

# Cascade Attitude Observer for the SLAM filtering problem

Elias Bjørne\* Edmund Førland Brekke\* Tor Arne Johansen\*

**Abstract**—This article presents an attitude observer that exploits both bearing and range measurements from landmarks, in addition to reference vectors such as magnetometer and accelerometer. It is a gyro bias observer in cascade with a simplified complementary filter, driven by a gyro measurement, in which the gyro bias is estimated by comparing the bearing dynamics with the gyro measurements. The observer is compared to a full complementary filter, and it is shown that it is more robust to initial gyro bias estimation error compared to the complimentary filter. The article also reveals how this new observer handles magnetometer failure and can use landmarks as reference vectors.

**Index Terms**—Attitude observer, Navigation, Nonlinear Observer, Sensor data fusion, Localisation, Mapping,

## I. INTRODUCTION

Robust navigation and positioning of autonomous vehicles are fundamental for any autonomous mission. A possible scenario is the use of autonomous vehicles for inspection of structures such as bridges, power lines, windmills etc., raising the need of high accuracy in position and attitude estimates, as the vehicles will have to work closely to the inspection target. In this case, the electromagnetic interference and the existence of ferromagnetic materials may degrade any magnetometer to the point of becoming unusable [1]. Moreover, global navigation satellite systems (GNSS) may be unreliable or severely degraded. Aided navigation techniques such as simultaneous localization and mapping (SLAM) algorithms can be used to handle these challenges. SLAM fuses the sensing of the surroundings with the inertial measurement unit IMU data to improve accuracy in the navigation. SLAM makes this possible by estimating the relative positions of landmarks, using ranges and/or bearing angles measurements between the vehicle and each stationary landmark. Over the past decades, the research community has devoted tremendous effort in the field of probabilistic SLAM, see [2] and [3], which includes several successful implementations of SLAM algorithms in experiments. A popular approach is to use the Extended Kalman filter (EKF) SLAM, although there is a consensus that EKF SLAM has a problem with consistency, especially related to error in the linearization due to wrong attitude estimates [4]. A proposed global exponentially stable SLAM solution was presented by Johansen et al. [5], with range and/or bearing SLAM.

Elias Bjørne\*, Edmund Førland Brekke\* and Tor Arne Johansen\* are with the Department of Engineering Cybernetics, Norwegian University of Science and Technology (NTNU), NO-7491 Trondheim, Norway. (Email: elias.bjorne@itk.ntnu.no, edmund.brekke@ntnu.no, tor.arne.johansen@itk.ntnu.no) and are associated with the NTNU Centre for Autonomous Marine Operations and Systems.

The system is represented as a linear time varying system (LTV), and thus globally solvable with the Kalman filter (KF) without any linearization. One feature is its dependence on an attitude heading reference system (AHRS) system, which implies that it may be dependent on a magnetometer. Thus, a primary motivation for the present paper was to make an AHRS potentially independent of the magnetometer. A similar work has been reported by Lourenco and Guerreiro et al. [6][1], in which a globally asymptotically stable GAS sensor-based SLAM solution is presented, for range, bearing and range and bearing measurements; where the system also was presented as a LTV, and solved with KF. In addition they are able to estimate the gyro bias using range and bearing measurements.

Attitude estimation is central to the navigation problem, and as well as being crucial for the SLAM problem when connecting the local estimates to a global map. A common approach for attitude estimation is the use of reference vectors. These can among others, come from a magnetometer and accelerometer measuring the earth's gravity. The attitude is then determined by finding the rotation matrix that maps the measured reference vectors in body coordinate system to the known or measured reference vectors in an earth-fixed coordinate system.

Common approaches for solving this problem are the QUEST and TRIAD algorithms, which are compared in et al. [7]. Problems with these solutions are A: the measurements are noisy, which implies that a solution should be filtered. B: the algorithm does not take the dynamics of the vehicle into account. Tackling this was first done using the EKF; however EKF has issues regarding stability, consistency and complexity due to the need of the Jacobian matrix [8]. Another solution was the multiplicative EKF (MEKF) presented by Markley [9], which gave several attitude error representations for Kalman filtering with quaternions. The computational load is still significant, especially for low-cost and lightweight applications. A computationally efficient complimentary filter was presented by Mahony et al. [10], which was proven to have almost global stability for constant reference vectors. Grip et al. [11] proved that with a minor modification, semi-global exponentially stability for time varying reference vectors could be achieved. An extension of this work was done by Fusini et al. [12], in which using optical flow and GNSS velocity provided new reference vectors. An attitude and gyro bias estimation scheme is presented in Batista et al. [13], where the dynamics are given as an LTV system and solved with a Kalman filter. A computational efficient gyro bias attitude observer is also presented by Batista

et al. [14], where the gyro bias is in cascade with the attitude estimation. The mentioned methods use reference vectors from global phenomena. A weakness of these observers is that they often rely on magnetometer for heading reference, which have often been proven inaccurate. A possible solution is to use reference vectors from local surroundings. An example is shown by Vasconcelos et al. [15] in which estimation is done by using known landmarks. Similar work is done by Hua [16] and Bras et al. [17], where the latter is for range-only measurements. These methods require prior knowledge of the landmarks position which makes them fall short on the SLAM problem.

### A. Contribution

The main contribution of this paper is the design of a new gyro bias estimator for the SLAM filter problem. This makes it possible to estimate gyro bias without magnetometer measurements, if velocity, range and bearing measurements from landmarks with *unknown* positions are available. It is intended to be used in the SLAM observer described in Johansen et al. [5].

The observer can both apply bearing and range measurements with velocity and reference vectors to estimate the gyro bias. The gyro bias estimate is then sent to a simplified complimentary filter from Mahony et al. [10]. This makes the gyro bias estimation and attitude estimation decoupled, as in Batista et al. [14], so that the erroneous initialization of the attitude does not affect the convergence of the gyro bias estimate, resulting in a smoother and faster transition phase for the attitude estimate, which becomes more apparent for large gyro biases. The performance is compared against the complimentary filter, where simulation results are made in  $\mathbb{R}^3$ . To illustrate how the observer performs without magnetometer, a scenario where the magnetometer is turned off is demonstrated. In addition, a method for using the vector between landmarks as reference vectors is briefly discussed.

The structure of this paper is as follows: Notation and preliminaries are presented in Section II; Section III presents the novel estimator and proof of its stability properties; while Section IV shows simulation results. Finally Section V concludes the paper.

## II. NOTATION AND PRELIMINARIES

### A. Notation

Scalars are in lower case  $a, x, \omega$ , vectors are lower case bold  $\mathbf{a}, \mathbf{x}, \boldsymbol{\omega}$ , sets are upper case  $A, X, \Omega$ , and matrices are bold upper case  $\mathbf{A}, \mathbf{X}, \mathbf{\Omega}$ . The 0 denotes the scalar zero, while  $\mathbf{0}$  is the matrix zero where dimensions are implicitly given by the context. The accents  $\hat{\bullet}, \tilde{\bullet}, \check{\bullet}, \bar{\bullet}$ , denotes estimate, estimate error, time derivative and upper bound. The subscript  $\bullet_m$  denotes the measured value. Some common mathematical expressions which will be used are: The euclidean norm for vectors and frobenius norm for matrices, denoted  $\|\bullet\|$ , absolute value, denoted  $|\bullet|$  and the transpose,

denoted  $\bullet^\top$ . The representation of index sets will be done with  $\{1, \dots, n\} = \{x \in \mathbb{Z} | x \leq n\}$ .

A vector can be represented in different coordinate systems, the representation is denoted with the superscripts  $\bullet^b, \bullet^n$  which represents the body-fixed and earth-fixed (inertial) coordinate systems respectively, and will be called body-frame and inertial-frame. Lower case will denote the indices of a landmark, vector or matrix  $\bullet_i$  and  $\bullet_{ij}$ .

