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ON THE PARETO CONTROL AND NO-REGRET CONTROL FOR DISTRIBUTED SYSTEMS WITH INCOMPLETE DATA*

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Abstract. We discuss the control of distributed systems with incomplete data following the notion of no-regret control (or, equivalently, Pareto control) used by Lions in [*C. R. Acad. Sci. Paris Ser. I Math.*, 302 (1986), pp. 223–227] and [*C. R. Acad. Sci. Paris Ser. I Math.*, 302 (1992), pp. 1253–1257]. We associate with the no-regret control a sequence of low-regret controls defined by a quadratic perturbation previously used by Nakoulima, Omrane, and Velin in [*C. R. Acad. Sci. Paris Ser. I Math.*, 330 (2000), pp. 801–806].

In the first part, we prove that the perturbed system corresponds to a sequence of standard control problems and converges to the no-regret (or Pareto) control for which we obtain a singular optimality system. We give also some applications.

In the second part, we show how the method can be extended to the evolution case. Equations of parabolic type, Petrowsky type, or hyperbolic type are considered.

Key words. Pareto control, no-regret control, low-regret control, systems with incomplete data, cost function, quadratic perturbation

AMS subject classifications. 49K40, 35B37, 35K55, 90C30, 93A15, 93D09

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1. Introduction.

Let $\mathcal{V}$ be a real Hilbert space of dual $\mathcal{V}'$, $A \in \mathcal{L}(\mathcal{V};\mathcal{V}')$ an elliptic (parabolic or hyperbolic in the sections below) differential operator modeling a distributed system, $\mathcal{U}$ the Hilbert space of controls, and $B \in \mathcal{L}(\mathcal{U};\mathcal{V}')$. Let G be a nonempty closed vector subspace of the Hilbert space of uncertainties F, and $\beta \in \mathcal{L}(F, \mathcal{V}')$.

For $f \in \mathcal{V}'$, the state equation related to the control $v \in \mathcal{U}$ and to the uncertainty $g \in G$ is given by

$$A\,y(v, g) \;=\; f \;+ B\,v + \beta\,g. \tag{1.1}$$

Supposing that A is an isomorphism from $\mathcal{V}$ to $\mathcal{V}'$, (1.1) is well posed in $\mathcal{V}$. Denote by $y = y(v, g)$ the unique solution to (1.1). For every $g \in G$ we have then a possible state for which we rely on a cost function given by

$$J(v, g) = \Big\|C\,y - z_d\Big\|^2_{\mathcal{H}} + N\Big\|v\Big\|^2_{\mathcal{U}}, \tag{1.2}$$

where $C \in \mathcal{L}(\mathcal{V};\mathcal{H})$, $\mathcal{H}$ is a Hilbert space, $z_d \in \mathcal{H}$ fixed, $N > 0$, and $\|.\|_X$ is the norm on the real Hilbert space X. We are concerned with the optimal control of the problem (1.1)–(1.2); i.e., we want to solve

$$\inf_{v\in\mathcal{U}} J(v, g) \qquad \forall g \in G,$$

which clearly makes no sense when $G \neq \{0\}$ (G being an infinite space).

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The idea is to look for a solution for the following minimization problem:

$$\inf_{v\in\mathcal{U}}\left(\sup_{g\in G} J(v,g)\right),$$

but J is not upper bounded as $\sup_{g\in G} J(v,g)=+\infty$.

Lions used the notions of Pareto control [12] and no-regret control [14] in application to control the system (1.1)–(1.2).

The concept of Pareto[1] is motivated by a number of applications in economics, and also in ecology. In his book [9], Kotarski discussed the Pareto optimum problem, where some results of geometrical and numerical interest are obtained in the case of optimal control. In [8], he generalized the well-known Dubovicki–Milutin theorem (on the feasible sets of Pareto) and applied it to obtain a necessary condition for the Pareto *minimum*, or necessary and sufficient conditions on the Pareto optimum, extending a geometrical work of Censor [3]. Lions [12], [13] used the concept to obtain controls for distributed problems with incomplete data.

