An Interval Arithmetic Approach to Input-Output Reachability^{*}

Ivan Perez Avellaneda

Luis A. Duffaut Espinosa

Abstract—The safety region of operation of a system is the subset of allowed outputs for which no undesirable outcome would occur. Knowing if a system would ever leave its safety regions of operation is important information for the planning and control of systems. The computation of the set of outputs also known as the reachable set of the system is, in many cases, intensive and a simple overestimation of it is preferred, instead. There are several perspectives to address this problem including set-based approaches, Mixed-Monotonicity, Hamilton-Jacobi reachability, neural networks, and recently reachability via Chen-Fliess series. In the present work, a Chen-Fliess series representing the input-output behavior of a dynamical system along with interval arithmetic is used to overestimate the reachable set of a system. The advantage of this combination of techniques is that it provides a closed-form of the overestimating set. Examples are presented to illustrate the results.

Index Terms—Nonlinear systems, Chen-Fliess series, Reachability, Interval arithmetic

I. INTRODUCTION

The complexity of the computation of the safety region of operation of systems increases as systems get more complex each day. An important tool to assess whether the output of a system under certain conditions lies in its safety region is the *reachable sets*. More precisely, it is defined as the set of all outputs of the system as a response to a given set of initial inputs, parameters, or initial states [4]. Practical applications are found throughout the engineering field. For example in aircraft auto-landing [5], stability of power systems [13] and aerobatic maneuvers of UAVs [8] to mention a few. Even for linear time-invariant systems, if the dimension is high, the computation of reachable sets turns easily burdensome. Such is the case of methods that rely on the Minkowski sum of polyhedra [12] and thus more efficient and fast alternatives are needed [9].

Among the most popular techniques used to analyze the reachability of systems are those based on set operations. In [2], the set of initial states and inputs are polytopes, zonotopes, and intervals and the arithmetic of these sets is used to compute reachable sets of linear, non-linear, and hybrid systems. In the Mixed-Monotonicity approach [6], [26], a decomposition function is associated with the original dynamics, which allows the system to decouple into monotone parts that recover the original dynamics. Then an *embedding system* is used to compute an overestimation of the system's reachable sets. In [6] a decomposition function is provided

in terms of an optimization problem. The overestimating sets produced are hypperrectangles determined by the northeast and southwest corners. In some cases, there exists a closed-form of the embedding system, but in general, this is not the case. The *Interval Reachability Analysis* (TIRA) toolbox [14] implements the Mixed-Monotonicity approach. Alternatively, the *Continuous Reachability Analyzer* software (CORA) package uses zonotopes for a faster computation of Minkowski sums [3]. The Hamilton-Jacobi reachability, Koopman operators, and machine learning are among other paradigms to tackle the problem of computing reachable sets [1], [8], [15], [22], [25].

A Chen-Fliess series is a sum of weighted iterative integrals which are functions of the input. For instance, any Volterra operator with analytic kernels and any controlaffine nonlinear system with analytic vector fields can be represented by this formalism. Recently, in [17] the Mixed-Monotonicity approach to computing reachable sets was extended to an input-output setting by use of Chen-Fliess series. This opened the door to a data-driven approach for reachability analysis. First, it was proved that all Chen-Fliess series are Mixed-Monotone according to a decomposition Chen-Fliess series, then this was used to compute an overestimation of the reachable set. Since this overestimation is not the smallest, in [18] the authors improved the result by optimizing the Chen-Fliess series of the system subject to input constraints. For numerical purposes regarding the optimization problem, the gradient descent algorithm was extended to Chen-Fliess series which provided the smallest bounding box (SBB) containing the reachable set. In [19], the second order approximation of the Chen-Fliess series along with the Newton-Raphson algorithm is used to optimize Chen-Fliess series. This approach also provided the SBB containing the reachable set. Although the optimization of Chen-Fliess series has proved to be successful to calculate the SBB containing reachable sets, a closed-form has not been obtained yet.

This manuscript contributes to the literature on inputoutput reachability by providing a theoretical framework and a simple methodology to compute an overestimation of reachable sets of input-output operators that can be described by the Chen-Fliess formalism and whose inputs lie in a hyperrectangle. In particular, a closed-form for the overestimation of the reachable set of a system represented by a Chen-Fliess series is provided in terms of interval arithmetic. It is shown that the reachable set is included within the boundaries produced by two specific convergent Chen-Fliess series.

The paper is organized as follows. Section II gives prelimi-

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The authors are affiliated with the Department of Electrical and Biomedical Engineering, University of Vermont, Burlington, Vermont 05405 USA.

naries on the Chen-Fliess formalism, and the standard theory of mixed-monotonicity with its application to reachable set calculation. Section III provides a closed-form of an overestimation of the reachable set of a Chen-Fliess series by means of interval arithmetics. First it is proved that an iterative integral is mixed-monotone, then the overestimation of this is obtained with the use of interval arithmetics, then the overestimation of the Chen-Fliess series is performed by summing the overestimation of each iterative integral. In Section IV, illustrative examples and simulations are presented. The conclusions and future work are given in the final section

NOTATION

A monoid is defined as the tuple (X, \cdot, \emptyset) where the finite set $X = \{x_0, x_1, \ldots, x_m\}$ is called *alphabet* and its elements are noncommuting symbols referred to as *letters*. The concatenation binary operation $\cdot : X \times X \to X$ is defined as the map that assigns x_{i_1} and x_{i_2} to $x_{i_1} \cdot x_{i_2}$ with identity element, \emptyset . This is, $\emptyset \cdot x_i = x_i \cdot \emptyset = x_i$. A word is defined as the recursive concatenation $\eta = x_{i_1} \cdots x_{i_k}$ of letters of X where the number of letters in a word η , written as $|\eta|$, is called its *length*. The empty word, \emptyset , is taken to have length zero. The collection of all words of length k is denoted by X^k . Define the Kleene closure of X as the set of all words of all lengths including the empty word and denote it $X^* = \bigcup_{k \ge 0} X^k$. Given a set $\mathcal{X} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : p(x)\}$, the projection to the *i*-th coordinate of \mathcal{X} is defined as Proy $_{x_i}(\mathcal{X}) = \{x_i \in \mathbb{R} : (x_1, \cdots, x_n) \in \mathcal{X}\}$.

