

$$\geq p_u^3 - \varepsilon/2. \tag{45}$$

We are left with analyzing the term $\mathbb{E}[A(x)A(x')B(y)B(y')]$. Fix the choice of v and w for now. The Fourier expansion along with standard arguments (analogous to those in earlier sections) yield,

$$\mathbb{E}\left[A(x)A(x')B(x)B(y')\right] = \sum_{\alpha,\beta} \widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\mathbb{E}\left[\chi_{\alpha}(xx')\chi_{\beta}(yy')\right]$$
$$= \sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)=\emptyset}} \widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\mathbb{E}\left[\chi_{\alpha}(xx')\right]\mathbb{E}\left[\chi_{\beta}(yy')\right]$$
$$+ \sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}} \widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\mathbb{E}\left[\chi_{\alpha}(xx')\chi_{\beta}(yy')\right].$$
(46)

It is easy to see that Lemma 5.2 implies that $\mathbb{E}[\chi_{\alpha}(xx')] \mathbb{E}[\chi_{\beta}(yy')] \ge -\varepsilon/2$. Thus, the first summation in the RHS of Equation (46) is at least $\widehat{A}_{\emptyset}^2 \widehat{B}_{\emptyset}^2 - (\varepsilon/2) \left(\sum_{\alpha} \widehat{A}_{\alpha}^2\right) \left(\sum_{\beta} \widehat{B}_{\beta}^2\right) = \widehat{A}_{\emptyset}^2 \widehat{B}_{\emptyset}^2 - \varepsilon/2$. Using this we obtain,

$$\mathbb{E}\left[A(x)A(x')B(x)B(y')\right] \ge \widehat{A}_{\emptyset}^{2}\widehat{B}_{\emptyset}^{2} + \sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\mathbb{E}\left[\chi_{\alpha}(xx')\chi_{\beta}(yy')\right] - \frac{\varepsilon}{2}$$

Taking a further expectation over v and w and applying Jensen's inequality we obtain,

$$\mathbb{E}_{\substack{x,x'\\y,y'}\\y,y'} \left[A(x)A(x')B(x)B(y') \right] \ge p_u^4 + \mathbb{E}_{v,w} \left[\sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha) \cap \pi_{wu}(\beta) \neq \emptyset}} \widehat{A}_{\alpha}^2 \widehat{B}_{\beta}^2 \mathbb{E} \left[\chi_{\alpha}(xx')\chi_{\beta}(yy') \right] \right] - \frac{\varepsilon}{2}.$$
(47)

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Combining the above inequality with our assumption on the probability of rejection of the verifier, along with Equations (40), (41), (45), and Lemma 5.3, yields,

$$\rho^{4} - \delta \geq \frac{1}{16} \mathbb{E}_{u} \left[4p_{u} + 4p_{u}^{2} + 2p_{u}^{2} - \varepsilon + 4p_{u}^{3} - 2\varepsilon + p_{u}^{4} - \frac{\varepsilon}{2} \right]$$

$$+ \frac{1}{16} \mathbb{E}_{u,v,w} \left[\sum_{\substack{\alpha,\beta \\ \pi_{vu}(\alpha) \cap \pi_{wu}(\beta) \neq \emptyset}} \widehat{A}_{\alpha}^{2} \widehat{B}_{\beta}^{2} \mathbb{E} \left[\chi_{\alpha}(xx')\chi_{\beta}(yy') \right] \right]$$

$$\geq \frac{1}{16} \mathbb{E}_{u} \left[(1 + p_{u})^{4} - 4\varepsilon \right] + \frac{1}{16} \mathbb{E}_{u,v,w} \left[\sum_{\substack{\alpha,\beta \\ \pi_{vu}(\alpha) \cap \pi_{wu}(\beta) \neq \emptyset}} \widehat{A}_{\alpha}^{2} \widehat{B}_{\beta}^{2} \mathbb{E} \left[\chi_{\alpha}(xx')\chi_{\beta}(yy') \right] \right]$$

$$\geq \frac{1}{16} \left[(1 + \mathbb{E}_{u}[p_{u}])^{4} - 4\varepsilon \right] + \frac{1}{16} \mathbb{E}_{u,v,w} \left[\sum_{\substack{\alpha,\beta \\ \pi_{vu}(\alpha) \cap \pi_{wu}(\beta) \neq \emptyset}} \widehat{A}_{\alpha}^{2} \widehat{B}_{\beta}^{2} \mathbb{E} \left[\chi_{\alpha}(xx')\chi_{\beta}(yy') \right] \right], \quad (48)$$

where Jensen's inequality is used to obtain the last inequality. Substituting the value ρ from Equation (39) in the above and simplifying we obtain,

$$\mathbb{E}_{u,v,w}\left[\sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\mathbb{E}\left[\chi_{\alpha}(xx')\chi_{\beta}(yy')\right]\right] \leq -16\delta + 4\varepsilon,\tag{49}$$

where the inner expectation is over the choice of x, x', y and y'. Before proceeding, we need the following lemma which follows from the way x, x', y, y' are chosen by the verifier.