The no-regret concept was introduced many years later in statistics by Savage [18]. In several works, Lions applied this notion and a related idea called "low-regret" control to problems of incomplete data (see [14], [15], [16], [7], [6]) for various applications. In [14], for example, he extends the results of the work of Allwright [1] to the infinite dimension case. In [7] with Gabay, a decision criterion is added to the uncertainties closed subspace; it improves by extending the notion of low-regret to many agents: each agent wishes to act with least-regret and all agents wish to have minimum exchanges of information, in order to make things as local as possible. The low-regret control is applied to systems where there are *controls* and *unknown perturbations*. One then looks for the control not making things worse with respect to a nominal control u_0 (or to then doing nothing, $u_0 = 0$), independently of the perturbations which may be of infinite number.

We will see in section 2 that Pareto controls and no-regret controls are actually the same.

In this work, we give a characterization of the no-regret (or the Pareto) control for problems of incomplete data, in both the stationary and evolution cases. We improve the results of the work in [14] (and also [12]) of Lions by giving the precise optimality system for the low-regret control and by describing a number of applications. Thanks to a quadratic perturbation used by the authors in [17], the optimality system for the no-regret control appears clearly as the limit of a standard control problem.

Some of the results in this paper are summarized in [17]. The proofs in the present paper and the treatment of the evolution cases are new.

The paper is organized as follows. In section 2, we see the main definitions, we verify the equivalence between the two approaches of Lions, and we introduce the low-regret control method. We then give the optimality system for the perturbed problem and prove that the optimal controls for the perturbed problem converge to the no-regret (Pareto) control of original problem. Moreover, by passing to the limit in the associated optimality system of the perturbed problem, we obtain a singular optimality system for the no-regret (Pareto) control. In section 3, we give several examples of elliptic type. Section 4 is devoted to the evolution case. Here, we give

[1] Wifredo Pareto (1848–1923) was an Italian economist and a political sociologist. He defined the efficient optimum, and in particular was the one who devised the law of *trivial many* and *critical few* known as the 80:20 rule.

the theoretical results for parabolic and hyperbolic distributed systems. An example involving a parabolic system is considered in the last section.

2. No-regret control for stationary problems.

2.1. Definitions and preliminary results. We give definitions for the Pareto and no-regret controls related to a given control here, as well as the preliminary results.

DEFINITION 2.1. *We say that* $u \in \mathcal{U}$ *is a Pareto control for* (1.1)–(1.2) *(cf. Lions* [12]*) iff* $J(u,g) \leq J(v,g)\ \forall v \in \mathcal{U}, \forall g \in G$, *and if there exists at least* $g_0 \in G$ *such that* $J(u,g_0) < J(v,g_0)\ \forall v \in \mathcal{U}$.

DEFINITION 2.2. *Let* $u \in \mathcal{U}$ *be a Pareto control. We say that* u *is related to a control* $u_0 \in \mathcal{U}$ *if*

$$J(u,g) \leq J(u_0,g) \qquad \forall g \in G.$$

DEFINITION 2.3. *We say that* $u \in \mathcal{U}$ *is a no-regret control for* (1.1)–(1.2) *related to a control* u_0 *if* u *is a solution to the following problem:*

$$\inf_{v\in\mathcal{U}} \sup_{g\in G} \left(J(v,g) - J(u_0,g)\right). \tag{2.1}$$

When $u_0 = 0$, Definition 2.3 reduces to the definition of no-regret control of Lions [14].

LEMMA 2.4. *For any* $u_0 \in \mathcal{U}$ *and* $v \in \mathcal{U}$ *we have*

$$J(v,g) - J(u_0,g) = J(v,0) - J(u_0,0) + 2\langle \beta^*\zeta(v-u_0), g\rangle_{G',G} \qquad \forall g \in G,$$

where $\zeta(v) \in \mathcal{V}$ *is defined for* $v \in \mathcal{U}$ *by*

$$A^*\zeta(v) = C^*C(y(v,0) - y(0,0)),$$

A^* *(resp.,* β^**) being the adjoint of* A *(resp.,* β*).*

Proof. We have in fact

$$J(v,g) - J(u_0,g) = J(v,0) - J(u_0,0) + 2\langle C(y(v-u_0,0) - y(0,0)), C(y(0,g) - y(0,0))\rangle_{\mathcal{H},\mathcal{H}}$$