II. PRELIMINARIES

In this section, Chen-Flies series are presented as a tool that provides an input-output representation of a type of nonlinear systems. Its definition is given in terms of iterated integrals indexed by words. Because of this, concepts of formal language theory such as words and formal power series along with the results that guarantee the Chen-Fliess series's convergence to the system's output are presented. Then, the theory of mixed-monotonicity is presented as a tool to approximate the set of outputs of a system as a result of a set of inputs and initial conditions acting on the system. This approximation is given in terms of an embedded system whose trajectory preserves certain partial order with respect to the inputs and initial conditions of the original system.

A. Chen-Fliess series

A formal power series is any mapping of the form $c : X^* \to \mathbb{R}^{\ell}$ that is written as the formal sum $c = \sum_{\eta \in X^*} (c, \eta) \eta$. Here, $(c, \eta) \in \mathbb{R}^{\ell}$ represents the value of c at $\eta \in X^*$ and is known as the *coefficient* of η . The set of all formal power series with domain X^* and coefficients in \mathbb{R}^{ℓ} is denoted by the symbol $\mathbb{R}^{\ell}\langle\langle X \rangle\rangle$. Given a formal power series $c \in \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$, the set of all words that map to nonzero coefficients is called the *support* of c and it is denoted by $\supp(c)$ and if c has finite support then it is called a polynomial. Denote the set of all polynomials with domain X^* mapping to \mathbb{R}^{ℓ} as $\mathbb{R}^{\ell}\langle X \rangle$.

Next, the type of input function that acts in a system is specified. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, this is, $u(t) = (u_1(t), \cdots, u_m(t))$, define the norm $||u||_{\mathfrak{p}} =$ $\max\{||u_i||_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $||u_i||_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$ -norm of the measurable real-valued coordinate function, u_i , defined on $[t_0, t_1]$. Let $L^m_{\mathfrak{p}}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $||\cdot||_{\mathfrak{p}}$ norm and define the closed ball of radius R as the set $B^m_{\mathfrak{p}}(R)[t_0, t_1] := \{u \in L^m_{\mathfrak{p}}[t_0, t_1] : ||u||_{\mathfrak{p}} \leq R\}$. Denote $C[t_0, t_1]$ the subset of continuous functions in $L^m_1[t_0, t_1]$. An *iterated integral* is defined inductively for each word $\eta = x_i \bar{\eta} \in X^*$ as the map $E_{\eta} : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_{\emptyset}[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t,t_0) = \int_{t_0}^t u_i(\tau) E_{\bar{\eta}}[u](\tau,t_0) \, d\tau, \tag{1}$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. A *Chen-Fliess* series, is a causal *m*-input, ℓ -output operator, F_c , associated with a formal power series $c \in \mathbb{R}^{\ell} \langle \langle X \rangle \rangle$ such that

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_{\eta}[u](t, t_0)$$
(2)

[7]. It is assumed hereafter without loss of generality that $t_0 = 0$, which allows denoting $E_{\eta}[u](t, t_0)$ as $E_{\eta}[u](t)$. When there exists $K, M \ge 0$ reals such that

$$|(c,\eta)| \le K M^{|\eta|} \, |\eta|!, \ \forall \eta \in X^*$$

 $(|z| := \max_i |z_i| \text{ when } z \in \mathbb{R}^\ell)$ then (2) converges absolutely and uniformly for sufficiently small R, T > 0 and constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R)[t_0, t_0+T]$ into $B_{\mathfrak{q}}^{\ell}(S)[t_0, t_0 + T]$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, +\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [10]. Any such mapping is called a locally convergent Chen-Fliess series, and the set of all locally convergent series is denoted $\mathbb{R}_{LC}^{\ell}\langle\langle X \rangle\rangle$. If instead $|(c,\eta)| \leq KM^{|\eta|}$, $\forall \eta \in X^*,$ then (2) converges on the extended space $L_{n,e}^{m}(t_0)$ into $C[t_0,\infty)$, where $L^m_{p,e}(t_0) := \bigcup_{T>0} L^m_p[t_0,t_0+T]$. Thus, (2) is well-defined for all times. Any such mapping is called a globally convergent Chen-Fliess series. The set of all globally convergent series is denoted $\mathbb{R}^{\ell}_{GC}\langle\langle X \rangle\rangle$. The more in-depth discussion of the convergence of Chen-Fliess series is presented in [24].

It was shown in [20], [21] that a locally convergent Chen-Fliess series F_c defined on $B_p^m(R)[t_0, t_0 + T]$ and having finite Lie Hankel rank is *realizable* by a system of the form

$$\dot{z}(t) = g_0(z(t)) + \sum_{i=1}^m g_i(z(t)) u_i(t), \ z(t_0) = z_0,$$
 (3a)

$$y_j(t) = h_j(z(t)), \quad j = 1, 2, \dots, \ell,$$
 (3b)

where each g_i is analytic on some neighborhood \mathcal{W} of $z_0 \in \mathbb{R}^n$, and each h_j is an analytic function on \mathcal{W} such that (3a) has a well defined solution z(t), $t \in [t_0, t_0 + T]$ for any given input $u \in B^m_p(R)[t_0, t_0 + T]$, and $y_j(t) = F_{c_j}[u](t) = h_j(z(t))$, $t \in [t_0, t_0 + T]$, $j = 1, 2, \ldots, \ell$. On the other hand, for any system as in (3), one can show that for any $\eta = x_{i_k} \cdots x_{i_1} \in X^*$ the coefficients of the corresponding Chen-

Fliess series can be written as

$$(c_j, \eta) = L_{g_\eta} h_j(z_0) := L_{g_{i_1}} \cdots L_{g_{i_k}} h_j(z_0), \qquad (4)$$

where $L_{q_i}h_j$ is the *Lie derivative* of h_j with respect to g_i [7], [11], [16]. Generally, a Chen-Fliess series can be defined independently of a state space model, and thus can be used for data-driven analysis and control [23].