### B. Rotation representation

Rotation is the attitude change between two coordinate systems, and a rotation from coordinate system  $b$  to  $n$  is denoted with subscript  $\bullet_{nb}$ . This can be represented as:

angle-axis

$$\Phi_{nb} = \theta \mathbf{u} \in \{\mathbb{R}^3 | |\theta| \leq \pi, \|\mathbf{u}\| = 1\},$$

Euler angles

$$\boldsymbol{\theta}_{nb} = [\rho, \phi, \psi]^\top \in \{\mathbb{R}^3 | |\rho| \leq \pi, |\phi| \leq \pi, |\psi| \leq \pi\},$$

quaternions

$$\mathbf{q}_{nb} = [s, \mathbf{r}^\top]^\top \in \{\mathbf{R}^4 | s \in \mathbb{R}, \mathbf{r} \in \mathbb{R}^3, \|\mathbf{q}\| = 1\}$$

rotation matrix

$$\mathcal{R}_{nb} \in \{\mathbb{R}^{3 \times 3} | \mathcal{R}_{nb} \mathcal{R}_{nb}^\top = \mathbf{I}, \det(\mathcal{R}_{nb}) = 1\}$$

which means  $\mathcal{R}_{nb} \in SO(3)$ . The rotational vector transformation is calculated with the rotation matrix  $\mathbf{x}^n = \mathcal{R}_{nb} \mathbf{x}^b$ . The cross product is represented in matrix form  $S(\mathbf{x})\mathbf{y} = \mathbf{x} \times \mathbf{y}$ , where  $S(\bullet)$  is a skew-symmetric matrix

$$S(\mathbf{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \quad (1)$$

which gives  $S(\bullet) = -S(\bullet)^\top$ ,  $\mathbf{x}^\top S(\bullet)\mathbf{x} = 0$ ,  $\forall \mathbf{x}$ ,  $S(\mathbf{x})\mathbf{y} = -S(\mathbf{y})\mathbf{x}$  and  $\mathcal{R}S(\mathbf{x})\mathcal{R}^\top = S(\mathcal{R}\mathbf{x})$ ,  $\forall \mathcal{R} \in SO(3)$ . Moreover the cross-product gives the difference in angle-axis between two vectors

$$S(\mathbf{x})\mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \sin(\theta) \mathbf{u} \quad (2)$$

where  $\theta$  is the angle between the vectors, and  $\mathbf{u}$  is the axis of the rotation, which is orthogonal to the two vectors.

More detailed information can be found in Sola [18] and Fossen [19]. Let the rotation matrix denote the rotation from the body-fixed frame to the inertia-fixed frame. The dynamics of the rotation matrix is described by

$$\dot{\mathcal{R}}_{nb} = \mathcal{R}_{nb} S(\boldsymbol{\omega}) \quad (3)$$

where  $\boldsymbol{\omega} = \boldsymbol{\omega}_{nb}^b$  is the angular velocity of the frame  $b$  relative to  $n$  decomposed in  $b$ . The  $\boldsymbol{\omega}$  is assumed bounded  $\|\boldsymbol{\omega}\| \leq \bar{\omega}$  and is measured. The measurement is assumed to be corrupted by a constant gyro bias

$$\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_\omega \quad (4)$$

*Remark 1:* The estimate of the rotation matrix is denoted  $\hat{\mathcal{R}}_{nb}$ , and will have the error defined as in Hamel [20]  $\tilde{\mathcal{R}}_{nb} = \hat{\mathcal{R}}_{nb} \mathcal{R}_{nb}^\top$ , which means the error is a rotation matrix in itself. The gyro-bias error is defined as  $\tilde{\mathbf{b}}_\omega = \mathbf{b}_\omega - \hat{\mathbf{b}}_\omega$

### C. Reference vectors

Assume that there are  $n$  vectors  $\mathbf{r}_j$  that are known or measurable in both body- and inertial-frame,  $\mathbf{r}_j^n = \mathcal{R}_{nb} \mathbf{r}_j^b$ ,  $j \in 1, \dots, n$ . These vectors are normalised  $\|\mathbf{r}_j\| = 1$ , and there is a standard assumption on the reference vectors to ensure they are not parallel.

*Assumption 1:* there exists a constant  $c_{obs} > 0$  so as, for each  $t \geq 0$ , the inequality  $\|\mathbf{r}_j^b \times \mathbf{r}_k^b\| > c_{obs}$  holds for at least two of the indices  $j, k \in \{1, \dots, n\}$

#### D. Landmark and vehicle dynamics

We assume that there is a vehicle with position  $\mathbf{p}^n$  and  $m$  stationary landmarks where the  $i$ th landmark has position  $\mathbf{p}_i^n$ . The landmark observations are the vectors between the vessel and the landmarks  $\delta_i^n = \mathbf{p}_i^n - \mathbf{p}^n$ . This vector can be represented by its range and bearing,

$$\varrho_i = \|\delta_i^n\|, \quad \mathbf{l}_i^n = \delta_i^n / \|\delta_i^n\| \quad (5)$$

where the range is the geometric distance, while the bearing is defined by the line of sight (LOS) vector on the unit ball, pointing at the landmark. These can also be represented in body-frame

$$\delta_i^b = \mathcal{R}_{nb}^\top \delta_i^n, \quad \mathbf{l}_i^b = \mathcal{R}_{nb}^\top \mathbf{l}_i^n \quad (6)$$

The kinematics of the position of the vehicle is

$$\dot{\mathbf{p}}^n = \mathbf{v}^n = \mathcal{R}_{nb} \mathbf{v}^b \quad (7)$$

The change in the landmark observation is as follows

$$\dot{\delta}_i^n = -\mathbf{v}^n \quad (8)$$

To find the dynamics of the landmark observation in body-frame, (3) and (6)-(8) was used with the product rule

$$\dot{\delta}_i^b = -\mathcal{S}(\boldsymbol{\omega})\delta_i^b - \mathbf{v}^b \quad (9)$$

From this, the dynamics of the range and bearing can be found

$$\dot{\varrho}_i = -(\mathbf{l}_i^n)^\top \mathbf{v}^n = -(\mathbf{l}_i^b)^\top \mathbf{v}^b \quad (10)$$

$$\dot{\mathbf{l}}_i^n = \frac{1}{\varrho_i} (\mathbf{l}_i^n (\mathbf{l}_i^n)^\top - \mathbf{I}) \mathbf{v}^n = \frac{1}{\varrho_i} \mathcal{S}(\mathbf{l}_i^n)^2 \mathbf{v}^n \quad (11)$$

$$\begin{aligned} \dot{\mathbf{l}}_i^b &= -\mathcal{S}(\boldsymbol{\omega})\mathbf{l}_i^b + \frac{1}{\varrho_i} (\mathbf{l}_i^b (\mathbf{l}_i^b)^\top - \mathbf{I}) \mathbf{v}^b \\ &= -\mathcal{S}(\boldsymbol{\omega})\mathbf{l}_i^b + \frac{1}{\varrho_i} \mathcal{S}(\mathbf{l}_i^b)^2 \mathbf{v}^b \end{aligned} \quad (12)$$

We will also use vector between the landmark observations,

$$\boldsymbol{\gamma}_{ij}^n = \delta_i^n - \delta_j^n \quad (13)$$

It should be noted that this vector does not change with time, and can therefore be used as a reference vector if it is known. It can also be directly computed from measurements in body coordinates if both range and bearing measurements are available

$$\boldsymbol{\gamma}_{ij}^b = \varrho_i \mathbf{l}_i^b - \varrho_j \mathbf{l}_j^b \quad (14)$$

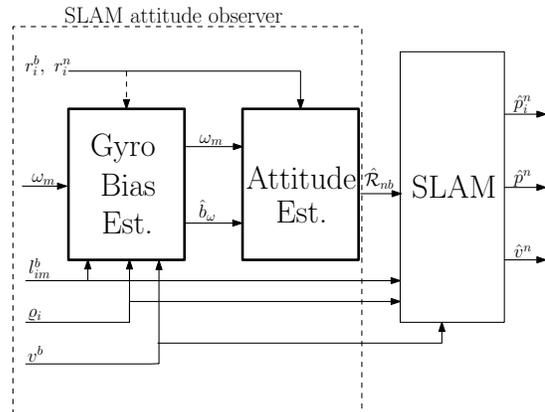
Finally we do a similar assumption on the position of the landmarks as done in assumption 1 to ensure that all the LOS vectors are not parallel.

