Comparison of Market Parameters for Jump-Diffusion Distributions Using Multinomial Maximum Likelihood Estimation

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Abstract— **Previously, we have shown that the proper method for estimating parameters from discrete, binned stock log returns is the multinomial maximum likelihood estimation, and its performance is superior to the method of least squares. Also, useful formulas have been derived for the density for jump-diffusion distributions. Numerically, the parameter estimation can be a large scale nonlinear optimization, but we have successfully implemented variants of multi-dimension direct search methods. In this paper, three jump-diffusion models using different jump-amplitude distributions are compared. These jump-amplitude distributions are the normal, uniform and double-exponential distribution. The parameters of all three models are fit to the Standard and Poor's 500 log-return market data, given the same first moment and second central moments. Our main results are first that uniform jump distribution has superior qualitative performance since it produces genuine fat tails that are typical of market data, whereas the other two have exponentially thin tails. Secondly, the uniform distribution is quantitatively better overall as measured by the closeness of both the skewness and kurtosis coefficients to the data, although the double-exponential is best on skewness while worst on kurtosis. However, the log-normal model has a big advantage in computational costs of parameter estimation compared with the others, while the double-exponential is most costly due to having one more model parameter to fit.**

I. INTRODUCTION

Despite the great success of Black-Scholes options model [2], in option pricing, this pure log-normal diffusion model fails to reflect the three empirical phenomena: (1) the large random fluctuations such as crashes or rallies; (2) the non-normal features, that is, negative skewness and leptokurtic (peakedness) behavior in the stock log-return distribution; (3) the implied volatility smile, that is, the implied volatility is not a constant as in the Black-Scholes model.

Therefore, many different models are proposed to modify the Black-Scholes model so as to represent the above three empirical phenomena. Some models are proposed to incorporate the *volatility smile*, for example, Andersen, Benzoni and Lund [1] have made elaborate estimations to fit jump-diffusion models with log-normal jump-amplitudes, stochastic volatility and other features. Some models are proposed to incorporate the asymmetric features of the stock log-return distributions. Merton [10] introduced the jump-diffusion model in financial modeling, using a Poisson process for the jump timing and a log-normal process for the jump-amplitudes to describe the market crashes or rallies. Recently, Kou [9] proposed a jump-diffusion model with a log-double-exponential process for the jump-amplitude. Since crashes or rallies are rare events, the Poisson process is reasonable for the timing of jumps. However, there is a problem in choosing the log-normal or log-double-exponential process for the jump-amplitude since the exponentially small tails of the log-normal and log-double-exponential distributions are contrary to the flat and thick tails of the long time financial market log-return data. Around the near-zero peak of the log-double-exponential and the log-normal, the jumps are small, so are not too different from the continuous diffusion fluctuations. When the jumps are large, then the density tails are exponentially small, but the large jumps of the data are more persistent. Moreover, an infinite jump domain is unrealistic, since the jumps should be bounded in a real world financial markets and an infinite domain leads to unrealistic restrictions in portfolio optimization [5].

So, Hanson and Westman [4] proposed one jump-diffusion model with log-uniform jump-amplitude. Most recently, Hanson, Westman and Zhu [8] showed that for IID simulations that the binned distribution is multinomial. They estimated the market parameters for this log-uniform model by subsequent multinomial maximum likelihood method to fit financial market distributions such as the Standard and Poor's 500 stock index. The estimation of the kurtosis differed by a very small amount, +0.78%, from the observed value. However, the estimation of the skewness differed significantly from the observed value, by -47%. In this paper, the value of the skewness of the log-uniform model is greatly improved using more accurate computations here.

The main purpose of this paper is to compare the performance of three jump-diffusion models whose jump-amplitudes are the log-normally, log-uniformly and log-double-exponentially distributed. The measures of performance are the skewness, kurtosis and computational costs.

II. SOME THEORETICAL RESULTS ABOUT THESE JUMP-DIFFUSION MODELS

A. *Stock Return Process,* $\mathbf{S(t)}$

The following stochastic differential equation (SDE) is used to model the dynamics of the asset price, $S(t)$:

$$dS(t) = S(t)\left(\mu_d dt + \sigma_d dW(t) + J(Q)dP(t)\right), \quad \text{(II.1)}$$

where μ_d is the drift coefficient, σ_d is the diffusive volatility, $W(t)$ is the stochastic diffusion process, $J(Q)$ is the Poisson

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jump-amplitude, Q is its underlying Poisson amplitude mark process, $P(t)$ is the standard Poisson jump process with joint mean and variance $\mathrm{E}[P(t)] = \lambda t = \mathrm{Var}[P(t)]$.

B. Stock Log-Return Process, $\ln(\mathbf{S}(\mathbf{t}))$

The stock log-return $\ln(S(t))$ can be transformed to a simpler jump-diffusion stochastic differential equation (SDE) upon use of the stochastic chain rule [7],

$$d[\ln(S(t))] = \mu_{ld}dt + \sigma_d dW(t) + QdP(t), \tag{II.2}$$

where $\mu_{ld} \equiv \mu_d - 0.5\sigma_d^2$ can be called the log-diffusive (ld) drift. For simplicity the log-transformed jump-amplitude is taken as the mark,

$$Q = \ln(J(Q) + 1)\,.$$

C. Log-Normal Jump Distribution

Let the density of the jump-amplitude mark Q be normal

$$\phi_Q(q) = \phi^{(n)}(q; \mu_j, \sigma_j^2), \tag{II.3}$$

where $\phi^{(n)}(q; \mu_j, \sigma_j^2)$ is the normal density with mean μ_j and variance σ_j^2. The log-normal jump-amplitude jump-diffusion model was used in [10], [1], [3] and others.