B. Mixed-Monotonicity of State Space Models

A partial order is a relation that satisfies the properties of reflexivity, transitivity, and antisymmetry. Let \leq be the componentwise partial order on \mathbb{R}^n . This is, for x = (x_1,\cdots,x_n) and $y=(y_1,\cdots,y_n)\in\mathbb{R}^n,\ x\leq y$ if and only if $x_i \leq y_i$ for $i \in \{1, \dots, n\}$. Consider the vectors $x, \hat{x}, y, \hat{y} \in \mathbb{R}^n$ and the concatenated vectors (x, \hat{x}) and (y, \hat{y}) in \mathbb{R}^{2n} . The southeast (SE) partial order \leq_{SE} on \mathbb{R}^{2n} is defined as $(x, \hat{x}) \leq_{SE} (y, \hat{y})$ if and only if $x \leq y$ and $\hat{y} \leq \hat{x}$. This is equivalently written in terms of the componentwise partial order as $(x, \hat{x}) \leq_{SE} (y, \hat{y})$ if and only if $(x, -\hat{x}) \leq (y, -\hat{y})$. Next, consider $a, b \in \mathbb{R}^n$, an extended hyper-rectangle is defined as the set $[a, b] := \{x \in \mathbb{R}^n : a \leq a \}$ $x \leq b \} \subset \mathbb{R}^n$. Another way used to define a hyper-rectangle in \mathbb{R}^n is by considering a single point $a = (b, \hat{b})$ in \mathbb{R}^{2n} that is the concatenation of the vectors $b, \hat{b} \in \mathbb{R}^n$ and setting $\llbracket a \rrbracket := \llbracket b, \hat{b} \rrbracket$. The SE partial order in \mathbb{R}^{2n} helps represent a partial order relation on the set of hyper-rectangles in \mathbb{R}^n . To observe this, consider the nested hyper-rectangles $[a,b] \subset [c,d] \subset \mathbb{R}^n$ where $c \leq a$ and $b \leq d$. This inclusion relation of hyper-rectangles in \mathbb{R}^n is equivalently written as $(c,d) \leq_{SE} (a,b)$ in \mathbb{R}^{2n} . Figure 1 illustrates the SE order as defined in this section.



Fig. 1. The hyper-rectangles [(3, 2), (5, 3)] and [(1, 1), (6, 4)] in \mathbb{R}^2 satisfy $((1,1),(6,4)) \leq_{SE} ((3,2),(5,3))$ in \mathbb{R}^4 .

Consider the hyper-rectangles $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$ and the locally Lipschitz continuous function in each argument $f: \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$. The continuous-time dynamical system is defined

$$\dot{x}(t) = f(x(t), u(t))$$

$$x(0) = x_0$$
(5)

where $x_0 \in \mathcal{X}$ and $u : [0,T] \rightarrow \mathcal{U}$. Given an input function u(t) and an initial condition x_0 , the trajectory function $\phi(t, u, x_0)$ of (5) satisfies the dynamical system and represents the state of the system at time t. Next, the set of states reached at a certain time by the dynamics is defined.

Definition 1: Consider the system described by (5). The reachable set of the system subject to a set of inputs \mathcal{U} = $[\underline{u}, \overline{u}]$ and a set of initial states $\mathcal{X}_0 = [\underline{x}, \overline{x}]$ is the set

$$\operatorname{Reach}(\mathcal{X}_0,\mathcal{U})(T) := \left\{ \phi(T,u,x_0) \in \mathbb{R}^n : \right.$$

for some $u : [0, t] \to \mathcal{U}, x_0 \in \mathcal{X}_0$ The next definition generalizes the concept of monotone system by embedding the vector field of the dynamics into the diagonal of a new function called *decomposition function*. This function has monotone properties in each argument.

Definition 2: [6] Let $d: \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ be a locally Lipschitz continuous function. The function d is said to be the decomposition function of (5) if the following holds:

- *i*. For all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$, d(x, u, x, u) = f(x, u).
- *ii.* For all $i, j \in \{1, \dots, n\}$ with $i \neq j$, $\frac{\partial d_i}{\partial x_j}(x, u, \hat{x}, \hat{u}) \ge 0$ for all $x, \hat{x} \in \mathcal{X}$ and for all $u, \hat{u} \in \mathcal{U}$ whenever the derivative exists.
- *iii.* For all $i, j \in \{1, \cdots, n\}$,

$$\frac{\partial d_i}{\partial \hat{x}_j}(x, u, \hat{x}, \hat{u}) \le 0$$

for all $x, \hat{x} \in \mathcal{X}$ and for all $u, \hat{u} \in \mathcal{U}$ whenever the derivative exists.

iv. For all $i \in \{1, \dots, n\}$ and all $k \in \{1, \dots, m\}$,

$$\frac{\partial d_i}{\partial u_k}(x,u,\hat{x},\hat{u}) \geq 0, \ \frac{\partial d_i}{\partial \hat{u}_k}(x,u,\hat{x},\hat{u}) \leq 0$$

for all $x, \hat{x} \in \mathcal{X}$ and for all $u, \hat{u} \in \mathcal{U}$ whenever the derivative exists.

