TECHNICAL RESEARCH REPORT

Global stability conditions for rate control with arbitrary communication delays

by Richard J. La, Priya Ranjan, and Eyad H. Abed

CSHCN TR 2003-25
(ISR TR 2003-53)

Global Stability Conditions for Rate Control with Arbitrary Communication Delays

Richard J. La, Priya Ranjan, and Eyad H. Abed
Department of Electrical & Computer Engineering and
Institute for Systems Research
University of Maryland, College Park
{hyongla, priya, abed}@eng.umd.edu

Abstract

We adopt the optimization framework for the rate allocation problem proposed by Kelly and investigate the stability of the system with *arbitrary* communication delays between network elements. It is shown that there is a natural underlying discrete time system whose stability is directly related to the stability of the given system. We first present general stability conditions of the system with arbitrary delays, and then apply these results to establish the stability of the system with a family of popular utility and resource price functions. The exponential stability of the system with the given utility and resource price functions is established. We also investigate discretized models that better approximate the packet level dynamics of the system and show that similar stability conditions can be obtained. Numerical examples are provided to validate our analyses.

1 Introduction

With continuing growth of the Internet with no signs of slowing down, proper management of congestion is emerging as an important issue. Poor management of congestion leads to a degradation in user experience and, potentially, inaccessibility of some parts of the network. Researchers have proposed many solutions to manage the congestion level inside the network and to allocate the available rates in a fair manner based on a variety of definitions of fairness. Some of examples of fairness include max-min fairness, proportional fairness [8], and (p, α)-fairness [16].

Kelly [8] has suggested that the problem of rate allocation for elastic traffic can be posed as one of achieving maximum aggregate utility of the users and proposed an optimization framework for rate allocation in the Internet. Here the utility of a user could either represent the true utility or preferences of the user or a utility function that is assigned to the user by the end user rate control algorithm, *e.g.*, Transmission Control Protocol (TCP) and Proportional-Fair Congestion Controller (PFCC). In the latter case the selection of the utility function determines the end user algorithm and the trade-off between the fairness among the users and system efficiency [1, 9, 12, 16]. The PFCC is such an example. Using the proposed framework he has shown that the system optimum is achieved at the equilibrium between the end users and resources. Based on this observation researchers have proposed various rate-based algorithms, in conjunction with a variety of active queue management (AQM) mechanisms, that solve the system optimization problem or its relaxation [8, 11, 12, 13].

The convergence of these algorithms, however, has been established only in the absence of feedback delay. Modeling the communication delay is especially important when the delay is non-negligible and/or the delay could be widely varying, *e.g.*, multi-hop mobile wireless network. Tan and Johari [7] have studied the case with homogeneous users, *i.e.*, same round-trip delays and same form of utility functions, and provided local stability conditions in term of users' gain parameters and communication delays. In general their results state that the product of gain parameter and communication delays should be no larger than some constant. Similar results have been obtained in [4, 15] in the context of single flow and single resource with more general utility functions and in [2] in the context of single bottleneck with multiple heterogeneous users. The given stability conditions are similar to those in [7] and state that the product of the delay and gain parameter of end user algorithms needs to be smaller than some constant. A more detailed discussion of these previous results is presented in Section 3. These results, however, focus on characterizing *sufficient* conditions on the communication delay and gain parameter for stability.

In this paper we study the problem of designing a robust rate control mechanism in the presence of communication delays between network resources and end users. However, unlike in the previous studies where the authors give the conditions on the delay and user's gain parameter for stability of the system, we are interested in establishing stability criteria in the presence of *arbitrary* delays and with arbitrary gain parameters of end users. In other words, our goal is to find a set of conditions, under which the rate control mechanism design can be carried out as if there were no feedback delays for the issue of stability of the system. Our approach is consistent with the philosophy that network protocols must be *simple* and *robust* given the complexity and scale of the Internet, and may prove to be more suitable for wireless ad hoc networks where delays are often expected to be unpredictable and widely varying. In particular, the stability conditions given in [2, 4] may be sufficient to ensure smooth operation of the network when the network is operating normally. However, a network can occasionally experience high congestion and behave unpredictably due to the presence of a large amount of nonresponsive traffic, *e.g.,* broadcast of a concert online, and/or a collapse of a part of network as a result of, for example, a link failure or routing instability. In such a scenario the system may temporarily deviate from the stable regime characterized by the conditions in [2, 4] because of the increased queueing delay or a larger number of flows, and the unstable behavior of the end user algorithm can aggravate the congestion level, eventually leading to a congestion collapse.

Our approach also provides a fresh way of looking at the issue of communication delay than traditional approaches. A natural question that arises in this setting is whether or not it is possible to design a system that is stable with an arbitrary communication delay. If possible, what are the necessary and/or sufficient conditions for the stability? These are the core issues addressed in this paper.

Our analysis is based on the invariance-based global stability results for nonlinear delay-differential equations [5, 6, 14]. This kind of global stability results are different from those based on Lyapunov or Razumikhin theorems for delayed differential equations used in [2, 4, 15, 21] or from passivity approach [22].

Generally speaking, our main results can be summarized as follows. First, we demonstrate that the stability of a system given by a set of delay-differential equations can be studied by looking at the stability of a natural underlying *discrete time* map, which is often easier to analyze and/or simulate. Second, if the user and resource price curves have a stable market equilibrium, which is captured by the underlying discrete time map, then the corresponding dynamical equation for flow optimization converges to the optimal point in the presence of arbitrary delays. This result essentially shows that stability is related to utility and price curves in a fundamental way. In particular, for a given resource price function, under certain conditions (to be stated precisely) it is possible to design a stable end user algorithm so that the ensuing dynamical system converges to the solution of the optimization problem, irrespective of the communication delays. Similarly, given the end user algorithm(s), under similar conditions one can design the resource price function(s) such that the system is stable regardless of the communication delays. Conversely, if the underlying market equilibrium is unstable then it is possible to find a large enough delay for which the equilibrium point loses its stability and gives way to oscillations. In practice, this gives rise to a fundamental trade-off between the responsiveness of end users and network resources.

It is worth noting that in general characterizing the exact necessary and sufficient conditions for stability with delays is difficult. Hence, our results provide a *simple* and *robust* way of dealing with the problem of widely varying feedback delay in communication networks through a clever choice of the users' utility functions and price functions.

This paper is organized as follows. Section 2 describes the optimization problem for rate control. An overview of the previous work on characterizing stability conditions in the presence of a communication delay is provided in Section 3. Section 4 describes the system model with communication delays. Section 5 studies a simple case where the feedback signal from the resources to a user is delayed by the same amount for each user. We apply our results in Sections 5 to establish the stability condition using a popular class of utility and resource price functions in Section 6. We investigate discretized systems that better approximate the packet level dynamics and/or a situation where measurements at the network resources are performed periodically over a window in Section 7 and establish exponential stability using the utility and price functions in Section 6. Numerical examples are given in Section 8, which is followed by a study of general models where the delays between sources and resources are captured in more details in Section 9. We conclude in Section 10.

2 Background

In this section we briefly describe the rate control problem in the proposed optimization framework. Consider a network with a set $\mathcal{L}$ of resources or links and a set $\mathcal{I}$ of users. Let C_l denote the finite capacity of link $l \in \mathcal{L}$. Each user has a fixed route r_i, which is a non-empty subset of $\mathcal{L}$. We define a zero-one matrix A, where $A_{i,l} = 1$ if link l is in user i's route r_i and $A_{i,l} = 0$ otherwise. When the throughput of user i is x_i, user i receives utility $U_i(x_i)$. As mentioned earlier, this utility function could represent either the user's true utility or some function assigned to the user for carrying out the trade-off between fairness among the users and system throughput [1, 9, 12, 16]. We take the latter view and assume that the utility functions of the users are used to select the desired rate allocation among the users (*i.e.*, the desired operating point of the system), which also determines the end user algorithms as will be shown shortly. The utility $U_i(x_i)$ is an increasing, strictly concave and continuously differentiable function of x_i over the range $x_i \geq 0$.[1] Furthermore, the utilities are additive so that the aggregate utility of rate allocation $x = (x_i, i \in \mathcal{I})$ is $\sum_{i \in \mathcal{I}} U_i(x_i)$. Let $U = (U_i(\cdot), i \in \mathcal{I})$ and $C = (C_l, l \in \mathcal{L})$. The rate control problem can be formulated as the following optimization problem:
SYSTEM(U,A,C):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in \mathcal{I}} U_i(x_i) \\ \text{subject to} \quad & A^T x \leq C \ , \ x \geq 0 \end{aligned} \tag{1}$$

The first constraint in the problem says that the total rate through a resource cannot be larger than the capacity of the resource. Instead of solving (1) directly, which is difficult for any large network, Kelly in [8] has proposed to consider the following two simpler problems.

Suppose that each user i is given the price per unit flow λ_i. Given λ_i, user i selects an amount to pay per unit time, w_i, and receives a rate $x_i = \frac{w_i}{\lambda_i}$.[2] Then, the user's optimization problem becomes the following [8].
$USER_i(U_i; \lambda_i)$:

$$\begin{aligned} \text{maximize} \quad & U_i\left(\frac{w_i}{\lambda_i}\right) - w_i \\ \text{over} \quad & w_i \geq 0 \end{aligned} \tag{2}$$

[1] Such a user is said to have elastic traffic.

[2] This is equivalent to selecting its rate x_i and agreeing to pay $w_i = x_i \cdot \lambda_i$.

The network, on the other hand, given the amounts the users are willing to pay, $w = (w_i, i \in \mathcal{I})$, attempts to maximize the sum of weighted log functions $\sum_{i\in\mathcal{I}} w_i \log(x_i)$. Then the network's optimization problem can be written as follows [8].

NETWORK(A,C;w) :

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in\mathcal{I}} w_i \log(x_i) \\ \text{subject to} \quad & A^T x \le C \ , \ x \ge 0 \end{aligned} \tag{3}$$

Note that the network does not require the true utility functions $(U_i(\cdot), i \in \mathcal{I})$, and pretends that user i's utility function is $w_i \cdot \log(x_i)$ to carry out the computation. It is shown in [8] that one can always find vectors $\lambda^* = (\lambda_i^*, i \in \mathcal{I})$, $w^* = (w_i^*, i \in \mathcal{I})$, and $x^* = (x_i^*, i \in \mathcal{I})$ such that w_i^* solves $USER_i(U_i; \lambda_i^*)$ for all $i \in \mathcal{I}$, x^* solves $NETWORK(A, C; w^*)$, and $w_i^* = x_i^* \cdot \lambda_i^*$ for all $i \in \mathcal{I}$. Furthermore, the rate allocation x^* is also the unique solution to $SYSTEM(U, A, C)$.

Assume that every user adopts rate-based flow control. Let $w_i(t)$ and $x_i(t)$ denote user i's willingness to pay per unit time and rate at time t, respectively. Now suppose that at time t each resource $l \in \mathcal{L}$ charges a price per unit flow of $\mu_l(t) = p_l(\sum_{i:l\in r_i} x_i(t))$, where $p_l(\cdot)$ is an increasing function of the total rate going through it. Consider the system of differential equations

$$\frac{d}{dt} x_i(t) = \kappa_i \left(w_i(t) - x_i(t) \sum_{l\in r_i} \mu_l(t) \right) \ . \tag{4}$$

These equations can be motivated as follows. Each user first computes a price per unit time it is willing to pay, namely $w_i(t)$. Then, it adjusts its rate based on the feedback provided by the resources in the network to equalize its willingness to pay and the total price. In [9] $w_i(t)$ is set to $x_i(t) \cdot U_i'(x_i(t))$. The feedback from a resource $l \in \mathcal{L}$ can also be interpreted as a congestion indicator, requiring a reduction in the flow rates going through the resource. For more detailed explanation of (4), refer to [9]. Since we assume that the utility functions of the users are selected to decide the rate allocation amongst the users, under (4) one can see that, in fact, both the users' utility functions and resource price functions can be utilized to decide the system operating point. Therefore, the design of rate control algorithms is equivalent to selecting the users' utility functions and the price functions of the resource in the network.

Kelly *et al.* have shown that under some conditions on $p_l(\cdot), l \in \mathcal{L}$, the above system of differential equations converges to a point that maximizes the following expression

$$\mathcal{U}(x) = \sum_i U_i(x_i) - \sum_l \int_0^{\sum_{i:l\in r_i} x_i} p_l(y)\, dy. \tag{5}$$

Note that the first term in (5) is the objective function in our *SYSTEM(U, A, C)* problem. Thus, the algorithm proposed by Kelly *et al.* solves a relaxation of the $SYSTEM(U, A, C)$ problem.

3 Previous Work

The analysis in [9] of the convergence of the rate control algorithm given by (4), however, does not model the communication delay that is present between the resources and the end users. There has been some previous work on studying the stability of the system in the presence of communication delay. Tan and Johari [7] have analyzed the case where every user has the same round-trip delay and utility function given by $w \log(\cdot)$, *i.e.*, $w(t) = w$. They have characterized the conditions on local stability in terms of the gain parameter κ and and communication delay d. Their results state that there exists some constant D such that the product of the gain parameter κ and communication delay d should be smaller than D. In addition, they have shown the convergence rate of the system in the case of single-user single-resource.

A similar sufficient condition is also obtained in [15] in the context of single-flow single-resource. Suppose that the end user algorithm is given by

$$\dot{x}(t) = \kappa(w - x(t-d)p(x(t-d))) \ ,$$

where $w, \kappa > 0$ and d is the communication delay. This models the end user algorithm with $U(x) = w \cdot \log(x)$ with a feedback delay of d. The authors show that, if $p(\cdot)$ is a function of class C^1 that is nonnegative, nondecreasing, and bounded in norm by 1 such that $p'(\cdot)$ is nonincreasing and $\lim_{x\to\infty} p(x) = 1$, then the system is stable provided that $0 \leq d \leq \frac{1}{4\kappa}$.

Recently Deb and Srikant [4] have investigated the stability of the system in the context of single flow and single resource with more general utility functions, and have provided a *sufficient* condition for stability. Assume that the rate x is constrained to $[l, M]$. Let d denote the feedback delay from the network resource to the single user. The resource price function is denoted by $p(\cdot)$, and x^* is the unique solution to (5). The proposed end user algorithm is given by

$$\dot{x}(t) = \kappa\left(w - \frac{1}{x(t)U'(x(t))} x(t-d)p(x(t-d))\right) \ .$$

Define

$$A(l,M) = 1 + \frac{w \min_{l\leq x\leq M}(-xU''(x) - U'(x))}{\max_{l\leq x\leq M} h(x)} \tag{6}$$

$$B(l,M) = \frac{w \max_{l\leq x\leq M}\left|xU''(x) + U'(x)\right| + \max_{l\leq x\leq M} h(x)}{\min_{l\leq x\leq M} xU'(x)} \tag{7}$$

where $h(x(t-d)) = \lim_{y\to x(t-d)} \frac{yp(y) - x^*p(x^*)}{y - x^*}$. Their main results state that if there exists some constant $q > 1$ such that $\sqrt{q}\kappa d < \frac{A(l,M)}{B(l,M)}$, then the system is globally exponentially stable. One can see from (6) - (7) that if the range $[l, M]$ is large, for some utility functions, *e.g.*, utility and price functions used in Section 6, the ratio $A(l,M)/B(l,M)$ will be very small, and thus, the given constraint may be very restrictive. In fact, we will show that this condition is not necessary for the utility functions in Section 6 under a mild condition.