For the density for this jump-diffusion model with log-normal jump-amplitude, Hanson and Westman [3] proved the following theorem:

Theorem: The probability density for the linear jump-diffusion log-return increment $\Delta[\ln(S(t))]$ with log-normal jump-amplitude is given by

$$\begin{aligned}\phi^{(\mathrm{jd})}(x) \;=\; & \sum_{k=0}^{\infty} p_k(\lambda\Delta t) \\ & \cdot\phi^{(n)}(x; \mu_{ld}\Delta t + k\mu_j, \sigma_d^2\Delta t + k^2\sigma_j^2),\end{aligned} \tag{II.4}$$

for $-\infty < x < +\infty$, where $p_k(\Lambda) = e^{-\Lambda}\Lambda^k/k!$ is the Poisson distribution with parameter Λ and k jumps, where $\Delta\mathbf{t}$ is the corresponding trading time increment.

This theorem is based upon the law of total probability [7] resulting in the sum over all k Poisson jumps, the convolution theorem [7] yielding the density of the log-jump-diffusion conditioned on there being k jumps, and the fact that the convolution of two normals is also normal [7]. Given k, the density of the jump term is simply $\phi_{kQ}(q) = \phi_Q(q/k)/k$. The theorem is posed as the log-return increment rather than for the infinitesimal, because the time between trading data is small but not infinitesimal. For the purpose of comparison, we use more terms of the expansion than we have in our other papers to provide more accurate estimations since we are dealing with small but not very small time steps and the scale these time steps can be magnified by a jump rate that includes many small jumps that are indistinguishable from the fluctuations of the diffusion process.

1) Basic Moments of Log-Return Increments $\Delta[\ln(\mathbf{S}(\mathbf{t}))]$ for Log-Normal Jumps::

- 1st moment:

$$M_1^{(\mathrm{jd})} \equiv \mathrm{E}[\Delta[\ln(S(t))]] = \mu_{ld}\Delta t + \mu_j\lambda\Delta t.$$

- 2nd moment:

$$\begin{aligned}M_2^{(\mathrm{jd})} &\equiv \mathrm{Var}[\Delta[\ln(S(t))]] \\ &= \sigma_d^2\Delta t + (\sigma_j^2(1+\lambda\Delta t) + \mu_j^2)\lambda\Delta t.\end{aligned}$$

- 3rd moment:

$$\begin{aligned}M_3^{(\mathrm{jd})} &\equiv \mathrm{E}\left[(\Delta[\ln(S(t))] - M_1^{(jd)})^3\right] \\ &= (3\mu_j\sigma_j^2 + \mu_j^3)\lambda\Delta t + 6\mu_j\sigma_j^2(\lambda\Delta t)^2.\end{aligned}$$

- 4th moment:

$$\begin{aligned}M_4^{(\mathrm{jd})} &\equiv \mathrm{E}\left[(\Delta[\ln(S(t))] - M_1^{(jd)})^4\right] \\ &= (\mu_j^4 + 3\sigma_j^4 + 6\mu_j^2\sigma_j^2)\lambda\Delta t \\ &\quad + (3\mu_j^4 + 21\sigma_j^4 + 30\mu_j^2\sigma_j^2)(\lambda\Delta t)^2 \\ &\quad + 6\sigma_d^2\Delta t(\sigma_j^2 + \mu_j^2)\lambda\Delta t \\ &\quad + 3(\sigma_d\Delta t)^2 + (6\mu_j^2\sigma_j^2 + 18\sigma_j^4)(\lambda\Delta t)^3 \\ &\quad + 6\sigma_d^2\Delta t\sigma_j^2(\lambda\Delta t)^2 + 3\sigma_j^4(\lambda\Delta t)^4.\end{aligned}$$

D. Log-Uniform Jump Distribution

Let the density of the jump-amplitude mark Q be uniform

$$\phi_Q(q) = (H(Q_b - q) - H(Q_a - q))/(Q_b - Q_a), \tag{II.5}$$

where $Q_a < 0 < Q_b$ and $H(x)$ is the Heaviside unit step function. The mark Q has moments, $\mu_j \equiv \mathrm{E}_Q[Q] = 0.5(Q_b + Q_a)$, $\sigma_j^2 \equiv \mathrm{Var}_Q[Q] = (Q_b - Q_a)^2/12$. The original jump-amplitude J has mean $\mathrm{E}[J(Q)] = (\exp(Q_b) - \exp(Q_a))/(Q_b - Q_a) - 1$ and log-uniform distribution

$$\Phi_J(x) = \ln((x+1)/(J_a+1))/\ln((J_b+1)/(J_a+1))$$

on $[J_a, J_b]$, where $J_a \equiv J(Q_a)$ and $J_b \equiv J(Q_b)$.

For the density of the jump-diffusion model with log-uniform jump-amplitude, the following theorem is given in [4].