The system (5) is said to be *MM* if there exists a decomposition function of its vector field. The definition suggests that the decomposition function of a system is not unique. In fact, a system may have several decompositions. The next theorem provides one decomposition function of any vector field described in (5).

Theorem 1: [6] Given an arbitrary system of the form (5). The system is MM with respect to the following decomposition function d:

$$d_{i}(x, u, \hat{x}, \hat{u}) = \begin{cases} \min_{\substack{y \in [x, \hat{x}] \\ y_{i} = x_{i} \\ z \in [u, \hat{u}] \\ m \in [\hat{x}, x] \\ y_{i} = x_{i} \\ z \in [\hat{u}, u] \end{cases}} f_{i}(y, z), \ \hat{x} \le x, \ \hat{u} \le u.$$
(6)

The decomposition function is used to compute an overapproximation of a reachable set of the system (5) by making it the vector field of system called *embedding system*.

Definition 3: Consider the MM system (5) with decomposition function d, the embedding system is defined as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \varepsilon(x, u, \hat{x}, \hat{u}) := \begin{bmatrix} d(x, u, \hat{x}, \hat{u}) \\ d(\hat{x}, \hat{u}, x, u) \end{bmatrix},$$
(7)

where $(x, \hat{x}) \in \mathcal{X} \times \mathcal{X} \subset \mathbb{R}^{2n}$ and input $(u, \hat{u}) \in \mathcal{U} \times \mathcal{U} \subset$ \mathbb{R}^{2m} .

The decomposition provided in Theorem 1 is the tightest in the sense that any other decomposition function generates no smaller hyper-rectangle that contains the true reachable set [6]. It is important to highlight that obtaining a closedform of (6) is nontrivial and in general, requires solving a non-convex optimization problem.

Let $\Phi^{\varepsilon}(t, (x_0, \hat{x}_0), (u, \hat{u}))$ denote the state trajectory of (7) at time t when the initial state is $(x_0, \hat{x}_0) \in \mathcal{X} \times \mathcal{X}$ and the inputs are $u, \hat{u} : [0, \infty[\rightarrow \mathcal{U} = [\underline{u}, \overline{u}]]$. When the inputs are of the form $u(t) = \underline{u}$ and $\hat{u}(t) = \overline{u}$, the state trajectory of (7) is denoted $\Phi^e(t, (x_0, \hat{x}_0), (\underline{u}, \overline{u}))$. The trajectory Φ^{ε} preserves the SE order [6]. The next theorem relates the embedding system with the reachable sets of system (5).

Theorem 2: [6] Assume system (5) is MM with respect to d. Consider $\mathcal{X}_0 = [\underline{x}, \overline{x}] \subset \mathcal{X}$ a hyperrectangle of initial states and the input functions $u_1 : [0, \infty[\rightarrow \mathcal{U}, u_2 : [0, \infty[\rightarrow \mathcal{U}, u_3 : [0, \infty[\rightarrow \mathcal{U$

$$\operatorname{Reach}([\underline{x},\overline{x}],[\underline{u},\overline{u}])(t) \subset \llbracket \Phi^e(t,(\underline{x},\overline{x}),(\underline{u},\overline{u})) \rrbracket$$

for all $t \ge 0$.

III. INTERVAL APPROACH TO REACHABLE SET OVERESTIMATION

In this section, an over-approximation of the reachable set produced by an arbitrary Chen-Fliess series is provided. First, the reachable set of a single iterated integral is computed by considering this as a system formed by a chain of integrators. Then, the corresponding decomposition function is obtained using MM, as described in Section II-B. It is proved that the closed-form of the reachable set of an iterated integral with input function bounded by a hyper-rectangle is written in terms of interval arithmetic. By summing all the reachable sets of all the iterated integrals weighted by their corresponding coefficient, an overestimation of the reachable set of the Chen-Fliess series is obtained.

Here, an interval is a set in \mathbb{R} defined as $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for $a, b \in \mathbb{R}$. In terms of section II-B, it is a hyper-rectangle over \mathbb{R} . In order to shorten the notation of hyper-rectangles defined by two vectors each with the same coordinates, let $[\mathbf{a}, \mathbf{b}]$ denote the hyper-rectangle in \mathbb{R}^n formed by taking n times the cartesian product of the interval $[a, b] \subset \mathbb{R}$. Equivalently, $[\mathbf{a}, \mathbf{b}] = [(a_1, \dots, a_n), (b_1, \dots, b_n)] \subset \mathbb{R}^n$ with $a_i = a$ and $b_i = b$ for all $i \in \{1, \dots, n\}$. Next, the product of intervals is defined.

Definition 4: Given the intervals $I_1 = [a_1, b_1] \subset \mathbb{R}$ and $I_2 = [a_2, b_2] \subset \mathbb{R}$, the product $I_1 \cdot I_2$ is defined as the interval $[\underline{I}, \overline{I}]$ where

$$\underline{I} = \min_{\substack{y_1 \in [a_1, b_1] \\ y_2 \in [a_2, b_2]}} y_1 y_2, \quad \overline{I} = \max_{\substack{y_1 \in [a_1, b_1] \\ y_2 \in [a_2, b_2]}} y_1 y_2.$$

Observe that by simple inspection the product of $[a_1, b_1]$ and $[a_2, b_2]$, with $a_1, a_2, b_1, b_2 \in \mathbb{R}$, is written as

$$[a_1, b_1] \cdot [a_2, b_2] = [\min\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}, \\ \max\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}].$$

In particular, when the intervals are the same $I_1 = I_2 = [a,b] \subset \mathbb{R}$, the product $[a,b] \cdot [a,b]$ is denoted as a power of intervals $[a,b]^2$. In general, the product of n times the same

interval [a, b] is denoted $\underline{[a, b] \cdots [a, b]} = [a, b]^n$.

