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A Bode Sensitivity Integral for Linear Time-Periodic Systems

Henrik Sandberg and Bo Bernhardsson

***Abstract*—Bode's sensitivity integral is a well-known formula that quantifies some of the limitations in feedback control for linear time-invariant systems. In this note, we show that there is a similar formula for linear time-periodic systems. The harmonic transfer function is used to prove the result. We use the notion of roll-off 2, which means that the first time-varying Markov parameter is equal to zero. It then follows that the harmonic transfer function is an analytic operator and a trace class operator. These facts are used to prove the result.**

***Index Terms*—Bode sensitivity integral, linear time-periodic systems, performance limitations.**

I. INTRODUCTION

In recent years, there has been an increased interest for the fundamental limitations in feedback control. One reason for this is that in many control design tools these limitations are not clearly visible, and an inexperienced designer can easily specify performance criteria that are not possible to attain. The articles [1] and [2] contain examples of this. There are many of these limitations in control. The connection between amplitude and phase of transfer functions and Bode's sensitivity integral formula are two examples. The limitations come from the fact that the transfer functions are analytic functions, and this has strong implications.

In this note, we focus on Bode's sensitivity integral. This is a standard result in control, see for example [3]. If the transfer function $\hat{G}(s)$ of an open-loop linear time-invariant system G has roll-off 2, and is stable, then we have in the multiple-input–multiple-output (MIMO) case that

$$\int_0^\infty \log|\det(I+\hat{G}(j\omega))^{-1}|\,d\omega = 0 \qquad (1)$$

see, for example, [3]. This is also called the waterbed effect. In particular, the modulus of the sensitivity, $|\det(I+\hat{G}(j\omega))^{-1}|$, cannot be less than 1 for all frequencies ω. This trade-off holds for time-invariant linear systems. It is known that there are limitations also for linear time-varying and nonlinear systems, see for example [4]. However, frequency-domain methods are then often not applicable. In the note [5], an analogue to (1) is developed for continuous-time time-varying linear systems. The sensitivity integral is interpreted as an entropy integral in the time domain, i.e., no frequency-domain representation is used. For discrete-time time-varying systems similar time-domain results are given in [6].

For time-periodic linear systems there do exist frequency-domain representations. Sampled-data systems are a special type of time-periodic systems. Fundamental limitations for sampled-data systems are studied in [7] using transfer function techniques. We study general time-periodic systems in this note and we use the *harmonic transfer function* (HTF), see [8]–[10], which formally is an MIMO transfer function $\hat{G}(s)$ with an infinite amount of inputs and outputs. Using the convergence and existence results for the harmonic transfer function that are developed in [10], we will be able to write (1) with $\hat{G}(j\omega)$ being the HTF. To do this we need to answer the following questions: What does roll-off 2 mean for a time-periodic system? In what sense is the HTF $\hat{G}(s)$ analytic? What does the determinant mean for the HTF?

We do not consider open-loop unstable systems in this note. This case is considered in [5] using exponential dichotomies. In the time-invariant case, when the open-loop system is unstable, the right-hand side of (1) is equal to $\pi\sum_i \operatorname{Re} p_i$, where p_i are the unstable open-loop poles, see [11]. During the completion of this article, the authors became aware of the independent work in [12]. The sensitivity integral derived there is similar to the one in this note. However, the result is derived using techniques from [5], and is restricted to state–space models.

The note is organized as follows: In Section II, we give some of the basic results for the HTF. The section ends with a definition of roll-off 2. In Section III, we derive Proposition 2, which shows that with roll-off 2 the HTF is an analytic operator. In Section IV, we review the definition of the trace class operators and the operator determinant. In Proposition 3, we see that the HTF indeed is a trace class operator and that the determinant is well defined. By using the propositions of the previous sections, we can in Section V state the main result, which is a direct analogue of (1) for periodic systems. In Section VI, we give an example of the result. This article is based on [13].

II. HARMONIC TRANSFER FUNCTION AND ROLL-OFF

We repeat some results from [10]. A linear time-periodic system G on impulse-response form is given by

$$y(t) = \int_{-\infty}^t g(t,\tau)u(\tau)\,d\tau, \qquad g(t,\tau) = g(t+T,\tau+T) \qquad (2)$$

for some period $T > 0$. We assume that the impulse response $g(t,\tau)$ is real and has uniform exponential decay of rate α

$$|g(t,\tau)| \le K\cdot e^{-\alpha(t-\tau)}, \qquad t\ge\tau$$

for some positive constants K and α. The operator G is then bounded on L_2. To define the HTF we expand the periodic impulse response in a Fourier series

$$g(t,\tau) = \sum_{l=-\infty}^{\infty} g_l(t-\tau)e^{jl\omega_0 t} \quad \omega_0 = \frac{2\pi}{T}$$
$$g_l(t-\tau) = \frac{1}{T}\int_0^T g(r, r-t+\tau)e^{-jl\omega_0 r}\,dr \qquad (3)$$

with convergence in L_2, see [10]. Hence, we expand the periodic impulse response into a sum of modulated time-invariant impulse responses $g_l(t)$. For exponentially stable systems, we can apply the Laplace transform on each time-invariant impulse response $g_l(t)$

$$\hat{g}_l(s) = \int_0^\infty g_l(t)e^{-st}\,dt \quad \operatorname{Re} s > -\alpha. \qquad (4)$$