Theorem: The probability density for the linear jump-diffusion, log-return increment $\Delta[\ln(S(t))]$ with log-uniform jump-amplitude is given by

$$\begin{aligned}\phi^{(\mathrm{jd})}(x) \;=\; & p_0(\lambda\Delta t)\phi^{(n)}(x; \mu_{ld}\Delta t, \sigma_d^2\Delta t) \\ & + \sum_{k=1}^{\infty} p_k(\lambda\Delta t) \\ & \cdot\frac{\Phi^{(n)}(x - kQ_b, x - kQ_a; \mu_{ld}\Delta t, \sigma_d^2\Delta t)}{k(Q_b - Q_a)},\end{aligned} \tag{II.6}$$

for $-\infty < x < +\infty$, where $p_k(\Lambda) = e^{-\Lambda}\Lambda^k/k!$ is the Poisson distribution with parameter Λ and k jumps and $\Phi^{(n)}(x_1, x_2; \mu, \sigma^2)$ is the normal distribution in interval $[x_1, x_2]$, where $\Delta\mathbf{t}$ is the corresponding trading time increment.

The justification of this theorem is similar to that as for the log-normal, except that the k-jump conditioned convolution leads to a combined jump-diffusion normal-uniform density given in (II.6) that we call the *secant-normal density* since the density is the secant of the normal distribution.

1) Basic Moments of Log-Return Increments $\Delta[\mathbf{ln}(\mathbf{S}(\mathbf{t}))]$ *for Log-Uniform Jumps::*

- 1st moment:

$$M_1^{(\mathrm{jd})} \equiv \mathrm{E}[\Delta[\ln(S(t))]] = \mu_{ld}\Delta t + \mu_j \lambda \Delta t.$$

- 2nd moment:

$$\begin{aligned} M_2^{(\mathrm{jd})} &\equiv \mathrm{Var}[\Delta[\ln(S(t))]] \\ &= \sigma_d^2 \Delta t + (\sigma_j^2(1+\lambda\Delta t) + \mu_j^2)\lambda\Delta t. \end{aligned}$$

- 3rd moment:

$$\begin{aligned} M_3^{(\mathrm{jd})} &\equiv \mathrm{E}\left[(\Delta[\ln(S(t))] - M_1^{(jd)})^3\right] \\ &= (3\mu_j\sigma_j^2 + \mu_j^3)\lambda\Delta t + 6\mu_j\sigma_j^2(\lambda\Delta t)^2. \end{aligned}$$

- 4th moment:

$$\begin{aligned} M_4^{(\mathrm{jd})} &\equiv \mathrm{E}\left[(\Delta[\ln(S(t))] - M_1^{(jd)})^4\right] \\ &= (\mu_j^4 + 1.8\sigma_j^4 + 6\mu_j^2\sigma_j^2)\lambda\Delta t \\ &\quad + (3\mu_j^4 + 12.6\sigma_j^4 + 30\mu_j^2\sigma_j^2)(\lambda\Delta t)^2 \\ &\quad + 6\sigma_d^2\Delta t(\sigma_j^2 + \mu_j^2)\lambda\Delta t \\ &\quad + 3(\sigma_d^2\Delta t)^2 + (6\mu_j^2\sigma_j^2 + 10.8\sigma_j^4)(\lambda\Delta t)^3 \\ &\quad + 6\sigma_d^2\Delta t(\sigma_j^2 + \mu_j^2)(\lambda\Delta t)^2 \\ &\quad + 1.8\sigma_j^4(\lambda\Delta t)^4. \end{aligned}$$

Note that the formulas for the first three moments are the same for both log-normal and log-uniform jumps.

E. Log-Double-Exponential Jump Distribution

Let the density of the jump-amplitude mark Q be double-exponential

$$\phi_Q(q) = \frac{p}{\mu_1} e^{\frac{q}{\mu_1}} I_{\{q<0\}} + \frac{(1-p)}{\mu_2} e^{\frac{-q}{\mu_2}} I_{\{q\geq 0\}}, \qquad \text{(II.7)}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are one-sided means, and $0 < p < 1$ represents the probability of downward jumps while $1-p$ is the probability of upward jumps. The set indicator function is $I_{\{\mathcal{S}\}}$ for set $\mathcal{S}$. The mark Q has moments, $\mu_j \equiv \mathrm{E}_Q[Q] = -p\mu_1 + (1-p)\mu_2$, $\sigma_j^2 \equiv \mathrm{Var}_Q[Q] = p(2-p)\mu_1^2 + 2p(1-p)\mu_1\mu_2 + (1-p^2)\mu_2^2$.

Similar to the theorem in [3], we get the following theorem:

Theorem: The probability density for the linear jump-diffusion log-return increment $\Delta[\ln(S(t))]$ with log-double-exponential jump-amplitude is given by

$$\begin{aligned} \phi^{(\mathrm{jd})}(x) &= p_0(\lambda\Delta t)\phi^{(n)}(x;\mu_{ld}\Delta t, \sigma_d^2\Delta t) \qquad \text{(II.8)} \\ &\quad + \sum_{k=1}^{\infty} \frac{p_k(\lambda\Delta t)}{k} \\ &\quad \cdot\left(\frac{p}{\mu_1}\exp\left(\frac{x - \mu_{ld}\Delta t + 0.5\sigma_d^2\Delta t/(k\mu_1)}{k\mu_1}\right)\right. \\ &\quad \cdot\Phi^{(n)}(-x;\bar{\mu}_1, \sigma_d^2\Delta t) \\ &\quad + \frac{1-p}{\mu_2}\exp\left(\frac{\mu_{ld}\Delta t - x + 0.5\sigma_d^2\Delta t/(k\mu_2)}{k\mu_2}\right) \\ &\quad \left.\cdot\left(1 - \Phi^{(n)}(-x;\bar{\mu}_2, \sigma_d^2\Delta t)\right)\right), \end{aligned}$$