Example 1: Consider the intervals [-2, 1] and [1, 3]. From Definition 4, the product $[-2, 1] \cdot [1, 3] = [-6, 3]$, because $\min\{-2, -6, 1, 3\} = -6$ and $\max\{-2, -6, 1, 3\} = 3$. Also, the second power of the interval [-2, 1] is $[-2, 1]^2 = [-2, 4]$.

Another operation used in this manuscript is the product of a real number and a set

Definition 5: Given a set $\mathcal{X} \subset \mathbb{R}^n$ and a number $\lambda \in \mathbb{R}$, the product of \mathcal{X} and λ is defined as $\lambda \mathcal{X} = \{\lambda x : x \in \mathcal{X}\}$. *Example 2:* Consider the interval [-2, 1] and $\lambda = 3$, then 3[-2, 1] = [-6, 3].

In order to distinguish between the reachable set of system (5) given by Definition 1 and the reachable set of a Chen-Fliess series, the next definition is provided.

Definition 6: Given the alphabet $X = \{x_0, \dots, x_m\}$, the formal power series $c \in \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$ and the hyper-rectangle $\mathcal{U} \subset \mathbb{R}^m$, the reachable set of the Chen-Fliess series $F_c[u](t)$ taking values in the set of inputs \mathcal{U} is the set

$$\operatorname{Reach}_{c}(\mathcal{U})(T) := \left\{ y = F_{c}[u](T) \in \mathbb{R}^{\ell} : for some \ u : [0, t] \to \mathcal{U} \right\}.$$

Next, a simple example is provided to illustrate this definition and the idea behind the use of interval arithmetic to compute overestimations of the reachable set of a Chen-Fliess series.

Example 3: Consider the alphabet $X = \{x_0, x_1\}$, the formal power series $c = x_1^2 \in \mathbb{R}\langle\langle X \rangle\rangle$ and the hyperrectangle $\mathcal{U} = [-2, 1] \subset \mathbb{R}$. The Chen-Fliess series $F_c[u](t)$ is represented by only one iterated integral $F_c[u](t) = E_{x_1^2}[u](t)$. Since the inputs are taken in the set $\mathcal{U} = [-2, 1]$, then

$$-2 \le u(\tau) \le 1$$

$$-2\tau_1 \le \int_0^{\tau_1} u(\tau) d\tau \le \tau_1$$

which means that $\int_0^{\tau_1} u(\tau) d\tau \in [-2\tau_1, \tau_1]$ and since $\tau_1 > 0$, then by Definition 5 $[-2\tau_1, \tau_1] = [-2, 1]\tau_1$. On the other hand, $u(\tau_1) \in [-2, 1]$, then

$$u(\tau_1) \int_0^{\tau_1} u(\tau) d\tau \in [-2,1] \cdot [-2,1]\tau_1$$

and

$$E_{x_1^2}[u](t) = \int_0^t u(\tau_1) \int_0^{\tau_1} u(\tau) d\tau d\tau_1 \in [-2,1]^2 \frac{t^2}{2!}.$$

Therefore, $\operatorname{Reach}_c(\mathcal{U})(t) \subset [-2,1]^2 \frac{t^2}{2!}.$

Remark :The challenge to computing reachable sets of more complex Chen-Fliess series by adding up the reachable sets of the iterated integrals is that in general

$$\min_{u \in \mathcal{U}} E_{\eta_1}[u](t) + \min_{u \in \mathcal{U}} E_{\eta_2}[u](t) \leq \min_{u \in \mathcal{U}} (E_{\eta_1}[u](t) + E_{\eta_2}[u](t))$$
$$\max_{u \in \mathcal{U}} (E_{\eta_1}[u](t) + E_{\eta_2}[u](t)) \leq \max_{u \in \mathcal{U}} E_{\eta_1}[u](t) + \max_{u \in \mathcal{U}} E_{\eta_2}[u](t)$$

then the sum of the reachable sets of the iterated integrals provides an overestimation of the reachable set of the Chen-Fliess series. To relate both reachable sets in Definition 1 and 6, notice that an arbitrary iterated integral has a dynamical system associated with it. To observe this, consider the formal power series consisting of only one word $c = \eta = x_{i_1} \cdots x_{i_n} \in X^*$. The associated Chen-Fliess series of c is the iterated integral $F_c[u](t) = E_{\eta}[u](t)$. Define the following set of functions

$$w_{1}(t) = E_{\eta}[u](t)$$

$$\vdots$$

$$w_{n-1}(t) = E_{x_{i_{n-1}}x_{i_{n}}}[u](t),$$

$$w_{n}(t) = E_{x_{i_{n}}}[u](t).$$
(8)

By differentiating the equations above, the following statespace represented dynamical system is obtained in \mathbb{R}^n :

$$\dot{w}_{1}(t) = u_{i_{1}}(t)w_{2}(t)
\dot{w}_{2}(t) = u_{i_{2}}(t)w_{3}(t)
\vdots
\dot{w}_{n}(t) = u_{i_{n}}(t)$$
(9)

with initial condition $w(0) = w_1(0) = \cdots = w_n(0) = 0$. Without loss of generality, assume that $i_n \neq 0$. Consider $\mathcal{W}_0 = \{(0, \cdots, 0)\} \subset \mathbb{R}^n$. According to Definition 1, for a hyper-rectangle $\mathcal{U} \subset \mathbb{R}^n$, the reachable set of system (9) is Reach $(\mathcal{W}_0, \mathcal{U})(t)$ and from Definition 6 the reachable set of the iterated integral $w_1(t) = E_{\eta}[u](t)$ is Reach $_{\eta}(\mathcal{U})(t)$. Therefore,

$$\operatorname{Reach}_{n}(\mathcal{U})(t) = \operatorname{Proy}_{w_{1}}(\operatorname{Reach}(\mathcal{W}_{0},\mathcal{U})(t)).$$

The closed-form of the reachable set of an iterated integral in terms of the interval defining the input set is given next.