*Assumption 2:* The vehicle and the landmarks are not all located on a line, and the vectors between them spans a space of dimension  $n \geq 2$

#### E. SLAM Attitude Problem Formulation

Let  $\mathcal{R}_{nb} \in SO(3)$ , and with the dynamics from (3). Then the goal of the observer is to estimate the rotation matrix  $\tilde{\mathcal{R}}_{nb} \in SO(3)$  and ensure the attitude error  $\tilde{\mathcal{R}}_{nb} \rightarrow \mathbf{I}$  as  $t \rightarrow \infty$ . In addition, the gyro bias error should go to zeros  $\tilde{\mathbf{b}}_\omega \rightarrow \mathbf{0}$ , and its estimate should not be dependent on the magnetometer. For the attitude estimate, measurements available for the observer will be the  $\boldsymbol{\omega}_m$ ,  $\varrho_{im}$ ,  $\mathbf{l}_{im}^b$  and  $\mathbf{v}_m^b$ , in addition to the reference vectors  $\mathbf{r}_i^b$  and  $\mathbf{r}_i^n$ .

*Remark 2:* The goal is to get the error to  $\tilde{\mathcal{R}}_{nb} \rightarrow \mathbf{I}$  as  $t \rightarrow \infty$ . In angle-axis, Euler angles and quaternions it will correspond to  $\tilde{\boldsymbol{\Phi}}_{nb} \rightarrow \mathbf{0}$ ,  $\tilde{\boldsymbol{\theta}}_{nb} \rightarrow \mathbf{0}$  and  $\tilde{\mathbf{q}}_{nb} \rightarrow [1, \mathbf{0}]^\top$



**Fig. 1:** Block Diagram of the structure of the SLAM attitude observer in cascade with the SLAM estimator from [5]

### III. SLAM ATTITUDE OBSERVER

The attitude and gyro bias estimator in this paper is based on the intuition that the dynamics of the bearing measurement is closely related to the angle rates  $\boldsymbol{\omega}$ , and thus will be useful for estimating the gyro bias. This will also make it possible to decouple the bias estimation and attitude estimation so that an erroneous attitude estimate does not interfere with the bias estimation. The gyro bias estimator for a vehicle with  $m$  landmark observations is

$$\dot{\hat{\mathbf{l}}}_i^b = -\mathcal{S}(\boldsymbol{\omega}_m - \hat{\mathbf{b}}_\omega + \boldsymbol{\sigma}_{li})\hat{\mathbf{l}}_i^b + \frac{1}{\varrho_{im}} \mathcal{S}(\hat{\mathbf{l}}_i^b) \mathcal{S}(\mathbf{l}_{im}^b) \mathbf{v}_m^b \quad (15)$$

$$\boldsymbol{\sigma}_{li} = k_i \mathcal{S}(\mathbf{l}_{im}^b) \hat{\mathbf{l}}_i^b \quad (16)$$

$$\dot{\hat{\mathbf{b}}}_\omega = -\sum_{i=1}^m \boldsymbol{\sigma}_{li} \quad (17)$$

The error  $\tilde{\mathbf{l}}_i^b = \mathcal{S}(\mathbf{l}_i^b) \hat{\mathbf{l}}_i^b = \frac{1}{k_i} \boldsymbol{\sigma}_{li}$  will be used, as well as  $\tilde{\theta}_{li}$ , which is the angle between  $\mathbf{l}_i^b$  and  $\hat{\mathbf{l}}_i^b$ . The relation between these errors can be seen in (2). We see that  $(\hat{\mathbf{l}}_i^b)^\top \dot{\hat{\mathbf{l}}}_i^b = 0$ , because  $(\hat{\mathbf{l}}_i^b)^\top \mathcal{S}(\hat{\mathbf{l}}_i^b) = 0$ , which ensures that  $\hat{\mathbf{l}}_i^b$  is maintained on the unit ball.

*Theorem 1:* Consider the dynamics of a vehicle with bearing and range measurements of landmarks, in addition to velocity measurement, and gyro measurements with a bias  $\mathbf{b}_\omega$ . Let assumption 2 hold, and assume  $k_i$  is large

enough; then the observer (15)-(17) is uniformly semi-globally asymptotically stable for all trajectories, and constant values of  $\tilde{\mathbf{b}}_\omega$

The proof is divided into three steps. The first step, A), is to verify that the LOS estimates  $\hat{\mathbf{l}}_i^b$  are uniformly bounded away from pointing in the opposite direction meaning  $|\tilde{\theta}_{li}| < \pi - \epsilon$ . The next step, B), is to prove that the system is stable, and that all the LOS estimate errors  $\tilde{\mathbf{l}}_i^b$  will converge to zero, implying  $\tilde{\theta}_{li}$  also converges to zero. The last step, C), is to show that this also implies that the gyro bias estimate error  $\tilde{\mathbf{b}}_\omega$  is UGAS, this is done using Matrosov's theorem.

*Proof:*

Comparing the vehicle dynamics to the estimator dynamics,

$$\dot{\mathbf{l}}_i^b = -\mathbf{S}(\boldsymbol{\omega})\mathbf{l}_i^b + \frac{1}{\rho_i}\mathbf{S}(\mathbf{l}_i^b)\mathbf{S}(\mathbf{l}_i^b)\mathbf{v}^b \quad (18)$$

$$\dot{\hat{\mathbf{l}}}_i^b = -\mathbf{S}(\boldsymbol{\omega} + \tilde{\mathbf{b}}_\omega + \boldsymbol{\sigma}_{li})\hat{\mathbf{l}}_i^b + \frac{1}{\rho_i}\mathbf{S}(\hat{\mathbf{l}}_i^b)\mathbf{S}(\mathbf{l}_i^b)\mathbf{v}^b \quad (19)$$

the error dynamics of  $\tilde{\mathbf{l}}_i^b = \mathbf{S}(\mathbf{l}_i^b)\hat{\mathbf{l}}_i^b = \frac{1}{k_i}\boldsymbol{\sigma}_{li}$  and  $\tilde{\mathbf{b}}_\omega = \mathbf{b}_\omega - \hat{\mathbf{b}}_\omega$  is then

$$\begin{aligned} \dot{\tilde{\mathbf{l}}}_i^b &= \mathbf{S}(\mathbf{l}_i^b)\dot{\hat{\mathbf{l}}}_i^b - \mathbf{S}(\hat{\mathbf{l}}_i^b)\dot{\mathbf{l}}_i^b \\ \dot{\tilde{\mathbf{l}}}_i^b &= (\mathbf{S}(\mathbf{l}_i^b)\mathbf{S}(\hat{\mathbf{l}}_i^b) - \mathbf{S}(\hat{\mathbf{l}}_i^b)\mathbf{S}(\mathbf{l}_i^b))\boldsymbol{\omega} - \mathbf{S}(\mathbf{l}_i^b)\mathbf{S}(\tilde{\mathbf{b}}_\omega + \boldsymbol{\sigma}_{li})\hat{\mathbf{l}}_i^b \\ &\quad - \frac{1}{\rho_i}(\mathbf{S}(\mathbf{l}_i^b)\mathbf{S}(\hat{\mathbf{l}}_i^b) - \mathbf{S}(\hat{\mathbf{l}}_i^b)\mathbf{S}(\mathbf{l}_i^b))\mathbf{S}(\mathbf{l}_i^b)\mathbf{v}^b \\ \dot{\tilde{\mathbf{l}}}_i^b &= \mathbf{S}(\tilde{\mathbf{l}}_i^b)\boldsymbol{\omega} + \frac{1}{\rho_i}\mathbf{S}(\tilde{\mathbf{l}}_i^b)\mathbf{S}(\mathbf{l}_i^b)\mathbf{v}^b - \mathbf{S}(\mathbf{l}_i^b)\mathbf{S}(\tilde{\mathbf{b}}_\omega + k_i\tilde{\mathbf{l}}_i^b)\hat{\mathbf{l}}_i^b \\ \dot{\tilde{\mathbf{b}}}_\omega &= \sum_{i=1}^m k_i\tilde{\mathbf{l}}_i^b \end{aligned}$$