for $-\infty < x < +\infty$, where $\bar{\mu}_1 \equiv \sigma_d^2\Delta t/(k\mu_1) - \mu_{ld}\Delta t$, $\bar{\mu}_2 \equiv -\sigma_d^2\Delta t/(k\mu_2) - \mu_{ld}\Delta t$, and $\mathbf{\Delta t}$ is the corresponding trading time increment.

1) Basic Moments of Log-Return Increments $\Delta[\mathbf{ln}(\mathbf{S}(\mathbf{t}))]$ *for Log-Double-exponential Jumps::*

- 1st moment:

$$\begin{aligned} M_1^{(\mathrm{jd})} &\equiv \mathrm{E}[\Delta[\ln(S(t))]] \\ &= \mu_{ld}\Delta t + (-p\mu_1 + (1-p)\mu_2)\lambda\Delta t. \end{aligned}$$

- 2nd moment:

$$\begin{aligned} M_2^{(\mathrm{jd})} &\equiv \mathrm{Var}[\Delta[\ln(S(t))]] \\ &= \sigma_d^2\Delta t + 2(p(\mu_1^2 - \mu_2^2) + \mu_2^2)\lambda\Delta t \\ &\quad + \left(p(2-p)\mu_1^2 + 2p(1-p)\mu_1\mu_2\right. \\ &\quad \left. + (1-p^2)\mu_2^2\right)(\lambda\Delta t)^2. \end{aligned}$$

- 3rd moment:

$$\begin{aligned} M_3^{(\mathrm{jd})} &\equiv \mathrm{E}\left[(\Delta[\ln(S(t))] - M_1^{(jd)})^3\right] \\ &= 2(\lambda\Delta t)^3(3p(p-1)\mu_1^3 - p^3(\mu_1^3 + \mu_2^3) \\ &\quad + 3p^2\mu_1\mu_2^2 - 3p^3\mu_1\mu_2(\mu_2 + \mu_1) \\ &\quad + \mu_2^3 + 3p(2p-1)\mu_1^2\mu_2) \\ &\quad + 6(\lambda\Delta t)^2(-p\mu_1^2\mu_2 + p\mu_2^2\mu_1 - p^2\mu_1\mu_2^2 \\ &\quad - p^2\mu_2^3 - p\mu_2^3 + 2\mu_2^3 + p^2\mu_1^3 + p^2 * \mu_1^2\mu_2 \\ &\quad - 3p\mu_1^3) + 6\lambda\Delta t((1-p)\mu_2^3 - p\mu_1^3); \end{aligned}$$

- 4th moment:

$$\begin{aligned} M_4^{(\mathrm{jd})} &\equiv \mathrm{E}\left[(\Delta[\ln(S(t))] - M_1^{(jd)})^4\right] \\ &\sim 24\lambda\Delta t((1-p)\mu_2^4 + p\mu_1^4) \\ &\quad + 24(7p\mu_1^4 - 5p\mu_2^4 + 6\mu_2^4 \\ &\quad + p\mu_1(\mu_2^3 - p\mu_1^3) - p^2\mu_1\mu_2(\mu_2^2 + \mu_1^2) \\ &\quad + p\mu_2(\mu_1^3 - p\mu_2^3))(\lambda\Delta t)^2 \\ &\quad + 12(p\mu_1^2 + (1-p)\mu_2^2)(\sigma_d^2\Delta t)(\lambda\Delta t) \\ &\quad + 3\sigma_d^4\Delta t^2. \end{aligned}$$

All $\mathrm{O}((\lambda\Delta t)^3)$ and $\mathrm{O}((\lambda\Delta t)^4)$ are omitted in $M_4^{(jd)}$.

F. Skewness and Kurtosis

In this paper, the skewness and kurtosis are the main benchmarks used to compare the three jump-diffusion models. Therefore, it is important to get M_3^{jd} and M_4^{jd} in order to get the theoretical skewness and kurtosis coefficient for these three models to sufficient accuracy for a satisfactory comparison.

- Skewness coefficient: $\beta_3^{(\mathrm{jd})} \equiv M_3^{(\mathrm{jd})} / \left(M_2^{(\mathrm{jd})}\right)^{1.5}$.
- Kurtosis coefficient: $\beta_4^{(\mathrm{jd})} \equiv M_4^{(\mathrm{jd})} / \left(M_2^{(\mathrm{jd})}\right)^{2}$.

Sometimes, the kurtosis is represented as the excess kurtosis coefficient by subtracting three from the above kurtosis coefficient definition so that the excess kurtosis coefficient is zero for the normal distribution.

III. Parameter Estimations

The basic point of view, here, is that the financial markets are considered to be a moderate size simulation of one of these three jump-diffusion processes.

A. Empirical Data

We use Standard and Poor's 500 (S&P500) stock index in the decade 1992-2001 [13] as the sample of the financial market since it is in general viewed as one big mutual fund so that it is less dependent on the peculiar behavior of any one stock.