Lemma 1: Consider the input function $u : \mathbb{R} \to \mathbb{R}^m$ with values in the hyperrectangle $[\mathbf{a}, \mathbf{b}]$ (i.e., $u_i(t) \in [a, b], \forall t \in [0, T]$ with T > 0). For $\eta = x_{i_1} \cdots x_{i_n} \in X^*$, the reachable set of the iterated integral $E_{\eta}[u](t)$ is bounded by

$$\operatorname{Reach}_{\eta}([\mathbf{a},\mathbf{b}])(t) \subset [a,b]^{|\eta|-|\eta|_{x_0}} \frac{t^{|\eta|}}{|\eta|!}, \forall \ t \in [0,T]$$

Proof: As shown by (8) and (9), there is a dynamical system associated to $E_{\eta}[u](t)$. From Theorem 1, the decomposition function associated to the embedding system of (9) is

$$d_n(a,b) = a, \ d_n(b,a) = b,$$

$$\begin{aligned} d_j(w_{j+1}, a, \hat{w}_{j+1}, b) &= \min_{\substack{y \in [w_{j+1}, \hat{w}_{j+1}] \\ z \in [a, b]}} zy, \text{ if } w_{j+1} \le \hat{w}_{j+1} \\ d_j(\hat{w}_{j+1}, b, w_{j+1}, a) &= \max_{\substack{y \in [w_{j+1}, \hat{w}_{j+1}] \\ z \in [a, b]}} zy, \text{ if } \hat{w}_{j+1} \le w_{j+1} \end{aligned}$$

for $j \in \{1, \dots, n-1\}$. Therefore, the embedding system corresponding to $E_n[u](t)$ is

$$\dot{w}_1 = d_1(w_2, a, \hat{w}_2, b), \cdots, \dot{w}_n = d_n(a, b),
\dot{\hat{w}}_1 = d_1(\hat{w}_2, b, w_2, a), \cdots, \dot{\hat{w}}_n = d_n(b, a).$$
(10)

According to Theorem 2, the states w_1 and \hat{w}_1 provide the evolution of the south-west and north-east corners of the overapproximating hyper-rectangle of the reachable set of (10). Given that (10) is also a chain of integrators, one can simply start solving for w_n and \hat{w}_n , and then solve

sequentially for n - 1, ..., 1. The solution to (10) is then

$$w_n = at, \ \hat{w}_n = bt$$

and, for any
$$j = 1, ..., n - 1$$
,
 $w_{j-1} = \min_{\substack{y \in [w_j, \hat{w}_j] \\ z \in [a,b]}} \int_0^t zy(\tau) d\tau, \hat{w}_{j-1} = \max_{\substack{y \in [w_j, \hat{w}_j] \\ z \in [a,b]}} \int_0^t zy(\tau) d\tau.$

Given the simple solution for w_n and \hat{w}_n , one can apply the change of variables $x = y/\tau^i$ in each w_{n-i} and \hat{w}_{n-i} for $i \in \{1, \dots, n-1\}$ so that the min and max are calculated over [a, b]. This together with the monotonicity of the integral operator gives

$$w_{n-1} = \min_{\substack{x \in [a,b] \\ z \in [a,b]}} \int_0^t \tau z x(\tau) d\tau = \min_{\substack{x \in [a,b] \\ z \in [a,b]}} z x \frac{t^2}{2},$$
$$\hat{w}_{n-1} = \max_{\substack{x \in [a,b] \\ z \in [a,b]}} \int_0^t \tau z x(\tau) d\tau = \max_{\substack{x \in [a,b] \\ z \in [a,b]}} z x \frac{t^2}{2}.$$

Continuing the recursion, it follows that

$$w_{1} = \min_{\substack{x_{i} \in [a,b] \\ i \in \{1, \cdots, r\}}} x_{1} \cdots x_{r} \frac{\iota}{n!},$$
$$\hat{w}_{1} = \max_{\substack{x_{i} \in [a,b] \\ i \in \{1, \cdots, r\}}} x_{1} \cdots x_{r} \frac{t^{n}}{n!},$$

 $_{\perp n}$

where $r = |\eta| - |\eta|_{x_0}$. Finally, from Definition 4,

 $\operatorname{Reach}_{\eta}([\mathbf{a},\mathbf{b}])(t) \subset [w_1,\hat{w}_1] = [a,b]^{|\eta|-|\eta|_{x_0}} \frac{t^{|\eta|}}{|\eta|!},$ which completes the proof.

Since the reachable set of iterated integrals with inputs taking values in a hyperrectangle is given in terms of interval products, the following lemma provides the closed-form of the n-th power of an interval.

Lemma 2: Given the interval $I = [a, b] \subset \mathbb{R}$, its *n*-th power I^n is given by

$$\begin{split} I^{n} = \\ \begin{cases} [a^{n}, b^{n}], & a, b > 0, n \text{ even} \\ [b^{n}, a^{n}], & a, b < 0, n \text{ even} \\ [\min\{a^{n-1}b, ab^{n-1}\}, & \max\{|a|, |b|\}^{n}], a < 0, b > 0, \\ & n \text{ even} \\ [a^{n}, b^{n}], & ab > 0, n \text{ odd} \\ [\min\{a^{n}, ab^{n-1}\}, & \max\{a^{n-1}b, b^{n}\}], & a < 0, b > 0, \\ & n \text{ odd} \\ \end{cases} \end{split}$$

Proof: The lemma is proved by induction. For n = 1, it follows directly that $I^1 = [a, b]$. For n = 2, one has that

$$I^{2} = \begin{cases} [a^{2}, b^{2}], & a, b > 0, \\ [b^{2}, a^{2}], & a, b < 0, \\ [ab, \max\{|a|, |b|\}^{2}], a < 0, b > 0. \end{cases}$$