where we have used that  $\mathbf{S}(\mathbf{l}_i^b)\mathbf{S}(\hat{\mathbf{l}}_i^b) - \mathbf{S}(\hat{\mathbf{l}}_i^b)\mathbf{S}(\mathbf{l}_i^b) = \mathbf{S}(\tilde{\mathbf{l}}_i^b)$  and that the bias is constant  $\dot{\mathbf{b}}_\omega = \mathbf{0}$ . We then see that this system has an equilibrium point  $\tilde{\mathbf{l}}_i^b = \mathbf{0}$  and  $\tilde{\mathbf{b}}_\omega = \mathbf{0}$ .

A) *Uniform Boundedness:* First we want to show that for any  $0 < \epsilon < \pi$ , a solution starting in  $\tilde{\theta}_{li} \in [-\pi + \epsilon, \pi - \epsilon] \forall i \in \{1, m\}$  will stay in this set for gain  $k_i$  large enough. We choose a Lyapunov function for every bearing measurement,  $V_i = 1 - (\hat{\mathbf{l}}_i^b)^\top \mathbf{l}_i^b = 1 - \cos(\tilde{\theta}_{li})$ , which is positive definite and increasing for  $\tilde{\theta}_{li} \in (-\pi, \pi)$ . It has the derivative

$$\dot{V}_i = \tilde{\mathbf{b}}_\omega^\top \tilde{\mathbf{l}}_i^b - k_i \|\tilde{\mathbf{l}}_i^b\|^2 \leq -\|\tilde{\mathbf{l}}_i^b\| (k_i \|\tilde{\mathbf{l}}_i^b\| - \tilde{b}_\omega)$$

where we recall that there is a bound on the bias  $\tilde{b}_\omega$  and that  $\|\tilde{\mathbf{l}}_i^b\| = |\sin(\tilde{\theta}_{li})|$ . We see that we can choose  $k_i > \frac{\tilde{b}_\omega}{|\sin(\epsilon)|}$ , so that for  $|\tilde{\theta}_{li}| \leq \pi - \epsilon \Rightarrow \|\tilde{\mathbf{l}}_i^b\| \leq |\sin(\epsilon)|$  we will have  $\dot{V}_i < 0$ , which implies that  $\tilde{\theta}_{li}$  is strictly decreasing. Consider  $\|\tilde{\mathbf{l}}_i^b(0)\|$  so that  $|\tilde{\theta}_{li}(0)| < \pi - \epsilon$ , and by the continuity of the solution we can guarantee that the solution will never exceed  $|\tilde{\theta}_{li}| > \pi - \epsilon$ , and will therefore utilize that this holds for the rest of our analysis.

B) *Uniform stability and convergence of  $\hat{\mathbf{l}}_i^b$ :* We continue to choose the Lyapunov function candidate for the whole system

$$V_b(\tilde{\mathbf{l}}_i^b, \tilde{\mathbf{b}}_\omega) = \sum_{i=1}^m k_i (1 - (\mathbf{l}_i^b)^\top \hat{\mathbf{l}}_i^b) + \frac{1}{2} \tilde{\mathbf{b}}_\omega^\top \tilde{\mathbf{b}}_\omega \quad (20)$$

We also see that  $V_b$  can be rewritten as

$$V_b(\tilde{\mathbf{l}}_i^b, \tilde{\mathbf{b}}_\omega) = \sum_{i=1}^m k_i (1 - \cos(\tilde{\theta}_{li})) + \frac{1}{2} \tilde{\mathbf{b}}_\omega^\top \tilde{\mathbf{b}}_\omega \quad (21)$$

where we see that  $V_b$  is positive definite for  $\tilde{\theta}_{li} \in [-\pi, \pi]$ . We calculate

$$\begin{aligned} ((\mathbf{l}_i^b)^\top \hat{\mathbf{l}}_i^b + (\mathbf{l}_i^b)^\top \dot{\hat{\mathbf{l}}}_i^b) &= (-\mathbf{S}(\boldsymbol{\omega})\mathbf{l}_i^b + \frac{1}{\rho_i}\mathbf{S}(\mathbf{l}_i^b)\mathbf{S}(\mathbf{l}_i^b)\mathbf{v}^b)^\top \hat{\mathbf{l}}_i^b \\ &\quad + (\mathbf{l}_i^b)^\top (-\mathbf{S}(\boldsymbol{\omega} + \tilde{\mathbf{b}}_\omega + \boldsymbol{\sigma}_{li})\hat{\mathbf{l}}_i^b + \frac{1}{\rho_i}\mathbf{S}(\hat{\mathbf{l}}_i^b)\mathbf{S}(\mathbf{l}_i^b)\mathbf{v}^b) \\ &= -(\mathbf{l}_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega + \boldsymbol{\sigma}_{li})\hat{\mathbf{l}}_i^b \end{aligned}$$

where we used that  $(\frac{1}{\rho_i}\mathbf{S}(\mathbf{l}_i^b)\mathbf{S}(\mathbf{l}_i^b)\mathbf{v}^b)^\top \hat{\mathbf{l}}_i^b + (\mathbf{l}_i^b)^\top \frac{1}{\rho_i}\mathbf{S}(\hat{\mathbf{l}}_i^b)\mathbf{S}(\mathbf{l}_i^b)\mathbf{v}^b = 0$ . The time derivative of  $V_b$  is then

$$\begin{aligned} \dot{V}_b(\mathbf{l}_i^b, \hat{\mathbf{l}}_i^b, \tilde{\mathbf{b}}_\omega) &= -\sum_{i=1}^m k_i ((\mathbf{l}_i^b)^\top \hat{\mathbf{l}}_i^b + (\mathbf{l}_i^b)^\top \dot{\hat{\mathbf{l}}}_i^b) + \tilde{\mathbf{b}}_\omega^\top \dot{\tilde{\mathbf{b}}}_\omega \\ &= \sum_{i=1}^m [k_i (\mathbf{l}_i^b)^\top \mathbf{S}(\boldsymbol{\sigma}_{li})\hat{\mathbf{l}}_i^b + k_i (\mathbf{l}_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega)\hat{\mathbf{l}}_i^b] + \tilde{\mathbf{b}}_\omega^\top \sum_{i=1}^m k_i (\mathbf{S}(\mathbf{l}_i^b)\hat{\mathbf{l}}_i^b) \\ &= \sum_{i=1}^m k_i^2 (\mathbf{l}_i^b)^\top \mathbf{S}(\mathbf{S}(\mathbf{l}_i^b)\hat{\mathbf{l}}_i^b)\hat{\mathbf{l}}_i^b = -\sum_{i=1}^m k_i^2 (\hat{\mathbf{l}}_i^b)^\top \mathbf{S}(\mathbf{l}_i^b)^\top \mathbf{S}(\mathbf{l}_i^b)\hat{\mathbf{l}}_i^b \end{aligned}$$

which is equal to

$$\dot{V}_b = -\sum_{i=1}^m k_i^2 \|\tilde{\mathbf{l}}_i^b\|^2 = -\sum_{i=1}^m k_i^2 \sin^2(\tilde{\theta}_{li}) < 0, \quad \tilde{\theta}_{li} \neq 0 \text{ or } \pm\pi$$

Hence  $\dot{V}_b$  is negative definite on the open set  $\tilde{\theta}_{li} \in (-\pi, \pi)$ . We can conclude that the system is stable, and that the Lyapunov function will converge to  $\dot{V}_b = 0$ , hence the trajectories will converge to the set  $E(\dot{V}_b = 0) = \{\tilde{\mathbf{l}}_i^b \mid \|\tilde{\theta}_{li}\| = 0 \forall i \in \{1, m\}\}$ , since  $|\tilde{\theta}_{li}| \neq \pi$ . We can also conclude that the Lyapunov function  $V_b$  converges to a constant, thus  $\|\tilde{\mathbf{b}}_\omega\|$  will also converge to a constant.