Let $n^{(\mathrm{sp})} = 2522$ be the number of daily closings $S_s^{(\mathrm{sp})}$ for $s = 1 : n^{(\mathrm{sp})}$, such that there are $ns = 2521$ log-returns,

$$\Delta\left[\ln\left(S_s^{(\mathrm{sp})}\right)\right] \equiv \ln\left(S_{s+1}^{(\mathrm{sp})}\right) - \ln\left(S_s^{(\mathrm{sp})}\right), \qquad \text{(III.1)}$$

for $s = 1 : ns$ log-returns, with

- Mean:
$$M_1^{(\mathrm{sp})} = \frac{1}{ns}\sum_{s=1}^{ns}\Delta\left[\ln\left(S_s^{(\mathrm{sp})}\right)\right] \simeq 4.015\text{e-}4\,.$$
- Variance:
$$M_2^{(\mathrm{sp})} = \frac{1}{ns-1}\sum_{s=1}^{ns}\left(\Delta\left[\ln\left(S_s^{(\mathrm{sp})}\right)\right] - M_1^{(\mathrm{sp})}\right)^2 \simeq 9.874\text{e-}5\,.$$
- Skewness coefficient:
$$\beta_3^{(\mathrm{sp})} \equiv \frac{M_3^{(\mathrm{sp})}}{\left(M_2^{(\mathrm{sp})}\right)^{1.5}} \simeq -0.2913 < 0,$$
where $\beta_3^{(n)} = 0$ is the normal distribution value and $M_3^{(\mathrm{sp})}$ is the 3rd central log-return moment of the data.
- Kurtosis coefficient:
$$\beta_4^{(\mathrm{sp})} \equiv \frac{M_4^{(\mathrm{sp})}}{\left(M_2^{(\mathrm{sp})}\right)^{2}} \simeq 7.804 > 3,$$
where $\beta_4^{(n)} = 3$ is the normal distribution value and $M_4^{(\mathrm{sp})}$ is the 4th central log-return moment of the data.

B. Multinomial Maximum Likelihood Estimation

In a previous paper [8], the multinomial maximum likelihood estimation of model parameters is justified for binned financial data, but applied to very general binned data. The main idea for this method is the following:

- Step 1: Sample Data is sorted into nb bins and get the sample frequency $f_b^{(sp)}$, for $b = 1 : nb$.
- Step 2: Get the theoretical jump-diffusion frequency with parameter vector $\boldsymbol{x}$:
$$f_b^{(\mathrm{jd})}(\boldsymbol{x}) \equiv ns\int_{B_b}\phi^{(\mathrm{jd})}(\eta;\boldsymbol{x})d\eta\,,$$
where B_b is the bth bin.
- Step 3: Minimize the objective function:
$$y(\boldsymbol{x}) \equiv -\sum_{b=1}^{nb}\left[f_b^{(\mathrm{sp})}\ln\left(f_b^{(\mathrm{jd})}(\boldsymbol{x})\right)\right], \qquad \text{(III.2)}$$
which is the negative of the likelihood. Getting the negative of the maximum likelihood corresponds to the minimizing `fminsearch` function implementation of the Nelder-Mead down-hill simplex direct search method in MATLAB. The Nelder-Mead method [12] is used to get the optimal parameters $\boldsymbol{x}^*$ for the three compared models, respectively. The Nelder-Mead is usually faster than other optimization methods when it works. Some comparisons with our multidimensional golden section search method for the financial parameter estimation problem are given in [8].

C. Jump-Diffusion Moment Estimation Constraints

For the jump-diffusion model with log-normal and log-uniform jump-amplitude, there are five (5) free jump-diffusion parameters:

$$\{\mu_{ld}, \sigma_d^2, \mu_j, \sigma_j^2, \lambda\}\,.$$

For the stock return jump-diffusion model with log-double-exponential jump-amplitude, there are six (6) free jump-diffusion parameters:

$$\{\mu_{ld}, \sigma_d^2, \mu_1, \mu_2, p, \lambda\}\,.$$

So, to reduce this set to a reasonable number, the multinomial maximum likelihood estimation is subjected to the mean and variance constraints:

$$M_1^{(\mathrm{sp})} = M_1^{(\mathrm{jd})} \qquad \text{(III.3)}$$

and

$$M_2^{(\mathrm{sp})} = M_2^{(\mathrm{jd})}\,. \qquad \text{(III.4)}$$

So, for the log-normal and log-uniform jump-diffusion model, the two diffusion parameters, μ_{ld} and σ_d, are eliminated by

$$\mu_{ld} = \left(M_1^{(sp)} - \mu_j\lambda\Delta t\right)/\Delta t \qquad \text{(III.5)}$$

and

$$\sigma_d^2 = \left(M_2^{(sp)} - \left(\sigma_j^2(1+\lambda\Delta t) + \mu_j^2\right)\lambda\Delta t\right)/\Delta t\,, \qquad \text{(III.6)}$$

the latter is subject to positivity constraints, for fixed and small $\Delta t \ll 1$. Hence, only three free parameters are left:

$$\boldsymbol{x} = \{\mu_j, \sigma_j^2, \lambda\}\,.$$

For the log-double-exponential jump-diffusion model, two parameters μ_{ld} and σ_d are eliminated by

$$\mu_{ld} = \left(M_1^{(sp)} - (-p\mu_1 + (1-p)\mu_2)\lambda\Delta t\right)/\Delta t \quad \text{(III.7)}$$

and

$$\begin{aligned}\sigma_d^2 \;=\; &\Big(M_2^{(sp)} - 2(p(\mu_1^2 - \mu_2^2) + \mu_2^2)\lambda\Delta t \quad \text{(III.8)}\\ &+(p(2-p)\mu_1^2 + 2p(1-p)\mu_1\mu_2\\ &+(1-p^2)\mu_2^2)(\lambda\Delta t)^2\Big)/\Delta t\,.\end{aligned}$$