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Furthermore, we have that $\hat{g}_l(s)$ is analytic in $\mathrm{Re}\, s > -\alpha$, and $\hat{g}_l \in H_2 \cap H_\infty$. Now, the HTF is defined by the infinite-dimensional matrix

$$\hat{G}(s) = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \hat{g}_0(s+j\omega_0) & \hat{g}_1(s) & \hat{g}_2(s-j\omega_0) & \\ \ddots & \hat{g}_{-1}(s+j\omega_0) & \hat{g}_0(s) & \hat{g}_1(s-j\omega_0) & \ddots \\ & \hat{g}_{-2}(s+j\omega_0) & \hat{g}_{-1}(s) & \hat{g}_0(s-j\omega_0) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \quad (5)$$

Since the HTF has a periodic structure, it is often enough to consider $\hat{G}(s)$ for complex numbers s in open regions J_ϵ

$$J_\epsilon = \{s: \mathrm{Re}\, s > -\epsilon, \mathrm{Im}\, s \in I_\epsilon\}$$
$$I_\epsilon = (-\omega_0/2 - \epsilon, \omega_0/2 + \epsilon), \qquad \omega_0 = 2\pi/T$$

where $\alpha \geq \epsilon \geq 0$. The HTF $\hat{G}(s)$ can be seen as an infinite-dimensional operator defined on the space of square-summable sequences ℓ_2. In [8], [9] it is shown that we can compute the induced L_2-norm as

$$\|G\|_{L_2 \to L_2} = \sup_{\|u\|_{L_2} \leq 1} \|Gu\|_{L_2} = \operatorname{ess\,sup}_{\omega \in I_0} \|\hat{G}(j\omega)\|_\infty \quad (6)$$

where $\|\cdot\|_\infty$ is the induced ℓ_2-norm.

A. *Roll-Off of Periodic Systems [10]*

For all $q \in \boldsymbol{R}$, we can rewrite (2) as

$$y(t)e^{-qt} = \int_{-\infty}^{t} \left[g(t,\tau)e^{-q(t-\tau)}\right] u(\tau)e^{-q\tau}\, d\tau. \quad (7)$$

We use the notation

$$y_q = G_q u_q$$

where the operator G_q has impulse response $g(t,\tau)e^{-q(t-\tau)}$ and maps input signals of the type $u_q(t) = u(t)e^{-qt}$ into signals $y_q(t) = y(t)e^{-qt}$. For every fixed $q > -\alpha$, we may apply the theory developed in [10].

In the following proposition, $g_t, g_{tt}, g_\tau, g_{\tau\tau}$ denote one and two partial derivatives of g with respect to the first and second argument, respectively, and $pu(t) = du(t)/dt$. Furthermore, the set $\mathcal{S}$ is the set of Schwartz functions, i.e., the set of infinitely differentiable functions $u(t)$ with $t^a p^b u(t)$ bounded for $t \in \boldsymbol{R}$ and all nonnegative a and b. The set $\mathcal{S}$ is dense in L_2.

Proposition 1: Assume that $g(t,t) = 0$ for all t, and that $g, g_t, g_{tt}, g_\tau, g_{\tau\tau}$ are continuous and have uniform exponential decay of rate $\alpha > 0$. Then, (7) can be expanded in either of the following ways:

$$\begin{aligned} y_q(t) = &-g_\tau(t,t)\frac{1}{(p+q)^2}u_q(t) \\ &+ \int_{-\infty}^{t}\left[g_{\tau\tau}(t,\tau)e^{-q(t-\tau)}\right]\frac{1}{(p+q)^2}u_q(\tau)\, d\tau \end{aligned} \quad (8)$$

$$\begin{aligned} y_q(t) = &\frac{1}{(p+q)^2}g_t(t,t)u_q(t) \\ &+ \frac{1}{(p+q)^2}\int_{-\infty}^{t}\left[g_{tt}(t,\tau)e^{-q(t-\tau)}\right]u_q(\tau)\, d\tau \end{aligned} \quad (9)$$

when $u_q \in \mathcal{S}$ and $q > -\alpha$.