Lemma 5.4. For $i \in [k]$, let $J \subseteq \pi_{vu}^{-1}(i)$ and $K \subseteq \pi_{wu}^{-1}(i)$ be non-empty subsets. Then,

$$\mathbb{E}\left[\chi_J(xx')\chi_K(yy')\right] = \begin{cases} (1-\varepsilon) & \text{if both } |J|, |K| \text{ even,} \\ -(1-\varepsilon) & \text{if both } |J|, |K| \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Combining the above lemma with Lemma 5.1, we obtain that,

$$\left| \mathbb{E} \left[\chi_{\alpha}(xx')\chi_{\beta}(yy') \right] \right| \leq \left[\left(1 - \frac{\varepsilon}{2} \right)^{\left| (\pi_{vu}(\alpha) \cup \pi_{wu}(\beta) \setminus (\pi_{vu}(\alpha) \cap \pi_{wu}(\beta)) \right|} \right] \cdot \left[(1 - \varepsilon)^{\left| \pi_{vu}(\alpha) \cap \pi_{wu}(\beta) \right|} \right]$$
$$\leq \left(1 - \frac{\varepsilon}{2} \right)^{\max\{|\pi_{vu}(\alpha)|, |\pi_{wu}(\beta)|\}}.$$
(50)

Let R and T $(R \ge T)$ be positive integers we shall fix later. Using the above we have,

$$\sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\left|\mathbb{E}\left[\chi_{\alpha}(xx')\chi_{\beta}(yy')\right]\right| \leq \sum_{\substack{|\alpha|< R, |\beta|< R\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2} + \sum_{\substack{|(\alpha|\geq R, |\pi_{vu}(\alpha)|< T)\\\nu(|\beta|\geq R, |\pi_{vu}(\alpha)|< T)\\\forall(|\beta|\geq R, |\pi_{wu}(\beta)|< T)\right],}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\left(1-\frac{\varepsilon}{2}\right)^{T} + \sum_{\substack{|(\alpha|\geq R, |\pi_{vu}(\alpha)|< T)\\\nu(|\beta|\geq R, |\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\left(1-\frac{\varepsilon}{2}\right)^{T}$$
(51)

By Parseval's, the second term in the RHS above is at most,

$$\sum_{\substack{|\alpha| \ge R, \\ \pi_{vu}(\alpha) < T}} \widehat{A}_{\alpha}^2 + \sum_{\substack{|\beta| \ge R, \\ \pi_{wu}(\beta) < T}} \widehat{B}_{\beta}^2,$$

and the third term is at most,

$$\left(1-\frac{\varepsilon}{2}\right)^T$$

We set $T = R^{c_0}$ where c_0 is the constant from Theorem 2.4, and using the above analysis and Equation (18), we obtain,

$$\left| \mathbb{E}_{u,v,w} \left[\sum_{\substack{\alpha,\beta\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}} \widehat{A}_{\alpha}^{2} \widehat{B}_{\beta}^{2} \mathbb{E} \left[\chi_{\alpha}(xx')\chi_{\beta}(yy') \right] \right] \right| \leq \mathbb{E}_{u,v,w} \left[\sum_{\substack{|\alpha|< R, |\beta|< R\\\pi_{vu}(\alpha)\cap\pi_{wu}(\beta)\neq\emptyset}} \widehat{A}_{\alpha}^{2} \widehat{B}_{\beta}^{2} \right] + \frac{2}{R^{c_{0}}} + \left(1 - \frac{\varepsilon}{2}\right)^{R^{c_{0}}}.$$
(52)

Let us set $R = \left(\frac{2}{\varepsilon} \log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{c_0}}$ and $\varepsilon = \delta$. Using the above equation in conjunction with Equation (49) yields,

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$$\mathbb{E}_{u,v,w}\left[\sum_{\substack{|\alpha|< R, |\beta|< R\\ \pi_{vu}(\alpha)\cap \pi_{wu}(\beta)\neq \emptyset}}\widehat{A}_{\alpha}^{2}\widehat{B}_{\beta}^{2}\right] \geq 10\delta.$$

This yields a randomized labeling as follows: for every vertex $v \in V$, choose $\alpha \subseteq [m]$ with probability $\widehat{A^v}_{\alpha}^2$ and select a random label from α . For a vertex $u \in U$, choose a random neighbor w of u and assign u the label $\pi_{wu}(j_w)$ to u where j_w is the label assigned to w. The expected fraction of edges of the \mathcal{L} satisfied by this labeling is,

$$(10\delta)\left(\frac{1}{R}\right)^2 = \Omega(\delta^{c'}),$$

for some constant c' > 0 depending on c_0 .

5.2.1 Choice of parameters

Analogous to previous sections, choosing $r = (\log \log N)/4$ in Theorem 2.4 we get that the reduction to MAX-E4-SET-SPLITTING is of size $n = N^{O(r)}2^{2^{3r}} \leq N^{O(\log \log N)}$. The soundness of \mathcal{L} is $2^{-\Omega(\log \log N)} = 2^{-\Omega(\log \log n)}$. Combining this with the above analysis in the NO Case, choosing $\delta = \frac{1}{(\log n)^c}$ for some positive constant c (depending on c_0 and γ_0) we obtain a contradiction to our assumption on the probability of rejection of the verifier.

Thus, in the NO Case, the verifier rejects with probability at least $\rho^4 - \frac{1}{(\log n)^c}$. This completes the proof of Theorem 1.8.

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