Then, four (4) free parameters are left:

$$\boldsymbol{x} = \{\mu_1, \mu_2, p, \lambda\}\,,$$

with significantly more computational cost.

IV. Numerical Results, Figures and Discussion

The multinomial maximum likelihood estimation given here is used to estimate the jump-diffusion parameters. The numerical optimization was performed using the MATLAB 6.5 [11] computing system's `fminsearch` function, an implementation of the down-hill simplex direct search method of Nelder and Mead [12].

For the log-normal and log-uniform model, the same starting point $\boldsymbol{x}_0$ is used. For the log-double-exponential model, the different starting point $\boldsymbol{x}_0$ is used: μ_1 and μ_2 are from the estimation of the μ_j of the log-uniform model, $p \simeq 0.6 > 0.5$ means more likely downward jump-amplitudes and the $\lambda\Delta t$ value are the same as the log-normal and log-uniform.

The empirical data used in the estimation are the S&P500 daily closing log-returns from the decade 1992-2001. In Figure 1 is the histogram of bin frequencies using 100 centered bins. Note the long, relatively thick tails signifying crashes in the negative tails and rallies in the positive tails, where the normal distribution or the double-exponential would have insignificant tail values. The ragged appearance of the histogram resembles the random simulation of a density using a moderate, but inadequate, sample size. The rare, larger jump events are difficult to see in the scale of the figure.

However, if the histogram frequencies are multiplied by the centered value of the bin log-return, then the larger jumps are clearly visible. This moment-histogram is called a *hysteriagram* since it magnifies the larger jumps and corresponds to the extreme behavioral reaction of some investors. The hysteriagram for the S&P500 is given in Figure 2 and clearly indicates the inadequacy of using a log-normal and the log-double-exponential to characterize significant large events.

Figure 3 shows that the log-normal jump-amplitude model hysteriagram exhibits too thin tails that decay too fast with the jump magnitude. From (II.4) it can be seen that the bin

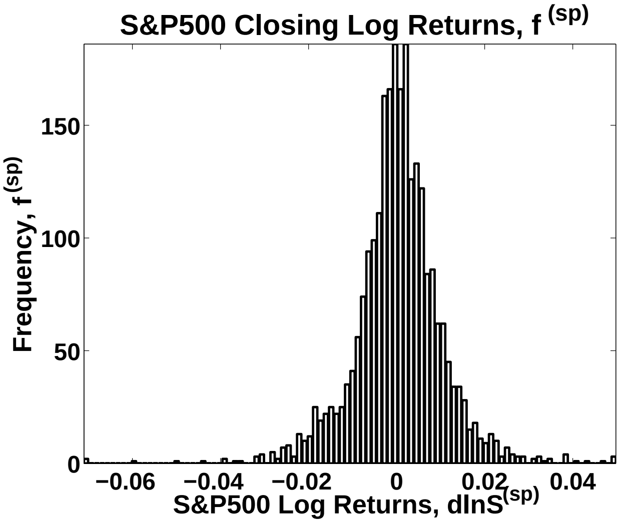

Fig. 1. Histogram of S&P500 log-return frequencies for the decade 1992-2001, using 100 bins.

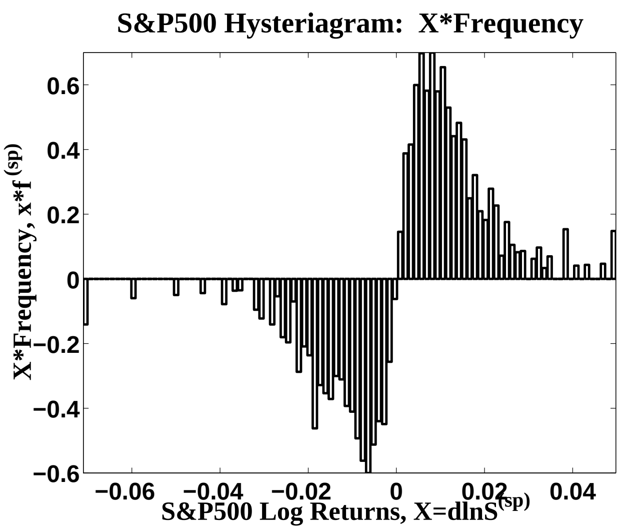

Fig. 2. Hysteriagram of S&P500 log-return frequencies multiplied by the average bin log-return value for the decade 1992-2001, using 100 bins.

distribution for sufficiently narrow bins will be a Poisson sum of normal distributions, so will have thin exponential Gaussian tails. The corresponding histogram for the log-normal, not shown here, does not show enough visual detail to sufficiently distinguish it from the other jump-amplitude models.