C) *UGAS using Matrosov:* What is left is to show that the states will be in the set  $E(\dot{V}_b = 0)$  in finite time, when  $\|\tilde{\mathbf{b}}_\omega\| \neq 0$ , which means  $\|\tilde{\mathbf{b}}_\omega\|$  has to converge to zero. For this, Matrosov theorem will be utilized [21]. We choose the auxiliary function

$$W = \sum_{i=1}^m \tilde{\mathbf{b}}_\omega^\top \tilde{\mathbf{l}}_i^b = \sum_{i=1}^m (\mathbf{l}_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega)\hat{\mathbf{l}}_i^b \quad (22)$$

First we see that  $W$  is bounded by the states. To find the

derivative of  $\dot{W}$  we use

$$\begin{aligned} (l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \dot{l}_i^b &= (l_i^b)^\top \mathbf{S}(\omega) \mathbf{S}(\tilde{\mathbf{b}}_\omega) \dot{l}_i^b + \frac{1}{\rho_i} (v^b)^\top \mathbf{S}(l_i^b) \mathbf{S}(l_i^b) \mathbf{S}(\tilde{\mathbf{b}}_\omega) \dot{l}_i^b \\ (l_i^b)^\top \mathbf{S}(\dot{\tilde{\mathbf{b}}}_\omega) \dot{l}_i^b &= (l_i^b)^\top \mathbf{S} \left( \sum_{i=1}^m k_i \tilde{l}_i^b \right) \dot{l}_i^b = -(\tilde{l}_i^b)^\top \sum_{i=1}^m k_i \tilde{l}_i^b \\ (l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \dot{l}_i^b &= -(l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \mathbf{S}(\omega) \dot{l}_i^b + (l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \mathbf{S}(\tilde{\mathbf{b}}_\omega) \dot{l}_i^b \\ &\quad + (l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \mathbf{S}(\sigma_{l_i}) \dot{l}_i^b + \frac{1}{\rho_i} (l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \mathbf{S}(l_i^b) \mathbf{S}(l_i^b) v^b \end{aligned}$$

Adding these together, along with some algebra gives

$$\begin{aligned} \dot{W} &= \sum_{i=1}^m [(l_i^b)^\top \mathbf{S}(\omega) \tilde{l}_i^b + (v^b)^\top \mathbf{S}(l_i^b) \mathbf{S}(\tilde{\mathbf{b}}_\omega) \tilde{l}_i^b \\ &\quad - (\tilde{l}_i^b)^\top \sum_{i=1}^m k_i \tilde{l}_i^b + (l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \mathbf{S}(\tilde{\mathbf{b}}_\omega) \dot{l}_i^b \\ &\quad + (l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \mathbf{S}(k_i \tilde{l}_i^b) \dot{l}_i^b] \end{aligned}$$

As  $\|\tilde{l}_i^b\| \rightarrow 0$ , we see that there is only one term that does not disappear, and that  $\dot{l}_i^b = l_i^b$ ; We then end up with

$$\begin{aligned} \lim_{\|\tilde{l}_i^b\| \rightarrow 0} \dot{W} &= \sum_{i=1}^m (l_i^b)^\top \mathbf{S}(\tilde{\mathbf{b}}_\omega) \mathbf{S}(\tilde{\mathbf{b}}_\omega) l_i^b = \sum_{i=1}^m (\tilde{\mathbf{b}}_\omega)^\top \mathbf{S}(l_i^b) \mathbf{S}(l_i^b) \tilde{\mathbf{b}}_\omega \\ &\leq -c \|\tilde{\mathbf{b}}_\omega\|^2 \end{aligned}$$

where  $c$  is the smallest singular value of the matrix  $\sum_{i=1}^m \mathbf{S}(l_i^b)^2$ . We have used that the eigenvalue of the matrix  $\mathbf{S}(l_i^b) \mathbf{S}(l_i^b)$  are  $\lambda = [0, -(l_i^b)^\top \tilde{l}_i^b, -(l_i^b)^\top \tilde{l}_i^b]$  which for  $\|\tilde{l}_i^b\| \rightarrow 0$  the matrix has eigenvalues  $\lambda = [0, -1, -1]$  with the zero vector  $l_i^b$ . The matrix  $\sum_{i=1}^m \mathbf{S}(l_i^b)^2$  is therefore negative definite, because from assumption 2, there are at least two  $l_i^b$  that are not parallel so that the matrix is full rank, negative definite. Hence  $\dot{W}$  is definitely not equal to zero when  $\|\tilde{l}_i^b\| = 0$  and  $\|\tilde{\mathbf{b}}_\omega\| > 0$ . Since we know that  $W$  is bounded by the states, we know that  $W$  converges to a constant. We therefore know that  $\dot{W} \rightarrow 0$  as  $t \rightarrow \infty$ , which implies that  $\tilde{\mathbf{b}}_\omega \rightarrow 0$  as  $t \rightarrow \infty$ . Hence we can conclude from Matrosov's theorem that the system will converge uniformly to  $\|\tilde{\mathbf{b}}_\omega\| \rightarrow 0$  and  $\|\tilde{l}_i^b\| \rightarrow 0, \forall i \in \{1, m\}$  as  $t \rightarrow \infty$ , for all trajectories starting with  $\tilde{\theta}(0) < \pi - \epsilon$ ; since  $\epsilon > 0$  can be chosen arbitrary small. ■

In practice, the estimate  $\hat{l}_i^b$  can be chosen from its direct measurement in the initialization, so that the bound on  $k_i$  is easier to fulfill. In simulations the observer has been tested with initialization  $\tilde{\theta}_{l_i} = \pi$  without any convergence problem. The bias estimated can then be used in cascade with the simplified complimentary filter from [16]

$$\dot{\hat{\mathcal{R}}}_{nb} = \hat{\mathcal{R}}_{nb} \mathbf{S}(\omega_m - \hat{\mathbf{b}}_\omega + \sigma_{\mathcal{R}}) \quad (23)$$

$$\sigma_{\mathcal{R}} = \sum_{i=1}^n c_i \mathbf{S}(r_i^b) \hat{\mathcal{R}}_{nb}^\top r_i^n \quad (24)$$

were semi-globally stability can be achieved.

*Theorem 2:* Consider the dynamics of a vehicle with bearing and range measurements of landmarks, in addition

to velocity measurement, and gyro measurements with a bias (4). Under the conditions of Theorem 1 and Assumption 1 satisfied and for  $c_i$  large enough; the observer with (15)-(17) in cascade with (23)-(24), will be uniformly semi-globally asymptotically stable for time varying reference vectors.