Alpcan and Basar [2] have studied the stability of a system with a single resource and multiple flows, using a delay based algorithm, and provided a *sufficient* condition for stability. Although the algorithm uses the estimated queueing delay as the feedback information, the authors assume that feedback delay is fixed. Denote the feedback delay of flow i by r_i. Assume x^* is the solution to (5) and $0 \leq x_i \leq x_{i,max}$, where $x_{i,max}$ is assumed not to exceed the minimum capacity of the links on the user's route. Let $\tilde{x} := x - x^*$ and $g_i(\tilde{x}) := \frac{dU_i(x)}{dx} - \frac{dU_i(x^*)}{dx}$. Define

$$k_{min} := \min_i \inf_{-x_i^* \leq \tilde{x}_i \leq x_{i,max} - x_i^*} \left|\frac{g(\tilde{x}_i)}{\tilde{x}_i}\right| \ . \tag{8}$$

The end user's rate evolves according to

$$\dot{x}_i(t) = \frac{dU_i(x_i(t))}{dx_i} - \alpha_i q(\underline{x}(t - r_i)) \ , \ i = 1, \ldots, I \ ,$$
$$\dot{q}(t) = \frac{\sum_{i=1}^{I} x_i(t - r_i)}{C} - 1 \ ,$$

where q is the queueing delay and $\alpha_i > 0$. They show that if $r_{max} := \max_i r_i < \frac{k_{min} C}{2I\alpha_{max}}$, then the system is asymptotically stable. These conditions are, however, not always easy to verify, and also become more restrictive with increasing number of flows I. Therefore, when a bottleneck is shared by many flows, for instance, in the core network, these may provide very conservative conditions.

All of the conditions presented above are imposed on the gain parameters of the end user algorithm and the feedback delay. However, these are sufficient conditions derived using well known techniques and are typically not necessary as we have shown in [17]. Moreover, it is not known how the system behaves when it leaves the stability region characterized in [2, 4] due to unforeseen events, such as a partial collapse of the network or routing instability. Clearly, in a large scale system such as the Internet, it is not desirable to use an algorithm whose stability region is limited to a small region. In this paper, rather than providing the sufficient conditions on the gain parameters and feedback delays after the utility and price functions are selected, we study the problem of selecting the utility and/or price functions that ensure system stability regardless of the communication delay or users' gain parameters κ_i. This is different from the approaches used in the previous work in [2, 4, 15]. For example, if the utility functions of the users are fixed through the end user algorithms, then our results provide the conditions on the network resource price functions for system stability and vice versa, hence providing a guideline for designing a stable AQM scheme. We study the trade-off between the responsiveness of resource price functions and end users' utility functions, which can be captured using the notion of price elasticity of demand.

4 Network Model with Delays

Although the previous studies [2, 4, 15] attempt to model the delays between the network resources and end users, their models are relatively simple due to the simplicity of the network that was considered. In this section we first describe the network model that captures the delays between the network resources and end users under the assumption that the delays are constant. Although in practice the delays are time-varying due to the varying queue sizes, we assume that the variation in the delays due to fluctuating queue sizes is not significant. For example, AQM mechanisms that attempt to either maintain very small queue sizes, *e.g.,* AVQ, or keep the queue sizes around some target queue sizes, *e.g.,* REM, can be well approximated by our model.

Consider a set $\mathcal{I} = \{1, \ldots, N\}$ of users sharing a network consisting of a set $\mathcal{L}$ of resources as described in Section 2. Let I_l be the set of users traversing resource $l \in \mathcal{L}$, *i.e.,* $I_l = \{i \in \mathcal{I} \mid l \in r_i\}$. We assume that $p_l(\cdot), l \in \mathcal{L}$, are strictly increasing and continuous. The feedback information from the resources to user i is delayed due to link propagation delays and transmission delays. For all $i \in \mathcal{I}$ and $l \in r_i$ let $T_{i,l}$ and $Z_{i,l}$ denote the delay of the feedback signal from resource l to user i and the delay user i packets experience before reaching resource l from the sender, respectively. If user i does not traverse resource l, then we assume that $T_{i,l} = Z_{i,l} = 0$. Suppose that the links in $r_i = \{l_{i,1}, \ldots, l_{i,R_i}\}$ are arranged in the order user i packets visit, where $R_i = |r_i|$. Define $T_i = Z_{i,l_{i,1}} + T_{i,l_{i,1}} = Z_{i,l_{i,k}} + T_{i,l_{i,k}}, k = 1, \ldots, R_i$, *i.e.,* the total delay before the receipt of the acknowledgement of a packet. We denote $\sum_{i \in \mathcal{I}} (R_i + 1)$ by Ξ. Under this general model, the end user dynamics are given by

$$\frac{d}{dt} x_i(t) = \kappa_i \left(x_i(t) U_i^{'}(x_i(t)) - x_i(t - T_i) (\sum_{l \in r_i} \mu_l(t - T_{i,l})) \right) \tag{9}$$

where

$$\mu_l(t - T_{i,l}) = p_l \left(\sum_{j \in I_l} x_j(t - (T_{i,l} + Z_{j,l})) \right) .$$

Under this model the price of resource l at time t depends on the rates of the users at time $t - Z_{i,l}$ due to the delay from the sources to the resource. The feedback signal generated by the resource price functions is then delayed by $T_{i,l}$ before user i receives it.

One thing to note is that when there is a unique path that is used by the users' routes between any two resources in the network, the above model can be simplified considerably. However, we consider more general cases, where multiple paths between any two resources can be utilized.

In this paper we are interested in studying the stability of the system given by the set of delay-differential equations in (9). In particular, our goal is to find a necessary and/or sufficient conditions on the utility and resource price functions that will ensure the convergence of $x_i(t), i \in \mathcal{I}$, to the solution of (5) regardless of the delays $T_{i,l}$ and $Z_{i,l}$.

5 General Network with Heterogeneous Delays

In this section we first study a simpler case to illustrate the basic techniques used in this paper for establishing the stability of the rate control mechanism in the presence of arbitrary feedback delays and to explain the intuition and insight behind the techniques. We assume that $Z_{i,l} = 0$ for all $i \in \mathcal{I}$ and $l \in \mathcal{L}$ and $T_i > 0$, *i.e.,* for each user $i \in \mathcal{I}$ there is no delay from the sender to its receiver and all of the feedback delay lies in the reverse path from the receiver to the sender, which equals $T_i > 0$. The general cases described in Section 4 where the feedback signals from resources to the users experience different delays are discussed in Section 9. We assume that there exists some small positive constant h such that $T_i = m_i \cdot h$, where m_i is a nonnegative integer, for all $i \in \mathcal{I}$. This does not pose any serious limitation as h can be arbitrarily small and typically devices are driven by local oscillators with fixed frequencies and hence periods. Thus, we can find the greatest common divisor of their periods.

Following Kelly's rate control formulation [8], in this simpler model eq. (9) can be simplified to

$$\frac{d}{dt}x_i(t) = \kappa_i \left(x_i(t)U_i^{'}(x_i(t)) - x_i(t - T_i)(\sum_{l \in r_i} \mu_l(t - T_i)) \right) \tag{10}$$

where $\mu_l(t - T_i) = p_l(\sum_{j \in I_l} x_j(t - T_i))$. We let $t = s \cdot h$ and normalize time in (10) by h:

$$\frac{1}{h}\frac{d}{ds}x_i(s) = \kappa_i \left(x_i(s)U_i^{'}(x_i(s)) - x_i(s - m_i)(\sum_{l \in r_i} \mu_l(s - m_i)) \right) . \tag{11}$$

Using the substitution $y_i = x_i U_i^{'}(x_i) := g_i(x_i)$, we obtain

$$x_i(t) = g_i^{-1}(y_i(t)) \ , \ \dot{x}_i(t) = \frac{\dot{y}_i(t)}{g_i^{'}(g_i^{-1}(y_i(t)))} \ ,$$

and (11) can be rewritten as

$$\nu \dot{y}_i(t) = \kappa_i g_i^{'} \left(g_i^{-1}(y_i(t)) \right) \left(y_i(t) - f_i(\overline{g}^{-1}(\overline{y}(t - m_i))) \right) \tag{12}$$

where $\nu = \frac{1}{h}$, $\overline{y}(t - m_i) = (y_1(t - m_i), \cdots, y_N(t - m_i))$, $\overline{g}^{-1}(\overline{y}) = (g_1^{-1}(y_1), \ldots, g_N^{-1}(y_N))$,[3] and

$$\begin{aligned} f_i(\overline{g}^{-1}(\overline{y}(t - m_i))) &= f_i(g_1^{-1}(y_1(t - m_i)), \ldots, g_N^{-1}(y_N(t - m_i))) \\ &= g_i^{-1}(y_i(t - m_i)) \left(\sum_{l \in r_i} p_l \left(\sum_{j \in I_l} g_j^{-1}(y_j(t - m_i)) \right) \right) . \end{aligned} \tag{13}$$

[3]Throughout the paper we assume that all vectors are column vectors.

We can write the above in the following matrix form:

$$\nu\dot{\overline{y}}(t) = \kappa(\overline{y}(t))\,(F(\overline{y}(t-m_1), \ldots, \overline{y}(t-m_N)) - \overline{y}(t)) \tag{14}$$

where $\kappa(\cdot)$ is a state dependent diagonal gain matrix with $\kappa_{ii}(\overline{y}(t)) = -\kappa_i g_i^{'}(g_i^{-1}(y_i(t)))$, and $F_i((\overline{y}_1, \ldots, \overline{y}_N))$ $= f_i(\overline{g}^{-1}(\overline{y}_i))$. This decomposition is possible due to the fact that the utility of a user is a function only of its own rate and does not depend on those of other users. The map $F(\cdot)$ given by (14) is a multidimensional nonlinear map, which is crucial for understanding the stability of the system. We note that this system of differential equations in (14) has the following natural underlying discrete time map with unit time of h.

$$\overline{y}_{n+1} = F(\tilde{y}_n)\ ,\ n \in Z_+ = \{1, 2, \ldots\} \tag{15}$$

where $\overline{y}_n \in \Re_+^N$, $\tilde{y}_n = (\overline{y}_{n-m_1+1}, \ldots, \overline{y}_{n-m_N+1}) \in \Re_+^{N^2}$, and

$$F_i(\tilde{y}_n) = f_i(g^{-1}(\overline{y}_{n-m_i+1})) = f_i(g_1^{-1}(y_{n-m_i+1,1}), \ldots, g_N^{-1}(y_{n-m_i+1,N}))\ .$$

The importance of this multidimensional map will be clear when we prove that the global stability of this map is a sufficient condition for the global stability of the delay-differential system given by (14) in the following subsection.

5.1 Convergence Results

In order to establish the stability of the system with multiple users of heterogenous utility functions we need to prove the global convergence in a multidimensional space. Our approach extends the basic approach used by Verriest and Ivanov [20]. The basic idea behind this approach is to use invariance and continuity properties of the underlying discrete time map given in (15) for the differential equations and find a sequence of bounds using convex sets, which converges to the singleton with the solution to (5). Hence, the convergence is derived from the underlying discrete time map, which provides the bounds for the trajectories of the delay-differential system. Following this plan, we first prove the invariance of the system given by (14) when the underlying map given by (15) has a convex invariance set that is a product space. The assumption of the existence of a convex invariant product space is natural in a rate control problem because the utility of a user depends only on its own rate. This will be illustrated in the next section using a family of popular utility and resource price functions. Then, we use this invariance property to establish the asymptotic stability of the system when the underlying discrete time map has a stable fixed point.

Before we present the convergence results, we state an assumption that we make on the functions $g(\cdot)$ and $f_i(\cdot)$, $i \in \mathcal{I}$. We denote by $\mathcal{J}^i$ the set of users that share a resource with user i, *i.e.*, $\mathcal{J}^i = \{j \in \mathcal{I} \mid r_i \cap r_j \neq \emptyset\} = \cup_{l \in r_i} I_l$.

Assumption 1 *(i) The function $g_i(x_i)$ is strictly decreasing with $g_i^{'}(x_i) < 0$ for all $x_i > 0$, (ii) the function $f_i(\overline{x})$ is strictly increasing in x_j for all $j \in \mathcal{J}^i$ and $\overline{x} > 0$, and does not depend on x_j for all $j \in \mathcal{I} \setminus \mathcal{J}^i$, and (iii) both $g_i(x_i)$ and $f_i(\overline{x})$ are Lipschitz continuous on $\Re_\epsilon$ and $\Re_\epsilon^N$, respectively, where $\Re_\epsilon = [\epsilon, \infty)$ and ϵ is an arbitrarily small positive constant.*

From (13) one can see that condition (ii) is equivalent to the assumption that the resource price functions $p_l(\cdot)$ are strictly increasing in the total rate traversing them.

It can be seen that under this assumption $\kappa(\cdot)$ is a positive definite matrix, which turns out to be an important property in proving the convergence results for the system given by (14). We first define the invariance of the map $F(\cdot)$ that will be used throughout the rest of the paper.

Definition 1 *A set $D \subset \Re_\epsilon^N$ is said to be invariant under the discrete time map $F(\cdot)$ defined in (14) if $F(\tilde{y}) \in D$ whenever $\tilde{y} \in D^N$, i.e., $\tilde{y} = (\overline{y}_1, \ldots, \overline{y}_N)$ and $\overline{y}_i \in D$ for all $i \in \mathcal{I}$.*

Let $m_{max} = \max_{i \in \mathcal{I}} m_i$. We denote by $C([-m_{max}, 0], D)$ the set of functions that are continuous over $[-m_{max},\ 0]$ with the range D.

Our first result states that the set $C([-m_{max}, 0], D)$ is invariant under the action generated by (14), provided that D is closed, convex and invariant under F in (15).

Theorem 1 *(Invariance) Suppose that $D \subset \Re_{\epsilon}^N$ is a closed, convex, product space, i.e., $D = \prod_{i=1}^N proj_i(D)$, that is invariant under $F(\cdot)$, where $proj_i(\cdot)$ denotes the i-th component projection operator. Then, for any initial function $\phi \in C([-m_{max}, 0], D) := Y_D$ the resulting $\overline{y}(t)$ from (14) belongs to the domain D for all $t \geq 0$ and $\nu > 0$.*

Proof: The proof is provided in Appendix A. ■

We next define a fixed point of the multidimensional map $F(\cdot)$.