$\forall g \in G$. We then introduce $\zeta(v) \in \mathcal{V}$ defined by $A^*\zeta(v) = C^*C(y(v,0) - y(0,0))$, where A^* is the adjoint of A. Then

$$\begin{aligned}\langle C(y(v,0) - y(0,0)), C(y(0,g) - y(0,0))\rangle_{\mathcal{H},\mathcal{H}} &= \langle A^*\zeta(v), y(0,g) - y(0,0)\rangle_{\mathcal{V}',\mathcal{V}},\\ &= \langle \zeta(v), \beta g\rangle_{\mathcal{V},\mathcal{V}'} = \langle \beta^*\zeta(v), g\rangle_{G',G}\end{aligned}$$

(notice that $A(y(0,g) - y(0,0)) = \beta\, g$).

Remark 1. For sake of simplicity, we denote by $S(v) = \beta^*\zeta(v)$ the linear function for $v \in \mathcal{U}$. Then we have

$$J(v,g) - J(u_0,g) = J(v,0) - J(u_0,0) + 2\langle S(v-u_0), g\rangle_{G',G} \qquad \forall g \in G. \tag{2.2}$$

In the applications below $\beta = Id$, and we have $S(v) = \zeta(v)\ \forall v \in \mathcal{U}$.

Remark 2. Of course the problem (2.1) is defined only for the controls $v \in \mathcal{U}$ such that

$$\sup_{g\in G} \left(J(v,g) - J(u_0,g)\right) < \infty.$$

From (2.2) this is realized for the no-regret control (and the Pareto control) v iff $v \in K + u_0$, where $K = \{w \in \mathcal{U}, \langle\, S(w), g\, \rangle = 0 \ \forall g \in G\}$.

PROPOSITION 2.5 (cf. Lions [12]). *Let be $u_0 \in \mathcal{U}$. Then there exists a unique Pareto control related to u_0. Moreover, it is the unique element of the set $K + u_0$, which minimizes the functional $J(v, 0)$ on $K + u_0$.*

We can now prove the following.

THEOREM 2.6. *Let $u_0 \in \mathcal{U}$ be a given control. Then we have the following: a control $u \in \mathcal{U}$ is a Pareto control related to u_0 iff u is a no-regret control related to u_0.*

Proof. Let u be a Pareto control related to u_0, and let be $v \in K + u_0$. Then $\langle S(u - u_0), g\rangle = 0 = \langle S(v - u_0), g\rangle \ \forall g \in G$, and we have $J(u, 0) \le J(v, 0)$ according to Proposition 2.5. Hence, using (2.2)

$$J(u,0) - J(u_0,0) + 2\langle S(u-u_0), g\rangle \le \sup_{g\in G} \left(J(v,g) - J(u_0,g)\right);$$

that is, $\sup_{g\in G} (J(u,g) - J(u_0,g)) \le \sup_{g\in G} (J(v,g) - J(u_0,g))$. So,

$$\sup_{g\in G} \left(J(u,g) - J(u_0,g)\right) = \inf_{v\in K+u_0} \Big(\sup_{g\in G} \left(J(v,g) - J(u_0,g)\right) \Big).$$

Now, let be $v \in \mathcal{U}\backslash \{K + u_0\}$. There is at least one $g_0 \in G$ such that $\langle S(v-u_0), g_0\rangle \neq 0$. Then we have

$$\sup_{g\in G} \left(J(v,g) - J(u_0,g)\right) = J(v,0) - J(u_0,0) + 2\sup_{g\in G}\langle S(v-u_0), g\rangle = +\infty.$$

(Note that G is a vector space, and henceforth we have the only two possibilities: $\sup_{g\in G}\langle S(w), g\rangle = 0$ or $\sup_{g\in G}\langle S(w), g\rangle = +\infty$. Indeed, $\lim_{t\to+\infty}\langle S(v-u_0), tg_0\rangle = +\infty$.)