*Proof:*

The gyro bias estimator is independent of the attitude estimates. We therfor have from *Theorem 1*, that the gyro bias estimate error  $\tilde{\mathbf{b}}_\omega$  is bounded and converges  $\|\tilde{\mathbf{b}}_\omega\| \rightarrow 0$  as  $t \rightarrow \infty$ , we also know that the gyro bias error is bounded by the Lyapunov function  $V_b(t) > \|\tilde{\mathbf{b}}_\omega(t)\|^2 \forall(t)$ , and that  $V_b$  goes to zero monotonically. What is left to show is that the attitude estimates in (23)-(24) are uniformly bounded away from  $|\tilde{\Phi}_{nb}| = \pi$ , subjected to a gyro bias error  $\tilde{\mathbf{b}}_\omega$ , and that as  $\|\tilde{\mathbf{b}}_\omega\| \rightarrow 0$ , the attitude estimate  $|\tilde{\Phi}_{nb}| \rightarrow 0$ . First we recall that the observer (23)-(24) with zero gyro bias, is shown in Mahony [10] and Hua [22] to have almost globally asymptotically stability properties for constant reference vectors and semi-globally exponentially stability propeties for time-varying reference vectors. This gives  $\hat{\mathcal{R}}_{nb} \rightarrow I$  for  $\|\tilde{\mathbf{b}}_\omega\| = 0$  for all trajectories starting with  $|\tilde{\Phi}_{nb}(0)| < \pi - \epsilon$ . As in Grip et al. [11], because of the dynamic of  $\hat{\mathcal{R}}_{nb}$  we can see that the estimates will be kept in  $\hat{\mathcal{R}}_{nb} \in SO(3)$ . In addition, Grip showed in the start of the proof of Theorem 1 in appendix B [11]; that if the gyro bias error is bounded, then an  $c_i$  can be chosen for system (23)-(24) so that the attitude estimate is bounded away from  $|\tilde{\Phi}_{nb}| < \pi - \epsilon$ , which means an error from the gyro bias estimator below the bound will not destabilize the attitude estimate. The proof is in quaternions, where the attitude error is  $\tilde{\mathbf{q}} = [\tilde{s}, \tilde{\mathbf{r}}^\top]^\top$ , and we recall that  $\tilde{s} = 1 \Leftrightarrow \hat{\mathcal{R}}_{nb} = I$ . The Lyapunov function is chosen  $V_R(\tilde{s}) = 1 - \tilde{s}^2$ , which for system system (23)-(24) with a gyro bias error  $\tilde{\mathbf{b}}_\omega$  has the derivative (from [11] appendix B)

$$\begin{aligned} \dot{V}_R &\leq -\tilde{s} \tilde{\mathbf{r}}^\top \hat{\mathcal{R}}_{nb} \tilde{\mathbf{b}}_\omega - k_p c_{obs}^2 \tilde{s}^2 (1 - \tilde{s}^2) \\ &\leq \sqrt{V_b} - k_p c_{obs}^2 \tilde{s}^2 (1 - \tilde{s}^2) \end{aligned}$$

where  $k_p$  is a lower bound on the  $c_i$  gains, and  $c_{obs}$  comes from the assumption on the reference vectors. For a given bound  $V_b(0)$  on the gyro bias at the initialization,  $k_p$  can be chosen  $k_p > \bar{b}_\omega / (c_{obs}^2 \epsilon^2 (1 - \epsilon^2))$  so that  $\dot{V}_R < 0$  for  $|\tilde{s}| \geq \epsilon$ . Which implies that  $\tilde{s}$  is increasing if  $|\tilde{s}| \geq \epsilon$  and because of continuity of  $\dot{V}_R$  and the solutions  $\tilde{s}(0) \geq \epsilon$  then  $\tilde{s}(t) > \epsilon \forall t > 0$ . Further on, there is a  $\gamma$  so that for  $\epsilon < |\tilde{s}| < \gamma$  the  $\dot{V}_R < 0$ , which by the ultimate boundedness Theorem 4.18 [23] ensures that  $|\tilde{s}| > \gamma, t \geq \tau + T$  in finite time. In addition, we see that as  $V_b \rightarrow 0$  the bound  $\gamma \rightarrow 1$  which implies that  $\hat{\mathcal{R}}_{nb} \rightarrow I$  as  $t \rightarrow \infty$ . Thus we can conclude that (15)-(17) in cascade with (23)-(24) is semi-globally uniformly asymptotically stable, making the estimates converge to  $\hat{\mathcal{R}}_{nb} \rightarrow I, \tilde{\mathbf{b}}_\omega \rightarrow 0$  as  $t \rightarrow \infty$  because  $\epsilon$  can be arbitrary small. ■

*Remark 3:* In the proof we assume that the rotation matrix is  $\hat{\mathcal{R}}_{nb} \in SO(3)$  for all time. To ensure this in the implementation, that the rotation matrices were kept

orthogonal, several methods can be used. In this paper, the singular value decomposition(svd) algorithm was used; if  $[U, S, D] = \text{svd}(\hat{\mathcal{R}})$  then the new orthogonalized rotation matrix is  $\mathcal{R} = UV^T$ . Optionally it can be implemented using quaternions as in Mahony et al. [10], or an algorithm presented in Grip et al. [24] can be used.

*Remark 4:* . It should be noted that the global reference vectors can also be used in the gyro bias observer, for  $l_i^b = r_i^b$  and  $\rho_i \rightarrow \infty$ .

The total observer is summarized in Table I.

**TABLE I:** Summary of the SLAM Attitude Observer

SLAM ATTITUDE OBSERVER	
Measurements: $\omega_m, \rho_{im}, v_m^b, l_i^b, v_i^b, r_i^b, r_i^n$	
$\dot{l}_i^b = -S(\omega_m - \hat{b}_\omega + \sigma_{li})\hat{l}_i^b + \frac{1}{\rho_{im}}S(\hat{l}_i^b)S(l_{im}^b)v_m^b$	(15)
$\sigma_{li} = k_i S(l_{im}^b)\hat{l}_i^b$	(16)
$\dot{\hat{b}}_\omega = -\sum_{i=1}^m \sigma_{li}$	(17)
$\dot{\hat{\mathcal{R}}}_{nb} = \hat{\mathcal{R}}_{nb}S(\omega_m - \hat{b}_\omega + \sigma_{\mathcal{R}})$	(23)
$\sigma_{\mathcal{R}} = \sum_{i=1}^n c_i S(r_i^b)\hat{\mathcal{R}}_{nb}^T r_i^n$	(24)

#### IV. SIMULATION RESULTS AND PERFORMANCE EVALUATION

The observer was tested with simulations, and compared to the complimentary filter. A vessel travelling in a 3D-space described by the dynamics of (3) and (7), with changing angular velocity

$\omega(t) = [r_1 \cos(f_1 t), r_2 \sin(f_2 t), r_3 \log(1 + f_3 t)]^T$  and constant speed  $v^b = [u, 0, 0]$ . The trajectory of the vessel, with the landmarks positions can be seen in Figure 2.

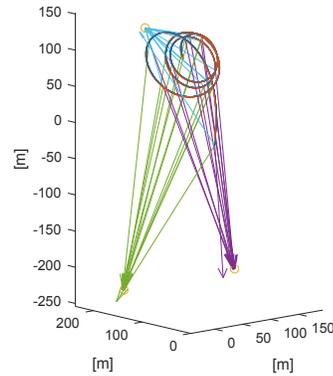
Three landmarks are placed randomly in a box 50[m] from the start point of the vessel. The  $\omega_m$  has a bias  $b_\omega = [0.8, 0.1, -0.5]^T$ , and is corrupted with a white noise with standard deviation  $\sigma_\omega = I0.001$ . The noise in the bearing and range measurements are  $\sigma_l = I0.01$  and  $\sigma_\rho = 0.005$ . The bearing noise is orthogonal to the bearing  $l_n = S(l_i^b)w_l$ , where the noise  $w_l$  is a white noise vector  $w_l = \mathcal{N}(0, \sigma_l)$ . The reference-vectors chosen were the normalized magnetometer and accelerometer gravity.