Assuming now that (11) holds for k odd, it follows that $I^k I^2 =$

$$\begin{split} & \bigl\{ [\min\{a^{k+2}, a^k b^2, b^k a^2, b^{k+2}\}, \\ & \max\{a^{k+2}, a^k b^2, b^k a^2, b^{k+2}\}], \\ & [\min\{a^{k+1}b, a^{k+2}, a^k b^2, a^{k+1}\}, \\ & \max\{a^{k+1}b, a^{k+2}, a^k b^2, a^{k+1}\}], a < 0, b > 0, |a| > |b|, \\ & [\min\{a^2 b^k, a b^{k+1}, b^{k+2}\}, \\ & \max\{a^2 b^k, a b^{k+1}, b^{k+2}\}], \\ & a < 0, b > 0, |b| > |a|. \end{split}$$

If ab > 0, then

$$\min\{a^{k+2}, a^k b^2, b^k a^2, b^{k+2}\} = a^{k+2}$$

and

$$\max\{a^{k+2}, a^k b^2, b^k a^2, b^{k+2}\} = b^{k+2}.$$

If a < 0, b > 0 and |a| > |b|, then

$$\min\{a^{k+1}b,a^{k+2},a^kb^2,a^{k+1}\}=a^{k+2}$$

and

$$\max\{a^{k+1}b, a^{k+2}, a^kb^2, a^{k+1}\} = a^{k+1}b.$$

If a < 0, b > 0 and |b| > |a|, then

$$\min\{a^2b^k, ab^{k+1}, b^{k+2}\} = ab^{k+1}$$

and

$$\max\{a^2b^k, ab^{k+1}, b^{k+2}\} = b^{k+2}$$

Hence, (11) holds for k odd. The case for k even follows in a similar manner.

Observe that (11) can be written compactly as

$$I^{n} = \begin{cases} [\min\{a^{n}, ab^{n-1}\}, \max\{b^{n}, a^{n-1}b\}], n \text{ odd} \\ [\min\{a^{n}, ab^{n-1}, a^{n-1}b, b^{n}\}, \max\{a^{n}, b^{n}\}], n \text{ even} \end{cases}$$

Since a Chen-Fliess series is a sum of weighted iterated integrals with coefficients in \mathbb{R}^{ℓ} , the reachable set obtained in Lemma 1 of each iterated integral is used in the next theorem to provide an overapproximation of the reachable set of a Chen-Fliess series.

Theorem 3: Consider the formal power series $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ and $u : \mathbb{R} \to \mathbb{R}^m$ be a function with image in the hyperrectangle $[\mathbf{a}, \mathbf{b}]$ for all t > 0. The reachable set of the Chen-Fliess series $F_c[u](t)$ satisfies

$$\operatorname{Reach}_{c}([\mathbf{a},\mathbf{b}])(t) \subset \left[F_{\underline{c}}[1](t), F_{\overline{c}}[1](t)\right], \ \forall t \in \mathbb{R},$$

where

$$(\underline{c},\eta) = \min\left\{(c,\eta)[a,b]^{|\eta|-|\eta|_{x_0}}\right\},\tag{12a}$$

$$(\overline{c},\eta) = \max\left\{(c,\eta)[a,b]^{|\eta|-|\eta|_{x_0}}\right\}.$$
 (12b)

Here, 1 as the input of a Chen-Fliess series refers to the vector of m ones. This is, $1 = (1, \dots, 1)$.

Proof: The result follows directly from adding up the reachable set of each iterated integral provided in Lemma 1. Given the minimum of a sum is not smaller than the sum of minimums and the maximum of a sum is not greater than the sum of maximums, it follows that

$$\sum_{\eta \in X^*} \min_{u \in [\mathbf{a}, \mathbf{b}]} (c, \eta) E_{\eta}[u](t) \leq \min_{u \in [\mathbf{a}, \mathbf{b}]} F_c[u](t)$$
 and

$$\max_{u \in [\mathbf{a}, \mathbf{b}]} F_c[u](t) \le \sum_{\eta \in X^*} \max_{u \in [\mathbf{a}, \mathbf{b}]} (c, \eta) E_{\eta}[u](t).$$

Then

R

$$F_c[u](t) \in \sum_{\eta \in X^*} (c,\eta) \mathrm{Reach}_\eta([\mathbf{a},\mathbf{b}])(t), \forall u \in [\mathbf{a},\mathbf{b}],$$

where the product of $(c, \eta) \in \mathbb{R}$ and the set $\operatorname{Reach}_{\eta}([\mathbf{a}, \mathbf{b}])(t)$ is as described in Definition 5. One now has that

$$\operatorname{each}_c([\mathbf{a},\mathbf{b}])(t) \subset \sum_{\eta \in X^*} (c,\eta) \operatorname{Reach}_\eta([\mathbf{a},\mathbf{b}])(t).$$

The result follows by replacing the expression of the reachable set of each iterated integral, from Lemma 1, and noticing that

$$\min\{(c,\eta)\operatorname{Reach}_{\eta}([\mathbf{a},\mathbf{b}])(t)\} = \\ \min\{(c,\eta)[a,b]^{|\eta|-|\eta|_{x_0}}\}E_{\eta}[1](t), \\ \max\{(c,\eta)\operatorname{Reach}_{\eta}([\mathbf{a},\mathbf{b}])(t)\} = \\ \max\{(c,\eta)[a,b]^{|\eta|-|\eta|_{x_0}}\}E_{\eta}[1](t).$$

Next, explicit expressions of the coefficients of the defining series of $\text{Reach}_c([\mathbf{a}, \mathbf{b}])(t)$ are obtained.