From another side, as u is a Pareto control we have $J(u,g) - J(u_0,g) \le 0 \ \forall g \in G$; hence

$$J(u,g) - J(u_0,g) \le 0 \le \sup_{g\in G} \left(J(v,g) - J(u_0,g)\right) \qquad \forall g \in G.$$

Finally,

$$\sup_{g\in G} \left(J(u,g) - J(u_0,g)\right) = \inf_{v\in \mathcal{U}\backslash(K+u_0)} \Big(\sup_{g\in G} \left(J(v,g) - J(u_0,g)\right) \Big).$$

In conclusion, u is a no-regret control related to u_0.

Conversely, let u be a no-regret control related to u_0. We have

$$\sup_{g\in G} \left(J(u,g) - J(u_0,g)\right) \le \sup_{g\in G} \left(J(v,g) - J(u_0,g)\right) \qquad \forall v \in \mathcal{U}.$$

Then for $v = u_0$,

$$J(u,0) + \sup_{g\in G}\langle S(u-u_0), g\rangle \le J(u_0,0) = c \quad \text{constant}.$$

As $J(u,0) \ge 0$, we have $\sup_{g\in G}\langle S(u-u_0), g\rangle \le c$. We deduce that $\sup_{g\in G}\langle S(u-u_0), g\rangle = 0$. Consequently, $\langle S(u-u_0), g\rangle \le 0 \ \forall g \in G$, and hence $\langle S(u-u_0), g\rangle = 0$. So, $u \in K + u_0$, and we have

$$J(u,0) \le J(v,0) \qquad \forall v \in K + u_0.$$

In conclusion, u is a Pareto control related to u_0. $\square$

Remark 3. By Proposition 2.5, we know that there exists a unique Pareto control related to u_0, and that is the only one which minimizes the functional $\inf_{v\in K+u_0} J(v,0)$. In the second part of Theorem 2.6, it is proved that the no-regret control related to u_0—if it exists—also minimizes this functional. As a matter of fact, that suffices to show the existence of a unique no-regret control related to u_0 and that the Pareto control and no-regret control for the problem (1.1)–(1.2) are actually the same.

We are interested in the existence and the characterization of the no-regret (or Pareto) control related to u_0. We follow the lines of [14] where Lions introduced the method of low-regret control.

2.2. Low-regret control. As in [17], we define the low-regret control by relaxing the problem (2.1) as follows:

$$\inf_{v\in\mathcal{U}} \sup_{g\in G} \left[J(v,g) - J(u_0,g) - \gamma\|g\|_G^2\right], \tag{2.3}$$

where $u_0 \in \mathcal{U}$ is a given control, and where γ is a strictly positive parameter. The solution to problem (2.3), if it exists, will be the low-regret control related to u_0, of the problem (1.1)–(1.2).

Remark 4 (cf. Lions [15]). With the "low-regret control," we admit the possibility of making a choice of controls v "slightly worse" $(J(v,g) - J(u_0,g) \le \gamma\|g\|_G^2$ and not $J(v,g) - J(u_0,g) \le 0$ as for the no-regret control) than by doing better than u_0—but not much better—if we choose γ small enough (compared to the worst things that could happen with the "pollution" g).

The best possible choice of v is then given by (2.3).

From (2.2) the problem (2.3) also writes

$$\inf_{v\in\mathcal{U}} \left[J(v,0) - J(u_0,0) + \sup_{g\in G}\left(\langle 2S(v-u_0), g\rangle - \gamma\|g\|_G^2\right)\right].$$

Remark 5. By the perturbation (2.3) we have explicitly the conjugate

$$\sup_{g\in G}\left(\langle 2S(v-u_0), g\rangle - \gamma\|g\|_G^2\right)$$

as we find

$$\sup_{g\in G}\left(\langle 2S(v-u_0), g\rangle - \gamma\|g\|_G^2\right) = \frac{1}{\gamma}\left\|S(v-u_0)\right\|_{G'}^2.$$

With this, if we identify G and G', the problem (2.3) takes the form

$$\inf_{v\in\mathcal{U}} \mathcal{J}^\gamma(v), \tag{2.4}$$

where

$$\mathcal{J}^\gamma(v) = J(v,0) - J(u_0,0) + \frac{1}{\gamma}\left\|S(v-u_0)\right\|_G^2. \tag{2.5}$$

We recognize then a standard optimization problem of a quadratic cost functional.