Figure 4 shows that the log-uniform jump-amplitude model hysteriagram exhibits much thicker tails that decay more slowly with the jump magnitude, but do not capture the largest negative jump in Figure 2. The secant-normal densities in (II.6) help counter the normal distribution tendency to having exponential thin tails, but not for beyond the largest jump values of the log-returns.

Figure 5 shows that the log-double-exponential jump-amplitude model hysteriagram exhibits too thin tails that decay too fast with the jump magnitude that is very similar

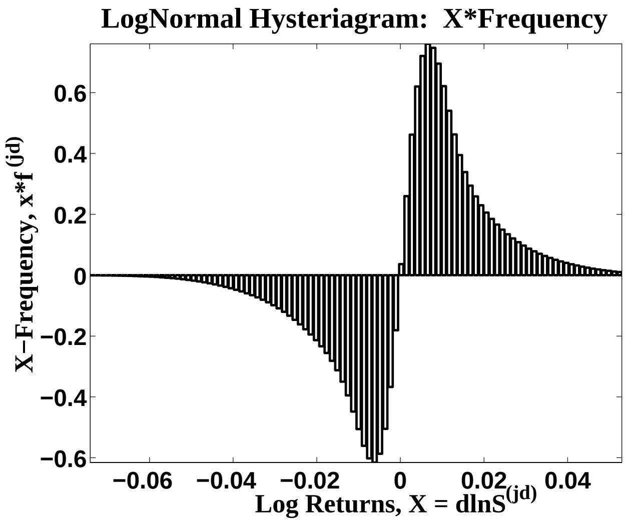

Fig. 3. Hysteriagram of the predicted log-returns frequencies multiplied by the average bin log-return value for the log-normal jump-amplitude jump-diffusion model, using 100 bins.

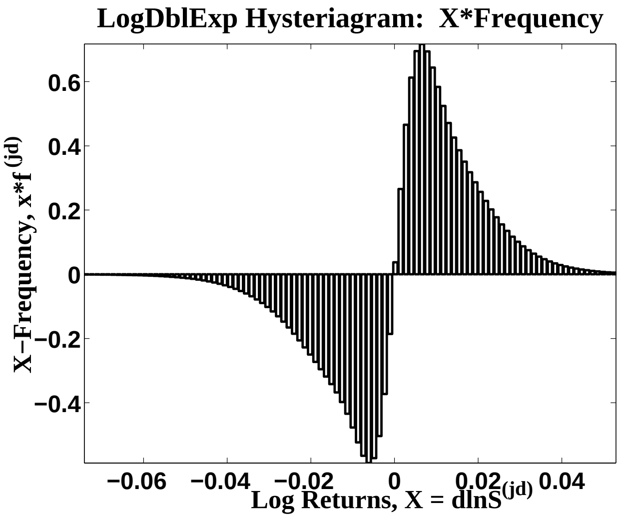

Fig. 5. Hysteriagram of the predicted log-returns frequencies multiplied by the average bin log-return value for the log-double-exponential jump-amplitude jump-diffusion model, using 100 bins.

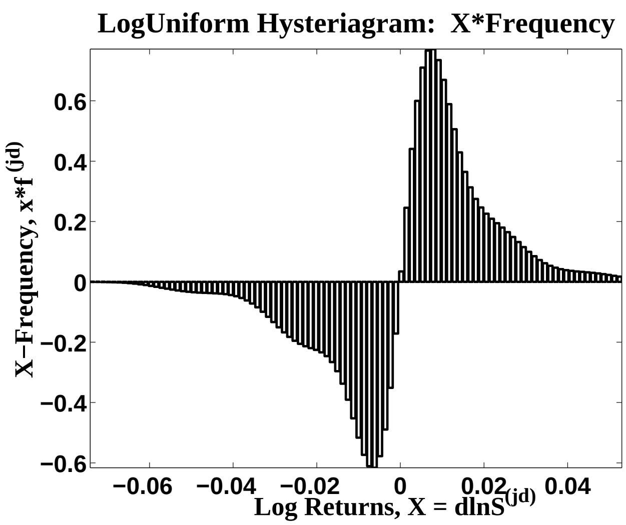

Fig. 4. Hysteriagram of the predicted log-returns frequencies multiplied by the average bin log-return value for the log-uniform jump-amplitude jump-diffusion model, using 100 bins.

to the log-normal jump-amplitude model. The convolution of normal and exponential distributions in (II.8), like the normal jump-amplitude model, can only lead to exponential thin tails.

Hence, the log-uniform model is a qualitatively better model for the S&P500 data, since the tails are thick enough to generate more of the larger jumps seen in the S&P500 data in Fig. 2 than the other two distributions.

From Table I, we can have a quantitative estimate of the derived distribution parameters μ_d, σ_d, μ_j, σ_j, λ. Since the trading days per year are about 250 days, it is not likely that the jumps rate is more than 100 per year because the finance market should be kept stable. So, $\lambda \simeq 59$ for the log-uniform is more reasonable, considering that the uniform jump distribution spans the crash to the rally data. The near-zero peaks of the normal and double-exponential lead to more than double the uniform jump rate. Note that the jump rate includes all size jumps, including those hidden by the log-normal part of the log-return distribution. In the table the overall jump mean μ_j is given for the purpose of comparison, but for the double-exponential, the negative jump mean is $\mu_{j,1} = -\mu_1 =$ -3.63e-3 and the positive jump mean is $\mu_{j,2} = \mu_2 =$ +3.24e-3. For the double-exponential, the probability of negative jumps is $p = 0.481$ and that for positive jumps is $(1-p) = 0.519$. For the other parameters, we can use the most-common value among these three models. Then, we get the other parameter estimations for the log-uniform as the following: $\mu_d \simeq 0.20$, $\sigma_d \simeq 0.085$, $\mu_j \simeq$ -1.6e-3 and $\sigma_j \simeq 0.015$. Hence, overall the log-uniform has a better estimation for these derived parameters.