Definition 2 *A vector $y^* \in \Re_{\epsilon}^N$ is said to be a fixed point of $F(\cdot)$ if $F((y^*, \ldots, y^*)) = y^*$.*

One can verify that $g^{-1}(y^*) = (g_1^{-1}(y_1^*), g_2^{-1}(y_2^*), \ldots, g_N^{-1}(y_N^*))$ is a solution to (5) from (13) and (14), *i.e.*, , $y^* = g(x^*)$, where x^* is a solution to (5).

Using the invariance property, we now turn to the asymptotic property of the delay-differential equation under the natural assumption that the map $F(\cdot)$ has a stable fixed point. Our approach is to find a sequence of convex coverings of the image $F(\tilde{D}_n)$ which are product spaces and converge to the single $\{y^*\}$. In other words, we construct a sequence of closed product spaces $\{D_n\}$, where $D_n = \prod_{i=1}^N D_{n,i} \subset \Re_{\epsilon}^N$, such that, under certain stability conditions, $F(D_n^N) \subset int(D_{n+1}) \subset D_{n+1} \subset int(D_n)$, where $int(D)$ denotes the interior of D, and $\cap_{n \geq 0} D_i = \{y^*\}$. If one can find such a sequence, then the solutions of the map given by (15) converges to y^* asymptotically provided that $\overline{y}_0, \ldots, \overline{y}_{-m_{max}+1} \in D_0$.

We now state the assumption under which the asymptotic stability is established.

Assumption 2 *The multidimensional map $F : \Re_+^{N^2} \to \Re_+^N$ has a fixed point $y^* \in \Re_{\epsilon}^N$, i.e., $F((y^*, \ldots, y^*)) = y^*$. Also, there is a sequence of closed convex product spaces $D_n, n \geq 0$, such that $F(D_n^N) \subset int(D_{n+1}) \subset D_{n+1} \subset int(D_n)$ and $\cap_{n \geq 0} D_n = \{y^*\}$.*

Let $Y_{D_0} = C([-m_{max}, 0], D_0)$ be a subset of initial functions and $\overline{y}_\phi^\nu$ a solution of (14) constructed through $\phi \in Y_{D_0}$.

Theorem 2 *(Asymptotic Stability) All solutions $\overline{y}_\phi(t)$ starting with initial functions $\phi \in Y_{D_0}$ converge to y^* as $t \to \infty$ for all $\nu > 0$ and for all $m_i \in Z_+, i \in \mathcal{I}$.*

Proof: The proof is given in Appendix B. ■

Theorem 2 tells us that the attracting fixed point of the map $F(\cdot)$ is stable in the set D_0. The basic tool these theorems give us is to look at a set of delay-differential equations as a discrete time map, which is typically more amenable to analysis and simulations. In addition, the study of the underlying discrete time maps typically provides us with more insight than the delay-differential equations themselves as will be shown in the following sections.

5.2 Comparison with Homogeneous Delay Case

In this subsection we discuss the relationship between a homogeneous delay system, *i.e.,* where every user has the same feedback delay, and a heterogeneous delay system discussed in subsection 5.1.

We first describe the system where all users have the same feedback delay $T > 0$. Eq. (10) can now be simplified to

$$\frac{d}{dt}x_i(t) = \kappa_i \left(x_i(t)U_i^{'}(x_i(t)) - x_i(t-T) \left(\sum_{l \in r_i} \mu_l(t-T) \right) \right) \tag{16}$$

where T is the common feedback delay of the users. We normalize time by T and use the substitution $y_i = x_i U_i^{'}(x_i) := g_i(x_i)$ as before to obtain

$$\nu \dot{y}_i(t) = \kappa_i g_i^{'} \left(g_i^{-1}(y_i(t)) \right) \left(y_i(t) - f_i(\overline{g}^{-1}(\overline{y}(t-1))) \right)$$

where $\nu = \frac{1}{T}$, $\overline{y}(t-1) = (y_1(t-1), \cdots, y_N(t-1))$, and

$$f_i \left(\overline{g}^{-1}(\overline{y}(t-1)) \right) = g_i^{-1}\left(y_i(t-1)\right) \left(\sum_{l \in r_i} p_l \left(\sum_{j \in I_l} g_j^{-1}(y_j(t-1)) \right) \right) .$$

We can write the above in the following simple matrix form:

$$\nu \dot{\overline{y}}(t) = \kappa(\overline{y}(t)) \left(\tilde{F}(\overline{y}(t-1)) - \overline{y}(t) \right) \tag{17}$$

where $\kappa(\cdot)$ is the state dependent diagonal gain matrix with $\kappa_{ii}(\overline{y}(t)) = -\kappa_i g_i^{'} \left(g_i^{-1}(y_i(t)) \right)$ and $\tilde{F}_i(\overline{y}) = f_i(\overline{g}^{-1}(\overline{y}))$. Note that since every user has the same delay, the multidimensional one step nonlinear map $\tilde{F}(\cdot)$ has the domain $\Re_+^N$ (as opposed to $\Re_+^{N^2}$ in subsection 5.1). The discrete time map corresponding to (15) for heterogeneous delay case is given by

$$\overline{y}_{n+1} = \tilde{F}(\overline{y}_n) \ , \ n \in Z_+ \tag{18}$$

where $\overline{y}_n \in \Re_+^N$, and $\tilde{F}_i(\overline{y}) = f_i \left(g_1^{-1}(y_1), ..., g_N^{-1}(y_N) \right)$.

Suppose that the multidimensional map $\tilde{F} : \Re_+^N \rightarrow \Re_+^N$ has some fixed point y^*, and that there is a sequence of closed, convex product spaces $\check{D}_n, n \geq 0$, such that $F(\check{D}_n) \subset int(\check{D}_{n+1}) \subset \check{D}_{n+1} \subset int(\check{D}_n)$ and $\cap_{n \geq 0} \check{D}_n = \{y^*\}$. Then, it is shown that if the initial functions $\phi \in \check{D}_0$, *i.e.*, $\overline{y}(s) \in \check{D}_0$ for all $-1 \leq s \leq 0$, then $\lim_{t \rightarrow \infty} \overline{y}(t) = \overline{y}^*$ for all $T > 0$ [17].

A key observation to be made here is the following: Consider a system consisting of a set of users and resources that satisfy Assumption 1. First, suppose that the users have the same delay T and the assumption in the previous paragraph holds for some sequence of $\check{D}_n, n \geq 0$, and hence the system is stable provided that the initial functions lie in $\check{D}_0$. Then, one can easily verify that even when the users have heterogeneous delays, the same sequence $D_n = \check{D}_n, n \geq 0$, satisfies Assumption 2 and, hence, the system is stable if the initial functions belong to $D_0 = \check{D}_0$. This observation is intuitive as follows. Suppose that in the homogeneous delay case, the delay of the flows is T, and in the heterogeneous delays case, the delays of the flows satisfy $T_i \leq T$ for all $i \in \mathcal{I}$. Then, since the communication delays of the users in the heterogeneous delay case are no larger than T of the homogeneous delay case, one would expect the system with heterogeneous delays to be stable if the system with homogeneous delay is stable. However, since our stability results for homogeneous delay case hold for any arbitrary T [17], one should expect the system with heterogeneous delays to be stable irrespective of $T_i, i \in \mathcal{I}$, as well.

6 An Application to a Family of Utility and Price Functions

In this section we apply the results in Section 5 to investigate the stability of the rate control problem described in Section 2. We consider the following class of users' utility functions:

$$U_a(x) = -\frac{1}{a}\frac{1}{x^a}\ ,\ a > 0. \tag{19}$$

In particular, $a = 1$ has been found useful for modeling the utility function of TCP algorithms [10]. This class of utility functions in (19) has been used extensively in engineering literature [1, 8, 10]. With the utility functions of the form in (19) one can easily show that the price elasticity of demand decreases with a as follows. Given a price per unit flow p, the optimal rate $x^*(p)$ of the user that maximizes the net utility $U_a(x) - p \cdot x$ is given by $p^{-\frac{1}{1+a}}$. The price elasticity of demand, which measures how responsive the demand is to a change in price, is defined to be the percent change in demand divided by the percent change in price [19]. In our case the price elasticity of demand is given by

$$\frac{p}{x^*(p)}\frac{dx^*(p)}{dp} = \frac{p}{p^{-\frac{1}{1+a}}} \cdot \frac{-1}{1+a} p^{-\frac{1}{1+a}-1} = \frac{-1}{1+a}\ . \tag{20}$$

Therefore, one can see that the price elasticity of demand decreases with a,[4] *i.e.,* the larger a is, the less responsive the demand is.

The class of resource price functions that we consider is of the form:

$$p(y) = \left(\frac{y}{C}\right)^b,\ \text{ where } b > 0 \tag{21}$$

This kind of marking function arises if the resource is modeled as an $M/M/1$ queue with a service rate of C packets per unit time and a packet receives a mark with a congestion indication signal if it arrives at the queue to find at least b packets in the queue. One can easily verify that these utility functions and resource price functions satisfy the assumptions in Section 5.

In order to establish the convergence of users' rates in a general network with heterogeneous delays, we use the observation stated in subsection 5.2, which states that the convergence of a system with arbitrary heterogeneous delays can be established by investigating the stability of the system with an arbitrary *homogeneous* delay. With the utility and resource price functions of (19) and (21), respectively, the underlying discrete time map from (18) for the homogeneous delay system is given by

$$\begin{aligned}
\frac{1}{x_{i,n+1}^{a_i}} &= x_{i,n}\left(\sum_{l \in r_i}\left(\frac{\sum_{j \in I_l} x_{j,n}}{C_l}\right)^{b_l}\right) \\
\Rightarrow x_{i,n+1} &= x_{i,n}^{-\frac{1}{a_i}}\left(\sum_{l \in r_i}\left(\frac{\sum_{j \in I_l} x_{j,n}}{C_l}\right)^{b_l}\right)^{-\frac{1}{a_i}} := \hat{F}_i(\overline{x}_n)
\end{aligned} \tag{22}$$

Note that $x_{i,n+1}$ is strictly decreasing in each of $x_{j,n}, j \in \mathcal{J}^i$.

We define $b^i_{max} = \max_{l \in r_i} b_l$ and $C^i = \min_{l \in r_i} C_l$ for all $i \in \mathcal{I}$, and assume that users are ordered by increasing a_i, *i.e.,* $a_1 \leq a_2 \leq \cdots \leq a_N$. Let x^* be the unique solution of the optimization problem in (5). We assume that $A^T x^* < C$. A sufficient condition for this is that $C_l > |I_l|$.

Suppose that $\tilde{D}_0 = \prod_{i=1}^N \tilde{D}_{0,i}$, where

$$\tilde{D}_{0,i} = [\overline{\beta} x_i^*\ ,\ \overline{\alpha} x_i^*]\ ,$$

[4] When comparing the price elasticity, typically the absolute value of (20) is used.

$\overline{\alpha}$ is some finite constant larger one, and $\overline{\beta}$ is a positive constant that satisfies

$$\hat{F}(\overline{\beta}x^*) < \overline{\alpha}x^* \text{ and } \overline{\beta}x^* < \hat{F}(\overline{\alpha}x^*) \; , \tag{23}$$

where $\hat{F}(\cdot) = g^{-1}(f(\cdot))$. Here we summarize the results shown in [17].

Lemma 1 *Suppose that $a_i > b^i_{max} + 1$ for all $i \in \mathcal{I}$. Define $\sigma = -\max_{i\in\mathcal{I}} \frac{b^i_{max}+1}{a_i} - \varepsilon$, where $0 < \varepsilon < 1 - \max_{i\in\mathcal{I}} \frac{b^i_{max}+1}{a_i}$. Then, any $\overline{\beta}$ such that $\overline{\alpha}^{1/\sigma} < \overline{\beta} < \overline{\alpha}^{\sigma}$ satisfies (23).*

We assume that $\overline{\beta}$ satisfies the condition in Lemma 1. Now, for $n = 1, 2, \cdots$, we define

$$\tilde{D}_n = \begin{cases} \prod_{i=1}^{N} [\overline{\alpha}^{\sigma^n} x_i^* \; , \; \overline{\beta}^{\sigma^n} x_i^*] \; , & n \text{ odd} \\ \prod_{i=1}^{N} [\overline{\beta}^{\sigma^n} x_i^* \; , \; \overline{\alpha}^{\sigma^n} x_i^*] \; , & n \text{ even} \end{cases} \tag{24}$$

Lemma 2 *Suppose that $a_i > b^i_{max} + 1$ for all $i \in \mathcal{I}$. Then, for all $n \geq 1$, $\hat{F}(\tilde{D}_{n-1}) \subset int(\tilde{D}_n) \subset \tilde{D}_n \subset int(\tilde{D}_{n-1})$, where $int(\tilde{D}_{n-1})$ is the interior of $\tilde{D}_{n-1}$, and $\cap_{n=0}^{\infty} \tilde{D}_n = \{x^*\}$.*

Theorem 3 *Suppose that $a_i > b^i_{max} + 1$ for all $i \in \mathcal{I}$. If the initial functions ϕ_x lie in $C([0, \; -1], \tilde{D}_0)$, then $x(t)$ produced by (16) converges to x^* asymptotically for all $T > 0$ and $\kappa_i > 0, i \in \mathcal{I}$.*

Now note that as $\overline{\alpha}$ increases, $\hat{F}(\overline{\alpha}x^*)$ goes to $\underline{0}$. Hence, since the rates of the users are in practice constrained from above by the link capacities and receiver buffer size, and limited from below from the fact that there is a lower bound on the transmission rate,[5] we can see that starting from any arbitrary rate vector satisfying the capacity constraint, the rates converge to x^* asymptotically from the above results.

Our results in this section state that if $a_i > b^i_{max} + 1$ for all $i \in \mathcal{I}$, then the users' rates converge to the solution of (5) regardless of the users' feedback delays T_i. In practice, if there is a lower bound on $a_i > 1$ of the end user algorithms, then the global stability of the system can be established by placing an upper bound on b_l of the resource price functions, regardless of the communication delays T_i of the users. Moreover, if all users and resources have the same parameters a and b, respectively, then, the condition $a > b + 1$ is again the necessary and sufficient condition for the stability of the system. The implications of a selection of utility and resource price functions is discussed in [17].

7 Discretized Models

The system of delay-differential equations given in (10) provides a good approximation to the end user behavior when the capacity of the link is high and/or the packet size is small, *e.g.,* ATM cells. However, in some cases, *e.g.,* low link capacity and/or large packet sizes, it may not be a very good approximation and the packet level dynamics may need to be modeled more accurately. In this section we introduce discretized models of (14) that capture the packet level dynamics more faithfully and characterize the conditions for asymptotic stability of these models.

[5] For instance, in the example of TCP the transmission rate of a connection cannot be smaller than one packet size divided by the round-trip time of the connection.