$$r_1^n = [1, 0, 0]^T, r_1^b = \mathcal{R}^T r_1^n \quad (25)$$

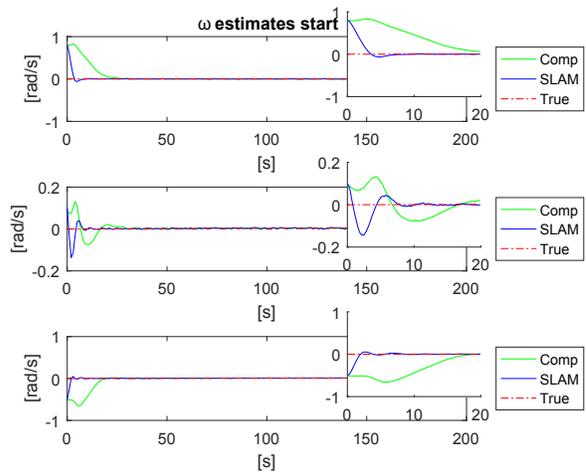
$$r_2^n = [0, 0, -1]^T, r_2^b = \mathcal{R}^T ([0, 0, -1]^T + a^n/g) \quad (26)$$

$$r_3^n = S(r_1^n)r_2^n, r_3^b = S(r_1^b)r_2^b \quad (27)$$

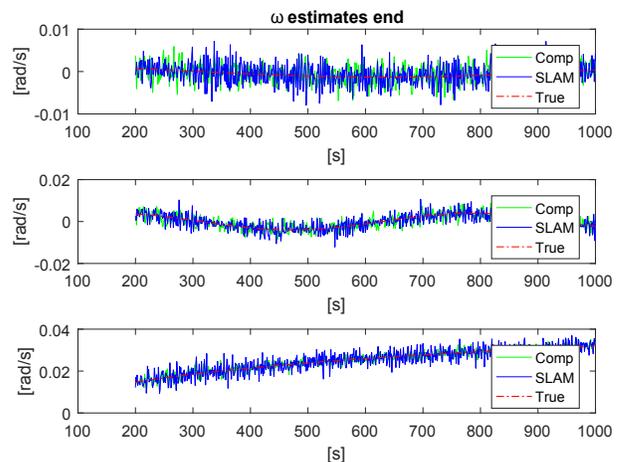
The LOS observer was tuned with  $k_i = \frac{1}{m} = \frac{1}{3}$ . The SLAM attitude observer in cascade with the LOS observer, was tuned as the complimentary filter:  $k_1 = 0.2, k_2 = 0.5, k_3 = 0.3$ . Where the weights  $k_i$  are weighted according quality of the measurements and gave the best results for both observers. The bias estimation gain of the complimentary filter was  $k_I = 0.15$ , which gave the best trade-off between the transient performance and variance of the  $\omega$  output. The starting attitude was  $\theta_{nb}(0) = \mathbf{0}$ , while the start estimate was  $\hat{\theta}_{nb}(0) = [\frac{5}{6}\pi, 0, \frac{3}{4}\pi]$  The results can be seen in Figures 3-5.



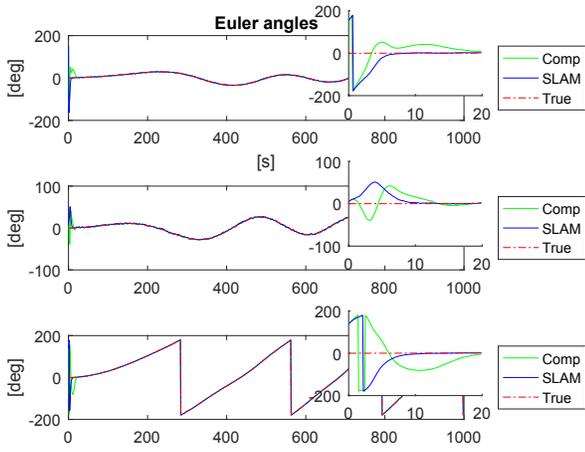
**Fig. 2:** The figure shows the trajectory of the vehicle, and the landmarks. The arrows represent the estimated distance vectors using  $\hat{l}_i^n$



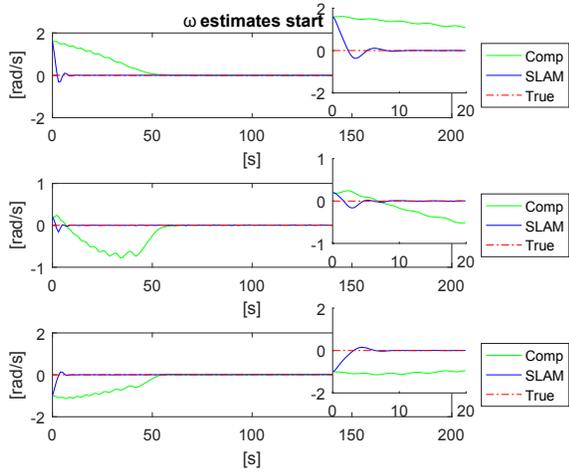
**Fig. 3:** The resulting  $\omega$  estimates transients from the complimentary filter and SLAM attitude observer



**Fig. 4:** The resulting  $\omega$  estimates from the complimentary filter and SLAM attitude observer, this shows the result for the steady state bias estimates



**Fig. 5:** The figure shows the attitude from the complimentary filter and SLAM observer. Both gets good estimated, but SLAM observer demonstrates less overshoot



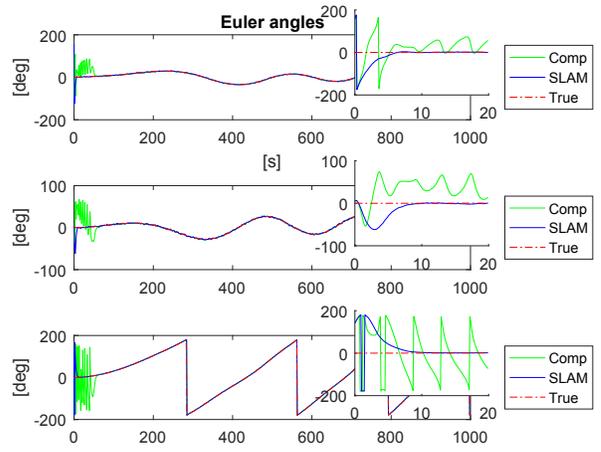
**Fig. 6:** The resulting  $\omega$  estimates transient from the complimentary filter and SLAM attitude observer for increased gyro bias

From the figures it is apparent that the SLAM observer has a faster convergence in both attitude and bias estimation, with no overshoots. This is a result of the decoupling of the estimation, so the bad attitude estimate does not affect the gyro bias estimation, as can happen with the complementary filter. It should be noted that this is the case if the gyro-bias is significant and there is a bad initial guess of the attitude estimate.

Another simulation was done for double gyro bias  $\mathbf{b}_\omega = [1.6, 0.2, -1]$ , and the results can be seen in Figures 6-7. From these simulations it is apparent that the SLAM attitude observer is much more robust against high gyro bias and bad attitude initialization, which again is the result of the decoupled system.

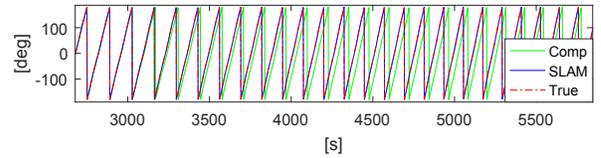
#### A. Magnetometer failure

A substantial goal for the SLAM attitude observer, is to make it less dependent on magnetometer measurements. To test if this is achieved, a scenario where the magnetometer is turned off is demonstrated. The compass reference vector

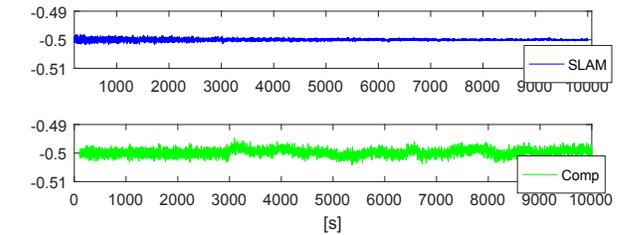


**Fig. 7:** The figure shows the attitude from the complimentary filter and SLAM observer for increased gyro bias. The SLAM observer demonstrates faster and smoother convergence

is set to the zero after  $t = 3000[s]$ , leaving the gravity as the only reference vector left. The results can be seen in Figures 8 and 9. The bias estimation of the complimentary filter starts being irregular, resulting in a drift in the attitude estimates with axis parallel to the gravity vector. The SLAM attitude observer still manages to estimate the bias, and thus the estimated attitude is hardly affected by the loss of the magnetometer reference vector, although it has some minor drift in the yaw axis, which is expected.