Proof: We prove (9). By the assumptions on $g(t,\tau)$ and since $u_q \in \mathcal{S}$, the output y_q is continuously differentiable and bounded. Differentiate (7) with respect to t, and we obtain

$$(p+q)y_q(t) = g(t,t)u_q(t) + \int_{-\infty}^{t}\left[g_t(t,\tau)e^{-q(t-\tau)}\right]u_q(\tau)\, d\tau. \quad (10)$$

By assumption, $g(t,t) = 0$, and the first term on the right-hand side disappears. If we divide by $(p+q)$ and repeat the procedure on the integral on the right hand side of (10), (9) follows. Equation (8) can be proven similarly. ■

Remark 1: The assumptions on $g(t,\tau)$ typically hold for smooth stable linear time-periodic systems. The expansions remain valid when $\alpha \leq 0$, but we only consider stable systems in this note. The operator $(1/(p+q))$ should be interpreted as multiplication with $(1/(j\omega + q))$ in the frequency domain.

Since the first terms in the expansions (8)–(9) contain double integrators when $q = 0$, we make the following definition.

Definition 1 [10]: If the first time-varying Markov parameter, $g(t,t)$, is zero for all t, then G is said to have roll-off 2.

Introduce P_Ω as an ideal (noncausal) low-pass filter with the frequency characteristic

$$\hat{P}_\Omega(j\omega) = \begin{cases} 1, & |\omega| \leq \Omega \\ 0, & |\omega| > \Omega \end{cases}.$$

Proposition 1 together with the facts that $\mathcal{S}$ is dense in L_2, and that the Fourier transform of a function in $\mathcal{S}$ is again in $\mathcal{S}$, implies that if we filter the input or the output of systems G_q there are, for all $\delta < \alpha$, positive constants C_1, C_2 (dependent on δ and α) such that

$$\|G_q(I - P_\Omega)\|_{L_2 \to L_2} \leq \frac{C_1}{|q + \delta + j\Omega|^2} \quad (11)$$

$$\|(I - P_\Omega)G_q\|_{L_2 \to L_2} \leq \frac{C_2}{|q + \delta + j\Omega|^2}. \quad (12)$$

To show (11) one uses (8), and to show (12), one uses (9). Similar bounds are derived in detail in [10]. In particular, we have that $\|G_q\|_{L_2 \to L_2} = O(q^{-2})$ as $q \to \infty$ and $\|G_q(I - P_\Omega)\|_{L_2 \to L_2} = O(\Omega^{-2})$ and $\|(I - P_\Omega)G_q\|_{L_2 \to L_2} = O(\Omega^{-2})$ for each fixed q as $\Omega \to \infty$.

The relation between the HTF of G and G_q is simple

$$\hat{G}(q + j\omega) = \hat{G}_q(j\omega), \qquad q > -\alpha$$

so it is enough to speak of $\hat{G}(s)$. The high-pass filtering of G_q with $(I - P_\Omega)$ means that rows or columns are truncated (replaced by zeros) in $\hat{G}(s)$. If we choose $\Omega = (N + 1/2)\omega_0$ for some nonnegative integer N, then $G_q(I - P_\Omega)$ has an HTF where the $2N+1$ middle columns of $\hat{G}(s)$ are replaced by zeros. $(I - P_\Omega)G_q$ has an HTF where the $2N+1$ middle rows of $\hat{G}(s)$ are replaced by zeros, see [10] for details. This has consequences for the roll-off of the individual transfer functions $\hat{g}_l(s)$ as is shown in the next section.

Remark 2: For a stable time-invariant system with smooth impulse response $g(t,\tau) = g_0(t - \tau)$, the Markov parameters are equal to $\{g_0(0), g_0'(0), g_0''(0), \ldots\}$. If $g(t,t) = g_0(0) = 0$ then we have that $|\hat{g}(s)| = O(|s|^{-2})$, as $|s| \to \infty$ and $\mathrm{Re}\, s > -\alpha$. This is called roll-off 2 for a time-invariant system.

III. Analytic Operators

To prove Bode's integral theorem for time-invariant systems, one uses that the transfer function is analytic and Cauchy's integral theorem. The HTF is an infinite-dimensional operator and therefore we need some of the theory for analytic operators. There are several equivalent definitions of an analytic operator, see for example [14]. We say that a bounded linear operator $\hat{G}(s)$ is analytic in an open set $\Omega \subseteq \boldsymbol{C}$ if it can be expanded in a power series around each $s_0 \in \Omega$

$$\hat{G}(s) = \sum_{k=0}^{\infty}(s - s_0)^k \hat{G}_k, \qquad s \in \Omega(s_0) \subseteq \Omega$$

with uniform convergence in the open disc $\Omega(s_0)$ in the induced ℓ_2-norm, $\|\cdot\|_\infty$. The constant operators $\hat{G}_k$ are linear bounded operators on ℓ_2. To prove that the HTF $\hat{G}(s)$ is an analytic operator in J_ϵ, we can check the following sufficient conditions [14].