7.1 Basic Model

In this subsection we start with the simplest form of discretized models and extend it in the following subsections. Suppose that δ is a small positive constant that divides h. We replace the time derivative $\dot{\overline{y}}(t)$ in (14) with an approximation $\frac{\overline{y}(t+\delta)-\overline{y}(t)}{\delta}$, *i.e.,*

$$\nu\frac{\overline{y}(t+\delta)-\overline{y}(t)}{\delta} = \kappa(\overline{y}(t))\left(F(\overline{y}(t-m_1),\ldots,\overline{y}(t-m_N)) - \overline{y}(t)\right)$$

or

$$\overline{y}(t+\delta) = (I - h\cdot\delta\cdot\kappa(\overline{y}(t)))\overline{y}(t) + h\cdot\delta\cdot\kappa(\overline{y}(t))F(\overline{y}(t-m_1),\ldots,\overline{y}(t-m_N))\ . \tag{25}$$

Here δ can be thought of the interarrival times of the feedback signal (normalized by h). In other words, $t = n\cdot\delta, n\geq 0$, can be interpreted as the times at which the acknowledgments from the receiver carrying the feedback information arrive at the sender. The updates of transmission rates take place upon the arrival of acknowledgments. Here we assume that each acknowledgment contains the precise value of the price. The effects of finite granularity of feedback information is discussed in [18]. In addition, this model implicitly assumes that the time between updates δ is smaller than the common divisor h of the round-trip times of the connections. This assumption will be lifted in the following subsections.

Denote $\alpha = \delta^{-1}$. Letting $t = n\cdot\delta$, we rewrite the continuous time system in (25) as the following discretized system:

$$\overline{y}_{n+1} = \left(I - h\cdot\delta\cdot\kappa(\overline{y}_n)\right)\overline{y}_n + h\cdot\delta\cdot\kappa(\overline{y}_n)\cdot F(\tilde{y}_n)\ , \tag{26}$$

where I is the $N\times N$ identity matrix, and

$$\tilde{y}_n := \left(\overline{y}_{n-\alpha\cdot m_1},\ldots,\overline{y}_{n-\alpha\cdot m_N}\right)\ .$$

Assumption 3 *Suppose that $D\subset\Re^N_\epsilon$ is a closed, convex product space that is invariant under $F(\cdot)$ in (14). In addition, assume that*

$$h\cdot\delta\cdot\kappa_{ii}(\overline{y}) < 1 \tag{27}$$

for all $i\in\mathcal{I}$ and for all $\overline{y}\in D$.

This assumption has the following implications. First, for a given invariant set the time step δ has an upper bound $\frac{1}{h\cdot\sup_{\overline{y}\in D,\ i\in\mathcal{I}}\kappa_{ii}(\overline{y})}$. Also, as h increases, the interarrival time (normalized by h) δ needs to become smaller to satisfy the assumption. This is intuitive in the sense that as the normalizing constant h increases, in order to maintain the same interarrival time before normalization, δ must decrease proportionally. Since (14) is the limiting case as the packet size and the interarrival times go to zero, this assumption can be removed in that case. Second, the larger the invariant set is, the smaller the value of h should be due to the fact that $\sup_{\overline{y}\in D,\ i\in\mathcal{I}}\kappa_{ii}(\overline{y})$ does not decrease with increasing invariant set D.

Assumption 3 is required to guarantee that the state-dependent gain $\gamma(\overline{y},h,\delta) := I - h\cdot\delta\cdot\kappa(\overline{y}) > 0$ for all $i\in\mathcal{I}$ and for all $\overline{y}\in D$. Given $\{\overline{y}_0,\overline{y}_{-1},\ldots,\overline{y}_{-\alpha\cdot m_{max}}\in\Re^N_\epsilon\}$ the solution $\overline{y}_n,\ n>0$, can be computed by successively iterating the map given by (14). We prove that the dynamical behavior of $\overline{y}_n, n>0$, generated by (26) can be described by the corresponding properties of map $\overline{y}_{n+1} = F(\tilde{y}_n)$ in (15). In particular, similar to our results for the delay-differential equations it is shown that if $\overline{y}_n\in D, n = 0,-1,\ldots,-\alpha\cdot m_{max}$, then $\overline{y}_n\in D$ for all $n\geq 0$, provided that D is invariant under $F(\cdot)$. Moreover, if there is a globally attracting fixed point $\overline{y}^*$ of the map $F(\cdot)$, then every solution $\overline{y}_n$ of (26) satisfies $\lim_{n\to\infty}\overline{y}_n = \overline{y}^*$, hence giving a sufficient condition for global stability of the discretized system.

Theorem 4 *(Invariance) Let a closed, convex product space* $D \subset \Re_{\epsilon}^N$ *be invariant under* F *and* $\overline{y}_n \in D$, $n = 0, -1, \ldots, -\alpha \cdot m_{max}$. *Suppose that* $\gamma(\overline{y}, h, \delta) > 0$ *for all* $i \in \mathcal{I}$ *and for all* $\overline{y} \in D$. *Then,* $\overline{y}_n \in D$ *for all* $n \geq 0$.

Proof: This can be shown directly from (26) as follows. Suppose that the theorem is false and there exists some $n^{'}$ such that $\overline{y}_{n'} \notin D$. Let $\tilde{n} = \inf\{n \geq 0 \mid \overline{y}_n \notin D\}$. This leads to a contradiction because both $\overline{y}_{\tilde{n}-1}$ and $F(\tilde{y}_{\tilde{n}-1})$ are assumed to be in D and $\overline{y}_{\tilde{n}}$ is a convex combination of them, *i.e.*, $\overline{y}_{\tilde{n}} = \left(I - h \cdot \delta \cdot \kappa(\overline{y}_{\tilde{n}-1})\right) \overline{y}_{\tilde{n}-1} + h \cdot \delta \cdot \kappa(\overline{y}_{\tilde{n}-1}) \cdot F(\tilde{y}_{\tilde{n}-1})$. ■

Our next result establishes the global stability of the discretized system under the assumption that there exists an attracting fixed point of $F(\cdot)$.

Theorem 5 *(Global Stability) Suppose that there is a sequence of closed convex product spaces* $D_n, n \geq 0$, *such that* $F(D_n^N) \subset int(D_{n+1}) \subset D_{n+1} \subset int(D_n)$ *and* $\cap_{n \geq 0} D_n = \{y^*\}$, *where* y^* *is a stable fixed point of the map* $F(\cdot)$, *and that* $\overline{y}_n \in D_0$, $n = 0, -1, \ldots, -\alpha \cdot m_{max}$. *Assume that* $\gamma(\overline{y}, h, \delta) > 0$ *for all* $i \in \mathcal{I}$ *and for all* $\overline{y} \in D_0$. *Then,* $\overline{y}_n \in D_0$ *for all* $n \geq 0$ *and* $\lim_{n \to \infty} \overline{y}_n = \overline{y}^*$.

Proof: The proof is provided in Appendix C. ■

7.2 Exponential Stability

In this subsection we study the discretized systems in the previous subsection with the utility and resource price functions given by (19) and (21), respectively. Let $\mathcal{I} = \{1, \ldots, N\}$ be the set of users sharing the network and $\mathcal{L} = \{1, \ldots, L\}$ the set of resources. The price function of the resources is given by $p_l(x) = \left(\frac{x}{C_l}\right)^{b_l}$, $b_l > 0$, and the utility function of the users is given by $U_i(x) = \frac{-1}{a_i \cdot x^{a_i}}$. We assume that $a_i > b_{max}^i + 1 = \max_{l \in r_i} b_l + 1$ so that the system is stable.

Assumption 4 *Suppose that the conditions in Theorem 5 are satisfied with* $D_k = \overline{g}(\tilde{D}_k)$ *and*

$$\tilde{D}_k = \begin{cases} \prod_{i=1}^N [\overline{\alpha}^{\sigma^k} x_i^* \ , \ \overline{\beta}^{\sigma^k} x_i^*] \ , & k \text{ odd} \\ \prod_{i=1}^N [\overline{\beta}^{\sigma^k} x_i^* \ , \ \overline{\alpha}^{\sigma^k} x_i^*] \ , & k \text{ even} \end{cases} ,$$

where $\overline{\alpha} > 1$, $\overline{\beta}$ *satisfies Lemma 1, and* $\max_{i \in \mathcal{I}} \frac{b_{max}^i + 1}{a_i} < -\sigma < 1$.

Theorem 7.1 *Suppose that Assumption 4 holds. Then, for each step size* $\delta > 0$ *there exist* $K(\delta) > 0$ *and* $\psi(\delta) > 0$ *such that,*

$$|x_{n,i} - x_i^*| \leq K(\delta) \cdot exp(-\psi(\delta) \cdot n) \cdot x_i^* \qquad \text{for all } i \in \mathcal{I} \ . \tag{28}$$

Proof: The proof is provided in Appendix D. ■

Corollary 1 *Suppose that Assumptions 1, 2, and 4 with initial functions* $\phi_x \in C([-\max_{i \in \mathcal{I}} T_i, \ 0], \tilde{D}_0)$ *hold. Then, there exist* $K > 0$ *and* $\psi > 0$ *such that,*

$$|x_i(t) - x_i^*| \leq K \cdot exp(-\psi t) \cdot x_i^* \qquad \text{for all } i \in \mathcal{I} \ , \tag{29}$$

where $\overline{x}(t)$ *is the solution to (10).*

Proof: The proof is provided in Appendix E. ■

7.3 Asynchronous Updates

In (25) we have assumed that users update their rates at the same fixed rate, *i.e.,* the time between updates is the same for all users. However, in some cases it may be more realistic to assume that the time between updates may be different from one user to another. For example, under the currently most popular transport layer protocol, TCP, the congestion window size, which controls the number of outstanding packets and hence the transmission rate, is updated upon the receipt of acknowledgements (ACKs). Since the round-trip times of the users are different, which leads to unfairness in rate allocation [3], the rate at which users update their window sizes depends on their round-trip times and may be different. In this subsection we consider the case where users update their rates at fixed rates, but rates are allowed to vary from one user to another. The case where the time between updates is a random variable is considered in the following subsection.

For a fixed $\delta > 0$ in (25) we assume that user i's update period is given by $\Gamma_i \in \{1, 2, \ldots, m_i \cdot \alpha\}$, *i.e.,* user i updates its rate every $\Gamma_i \cdot \delta$ amount of time, starting at some random time $\tau_i \cdot \delta$, where $\tau_i \in \{0, 1, \ldots, m_i \cdot \alpha - 1\}$. Here the constant δ can be interpreted as the smallest time scale at which the dynamics of the network evolve. Since devices are driven by local oscillators with fixed frequencies, one can view δ as a common divisor of the periods of these local oscillators. Since typically each source updates its transmission rate at least once per round-trip, we assume that $\Gamma_i \leq m_i \cdot \alpha$. However, this assumption can be replaced by any finite positive integer. User i updates its rate at discretized times $n \in \{\tau_i + k \cdot \Gamma_i;\ k = 0, 1, \ldots\} := \mathcal{T}_i$. This model also removes an implicit assumption we have imposed in subsection 7.1 that the rate update period is smaller than h, a common divisor of the round-trip delays, because Γ_i can be larger than α.

In order to define a discrete time map similar to (26), we first need to define a family of discrete time multi-dimensional maps. Let

$$\Theta = \{(\xi_1, \ldots, \xi_N),\ \xi_i \in \{0, 1\}\}\ . \tag{30}$$

The vectors $\xi(n) \in \Theta$, $n = 0, 1, \ldots$, indicate which users update their rates at time n. In other words, if $\xi_i(n) = 1$, *i.e.,* $n \in \mathcal{T}_i$, then user i updates its rate at time n, and $\xi_i(n) = 0$ indicates that user i does not update its rate. Now we define a family of multi-dimensional map: for each $\xi \in \Theta$, define $F^{\xi}(\tilde{y}_n)$, where $\tilde{y}_n = (\overline{y}_n, \overline{y}_{n-\alpha\cdot m_1}, \ldots, \overline{y}_{n-\alpha\cdot m_N}) \in \Re^{N(N+1)}$ and

$$F_i^{\xi}(\tilde{y}_n) = \begin{cases} f_i(g^{-1}(\overline{y}_{n-\alpha\cdot m_i})) = F_i(\overline{y}_{n-\alpha\cdot m_1}, \ldots, \overline{y}_{n-\alpha\cdot m_N}) & \text{if } \xi_i = 1 \\ y_{n,i} & \text{otherwise} \end{cases} \tag{31}$$

The user rates now evolve according to

$$\overline{y}_{n+1} = (I - h \cdot \delta \cdot \kappa(\overline{y}_n))\, \overline{y}_n + h \cdot \delta \cdot \kappa(\overline{y}_n) \cdot F^{\xi(n)}(\tilde{y}_n)\ , \tag{32}$$

where I is the $N \times N$ identity matrix. In other words, for the users with $\xi_i(n) = 1$, the rates are updated as before, while for the remaining users the rates remain unchanged.

Theorem 7.2 *Suppose that the conditions in Theorem 5 hold, i.e., (i) there is a sequence of closed convex product spaces $D_n, n \geq 0$, such that $F(D_n^N) \subset int(D_{n+1}) \subset D_{n+1} \subset int(D_n)$ and $\{y^*\} = \cap_{n\geq 0} D_n$, (ii) $\overline{y}_n \in D_0$, $n = 0, -1, \ldots, -\alpha \cdot m_{max}$, and (iii) $\gamma(\overline{y}, h, \delta) > 0$ for all $i \in \mathcal{I}$ and for all $\overline{y} \in D_0$. Then, the sequence generated by (32) satisfies that $\overline{y}_n \in D_0$ for all $n \geq 0$ and $\lim_{n\to\infty} \overline{y}_n = \overline{y}^*$.*

Proof: The proof of the theorem is similar to those of Theorems 4 and 5, and is omitted. ∎

The above theorem establishes the convergence of the system when the update rates of the users are relaxed to be heterogeneous. In addition, if the utility and resource price functions are of (19) and (21), respectively, then one can prove the exponential stability similar to Theorem 7.1.

7.4 Random Update Times

In the previous subsection we have considered the case where users are allowed to update their transmission rates at different update rates. However, the time between the updates is fixed. In this subsection we extend the model and investigate the case where the update times of the transmission rates are not deterministic. Again, because each source typically updates the transmission rate at least once per round-trip, *e.g.,* TCP, we assume that there exists some finite upper bound on the time between two consecutive updates of a user's rate. We denote this upper bound for user i by Z_i. Here we assume that $Z_i \leq \alpha \cdot m_i$, although this assumption can be easily replaced with any finite positive integer. For each k, $k = 0, 1, \ldots$, we denote the time between k-th and $(k+1)$-st rate updates of user i by random variable (rv) V_k^i. The rvs V_k^i, $i \in \mathcal{I}$ and $k = 0, 1, \ldots$, are assumed to be independent of the past given the current system state, with some distribution $\mathcal{V}_k^i$ whose support is $\{1, 2, \cdots, Z_i\}$, *i.e.,* $\sum_{v=1}^{Z_i} P(V_k^i = v) = 1$. In fact, these distributions $\mathcal{V}_k^i$ can depend on the state of the system including the transmission rate of the users. The only crucial assumption required for the stability of the system is that these rvs are upper bounded by some constant. The rvs V_k^i capture the fact that in the Internet the interarrival times of acknowledgments (ACKs) at the source from the receiver of a flow are random, for instance due to ACK compression on the reverse path, and there may be some small random delays before the source can process the ACKs.