**Fig. 8:** The figure shows the yaw angle from the complimentary filter and SLAM attitude observer. A compass failure is introduced in  $t = 3000$  and the result is a drift in the yaw estimates for the complimentary filter, while the SLAM attitude observer does not get affected.



**Fig. 9:** The resulting gyro bias estimates from the complimentary filter and SLAM attitude observer in yaw. A compass failure is introduced in  $t = 3000$  and the result is oscillations for the complimentary filter

#### B. Landmark reference-vectors

A scenario was tested, where the magnetometer reference-vector was turned off and replaced by landmark reference-vectors. Moreover, the vectors  $\gamma_{ij}$  from (14) are used as reference-vectors, by setting

$$\begin{aligned} \mathbf{r}_1^n &= \hat{\mathcal{R}}_b^n \hat{\gamma}_{12}^b, & \mathbf{r}_2^n &= \hat{\mathcal{R}}_b^n \hat{\gamma}_{23}^b & (28) \\ \mathbf{r}_1^b &= \hat{\gamma}_{12}^b = \varrho_2 \mathbf{l}_2^b - \varrho_1 \mathbf{l}_1^b, & \mathbf{r}_2^b &= \hat{\gamma}_{23}^b = \varrho_3 \mathbf{l}_3^b - \varrho_2 \mathbf{l}_2^b & (29) \end{aligned}$$

In the simulations, the attitude error is reset to zero  $\tilde{\mathcal{R}} = \mathbf{I}$ , when the magnetometer was turned off and reference-vectors switched, to make it easier to notice a drift. As expected the result was drift-less, which shows that it is possible to have drift-less estimates of the attitude, without the use of global reference-vectors, if the bearing and range measurements or estimates are accurate enough.

## V. CONCLUSION

In this paper, we have discussed the SLAM problem, in addition to attitude and gyro bias estimation. We presented an observer that has decoupled the gyro bias and the attitude estimation to avoid that bad initialization interfere with the bias estimations. It applies bearing and range measurements of unknown landmarks for the bias estimation, in addition to reference vectors measurements. The performance of the observer is compared to the complimentary filter, where the advantageous behaviour of the observer is seen. A scenario in which there is a failure in the magnetometer has been demonstrated, and thus also the benefits of redundant measurements are shown. In addition, a method for using landmarks as reference vectors is briefly discussed.

## ACKNOWLEDGEMENT

This work was supported by the Research Council of Norway through the FRINATEK funding scheme, project number 250725, and Centre for autonomous marine operations and systems, project number 223254.

## REFERENCES

- [1] B. J. Guerreiro, P. Batista, C. Silvestre, and P. Oliveira, "Globally asymptotically stable sensor-based simultaneous localization and mapping," *IEEE Transactions on Robotics*, vol. 29, no. 6, pp. 1380–1395, 2013.
- [2] H. Durrant-Whyte and T. Bailey, "Simultaneous localization and mapping(SLAM): part i," *IEEE robotics & automation magazine*, vol. 13, no. 2, pp. 99–110, 2006.
- [3] T. Bailey and H. Durrant-Whyte, "Simultaneous localization and mapping(SLAM): Part ii," *IEEE Robotics & Automation Magazine*, vol. 13, no. 3, pp. 108–117, 2006.
- [4] J. A. Castellanos, J. Neira, and J. D. Tardós, "Limits to the consistency of EKF-based SLAM 1," 2004.
- [5] T. A. Johansen and E. Brekke, "Globally exponentially stable Kalman filtering for SLAM with AHRS," in *Information Fusion (FUSION), 2016 19th International Conference on*, pp. 909–916, IEEE, 2016.
- [6] P. Lourenço, B. J. Guerreiro, P. Batista, P. Oliveira, and C. Silvestre, "3-d inertial trajectory and map online estimation: Building on a GAS sensor-based SLAM filter," in *Control Conference (ECC), 2013 European*, pp. 4214–4219, IEEE, 2013.
- [7] M. D. Shuster and S. Oh, "Three-axis attitude determination from vector observations," *Journal of Guidance, Control, and Dynamics*, 2012.
- [8] S. J. Julier and J. K. Uhlmann, "Unscented filtering and nonlinear estimation," *Proceedings of the IEEE*, vol. 92, no. 3, pp. 401–422, 2004.
- [9] F. L. Markley, "Attitude error representations for Kalman filtering," *Journal of guidance, control, and dynamics*, vol. 26, no. 2, pp. 311–317, 2003.
- [10] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the special orthogonal group," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, 2008.
- [11] H. F. Grip, T. I. Fossen, T. A. Johansen, and A. Saberi, "Attitude estimation based on time-varying reference vectors with biased gyro and vector measurements," *IFAC World Congress*, vol. 44, no. 1, pp. 8497–8502, 2011.
- [12] P. Fusini, T. I. Fossen, and T. A. Johansen, "A uniformly semiglobally exponentially stable nonlinear observer for GNSS-and camera-aided inertial navigation," in *Proceedings of the 22nd Mediterranean Conference of Control and Automation, Palermo, Italy, 2014*.
- [13] P. Batista, C. Silvestre, and P. Oliveira, "Sensor-based globally asymptotically stable filters for attitude estimation: Analysis, design, and performance evaluation," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2095–2100, 2012.
- [14] P. Batista, C. Silvestre, and P. Oliveira, "Globally exponentially stable cascade observers for attitude estimation," *Control Engineering Practice*, vol. 20, no. 2, pp. 148–155, 2012.
- [15] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira, "A nonlinear position and attitude observer on SE(3) using landmark measurements," *Systems & Control Letters*, vol. 59, no. 3, pp. 155–166, 2010.
- [16] M.-D. Hua, M. Zamani, J. Trumppf, R. Mahony, and T. Hamel, "Observer design on the special euclidean group SE(3)," in *2011 50th IEEE Conference on Decision and Control and European Control Conference*, pp. 8169–8175, IEEE, 2011.
- [17] S. Brás, R. Cunha, C. J. Silvestre, and P. J. Oliveira, "Nonlinear attitude observer based on range and inertial measurements," *IEEE Transactions on Control Systems Technology*, vol. 21, no. 5, pp. 1889–1897, 2013.
- [18] J. Sola, "Quaternion kinematics for the error-state kf," *Laboratoire d'Analyse et d'Architecture des Systemes-Centre national de la recherche scientifique (LAAS-CNRS), Toulouse, France, Tech. Rep.*, 2012.
- [19] T. I. Fossen, *Handbook of Marine Craft Hydrodynamics and Motion Control*. Wiley, 2011.
- [20] T. Hamel and R. Mahony, "Attitude estimation on SO(3) based on direct inertial measurements," in *Proceedings 2006 IEEE International Conference on Robotics and Automation, 2006. ICRA 2006.*, pp. 2170–2175, IEEE, 2006.
- [21] V. Matrosov, "On the stability of motion," *Journal of Applied Mathematics and Mechanics*, vol. 26, no. 5, pp. 1337–1353, 1962.
- [22] M.-D. Hua, "Attitude estimation for accelerated vehicles using gps/ins measurements," *Control Engineering Practice*, vol. 18, no. 7, pp. 723–732, 2010.
- [23] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 2002.
- [24] H. F. Grip, T. I. Fossen, T. A. Johansen, and A. Saberi, "Globally exponentially stable attitude and gyro bias estimation with application to GNSS/INS integration," *Automatica*, vol. 51, pp. 158–166, 2015.