TABLE I

Comparison summary of derived distribution parameters for the log-normal, log-uniform and log-double-exponential jump-diffusion models, respectively.

Model	μ_d	σ_d	μ_j	σ_j	λ
Normal	0.199	0.0907	-7.28e-4	9.25e-3	128.
Uniform	0.198	0.0865	-1.63e-3	1.53e-2	59.4
Dbl-Exp	0.125	0.0791	-6.15e-5	4.86e-3	338.

From Table II, the difference of skewness and kurtosis between the estimate value and the observed are 16% and -2.2% for the log-uniform model. These results are better when considering both coefficients than the other two models' results, except the difference skew for the double-exponential is the lowest, though the difference in the kurtosis is highest for the double-exponential. Also, given is the terminating multinomial maximum likelihood using the negative of minimum of the objective in (III.2), essentially the same for all models with the same stopping criterion being used.

TABLE II

The skewness and kurtosis coefficients for the three models are compared to S&P500 values, respectively, and Multinomial Maximum Likelihood (MML $\simeq -\min[y(\mathbf{x})]$).

Model	β_3	%	β_4	%	MML
Normal	-0.196	-32.7	8.90	14.0	1.118e4
Uniform	-0.387	+16.3	7.63	-2.21	1.117e4
Dbl-Exp	-0.279	-4.28	12.2	57.0	1.119e4
S&P500	-0.291	0.0	7.80	0.0	—

From the Table III, we can see that the log-normal and log-uniform models take the same order of magnitude of iterations and function evaluation, the log-normal model parameter estimate takes $1/7$ of the time to execute. One reason is that the log-normal requires only one normal distribution calculation for each jump k in (II.4), while the others required the calculation of either an integral of the secant-normal or of several normal distributions. However, the extra parameter needed for the double-exponential means the iteration count, the function evaluation count and the timings will be much greater for the double-exponential. The computational efforts for the uniform and double-exponential models were reduced by using integration by parts to reduce the original double bin distribution integrals to single integrals. An added advantage of such a reduction also can improve accuracy and speed of computation. The reduced formulas are too lengthy to report here. The reduction of the original double-exponential bin integrals to single integrals led to *exponential catastrophic cancellation* problems, in that exponential factors of that model interfered with the absolute error threshold of the MATLAB integration function *quadl* for the most negative bin locations causing small violations of positive probability properties. Absorption of these exponential factors into the integrands accurately corrected the error threshold problem.

TABLE III

Comparison summary of computational performance measures:

Model Used	Number Parms.	Number Iters.	Function Evals.	Timings (sec)
Normal	3	69	129	28.7
Uniform	3	54	101	211
Dbl-Exp	4	200*	342	2638

Combined Legend for Table I, Table II and Table III:

- Normal: Log-normal jump-amplitude.
- Uniform: Log-uniform jump-amplitude.
- Dbl-Exp: Log-double-exponential jump-amplitude.
- Maximum Number of Iterations: 200*.
- Using same tolerances: tolx = 5e-6 and toly = 5e-6.
- Using P4@1.6GHz CPU computer processor with MATLAB.

V. SUMMARY AND CONCLUSION

From the above theoretical and data analysis, we can get the following conclusions:

- The log-uniform model is the best overall among the three models, qualitatively in terms of genuinely representing the fat tail property of real-world market distributions and quantitatively in terms of reasonable overall higher moments, i.e., both skewness and kurtosis.
- The log-normal model runs faster than the other two models. The reason is that the optimization algorithm needs only single bin integrals over a normal density for the log-normal model. On the other hand, the integration by parts technique can be used to reduce the computational effort for the log-uniform and log-double-exponential models. However, the deficiencies of the log-normal model demonstrates that the distribution that is better analytically is not necessarily a better model for financial markets, i.e., finding a better model may be counter to the desire to obtained closed form solutions.
- The results for the log-normal and log-double-exponential jump amplitude models are qualitatively similar. Both of them have exponentially small tails and peaks in the center making small jumps more likely. If there are small jumps, they are not much different from diffusion fluctuations since the diffusion part of the jump-diffusion model dominates the stock log-Return process (II.2) in this case. If there are large jumps, then the exponentially small tails of their distributions can not contribute too much to the flat and thick tails of the real world financial markets. Therefore, these two models has some intrinsic defects and are not not recommended for monitoring the dynamics of the finance markets.
- For the future research and considerations.
    1) To develop better way to the fit rare, jump events.
    2) To improve the log-uniform model, the stochastic volatility should be considered since in the real world the implied volatility curve is not a constant, but 'smile' curve.
    3) To consider the option price problems based on the log-uniform model and try to get the exact or approximate solutions to these problems if it is possible. Now the option market grows very fast, we must face these problems and put the model under the real-world finance markets' test.

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