We assume that the first update of user i's rate takes place at $V_1^i \in \{1, \ldots, Z_i\}$, and the set of times at which user i updates its rate is denoted by $\mathcal{T}_i = \{\sum_{m=1}^{k} V_m^i;\ k = 1, 2, \ldots\}$. Under these assumptions the evolution of user rates is determined by

$$\overline{y}_{n+1} = (I - h \cdot \delta \cdot \kappa(\overline{y}_n))\, \overline{y}_n + h \cdot \delta \cdot \kappa(\overline{y}_n) \cdot F^{\xi(n)}(\tilde{y}_n) \ , \tag{33}$$

where $\tilde{y}_n = (\overline{y}_n, \overline{y}_{n-\alpha \cdot m_1}, \ldots, \overline{y}_{n-\alpha \cdot m_N}) \in \Re^{N(N+1)}$,

$$\xi_i(n) = \begin{cases} 1 & \text{if } n \in \mathcal{T}_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_i^{\xi}(\tilde{y}_n) = \begin{cases} f_i(g^{-1}(\overline{y}_{n-\alpha \cdot m_i})) = F_i(\overline{y}_{n-\alpha \cdot m_1}, \ldots, \overline{y}_{n-\alpha \cdot m_N}) & \text{if } \xi_i = 1 \\ y_{n,i} & \text{otherwise} \end{cases} . \tag{34}$$

Theorem 7.3 *Suppose that the conditions in Theorem 5 hold, i.e., (i) there is a sequence of closed convex product spaces $D_n, n \geq 0$, such that $F(D_n^N) \subset int(D_{n+1}) \subset D_{n+1} \subset int(D_n)$ and $\{y^*\} = \cap_{n \geq 0} D_n$, (ii) $\overline{y}_n \in D_0$, $n = 0, -1, \ldots, -\alpha \cdot m_{max}$, and (iii) $\gamma(\overline{y}, h, \delta) > 0$ for all $i \in \mathcal{I}$ and for all $\overline{y} \in D_0$. Then, the sequence generated by (33) satisfies that $\overline{y}_n \in D_0$ for all $n \geq 0$ and $\lim_{n \to \infty} \overline{y}_n = \overline{y}^*$.*

Proof: The proof is provided in Appendix F. ■

Similarly as in the previous subsection, if we take the utility and resource price functions given by (19) and (21), respectively, then we can establish the exponential stability of the system, following the same steps in the proofs.

8 Numerical Examples

In this section we present several numerical examples to illustrate the results in the previous sections. First, we consider the system given by (10) with utility and resource price functions given in (19) and (21). We demonstrate that under the stability condition given in Theorem 3 indeed the user rates converge to the solution of (5), while when the condition does not hold, for sufficiently large delays, the system becomes unstable. Second, we study the discretized system given by (25) and show the convergence of the user rates under Assumption 3. We also show that when (27) is violated, the system can become unstable where the user rates diverge from the optimal rates.

8.1 Delay-Differential System

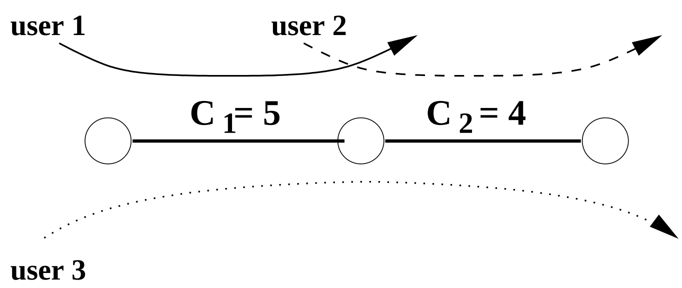


Figure 1: Topology of the example network.

In this subsection we consider a simple network consisting of two links. The first link is shared by users 1 and 3, while the second link is share by users 2 and 3. This is shown in Fig. 1. The capacities of the links are set to $[5\ 4]^T$. The utility functions are of the form in (19) with $a_1 = a_2 = 3$ and $a_3 = 4$, and the resource price functions are of (21). We select two different sets of parameters $b_l, l = 1, 2$, to create both a stable and an unstable system. The initial functions are given by $\phi(s) = [3\ 1\ 2]^T$ for all $s \in [-m_{max},\ 0]$. The gain parameters are set to $\kappa_i = 0.1$ for all users.

8.1.1 Stable System

The resource price parameters for the first case are $b_1 = b_2 = 1.9$. Since $a_i > b^i_{max} + 1$ for all users, the system is stable. One can solve the optimization problem in (5) and show that the solution is given by $x^* = [1.392\ 1.2809\ 1.0944]^T$. The feedback delays are given by $[T_1\ T_2\ T_3] = [28\ 43\ 77]$ unit times.

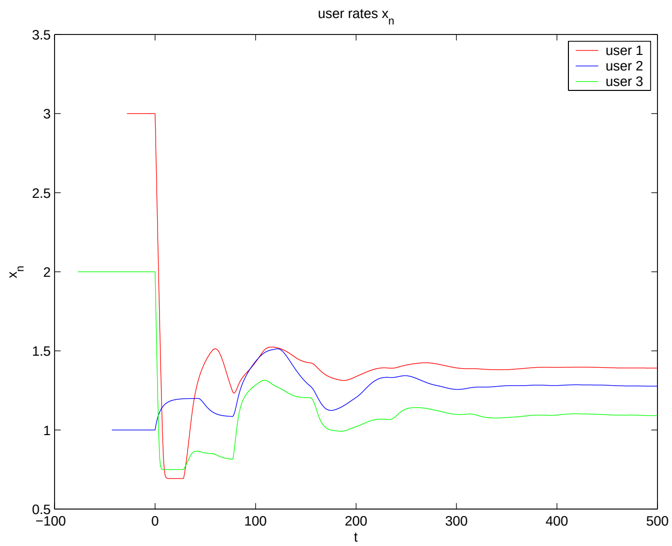


Figure 2: User rate evolution.

Fig. 2 plots the evolution of user rates according to (10). As one can see the user rates converge to the optimal rates.

8.1.2 Unstable System

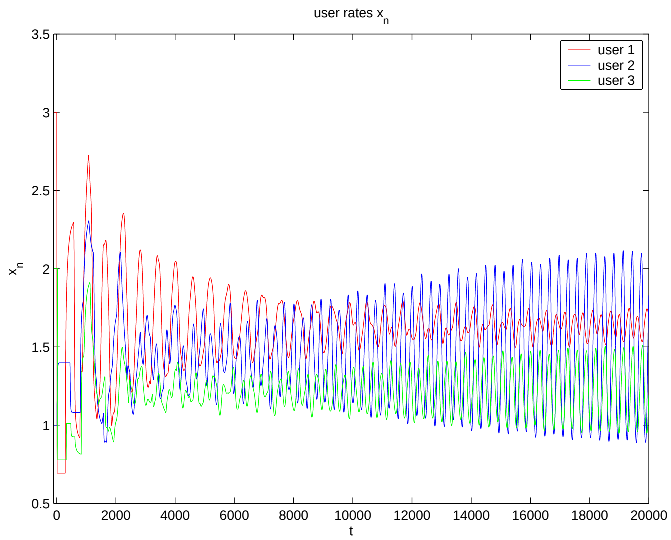


Figure 3: User rate evolution of an unstable system.

In the second case the resource price parameters are increased to $b_1 = b_2 = 3.5$. One can easily see that the stability conditions in Theorem 3 do not hold. We increase the feedback delays by ten folds from the previous case, *i.e.*, $[T_1\ T_2\ T_3] = [280\ 430\ 770]$ unit times. Fig. 3 shows the unstable behavior of the system as the user rates show no signs of settling down, resulting in large oscillations in the rates.

8.2 Discretized Systems

In this subsection we consider the discretized model of the system used in subsubsection 8.1.1. We set $\delta = 0.2$ or $\alpha = 5$. The rest of the parameters remain the same. Since the greatest common divisor of $T_i, i = 1, 2, 3$, is one, we set $h = 1$.

Fig. 4(a) shows the evolution of the user rates. Here we assume that $\tilde{D}_0 = \prod_{i=1}^{3} [\overline{\beta} x_i^*,\ \overline{\alpha} x_i^*]$, where $\overline{\alpha} = 1.5$ and $\overline{\beta} = 2/3$. The initial functions are set to $\overline{x}_n = [2\ 1\ 1.5]^T \in \tilde{D}_0$, $n = 0, -1, \ldots, -\alpha \cdot m_{max}$. One can verify that (27) holds as follows. From its definition the state dependent gains are given

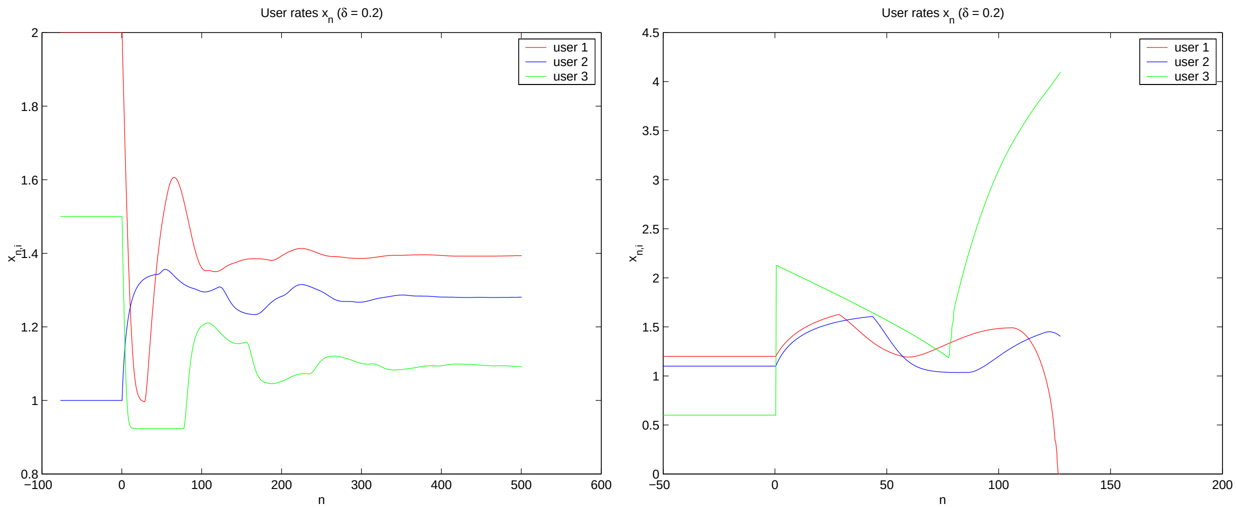
$$\kappa_{ii}(\overline{y}) = -\kappa_i g_i^{'}(g_i^{-1}(y_i)) = \kappa_i \frac{a_i}{x_i^{a_i+1}} \tag{35}$$

where we have used $x_i = g_i^{-1}(y_i)$. Therefore, the gain parameter is decreasing in x_i. Hence, it suffices to check the condition with $\overline{y} = \overline{\beta} x^*$.

In the second case we have expanded $\tilde{D}_0$ by fixing $\overline{\alpha} = 3$ and $\overline{\beta} = 0.33$ and set the initial functions to $\phi(s) = [1.2\ 1.1\ 0.6]^T$. From (35), we have

$$h \cdot \delta \cdot \kappa_{ii} = 0.2 \cdot 0.1 \cdot \frac{a_i}{x_i^{a_i+1}} \ .$$

For user 3, this equals $\frac{0.2 \cdot 0.1 \cdot 4}{0.6^5} = 1.0288$ at the initial value of 0.6, which violates the condition in Theorem 4. We plot the evolution of the user rates in Fig. 4(b). Clearly, the user rates do not converge to the optimal rates. In fact, the user rates diverge from the optimal rates due to the incorrect sign of the gains.



(a) (b)

Figure 4: User rate evolution in a discretized system.

9 General Models with Delays

In this section we build upon the simple model discussed in Section 5 and study the stability of the general network model described in Section 4. Recall that $T_{i,l}$ and $Z_{i,l}$, $i \in \mathcal{I}, l \in r_i$, denote the delay of the feedback signal from resource l to user i and the delay user i packets experience from the source to resource l, respectively, and they are assumed to be zero if user i does not traverse resource l, *i.e.*, $l \notin r_i$.

Define $y_i = x_i \cdot U_i^{'}(x_i) := g_i(x_i)$. Similarly as in Section 5 we assume that there exists some common divisor h such that, for all $i \in \mathcal{I}$ and $l \in \mathcal{L}$, $m_{i,l} = T_{i,l}/h$ and $z_{i,l} = Z_{i,l}/h$, where $m_{i,l}, z_{i,l} \in Z_+$. Let $d_{max} = \max_{i \in \mathcal{I}, l \in r_i}(\max_{j \in I_l} m_{i,l} + z_{j,l})$. By normalizing the time by h, we rewrite (9).

$$\nu \dot{y}_i(t) = \kappa_i g_i^{'}(g_i^{-1}(y_i(t))) \left(y_i(t) - \check{f}_i \left(g^{-1}(\overline{y}(t - m_i)), (g^{-1}(\hat{y}_{(i,l)}(t)), l \in r_i) \right) \right) \tag{36}$$

where $m_i = \frac{T_i}{h}$, $\hat{y}_{(i,l)}(t) = (y_j(t - z_{j,l} - m_{i,l}), j \in \mathcal{I})$,

$$\begin{aligned} & \check{f}_i \left(g^{-1}(\overline{y}(t - m_i)), (g^{-1}(\hat{y}_{(i,l)}(t)), l \in r_i) \right) \\ &= \check{f}_i \left(g^{-1}(\overline{y}(t - m_i)), g^{-1}(\hat{y}_{(i,l_{i,1})}(t)), \ldots, g^{-1}(\hat{y}_{(i,l_{i,R_i})}(t)) \right) \\ &= g_i^{-1}(y_i(t - m_i)) \left(\sum_{l \in r_i} p_l \left(\sum_{j \in I_l} g_j^{-1}(y_j(t - m_{i,l} - z_{j,l})) \right) \right) \end{aligned} \tag{37}$$

Let us define $Y^i(t) = (\overline{y}(t - m_i), \hat{y}_{i,l_{(i,1)}}(t), \ldots, \hat{y}_{i,l_{(i,R_i)}}(t)) \in \Re_\epsilon^{N \cdot (R_i + 1)}$, $Y(t) = (Y^1(t), \ldots, Y^N(t))$, and $G_i(Y^i(t)) = \left(g^{-1}(\overline{y}(t - m_i)), g^{-1}(\hat{y}_{(i,l_{i,1})}(t)), \ldots, g^{-1}(\hat{y}_{(i,l_{i,R_i})}(t)) \right)$. Now we can write (36) in the following matrix form:

$$\nu \dot{\overline{y}}(t) = \kappa(\overline{y}(t)) \left(\check{F}(Y(t)) - \overline{y}(t) \right) \tag{38}$$

where $\check{F}_i(Y(t)) = \check{f}_i(G_i(Y^i(t)))$, and $\kappa(\cdot)$ is a state dependent diagonal gain matrix with $\kappa_{ii}(\overline{y}(t)) = -\kappa_i g_i^{'}(g_i^{-1}(y_i(t)))$. We can define an underlying discrete time map, where the unit time is h:

$$\overline{y}_{n+1} = \check{F}(Y_n) \, , \tag{39}$$

where

$$n \in Z_+,\ \overline{y}_n = (y_1^n, \dots, y_N^n) \in \Re_+^N,\ Y_n = (Y_n^1, \dots, Y_n^N),\ Y_n^i = (\overline{y}_{n-m_i}, \hat{y}^n_{i,l_{i,1}}, \dots, \hat{y}^n_{i,l_{i,R_i}}),$$
$$\hat{y}^n_{i,l_{i,k}} = (y_1^{n-z_{1,l_{i,k}}-m_{i,l_{i,k}}}, \dots, y_N^{n-z_{N,l_{i,k}}-m_{i,l_{i,k}}}),\ \text{and}\ \check{F}_i(Y_n) = \check{f}_i(G_i(Y_n^i))\ .$$

Similarly as in Section 5 we use this discrete time map to study the stability of the system given by (38). The approach and steps taken for establishing the stability of the system in (38) are similar to those in Section 5.