Corollary 1: Letting $n = |\eta| - |\eta|_{x_0}$, and defining the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ as *i*. a, b > 0

$$f((c,\eta),n) = \begin{cases} (c,\eta)a^n, & (c,\eta) > 0, \\ (c,\eta)b^n, & (c,\eta) < 0 \end{cases}.$$

ii. a, b < 0

$$f((c,\eta),n) = \begin{cases} (c,\eta)b^n, & (c,\eta) > 0, \ n \text{ even} \\ & \text{or } (c,\eta) < 0, \ n \text{ odd} \\ (c,\eta)a^n, & (c,\eta) > 0, \ n \text{ odd} \\ & \text{or } (c,\eta) < 0, \ n \text{ even} \end{cases}$$

iii.a<0,b>0, |a|<|b|

$$f((c,\eta),n) = \begin{cases} (c,\eta)ab^{n-1}, & (c,\eta) > 0, \\ (c,\eta)b^n, & (c,\eta) < 0, \end{cases}$$

iv.
$$a < 0, b > 0, |b| < |a|$$

$$f((c,\eta),n) = \begin{cases} (c,\eta)a^{n-1}b, & (c,\eta) > 0, \ n \text{ even} \\ & \text{ or } (c,\eta) < 0, \ n \text{ odd} \\ (c,\eta)a^n, & (c,\eta) > 0, \ n \text{ odd} \\ & \text{ or } (c,\eta) < 0, \ n \text{ even} \end{cases}$$

then the coefficients of \overline{c} and \underline{c} in Theorem 3 are $(\underline{c}, \eta) = f((c, \eta), n)$ and $(\overline{c}, \eta) = -f(-(c, \eta), n)$.

Proof: The proof follows by using Lemma 2 and taking into account that the bounds of $\operatorname{Reach}_c([\mathbf{a}, \mathbf{b}])(t)$ switch when multiplying by a negative number.

IV. SIMULATIONS

Example 4: To illustrate the proposed Chen-Fliess overapproximation, consider the following single-input singleoutput system

$$\dot{x} = xu, \quad y = x, \quad x_0 = 1.$$
 (13)

Assume the input u is constrained to the interval $\mathcal{U} = [1, 2.8]$. Using (4), one can compute the coefficients of the Chen-Fliess series. Thus, one has that

$$F_{c}[u] = 1 + \sum_{k=1}^{\infty} E_{x_{1}^{k}}[u](t),$$

where $(c, \eta) = 1$ for all $\eta \in X^*$ which implies that $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$. This is, $F_c[u]$ converges for all t > 0. Also, from Definition 3, it is not difficult to see that the embedding system for (13) is

$$\dot{x} = xu, \ \dot{\hat{x}} = \hat{x}\hat{u},$$

with initial set of states equal to $(x_0, \hat{x}_0) = (1, 1)$. Solving for this embedding system (Theorem 2) provides the reachable set of (13) for inputs in $\mathcal{U} = [1, 2.8]$. On the other hand, from Theorem 3 and Corollary 1, one has that

$$(\bar{c}, \eta) = 2.8^k$$
 and $(c, \eta) = 1$

for $|\eta| = k$. Therefore,

$$\begin{aligned} \operatorname{Reach}_{c}([1,2.8])(t) &\subset [F_{\underline{c}}[1](t), F_{\overline{c}}[1](t)] \\ &= \Bigg[1 + \sum_{k=1}^{\infty} \frac{t^{k}}{k!} \ , \ 1 + \sum_{k=1}^{\infty} 2.8^{k} \frac{t^{k}}{k!} \Bigg]. \end{aligned}$$



Fig. 2. Comparison of the overestimation of the reachable set of the system in Example 4 with initial state $x_0 = 1$ by the Chen-Fliess series interval arithmetic (CFS-IA) procedure, for an input in $\mathcal{U} = [1, 2.8]$ and word truncation size N = 3 and the Mixed-Monotonicity (MM) approach.

Example 5: Consider the following MISO Lotka-Volterra system given by

 $\dot{x}_1 = -x_1x_2 + x_1u_1, \quad \dot{x}_2 = x_1x_2 - x_2u_2, \quad y = x_1$ (14)

with initial condition $x_0 = (1/6, 1/6)^{\top}$. The embedding system obtained using Definition 3 is

$$\begin{aligned} \dot{x}_1 &= -x_1 \hat{x}_2 + x_1 u_1, \ \dot{x}_2 &= x_2 x_1 - x_2 \hat{u}_2, \\ \dot{\hat{x}}_1 &= -\hat{x}_1 x_2 + \hat{x}_1 \hat{u}_1, \ \dot{\hat{x}}_2 &= \hat{x}_2 \hat{x}_1 - \hat{x}_2 u_2. \end{aligned}$$

The reachable set on the initial set $(x_{1,0}, x_{2,0}, \hat{x}_{1,0}, \hat{x}_{2,0}) = (1/6, 1/6, 1/6, 1/6)$ is given in Figure 3 together with the result obtained from the interval arithmetic procedure from Theorem 3.

V. CONCLUSIONS

This paper provides a closed-form of an overapproximation of the reachable set of the output of a non-linear affine



Fig. 3. Comparison of the overestimation of the reachable set of the system in Example 5 with initial state $x_0 = (1/6, 1/6)$ by the Chen-Fliess series interval arithmetic (CFS-IA) procedure, for inputs in $\mathcal{U} = [-1, 1]$ and word truncation size N = 3 and the Mixed-Monotonicity (MM) approach.

system represented by the Chen-Fliess operator. For this, interval arithmetic was used to compute an overestimation of the reachable set of an iterated integral, then the result is obtained by adding up all the overestimating sets of the defining Chen-Fliess series. The advantage of this method is its closed-form which makes computation faster than solving a non-convex optimization problem. Also, the examples show very good accuracy for short time horizons, but as time horizon gets bigger, the accuracy decreases.

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