K1) All the elements of $\hat{G}(s)$ are analytic functions in J_ϵ.

K2) There is a positive constant K such that $\|\hat{G}(s)\|_\infty \leq K$ for all $s \in J_\epsilon$.

We have the following statement.

Proposition 2: If the periodic system G fulfills the assumptions of Proposition 1, then its harmonic transfer function $\hat{G}(s)$ is an analytic operator in J_ϵ, where $\epsilon < \alpha$.

Proof: Property K1) follows from (4) and (5). Property K2) needs some extra attention. By using the roll-off formulas and the discussion about the truncation of rows and columns in Section II-A, we can conclude that for all positive integers N and $s \in J_\epsilon$

$$|\hat{g}_l(s)| \leq \frac{C_1 + C_2}{N^2\omega_0^2}, \qquad l \in \boldsymbol{Z} \quad |l| \geq 2N+1 \tag{13}$$

$$|\hat{g}_l(s)| \leq \frac{C_1}{|\delta + s|^2}, \qquad l \in \boldsymbol{Z}. \tag{14}$$

The first bound follows since $\|G_q - P_\Omega G_q P_\Omega\|_{L_2\to L_2} \leq \|(I - P_\Omega)G_q)\|_{L_2\to L_2} + \|G_q(I - P_\Omega)\|_{L_2\to L_2} \leq (C_1 + C_2)/(N^2\omega_0^2)$ when $\Omega = (N + 1/2)\omega_0$. The modulus of the analytic elements of the HTF of $G_q - P_\Omega G_q P_\Omega$ must be less or equal to the L_2-induced norm according to (6). Since the transfer functions $\hat{g}_l(s)$, $|l| \geq 2N+1$, are not truncated with this choice of Ω, (13) follows. The bound (14) follows since the modulus of the analytic functions $\hat{g}_l(s)$ must be less than the L_2-induced norm bound in (11). Hence, roll-off 2 for a time-periodic system implies that the transfer functions $\hat{g}_l(s)$ on the diagonals of $\hat{G}(s)$ have roll-off 2 in the classical sense (see Remark 2).

The Hilbert-Schmidt norm $\|\cdot\|_2$ gives an upper bound to the induced ℓ_2-norm, i.e., $\|\hat{G}(s)\|_\infty \leq \|\hat{G}(s)\|_2$. Now, by definition

$$\|\hat{G}(s)\|_2^2 = \sum_{k,l=-\infty}^{\infty} |\hat{g}_l(s + jk\omega_0)|^2$$

$$D_l(s) = \sum_{k=-\infty}^{\infty} |\hat{g}_l(s + jk\omega_0)|^2. \tag{15}$$

From (14), $D_{-1}(s), D_0(s), D_1(s)$ are bounded for $\mathrm{Re}\, s > -\epsilon > -\delta$. We bound the remaining diagonals $D_l(s)$ next.

From (13) and (14), we have for fixed $N > 0$ that

$$\left|\hat{g}_{\pm(2N+1)}(q + j\omega + jk\omega_0)\right| \leq (C_1 + C_2)\min\left\{\frac{1}{N^2\omega_0^2}, \frac{1}{|\omega + k\omega_0|^2}\right\}.$$

For $s = q + j\omega \in J_\epsilon$, we then obtain

$$D_{\pm(2N+1)}(s) \leq (C_1 + C_2)^2\left(\frac{2N-1}{N^4\omega_0^4} + 2\sum_{k=N}^{\infty}\frac{1}{|\omega + k\omega_0|^4}\right) \leq \frac{C}{N^3}$$

where C is a constant. We can derive a similar bound for $D_{\pm 2(N+1)}(s)$. Hence, we have that $D_l(s) = O(|l|^{-3})$ uniformly in s as $|l| \to \infty$, and there is a positive K such that $\|\hat{G}(s)\|_2^2 = \sum_l D_l(s) \leq K^2$, for all $s \in J_\epsilon$. Since the calculations hold for all $0 < \epsilon < \delta < \alpha$, the proposition follows. ∎