We first make the following assumption on $g_i(\cdot), i \in \mathcal{I}$, and the resource price functions $p_l(\cdot), l \in \mathcal{L}$.

Assumption 5 *(i) The function $g_i(x_i)$ is strictly decreasing with $g_i^{'}(x_i) < 0$ for all $x_i > 0$, (ii) the price functions $p_l(x)$ are strictly increasing in x for all $l \in \mathcal{L}$, and (iii) $g_i(x_i)$ and $p_l(x)$ are Lipschitz continuous on $\Re_\epsilon$, where $\Re_\epsilon = [\epsilon, \infty)$ and ϵ is an arbitrarily small positive constant.*

This assumption ensures that the state dependent gain matrix $\kappa(\cdot)$ is a positive definite matrix. Let us first define the invariance and a fixed point of the map $\check{F}(\cdot)$.

Definition 3 *A set $D \subset \Re_\epsilon^N$ is said to be invariant under the discrete time map $\check{F}(\cdot)$ defined in (38) if $\check{F}(Y) \in D$ whenever $Y \in D^\Xi$, i.e., $Y = (Y^1, \dots, Y^N)$ and $Y^i \in D^{R_i+1}$ for all $i \in \mathcal{I}$. A vector $y^* \in \Re_\epsilon^N$ is said to be a fixed point of $\check{F}(\cdot)$ if $\check{F}((y^*, \dots, y^*)) = y^*$.*

We now state the assumption under which the asymptotic stability of (38) is established.

Assumption 6 *Multidimensional map $\check{F} : \Re_+^{N\cdot\Xi} \to \Re_+^N$ has a fixed point $y^* \in \Re_\epsilon^N$, where $g^{-1}(y^*)$ is the solution to (5). Also, assume that there is a sequence of closed, convex product spaces $D_k, k \geq 0$, such that $\check{F}(D_k^\Xi) \subset int(D_{k+1}) \subset D_{k+1} \subset int(D_k)$ and $\cap_{k\geq 0} D_k = \{y^*\}$.*

Let $Y_{D_0} = C([-d_{max}, 0], D_0)$ be a subset of initial functions and $\overline{y}^\nu_\phi(t)$ a solution of (38) constructed through $\phi \in Y_{D_0}$.

Theorem 9.1 *(Asymptotic Stability) All solutions $\overline{y}^\nu_\phi(t)$ starting with initial functions $\phi \in Y_{D_0}$ converge to y^* as $t \to \infty$ for all $\nu > 0$, $m_{i,l} \in Z_+$ and $z_{i,l} \in Z_+$.*

Proof: The proof is provided in Appendix G. ∎

Following the steps used in Section 7 one can define similar discretized models, and using these discretized models, derive the asymptotic and exponential stability of the corresponding systems presented in Section 7. Their proofs are simple extensions of the proofs in appendices for these results. Moreover, one can verify that if a system is stable with a homogeneous feedback delay as described in subsection 5.2, then the system with more general delays as described in this section is also stable with the appropriate initial condition. Therefore, this demonstrates that the *delay-independent* stability of the system studied in this paper depends critically on the selection of users' utility functions and resource price functions, but not on the detailed delays between network elements, *e.g.,* end users and resources. Hence, the stability of the system with a given set of users and network resources can be studied by considering a simple homogeneous delay system discussed in subsection 5.2 with suitable initial conditions.

10 Conclusions

In this paper we have studied the problem of designing a robust congestion control mechanism in the presence of arbitrary delays between the end users and network resources. We have demonstrated that the stability of the system given as a set of delay-differential equations can be studied by looking at the corresponding discrete time system. We have provided a condition for the stability of the system, which is applied to a family of popular utility and resource price functions to establish the stability of the system. We have also investigated the discretized models of the system given as delay-differential equations. These models model the packet level dynamics more accurately. We have shown that under some conditions, which depend on the region in which the initial values lie, the system is stable. We have also shown the exponential stability of the system with the family of utility and price functions.

Although the models described in Section 4 are general and capture the *fixed* delays between various elements in the network, they do not capture the time-varying nature of the delays. If the queue sizes are large, then the fluctuations in queueing delays and hence the feedback delays may be significant compared to the fixed propagation and transmission delays. We are currently working on extending our model to incorporate the time-varying feedback delays and to characterize similar stability conditions.

References

[1] R. Agrawal, A. Bedekar, R. J. La, and V. G. Subramanian. C3wpf scheduler. In *Proc. ITC*, 2001.

[2] T. Alpcan and T. Basar. A utility-based congestion control scheme for Internet-style networks with delay. In *Proc. of INFOCOM*, 2003.

[3] D. Chiu and R. Jain. Analysis of the Increase and Decrease Algorithms for Congestion Avoidance in Computer Networks. *Computer Networks and ISDN Systems*, 17:1–14, 1989.

[4] S. Deb and R. Srikant. Global stability of congestion controllers for the internet. In *Proc. of IEEE CDC*, 2002.

[5] A. F. Ivanov, M. A. Pinto, and S. I. Trofimchuk. Global behavior in nonlinear systems with delayed feedback. *Proc. Conference of Decison and Control, Sydney*, 2000.

[6] A. F. Ivanov and A. N. Sharkovsky. Oscillations in singularly perturbed delay equations. *Dynamics Reported*, 1:165–224, 1991.

[7] R. Johari and D. Tan. End-to-end congestion control for the internet: Delays and stability. *IEEE Transactions on Networking*, 2000.

[8] F. Kelly. Charging and rate control for elastic traffic. *European Transactions on Telecommunications*, 8(1):33–7, January 1997.

[9] F. Kelly, A. Maulloo, and D. Tan. Rate control for communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49(3):237–252, March 1998.

[10] S. Kunniyur and R. Srikant. End-to-end congestion control schemes: Utility functions, Random losses and ECN marks. In *Proc. of IEEE INFOCOM*, Tel-Aviv, Israel, 2000.

[11] S. Kunniyur and R. Srikant. A time scale decomposition approach to adaptive ecn marking. In *Proc. of IEEE INFOCOM*, Anchorage, AK, 2001.

[12] R. J. La and V. Anantharam. Utility-based rate control in the Internet for elastic traffic. *IEEE/ACM Transactions on Networking*, 10(2):272–86, April 2002.

[13] S. Low and D. Lapsley. Optimization flow control. *IEEE/ACM Transactions on Networking*, 7(6):861–874, June 1997.

[14] J. Mallet-Paret and R. D. Nussbaum. Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation. *Annali di Matematica Pura ed Applicata*, 145(4):33–128, 1986.

[15] F. Mazenc and S.-I. Niculescu. Remarks on the stability of a class of tcp-like congestion control models. In *Proc. of IEEE CDC*, 2003.

[16] J. Mo and J. Walrand. Fair end-to-end window-based congestion control. *IEEE/ACM Transactions on Networking*, 8(5):556–567, October 2000.

[17] P. Ranjan, E. H. Abed, and R. J. La. Delay, elasticity, and stability trade-offs in rate control. Technical report, ISR, University of Maryland at College Park, 2003.

[18] P. Tinnakornsrisuphap and Richard J. La. Characterization of queue fluctuations in probabilistic AQM mechanisms. available at http://www.enee.umd.edu/ hyongla, October 2003.

[19] Hal R. Varian. *Intermediate Microeconomics*. Norton, New York, 1996.

[20] E. I. Verriest and A. F. Ivanov. Robust stability of systems with delayed feedback. *Circuits, Systems and Signal Processing*, 13(2-3):213–222, 1994.

[21] Z. Wang and F. Paganini. Global stability with time-delay in network congestion control. In *Proc. of IEEE CDC 2002*, Las Vegas, 2002.

[22] J. Wen and M. Arcak. A unifying passivity framework for network flow control. In *Proc. of IEEE INFOCOM*, 2003.

A Proof of Theorem 1

We prove the theorem by contradiction. Suppose that the claim is false. Then, there exist some initial function $\phi \in X_D$ and $t,\ t \geq 0$, such that $y(t) \notin D$. Define

$$t_0 = \inf\{t \geq 0 \mid \text{ every interval } [t,\ t^{'}),\ t^{'} > t,\ \exists\, t_1,\ t < t_1 < t^{'},\text{ such that } \overline{y}(t_1) \notin D\}\ .$$

Then, there is $i \in \mathcal{I}$ such that for all $(t_0, t^{'})$, where $t^{'} > t_0$, there exists $\tilde{t}_0, t_0 < \tilde{t}_0 < t^{'}$, such that $y_i(\tilde{t}_0) \notin D_i := proj_i(D)$. We assume that $y_i(t)$ leaves through the right end, *i.e.*, $y_i(t_0) = \sup\ D_i$. Then, for all $(t_0, t^{'})$ there exists $\tilde{t}, t_0 < \tilde{t} < t^{'}$, such that $y_i(\tilde{t}) > \sup\ D_i$ and $\dot{y}_i(\tilde{t}) > 0$. This, however, leads to a contradiction as follows. From (14) we have $\nu\dot{y}_i(\tilde{t}) = \kappa_{ii}(\overline{y}(\tilde{t}))\left(f_i(\overline{g}^{-1}(\overline{y}(\tilde{t} - m_i))) - y_i(\tilde{t})\right) < 0$ because $\kappa_{ii}(\overline{y}(\tilde{t})) > 0$ and $f_i(\overline{g}^{-1}(\overline{y}(\tilde{t} - m_i))) \in D_i$ and, hence, is less than or equal to $\sup D_i\ (< y_i(\tilde{t}))$, which contradicts the earlier assumption that $\dot{y}_i(\tilde{t}) > 0$. The other case that $y_i(t)$ leaves D_i through the left end, *i.e.*, $y_i(t_0) = \inf\ D_i$, can be shown to lead to a similar contradiction. Therefore, the theorem follows.

B Proof of Theorem 2

In this proof we omit the subscript ϕ since there is no confusion. Our proof utilizes the following lemmas.

Lemma 3 *Fix $n, n \geq 0$. Let $\tilde{D}$ be an open, convex product space that contains $F(D_n^N)$ and whose closure is contained in $int(D_n)$, i.e., $cl(\tilde{D}) \subset int(D_n)$. Suppose that the initial functions $\phi \in C([-m_{max}, 0], D_n)$. Then, there exists a finite $\tilde{t},\ \tilde{t} \geq 0$, such that, for all $t \geq \tilde{t}$, $y(t) \in \tilde{D}$.*

In order to prove the lemma, we first prove the following coordinate-wise invariance

Lemma 4 *(Coordinate-wise Invariance) If $y_i(\tilde{t}) \in \tilde{D}_i = proj_i(\tilde{D})$ for some $\tilde{t} \geq 0$, then $y_i(t) \in \tilde{D}_i$ for all $t \geq \tilde{t}$.*

Proof: Suppose that the lemma is not true, and there exists $\bar{t} > \tilde{t}$ at which $y_i(\bar{t}) = \inf \tilde{D}_i$ or $y_i(\bar{t}) = \sup \tilde{D}_i$. We assume that $\bar{t}$ is the smallest such time and $y_i(\bar{t}) = \sup \tilde{D}_i > \sup proj_i(F(D_n^N))$. Then, we can find $\bar{t}_1 < \bar{t}$ such that for all $t \in (\bar{t}_1, \bar{t})$, $y_i(t) \in \tilde{D}_i \setminus proj_i(F(D_n^N))$. This implies that $\dot{y}_i(t) < 0$ for all $t \in (\bar{t}_1, \bar{t})$ from (14) because $F_i((\overline{y}(t - m_i), i \in \mathcal{I})) < \sup proj_i(F(D_n^N))$ and, thus, $y_i(\bar{t}) < \sup \tilde{D}_i$, leading to a contradiction. A similar argument can be used for the case $y_i(\bar{t}) = \inf \tilde{D}_i$. ∎

Now let us proceed with the proof of Lemma 3.

Proof: Suppose that the lemma is false. Then, from Lemma 4 there exists $i \in \mathcal{I}$ such that for all $t \geq 0$, $y_i(t) \notin proj_i(\tilde{D}) = \tilde{D}_i$. We show that this leads to a contradiction. Suppose that $y_i(t) \geq \sup \tilde{D}_i$ for all $t \geq 0$. Then, one can see that $\dot{y}_i(t) < 0$ from (14). Combined with the assumption that $y_i(t) \geq \sup \tilde{D}_i$ for all $t \geq 0$, this implies that $y_i(t)$ converges to some $\tilde{y}_i \geq \sup \tilde{D}_i$. Since $\sup \tilde{D}_i > \sup proj_i(F(D^N))$ with $\delta := \sup \tilde{D}_i - \sup proj_i(F(D^N)) > 0$, there exists some positive constant ε such that $\dot{y}_i(t) \leq -\varepsilon \cdot \delta < 0$ for all sufficiently large t from (14). This, however, implies that $y_i(t) \downarrow -\infty$ as $t \uparrow \infty$, contradicting the assumption that $y_i(t) \geq \sup \tilde{D}_i$ for all $t \geq 0$. A similar contradiction can be shown when we assume $y_i(t) \leq \inf \tilde{D}_i$ for all $t \geq 0$. This completes the proof of the lemma. ∎

Lemma 5 *Let D be a closed, invariant product space and $\tilde{D}$ an open, convex product space that contains $F(D^N)$ and whose closure is contained in $int(D)$, i.e., $cl(\tilde{D}) \subset int(D)$. Suppose that the initial functions $\phi \in C([-m_{max}, 0], D)$ and $y(t_1) \in \tilde{D}$ for some $t_1 \geq 0$. Then, $y(t) \in \tilde{D}$ for all $t \in [t_1, t_1 + m_{max}]$.*

Proof: The lemma follows directly from Lemma 4. ∎

Now we are ready to proceed with the proof of Theorem 2. By repeatedly applying Lemmas 3 and 5 and Theorem 1 one can find a sequence of finite $t_n, n = 1, 2, \ldots$, such that $y(t) \in D_n$ for all $t \geq t_n$. The theorem now follows from the assumption that $\bigcap_{n=1}^{\infty} D_n = \{y^*\}$.