IV. Trace Class Operators and Determinants

We need to define a determinant for infinite-dimensional operators. This can be done for so-called trace class operators; see [15] and [16]. For a trace class operator $\hat{G}$, the determinant is defined as

$$\det(I + \hat{G}) = \prod_k (1 + \lambda_k(\hat{G})) \tag{16}$$

where $\lambda_k(\hat{G})$ are the eigenvalues of $\hat{G}$. Trace class operators are compact operators and have a countable number of eigenvalues. Note that for finite matrices, (16) coincides with the regular determinant. For the definition of a trace class operator, we need the s-numbers (or singular numbers) of $\hat{G}$

$$s_k(\hat{G}) = \inf\{\|\hat{G} - \hat{G}_k\|_\infty : \text{rank}\, \hat{G}_k \leq k\}.$$

The numbers s_k tell how well $\hat{G}$ may be approximated by a finite-rank operator. If $\hat{G}$ is compact, we have that $s_k \to 0$ as $k \to \infty$. The trace class operators are those operators for which

$$\|\hat{G}\|_1 = \sum_{k=0}^{\infty} s_k < \infty. \tag{17}$$

With the norm $\|\cdot\|_1$, the trace class operators form a complete normed space; see [15]. We have that $\mathrm{trace}(\hat{G}) = \sum_k \lambda_k(\hat{G}) \leq \|\hat{G}\|_1$, and

$$|\det(I + \hat{G})| \leq \exp(\|\hat{G}\|_1). \tag{18}$$

Next, we see that under the assumptions of Proposition 1, the HTF $\hat{G}(s)$ is in fact a trace class operator. We have the following proposition.

Proposition 3: If the periodic system G fulfills the assumptions of Proposition 1, then its harmonic transfer function $\hat{G}(s)$ is a bounded trace class operator in J_ϵ, where $\epsilon < \alpha$, and

$$\|\hat{G}(q + j\omega)\|_1 \leq \frac{K_1}{K_2 + q}, \qquad q + j\omega \in J_\epsilon$$

for some positive constants K_1 and $K_2 > \epsilon$.

Proof: Since $\hat{G}(s)$ is analytic from Proposition 2, $\hat{G}(q + j\omega)$ is continuous in ω, and

$$s_0(\hat{G}(q + j\omega)) = \|\hat{G}(q + j\omega)\|_\infty \leq \|G_q\|_{L_2\to L_2} \leq \frac{C_1}{(\delta + q)^2}.$$

The remaining singular numbers can be bounded as follows. The HTF of $G_q P_\Omega$, with $\Omega = (N + 1/2)\omega_0$, has elements equal to zero everywhere except for its $2N+1$ middle columns which are identical to the $2N+1$ middle columns of $\hat{G}(s)$ defined by (5). Hence, the truncated HTF has at most rank $2N+1$. We know that $G_q P_\Omega$ converges to G_q as $O(\Omega^{-2}) = O(N^{-2})$ from (11). We conclude that for each $q + j\omega \in J_\epsilon$ we have that

$$\begin{aligned} s_{2N+1}(\hat{G}_q(j\omega)) &\leq \|\hat{G}_q(j\omega)(I - \hat{P}_\Omega(j\omega))\|_\infty \\ &\leq \|G_q(I - P_\Omega)\|_{L_2\to L_2} \\ &\leq \frac{C_1}{|\delta + q + j\Omega|^2} \leq \frac{C_1}{(\delta + q)^2 + N^2\omega_0^2}. \end{aligned} \tag{19}$$

The singular numbers form a decreasing sequence and, hence, we can make the upper estimate

$$s_{2N+2}(\hat{G}(q + j\omega)) \leq s_{2N+1}(\hat{G}(q + j\omega)).$$

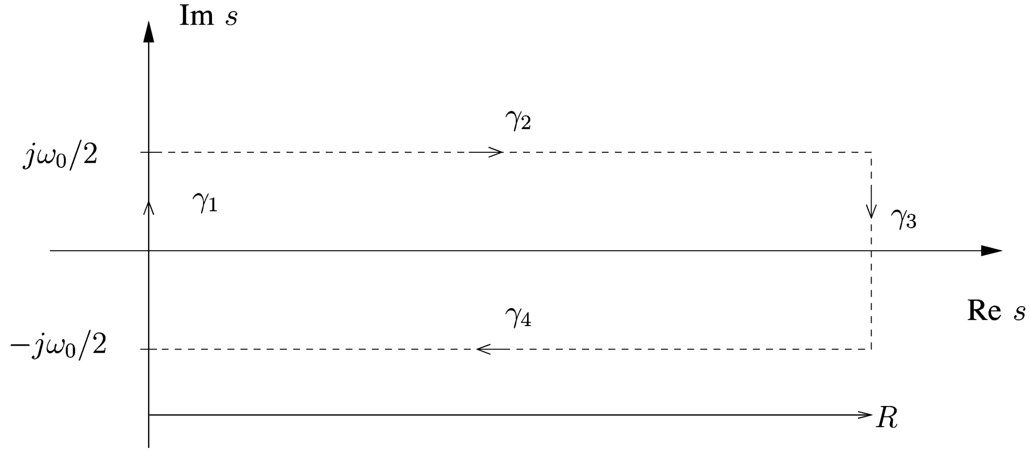

Fig. 1. Integration path Γ_R.

Hence, for each fixed s, the singular numbers $s_k(\hat{G}(s))$ decay as $O(k^{-2})$ for systems with roll-off 2. Now, we can use these estimates to bound the trace norm (17)

$$\|\hat{G}(q+j\omega)\|_1 \leq \sum_{k=0}^{\infty} \frac{2C_1}{(\delta+q)^2+\omega_0^2 k^2} \leq \frac{K_1}{K_2+q}$$

for some constants K_1 and $K_2 > \epsilon$ (since $\delta > \epsilon$). ■

Before stating the main result, we need the following lemma.

Lemma 1 [15]: If J_ϵ is an open set in $\boldsymbol{C}$ and if $\hat{G}(s)$ is an analytic trace-class-operator-valued function for $s \in J_\epsilon$, then $\det(I+\hat{G}(\cdot))\colon J_\epsilon \to \boldsymbol{C}$ is an analytic function.

V. Main Result

Using Propositions 1–3 and Lemma 1, we are finally ready to state the analogue of Bode's sensitivity integral, applicable to time-periodic systems.

Theorem 1 (Sensitivity Integral): Assume that a (real) linear time-periodic system G satisfies the assumptions of Proposition 1. Assume furthermore that the sensitivity operator $(I+G)^{-1}$ is exponentially stable, i.e., there are strictly positive ν and ϵ such that

$$|\det(I+\hat{G}(s))| \geq \nu, \qquad s \in J_\epsilon. \tag{20}$$

Then

$$\int_0^{\omega_0/2} \log|\det(I+\hat{G}(j\omega))^{-1}|\, d\omega = 0. \tag{21}$$

Proof: We have that $\det(I+\hat{G}(s))^{-1} = 1/\det(I+\hat{G}(s))$, see [16]. From Proposition 3 we know that $\|\hat{G}(s)\|_1 \leq K_1/(K_2-\epsilon)$ in J_ϵ. Using (18) and (20), we then have that

$$\frac{1}{\exp(K_1/(K_2-\epsilon))} \leq |\det(I+\hat{G}(s))^{-1}| \leq \frac{1}{\nu}$$

and, hence, $\det(I+\hat{G}(s))^{-1}$ is a bounded function that does not become zero for $s \in J_\epsilon$. Because of this, we can take the complex logarithm, and

$$\log\det(I+\hat{G}(s))^{-1} = -\log\det(I+\hat{G}(s)).$$

From Propositions 1–3 and Lemma 1, we know that $\det(I+\hat{G}(s))$ is an analytic function in J_ϵ. Then for any simply closed curve $\Gamma \subset J_\epsilon$

$$\int_\Gamma \log\det(I+\hat{G}(s))^{-1}\, ds = 0 \tag{22}$$

by Cauchy's integral formula. To prove the theorem we choose the curve Γ_R shown in Fig. 1 and let $R \to \infty$. First, we evaluate the integral (22) along γ_2 and γ_4. Notice that

$$\int_0^R \log\det(I+\hat{G}(q+j\omega_0/2))^{-1}\, dq + \int_R^0 \log\det(I+\hat{G}(q-j\omega_0/2))^{-1}\, dq = 0$$

for all R. The cancellation is because

$$\det(I+\hat{G}(q-j\omega_0/2)) = \det(I+\hat{G}(q+j\omega_0/2))$$

for all q. This follows by the structure (5) of the HTF and the definition of the determinant. Next, we evaluate the integral along γ_3. The complex logarithm is defined as

$$\log\det(I+\hat{G}(s)) = \log|\det(I+\hat{G}(s))| + j\arg\det(I+\hat{G}(s)).$$

Since the impulse response $g(t,\tau)$ is real, we have that $\hat{g}_l(s) = \overline{\hat{g}_{-l}(\bar{s})}$, where $\bar{\cdot}$ denotes complex conjugate. By the structure (5) and the definition of the determinant, it then holds that

$$\begin{aligned} \arg\det(I+\hat{G}(s)) &= -\arg\det(I+\hat{G}(\bar{s})) \\ |\det(I+\hat{G}(s))| &= |\det(I+\hat{G}(\bar{s}))|. \end{aligned} \tag{23}$$