C Proof of Theorem 5

First, the invariance property that $\overline{y}_n \in D_0$ for all $n \geq 0$ follows directly from Theorem 4.

The asymptotic convergence of $\overline{y}_n$ can be shown as follows. First, note from (26) that if $\overline{y}_n \in D_k, n = 0, -1, \cdots, -\alpha \cdot m_{max}$ and $y_{n',i} \in D_{k+1,i}$ for some $n' \geq 1$, then, $y_{n,i} \in D_{k+1,i}$ for all $n \geq n'$. Hence, it suffices to show that, for all $k \geq 1$, if $\overline{y}_n \in D_k$ for $n = 0, -1, \ldots, -\alpha \cdot m_{max}$, then there exists a finite $M(k)$ such that, for all $n \geq M(k)$, $\overline{y}_n \in D_{k+1}$.

We prove this claim by contradiction. If the claim is not true, then there exists some $k \geq 1$ and $i \in \mathcal{I}$ such that $y_{n,i} \notin D_{k+1,i} = proj_i(D_{k+1})$ for all $n \geq 0$. We assume that $y_{n,i} > \sup D_{k+1,i}$. The other case $y_{n,i} < \inf D_{k+1,i}$ can be handled similarly. From the assumption that $F(D_k) \subset int(D_{k+1})$, one can easily see that there exists some positive constant ε such that, for all $n \geq 0$, $\sup D_{k+1,i} - F_i(\overline{y}_n) \geq \varepsilon$ because $F(\overline{y}_n) \in F(D_k)$. Thus, if $y_{n,i} > \sup D_{k+1,i}$ for all $n \geq 0$, then from (26) this implies that there exists $\varepsilon' > 0$ such that $y_{n+1,i} - y_{n,i} \leq -\varepsilon' < 0$ and, hence, $y_{n,i} \to -\infty$ as $n \to \infty$, contradicting the earlier assumption that $y_{n,i} > \sup D_{k+1,i}$ for all $n \geq 0$. Thus, there exists some finite $M(k)$ such that $\overline{y}_n \in D_{k+1}$ for all $n \geq M(k)$.

D Proof of Theorem 7.1

We first show that there exist some $K_1 > 0$ and $\psi_1 > 0$ such that, for all $k \geq 1$,

$$1 - \overline{\beta}^{\sigma^{2k}} \leq K_1 exp(-\psi_1 \cdot 2k) \;\text{ and }\; \overline{\alpha}^{\sigma^{2k}} - 1 \leq K_1 exp(-\psi_1 \cdot 2k) \tag{40}$$

and

$$1 - \overline{\alpha}^{\sigma^{2k-1}} \leq K_1 exp(-\psi_1(2k-1)) \text{ and } \overline{\beta}^{\sigma^{2k-1}} - 1 \leq K_1 exp(-\psi_1(2k-1)) \ . \tag{41}$$

Lemma 6 *Suppose that* $0 < \psi_1 \leq \log(-\sigma^{-1})$ *and* $K_1 = exp(\psi_1) \cdot (exp(\overline{\alpha}^{-\sigma^{-1}} - 1) - 1)$. *Then, (40) and (41) hold.*

Proof: We first prove (40). Define $K^{'} = exp(\overline{\alpha}^{-\sigma^{-1}} - 1) - 1 = K_1 \cdot exp(-\psi_1) < K_1$. Let us first prove the second condition in (40) holds. Add one to both sides and then take the natural logarithm $\log(\cdot)$ of both sides:

$$\begin{aligned}
\log(\overline{\alpha})\sigma^{2k} &= \log(\overline{\alpha}) exp(-2k \log(-\sigma^{-1})) \\
&\leq (\overline{\alpha} - 1) exp(-2k \log(-\sigma^{-1})) \\
&\leq \log(1 + (exp(\overline{\alpha} - 1) - 1) exp(-2k \log(-\sigma^{-1}))) \\
&\leq \log(1 + K^{'} exp(-2k\psi_1)) \\
&\leq \log(1 + K_1 exp(-2k\psi_1)) \ ,
\end{aligned}$$

where the first inequality follows from $\log(1 + x) \leq x$, and the second inequality can be seen from Fig. 5 with $K^{''} = exp(\overline{\alpha} - 1) - 1 < K^{'}$ and the fact that $\log(-\sigma^{-1}) > 0$ because $-\sigma^{-1} > 1$.

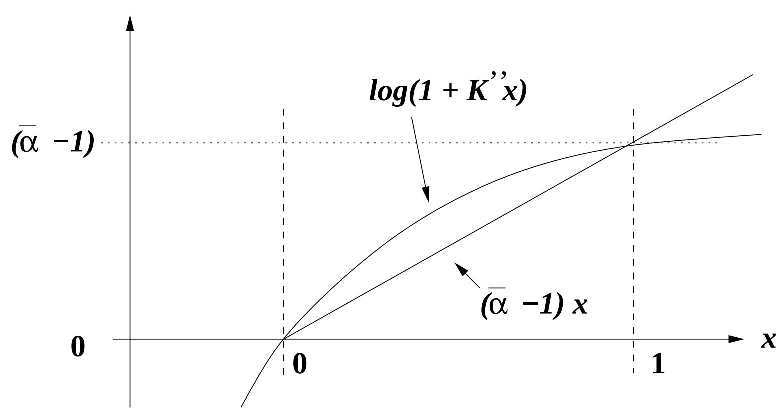


Figure 5: Proof of Lemma 6.

Similarly, in order to show that the first condition in (40) holds, we show that

$$\overline{\beta}^{\sigma^{2k}} \geq 1 - K_1 exp(-2k\psi_1) \ . \tag{42}$$

If we take the natural logarithm of the left-hand side,

$$\begin{aligned}
\sigma^{2k} \log(\overline{\beta}) &= \log(\overline{\beta}) exp(-2k \log(-\sigma^{-1})) \\
&= -\log(\overline{\beta}^{-1}) exp(-2k \log(-\sigma^{-1})) \ ,
\end{aligned}$$

and the right-hand side yields

$$\log(1 - K_1 exp(-2k\psi_1)) = -\log\left((1 - K_1 exp(-2k\psi_1))^{-1}\right) \ .$$

Hence, it suffices to show that

$$\log(\overline{\beta}^{-1}) exp(-2k \log(-\sigma^{-1})) \leq \log\left((1 - K_1 exp(-2k\psi_1))^{-1}\right) \ . \tag{43}$$

We can use a series of inequalities to show (43) as follows.

$$\begin{aligned}\log(\overline{\beta}^{-1})exp(-2k\log(-\sigma^{-1})) &\leq (\overline{\beta}^{-1}-1)exp(-2k\log(-\sigma^{-1}))\\ &\leq \log\left(1+(exp(\overline{\beta}^{-1}-1)-1)exp(-2k\log(-\sigma^{-1}))\right)\\ &\leq \log(1+K^{'}exp(-2k\psi_1))\\ &\leq \log(1+K_1 exp(-2k\psi_1))\\ &\leq \log\left((1-K_1 exp(-2k\psi_1))^{-1}\right)\end{aligned}$$

where the second inequality follows from Fig. 5 with $K^{''}=exp(\overline{\beta}^{-1}-1)-1$ and $\overline{\alpha}$ replaced by $\overline{\beta}^{-1}$, and the third inequality is a consequence of $\overline{\beta}^{-1}<(\overline{\alpha}^{\sigma^{-1}})^{-1}=\overline{\alpha}^{-\sigma^{-1}}$ from the assumption on $\overline{\beta}$ in Lemma 1.

Conditions in (41) follow from the fact that $\overline{\beta}^{\sigma^{2k-2}}<\overline{\alpha}^{\sigma^{2k-1}}<\overline{\beta}^{\sigma^{2k-1}}<\overline{\alpha}^{\sigma^{2k-2}}$, which is a consequence of Lemma 2. Here we only prove the second condition in (41). The first condition can be proved similarly. First, note that $\overline{\beta}^{\sigma^{2k-1}}<(\overline{\alpha}^{\sigma^{-1}})^{\sigma^{2k-1}}=\overline{\alpha}^{\sigma^{-1}\cdot\sigma^{2k-1}}=\overline{\alpha}^{\sigma^{2k-2}}$. We add one to the left-hand side of the second condition in (41) and then take the natural logarithm:

$$\begin{aligned}\log(\overline{\beta})\sigma^{2k-1} &< \log(\overline{\alpha})\sigma^{2k-2}\\ &=\log(\overline{\alpha})exp(-2(k-1)\log(\sigma^{-1}))\\ &\leq (\overline{\alpha}-1)exp(-2(k-1)\log(\sigma^{-1}))\\ &\leq \log\left(1+(exp(\overline{\alpha}-1)-1)exp\left(-2(k-1)\log(\sigma^{-1})\right)\right)\\ &\leq \log\left(1+K^{'}exp(-2(k-1)\psi_1)\right)\\ &=\log\left(1+K^{'}exp(\psi)exp(-(2k-1)\psi_1)\right)\\ &=\log\left(1+K_1 exp(-(2k-1)\psi_1)\right)\ .\end{aligned}$$

Hence, the condition is satisfied. ■

Now in order to complete the proof of the theorem, it suffices to show that there exists a finite positive integer M such that, for all $k\geq 0$, if $\overline{x}_n:=(x_{n,1},\ldots,x_{n,N})\in\tilde{D}_k, n=0,-1,\cdots,-\alpha\cdot m_{max}$, then $\overline{x}_n\in\tilde{D}_{k+1}$ for all $n\geq M$. This implies that $K(\delta)=K_1\cdot exp(\psi_1)$ and $\psi(\delta)=\frac{\psi_1}{M+\alpha\cdot m_{max}}$ satisfy (28), where K_1 and ψ_1 are from Lemma 6. First, note from (26) and Lemma 2 that if $\overline{x}_n\in\tilde{D}_k$ and $x_{n,i}>\sup\ \tilde{D}_{k,i}$, then $x_{n+1,i}\geq\inf\ \tilde{D}_{k+1,i}$, and similarly, if $\overline{x}_n\in\tilde{D}_k$ and $x_{n,i}<\inf\ \tilde{D}_{k,i}$, then $x_{n+1,i}\leq\sup\ \tilde{D}_{k+1,i}$. We prove the existence of such M in two steps:

(i) Suppose that $\overline{x}_n\in\tilde{D}_k$, $n=0,-1,\ldots,-\alpha\cdot m_{max}$. If $x_{0,i}\notin\tilde{D}_{k+1,i}:=proj_i(\tilde{D}_{k+1})$, where $proj_i(\cdot)$ denotes the i-th component projection operator, the number of steps it takes for $x_{n,i}$ to get inside $\tilde{D}_{k+1,i}$ is upper bounded by some finite $M(k)$ for all $i\in\mathcal{I}$, *i.e.*,

$$\inf\{n\geq 1\mid \overline{x}_n\in\tilde{D}_{k+1}\}\leq M(k)<\infty\ .$$

(ii) One can find a non-increasing sequence of $M(k), k\geq 0$, that satisfies (i).

We first prove (i). We assume that k is even and $x_{0,i}>\sup\ \tilde{D}_{k+1,i}$. Other cases can be handled similarly. In order to prove (i), we show that if $x_{n,i}>\sup\ \tilde{D}_{k+1,i}$, then there exists some constant $\varsigma>0$ such that $x_{n,i}-\hat{F}_i(\tilde{x}_n)>\overline{\beta}^{\sigma^{2k+1}}x_i^*-\hat{F}_i(\tilde{x}_n)>\varsigma\cdot x_i^*$, where $\tilde{x}_n=(\overline{x}_{n-\alpha\cdot m_1},\ldots,\overline{x}_{n-\alpha\cdot m_N})$ and

$$\hat{F}_i(\tilde{x}_n)=(x_{n-\alpha\cdot m_i,i})^{-1/a_i}\left(\sum_{l\in r_i}\left(\frac{\sum_{j\in I_l}x_{n-\alpha\cdot m_i,j}}{C_l}\right)^{b_l}\right)^{-1/a_i}\ .$$

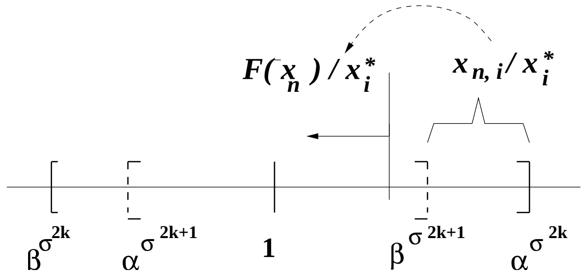


Figure 6: Illustration.