The argument is an antisymmetric function, so when we integrate it over the symmetric interval γ_3, it disappears from the logarithm

$$\begin{aligned} &\left|\int_{\omega_0/2}^{-\omega_0/2} \log\det(I+\hat{G}(R+j\omega))d(j\omega)\right| \\ &= \left|\int_{\omega_0/2}^{-\omega_0/2} \log|\det(I+\hat{G}(R+j\omega))|d(j\omega)\right| \\ &\leq \int_{\omega_0/2}^{-\omega_0/2} \|\hat{G}(R+j\omega)\|_1\, d\omega \end{aligned}$$

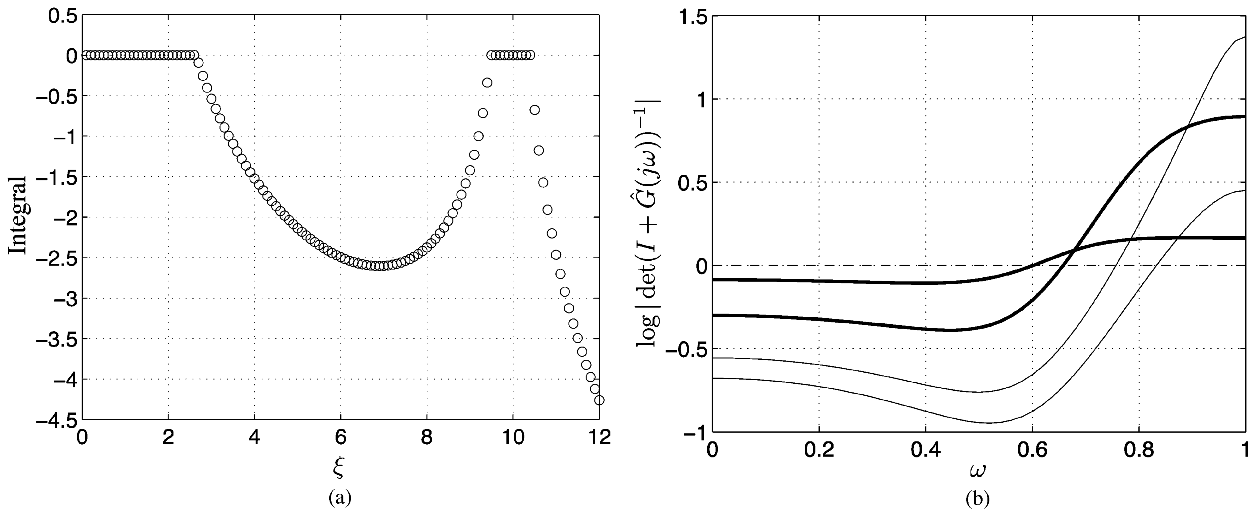

Fig. 2. (a) Values of the integral (21) for different values of ξ in (24). By Theorem 1, the integral must equal zero for stable closed-loop systems. It can be verified by, for instance, Floquet analysis that the system indeed is stable for $\xi \in [0, 2.6] \cup [9.6, 10.4]$. (b) Logarithm of the sensitivity function is plotted for $\xi = 1.0, 2.0, 3.0$, and 3.5. For the bold curves (the stable systems) the conservation law in Theorem 1 applies. When ξ increases the sensitivity decreases for low frequencies. The sensitivity must then increase for high frequencies to keep the areas below and above the zero level equal. This is the waterbed effect.

for each fixed R. The last bound follows by (18). Now, $\|\hat{G}(R+j\omega)\|_1$ converges uniformly to zero as $R \to \infty$ according to Proposition 3. The integral along γ_3 then goes to zero as $R \to \infty$. The only term remaining of (22) is the integral along γ_1

$$\int_{-\omega_0/2}^{\omega_0/2} \log\det(I+\hat{G}(j\omega))^{-1} d(j\omega) = 0.$$

Using (23) on the interval $[-\omega_0/2, \omega_0/2]$, we obtain (21). ∎

Remark 3 (Time-Invariant Systems): The integral in (1) is over the interval $[0,\infty)$ whereas the integral in (21) is over $[0, \omega_0/2]$. This might seem strange, but notice that for a time-invariant system with transfer function $\hat{g}(s)$, the HTF is given by

$$\hat{G}(s) = \operatorname{diag}\{\ldots, \hat{g}(s+j\omega_0), \hat{g}(s), \hat{g}(s-j\omega_0), \ldots\}$$

for any $\omega_0 > 0$, and we see that (1) and (21) are identical if we use that $\hat{g}(\bar{s}) = \overline{\hat{g}(s)}$.

Remark 4 (Sampled-Data Systems): The HTF can be calculated directly, without using the impulse response, for sampled-data systems; see [17]. If one can show that the HTFs also in these cases are analytic and of trace class, Theorem 1 still holds. In [7], another sensitivity integral is derived for sampled-data systems. There the integral satisfies an inequality constraint instead of an equality. The reason for this is that in [7] only the main diagonal ("the time-invariant component") of the HTF is integrated.

VI. Example: The Mathieu Equation

Now, we verify the main result on an example. We choose an open-loop system G with dynamics given by

$$\ddot{y}(t) + 0.4\dot{y}(t) + 2y(t) = \xi\cos(2t)w(t) \tag{24}$$

where ξ is a parameter and $w(t)$ the input. The impulse response is given by

$$g(t,\tau) = \frac{\xi}{1.4} e^{-0.2(t-\tau)} \sin(1.4(t-\tau))\cos(2\tau).$$

Clearly, the system has roll-off 2, and it is exponentially stable. The sensitivity operator is obtained by applying the feedback $w(t) = -(y(t) + u(t))$. Notice that when $u(t) = 0$, the dynamics of the closed-loop system is given by a damped Mathieu equation; see, for example, [8].

Next, we compute the HTF of G using (3)–(5). Here, $\omega_0 = 2$. After this, we may compute the integral (21) for different values of ξ. For $\xi \in [0, 2.6] \cup [9.6, 10.4]$, the closed loop is stable. This can be shown by, for instance, Floquet analysis. According to Theorem 1 the integral should then equal zero. In Fig. 2, this is verified. It is also seen that when the closed loop is unstable, the integral is strictly less than zero. Furthermore, we can visualize the waterbed effect for periodic systems. When the sensitivity decreases for some frequencies, it must increase for other frequencies.

VII. Conclusion

We have seen that there are fundamental limitations for feedback control of linear time-periodic systems. The modulus of the determinant of the harmonic transfer function $(I+\hat{G}(j\omega))^{-1}$ cannot be made small for all frequencies ω. The result is a direct generalization of Bode's sensitivity integral. To prove the result, we have defined roll-off 2 for a time-periodic system, and used some of the theory for analytic operators and trace class operators.

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Optimal Selection of the Forgetting Matrix Into an Iterative Learning Control Algorithm

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***Abstract*—A recursive optimal algorithm, based on minimizing the input error covariance matrix, is derived to generate the optimal forgetting matrix and the learning gain matrix of a P-type iterative learning control (ILC) for linear discrete-time varying systems with arbitrary relative degree. This note shows that a forgetting matrix is neither needed for boundedness of trajectories nor for output tracking. In particular, it is shown that, in the presence of random disturbances, the optimal forgetting matrix is zero for all learning iterations. In addition, the resultant optimal learning gain guarantees boundedness of trajectories as well as uniform output tracking in presence of measurement noise for arbitrary relative degree.**

***Index Terms*—Iterative learning control, optimal control, stochastic systems.**

I. Introduction

The original idea for using the forgetting factor into iterative learning control (ILC) is due to Heinzinger *et al.* [1]. However, this was introduced to D-type ILC. Subsequently, Arimoto *et al.* [2], [3] used the forgetting factor into P-type ILC. In [4], a forgetting factor was considered for generating the initial guess for the input to be learned whereby the speed of convergence can be accelerated. Thereafter, the inclusion of a forgetting factor into ILC algorithms, addressing different tracking problems, has been vastly considered, e.g., [5]–[11].

Most of ILC algorithms, which are considered in the ILC literature, incorporating a forgetting factor can be described as follows:

$$u(t,k+1) = (1-\alpha)u(t,k) + g(e(t,k))$$

where $0 < \alpha < 1$ is the forgetting factor, $u(t,k)$ is the control input of kth iteration trial, $g(\cdot)$ is the learning operator which depends on the previous measurement output error $e(t,k)$. For example, if the ILC algorithm is of P-type, then $g(e(t,k)) = K(t)e(t,k)$ where $K(t)$ is the learning gain matrix.

The main objective of the forgetting factor is found to increase the robustness of algorithm against random disturbances. The motive behind this attribute is as follows: When summing the signal over k learning iterations indicates that the scalar $(1-\alpha)$ is taken to kth power for input error further back in the iteration domain for every fixed t. Since $0 < (1-\alpha) < 1$, this could lead to more reduction in older errors than recent errors in the iteration domain. However, since the learning operator converges to zero as the error tends to zero, then by fixing $0 << \alpha < 1$, the control can only converge to a neighborhood of the desired control. Consequently, it is necessary to have the forgetting factor somehow converge to zero as the number of learning iterations increase in order to possibly have the control input converge to its desired control.

In this note a P-type ILC control algorithm is considered using a forgetting matrix, $\Lambda(t,k)$, which is described by $u(t,k+1) = [I - \Lambda(t,k)]u(t,k) + K(t,k)e(.,k)$. The optimal forgetting matrix and learning gain matrix are obtained by minimizing the trace of the input

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