The existence of such a constant ς can be shown as follows.

$$
\begin{aligned}
\hat{F}_i(\tilde{x}_n) &= (x_{n-\alpha\cdot m_i,i})^{-1/a_i}\left(\sum_{l\in r_i}\left(\frac{\sum_{j\in I_l} x_{n-\alpha\cdot m_i,j}}{C_l}\right)^{b_l}\right)^{-1/a_i} \\
&\le (\overline{\beta}^{\sigma^{2k+1}} x_i^*)^{-1/a_i}\left(\sum_{l\in r_i}\left(\frac{\sum_{j\in I_l} x_{n-\alpha\cdot m_i,j}}{C_l}\right)^{b_l}\right)^{-1/a_i} \\
&\le (\overline{\beta}^{\sigma^{2k+1}} x_i^*)^{-1/a_i}\left(\sum_{l\in r_i}\left(\frac{\sum_{j\in I_l} \overline{\beta}^{\sigma^{2k}} x_j^*}{C_l}\right)^{b_l}\right)^{-1/a_i} \\
&\le (\overline{\beta}^{\sigma^{2k+1}} x_i^*)^{-1/a_i}\left(\sum_{l\in r_i}\left(\overline{\beta}^{\sigma^{2k}}\right)^{b^i_{max}}\left(\frac{\sum_{j\in I_l} x_j^*}{C_l}\right)^{b_l}\right)^{-1/a_i} \\
&= \overline{\beta}^{-\sigma^{2k+1}/a_i}(x_i^*)^{-1/a_i}\left(\overline{\beta}^{\sigma^{2k}}\right)^{-b^i_{max}/a_i}\left(\sum_{l\in r_i}\left(\frac{\sum_{j\in I_l} x_j^*}{C_l}\right)^{b_l}\right)^{-1/a_i} \\
&= \overline{\beta}^{-\sigma^{2k+1}/a_i}\cdot\overline{\beta}^{-\sigma^{2k}\cdot b^i_{max}/a_i} x_i^* \\
&< (\overline{\beta}^{-1})^{\sigma^{2k} b^i_{max}/a_i} x_i^* && \text{(because } \overline{\beta}^{-\sigma^{2k+1}/a_i} < 1\text{)} \\
&< (\overline{\beta}^{-1})^{\sigma^{2k}(-\sigma-\frac{1}{a_i})} x_i^* && \text{(because } -1<\sigma< -\tfrac{b^i_{max}+1}{a_i}\text{)} \\
&= \overline{\beta}^{\sigma^{2k}(\sigma+\frac{1}{a_i})} x_i^* \\
&= \overline{\beta}^{\sigma^{2k+1}}\overline{\beta}^{\sigma^{2k}/a_i} x_i^*
\end{aligned}
$$

where the first three inequalities follow from the monotonicity of the map $\hat{F}(\cdot)$, the assumption that $x_{n,i} > \sup\ \tilde{D}_{k+1} = \overline{\beta}^{\sigma^{2k+1}} x_i^*$, and $\overline{\beta} < 1$. Note that $\overline{\beta}^{\sigma^{2k}/a_i} < 1$ because $\overline{\beta} < 1$. Therefore,

$$
\overline{\beta}^{\sigma^{2k+1}} x_i^* - \hat{F}_i(\tilde{x}_n) > \overline{\beta}^{\sigma^{2k+1}} x_i^* - \overline{\beta}^{\sigma^{2k+1}}\overline{\beta}^{\sigma^{2k}/a_i} x_i^* \tag{44}
$$

and

$$
\begin{aligned}
\inf\{n\ge 1 \mid x_{n,i}\in\tilde{D}_{k+1,i}\} &\le \frac{\overline{\alpha}^{\sigma^{2k}} - \overline{\beta}^{\sigma^{2k+1}}}{h\cdot\delta\cdot\varepsilon\cdot(\overline{\beta}^{\sigma^{2k+1}} - \overline{\beta}^{\sigma^{2k+1}}\overline{\beta}^{\sigma^{2k}/a_i})} \\
&= \frac{\overline{\alpha}^{\sigma^{2k}} - \overline{\beta}^{\sigma^{2k+1}}}{h\cdot\delta\cdot\varepsilon\cdot\overline{\beta}^{\sigma^{2k+1}}(1-\overline{\beta}^{\sigma^{2k}/a_i})}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\overline{\alpha}^{\sigma^{2k}} - (\overline{\alpha}^{\sigma})^{\sigma^{2k+1}}}{h \cdot \delta \cdot \varepsilon \cdot (\overline{\alpha}^{\sigma})^{\sigma^{2k+1}} (1 - (\overline{\alpha}^{\sigma})^{\sigma^{2k}/a_i})} \\
&= \frac{\overline{\alpha}^{\sigma^{2k}} - \overline{\alpha}^{\sigma^{2k+2}}}{h \cdot \delta \cdot \varepsilon \cdot \overline{\alpha}^{\sigma^{2k+2}} (1 - \overline{\alpha}^{\sigma^{2k+1}/a_i})} \\
&:= M_i(k) \ ,
\end{aligned} \tag{45}
$$

where the first inequality is a consequence of (44), the second inequality follows from the assumption $\overline{\beta} < \overline{\alpha}^{\sigma} < 1$, and ε is some positive constant that satisfies $\inf_{\overline{x} \in \tilde{D}_k} \kappa_{ii}(\overline{g}(\overline{x})) \ge \varepsilon$. The existence of such ε is guaranteed because $\tilde{D}_k$ is a compact set. If we define $M(k) = \max_{i \in \mathcal{I}} M_i(k)$, then $M(k)$ satisfies (i).

Claim (ii) can be established by showing that $M_i(k)$ is non-increasing in $k, k \ge 0$. Let $\xi(k) = \overline{\alpha}^{\sigma^{2k}}$, which is decreasing in k, and rewrite (45) as

$$
M_i(k) = \frac{1}{h \cdot \delta \cdot \varepsilon} \frac{\xi(k) - \xi(k)^{\sigma^2}}{\xi(k)^{\sigma^2} (1 - \xi(k)^{\sigma/a_i})} \ ,
$$

where ε is a positive constant satisfying $\inf_{\overline{x} \in \tilde{D}_0} \kappa_{ii}(\overline{g}(\overline{x})) \ge \varepsilon$. Then, one can show that $M_i(k)$ is non-decreasing in $\xi(k), \ \xi(k) > 1$. This completes the proof of (ii).

E Proof of Corollary 1

Note that $M_i(k) \propto \delta^{-1}$ from the proof of Theorem 7.1, *i.e.*, the number of steps n it takes for $x_{n,i}$ to get inside $\tilde{D}_{k+1,i}$ is inversely proportional to the step size δ. Hence $\delta \cdot M_i(k)$ is constant irrespective of δ. Now, the corollary is a simple consequence of the fact that (14) is a limit of (25) as $\delta \downarrow 0$.

F Proof of Theorem 7.3

The general basic idea of the proof is similar to that of Theorem 5. An invariant set and a fixed point of the map $F^{\xi}(\cdot)$ are defined similarly as in subsection 5.1 for the map $F(\cdot)$. First, one can establish the invariance property given in Theorem 4, following the same steps in its proof. Second, following the same argument in the proof of Theorem 5 one can show the following. For all $k \in \{0, 1, \dots\}$, (i) from (33) and (34) if $\overline{y}_n \in D_k, n = 0, -1, \cdots, -\alpha \cdot m_{max}$ and $y_{n',i} \in D_{k+1,i}$ for some $n' \ge 1$, then, $y_{n,i} \in D_{k+1,i}$ for all $n \ge n'$, and (ii) there exists a finite $M(k)$ such that, for all $n^{'} \ge M(k)$, $\overline{y}_{n'} \in D_{k+1}$. We first prove (i). Suppose that (ii) is false. Then, by (i) this implies that there exists some $k \ge 0$ and $i \in \mathcal{I}$ such that $y_{n,i} \notin D_{k+1,i} = proj(D_{k+1})$ for all $n \ge 0$. Following the similar steps in the proof of Theorem 5 one can draw a contradiction as follows. We assume that $y_{n,i} > \sup \ D_{k+1,i}$ for all $n = 0, 1, \dots$. From the assumption that $F(D_k) \subset int(D_{k+1})$ there is some positive constant ε such that, for all $n \ge 0$, $\sup \ D_{k+1,i} - F_i(\overline{y}_n) \ge \varepsilon$ because $F(\overline{y}_n) \in F(D_k)$. This implies that there is some positive constant $\varepsilon^{'}$ such that, for all $n \in \mathcal{T}_i$, $y_{n+1,i} - y_{n,i} \le -\varepsilon^{'} < 0$. Therefore, from the assumption that there is an upper bound on the time between updates, one can see that $y_{n,i} \to -\infty$ as $n \to \infty$, contradicting the earlier assumption that $y_{n,i} > \sup \ D_{k+1}$ for all $n \ge 0$. The other case that $y_{n,i} < \inf \ D_{k+1}$ can be shown to lead to a similar contradiction. Now, since we can find a finite $M(k)$ for all $k \ge 0$, one can construct a sequence $n(k) = \sum_{l=0}^{k} (\alpha \cdot m_{max} + M(l))$ such that, for all $n \ge n(k)$, $\overline{y}_n \in D_{k+1}$, and the convergence follows from the assumption that $\cap_{k \ge 0} D_k = \{y^*\}$.

G Proof of Theorem 9.1

The basic idea of the proof is similar to that of Theorem 2. We first show that the invariance property of the discrete time map also implies the invariance property of (38). Then, we prove that Assumption 6 implies the stability of (38) as well. We use the following three lemmas.

Lemma 7 *(Invariance) Suppose that $D \subset \Re^N_\epsilon$ is a closed, convex invariant product space under $\check{F}(\cdot)$, i.e., $D = \prod_{i=1}^N proj_i(D)$, where $proj_i(\cdot)$ denotes the i-th component projection operator. Then, for any initial function $\phi \in C([-d_{max}, 0], D) := X_D$ the resulting $\overline{y}(t)$ from (38) belongs to the set D for all $t \geq 0$ and $\nu > 0$.*

Proof: We prove the lemma by contradiction. Suppose that the lemma is false. Then, there exist some initial function $\phi \in X_D$ and $t,\ t \geq 0$, such that $y(t) \notin D$. Define

$$t_0 = \inf\{t \geq 0 \mid \text{ for every interval } [t,\ t^{'}),\text{ where } t^{'} > t,\ \exists\, t_1,\ t < t_1 < t^{'},\text{ such that } \overline{y}(t_1) \notin D\}\ .$$

Then, there is $i \in \mathcal{I}$ such that for all $(t_0, t^{'})$, where $t^{'} > t_0$, there exists $\tilde{t}_0, t_0 < \tilde{t}_0 < t^{'}$, such that $y_i(\tilde{t}) \notin D_i := proj_i(D)$. We assume that $y_i(t)$ leaves the interval D_i through the right end, *i.e.*, $y_i(t_0) = \sup\ D_i$. Then, for all $(t_0, t^{'})$ there exists $\tilde{t}_0, t_0 < \tilde{t}_0 < t^{'}$, such that $y_i(\tilde{t}_0) > \sup\ D_i$ and $\dot{y}_i(\tilde{t}_0) > 0$. This, however, leads to a contradiction as follows. From (36) we have $\nu \dot{y}_i(\tilde{t}_0) = \kappa_i g_i^{'}(g_i^{-1}(y_i(\tilde{t}_0)))$ $\left(y_i(\tilde{t}_0) - \check{f}_i\left(g^{-1}(\overline{y}(\tilde{t}_0 - m_i)), g^{-1}(\hat{y}_{(i,l)}(\tilde{t}_0)), l \in r_i\right)\right) < 0$ because $\kappa_{ii}(\overline{y}(\tilde{t}_0)) = -\kappa_i g_i^{'}(g_i^{-1}(y_i(\tilde{t}_0))) > 0$ and $\check{f}_i(g^{-1}(\tilde{t}_0)), g^{-1}(\hat{y}_{(i,l)}(\tilde{t}_0)), l \in r_i) \in D_i$ and, hence, is less than or equal to $\sup D_i\ (< y_i(\tilde{t}_0))$. This contradicts the earlier assumption that $\dot{y}_i(\tilde{t}_0) > 0$. The other case that $y_i(t)$ leaves D_i through the left end, *i.e.*, $y_i(t_0) = \inf\ D_i$, can be shown to lead to a similar contradiction. Therefore, the lemma follows. ∎

Lemma 8 *Fix $k, k \geq 0$. Let $\tilde{D}$ be an open product space that contains $F(D^{\Xi}_k)$ and whose closure is contained in $int(D_k)$, i.e., $cl(\tilde{D}) \subset int(D_k)$. Suppose that the initial functions $\phi \in C([-d_{max}, 0], D_k)$. Then, there exists a finite $\tilde{t},\ \tilde{t} \geq 0$, such that, for all $t \geq \tilde{t}$, $y(t) \in \tilde{D}$.*

In order to prove the lemma, we first prove the following coordinate-wise invariance.

Lemma 9 *(Coordinate-wise Invariance) If $y_i(\tilde{t}) \in \tilde{D}_i = proj_i(\tilde{D})$ for some $\tilde{t} \geq 0$, then $y_i(t) \in \tilde{D}_i$ for all $t \geq \tilde{t}$.*

Proof: Suppose that the lemma is not true, and there exists $\bar{t} > \tilde{t}$ at which $y_i(\bar{t}) = \inf \tilde{D}_i$ or $y_i(\bar{t}) = \sup \tilde{D}_i$. We assume that $\bar{t}$ is the smallest such time, *i.e.*,

$$\bar{t} = \inf\{t \geq \tilde{t} \mid y_i(t) \in \partial D_i\}\ ,$$

where ∂D_i is the boundary of the set D_i, and $y_i(\bar{t}) = \sup \tilde{D}_i > \ \sup proj_i(F(D^{\Xi}_k))$. Then, we can find $\bar{t}_1 < \bar{t}$ such that for all $t \in (\bar{t}_1, \bar{t})$, $y_i(t) \in \tilde{D}_i \setminus proj_i(F(D^{\Xi}_k))$. This implies that $\dot{y}_i(t) < 0$ for all $t \in (\bar{t}_1, \bar{t})$ from (36) because $F_i(Y(t)) \leq \sup proj_i(F(D^{\Xi}_k))$ and, thus, $y_i(\bar{t}) < \sup\ \tilde{D}_i$, leading to a contradiction. A similar argument can be used for the case $y_i(\bar{t}) = \inf\ \tilde{D}_i$. ∎

Now let us proceed with the proof of Lemma 8.

Proof: Suppose that the lemma is false. Then, from Lemma 9 there exists $i \in \mathcal{I}$ such that for all $t \geq 0$, $y_i(t) \notin \tilde{D}_i$. We show that this leads to a contradiction. Suppose that $y_i(t) \geq \sup\ \tilde{D}_i$ for all $t \geq 0$. Then, one can see that $\dot{y}_i(t) < 0$ from (36). Combined with the assumption that $y_i(t) \geq \sup\ \tilde{D}_i$ for all $t \geq 0$, this implies that $y_i(t)$ converges to some $\tilde{y}_i \geq \sup\ \tilde{D}_i$. Since $\sup\ \tilde{D}_i > \sup\ proj_i(\check{F}(D^{\Xi}))$ with $\delta := \sup\ \tilde{D}_i - \sup\ proj_i(\check{F}(D^{\Xi})) > 0$, there exists some positive constant ε such that $\dot{y}_i(t) \leq -\varepsilon \cdot \delta < 0$

for all sufficiently large t from (36). This, however, implies that $y_i(t) \downarrow -\infty$ as $t \uparrow \infty$, contradicting the assumption that $y_i(t) \geq \sup\ \tilde{D}_i$ for all $t \geq 0$. A similar contradiction can be shown when we assume $y_i(t) \leq \inf\ \tilde{D}_i$ for all $t \geq 0$. This completes the proof of the lemma. ∎

Lemma 10 *Let D be a closed, invariant product space and $\tilde{D}$ an open product space that contains $\check{F}(D^{\Xi})$ and whose closure is contained in $int(D)$, i.e., $cl(\tilde{D}) \subset int(D)$. Suppose that the initial functions $\phi \in C([-d_{max}, 0], D)$ and $y(t_1) \in \tilde{D}$ for some $t_1 \geq 0$. Then, $y(t) \in \tilde{D}$ for all $t \in [t_1,\ t_1 + d_{max}]$.*

Proof: The lemma follows directly from Lemma 9. ∎

The proof of the theorem can now be completed as follows. By repeatedly applying Lemmas 7, 8, and 10, one can find a sequence of finite $t_k, k = 1, 2, \ldots$, such that $y(t) \in D_k$ for all $t \geq t_k$. The theorem now follows from Assumption 6 that $\cap_{k=1}^{\infty} D_k = \{y^*\}$.