2.3. Approached optimality system. Now we give the optimality system for the low-regret control u_γ.

PROPOSITION 2.7. *The problem* (2.4)–(2.5) *admits a unique solution* u_γ *called the low-regret control related to* u_0.

Proof. We have $\mathcal{J}^\gamma(v) \geq -J(u_0, 0)\ \forall v \in \mathcal{U}$. Then $d_\gamma = \inf_{v\in\mathcal{U}} \mathcal{J}^\gamma(v)$ exists. Let then $v_n = v_n(\gamma)$ be a minimizing sequence such that $d_\gamma = \lim_{n\to\infty} \mathcal{J}^\gamma(v_n)$. We have

$$-J(u_0, 0) \leq \mathcal{J}^\gamma(v_n) = J(v_n, 0) - J(u_0, 0) + \frac{1}{\gamma}\Big\|S(v_n)\Big\|_G^2 \leq d_\gamma + 1.$$

Then we deduce the bounds

$$\Big\|v_n\Big\|_{\mathcal{U}} \leq c_\gamma, \qquad \frac{1}{\sqrt{\gamma}}\Big\|S(v_n - u_0)\Big\|_G \leq c_\gamma, \qquad \Big\|Cy(v_n, 0) - z_d\Big\|_{\mathcal{H}} \leq c_\gamma,$$

where the constant c_γ (independent of n) is not the same each time.

There exists $u_\gamma \in \mathcal{U}$ such that $v_n \rightharpoonup u_\gamma$ weakly in the Hilbert space $\mathcal{U}$. Also, $y(v_n, 0) \to y(u_\gamma, 0)$ (continuity w.r.t the data). We also deduce from the strict convexity of the cost function $\mathcal{J}^\gamma$ that u_γ is unique. ∎

THEOREM 2.8. *The solution* u_γ *of the relaxed problem* (2.4)–(2.5) *weakly converges in* $\mathcal{U}$ *as* $\gamma \to 0$ *to the unique no-regret control related to* u_0.

Proof. Let u_γ be the solution to (2.4)–(2.5). Then

$$J(u_\gamma, 0) - J(u_0, 0) + \frac{1}{\gamma}\Big\|S(u_\gamma - u_0)\Big\|_G^2 \leq J(v, 0) - J(u_0, 0) + \frac{1}{\gamma}\Big\|S(v - u_0)\Big\|_G^2 \quad \forall v \in \mathcal{U}.$$

Particularly for $v = u_0$, we have

$$J(u_\gamma, 0) - J(u_0, 0) + \frac{1}{\gamma}\Big\|S(u_\gamma - u_0)\Big\|_G^2 \leq 0,$$

and the structure of $J(u_\gamma, 0)$ in (1.2) gives

$$\Big\|Cy(u_\gamma, 0) - z_d\Big\|_{\mathcal{H}}^2 + N\Big\|u_\gamma\Big\|_{\mathcal{U}}^2 + \frac{1}{\gamma}\Big\|S(u_\gamma - u_0)\Big\|_G^2 \leq J(u_0, 0). \tag{2.6}$$

We deduce that $\|u_\gamma\|_{\mathcal{U}} \leq c$. Then we can extract a subsequence u_γ which weakly converges towards $u \in \mathcal{U}$, the solution to (2.4).

Now for $v \in \mathcal{U}$ we have

$$J(v, g) - J(u_0, g) - \gamma\|g\|^2 \leq J(v, g) - J(u_0, g) \quad \forall g \in G.$$

Then

$$J(u_\gamma, g) - J(u_0, g) - \gamma\|g\|^2 \leq \sup_{g\in G}\left(J(v, g) - J(u_0, g)\right) \quad \forall g \in G,$$

and passing to the limit in γ we obtain

$$J(u, g) - J(u_0, g) \leq \sup_{g\in G}\left(J(v, g) - J(u_0, g)\right) \qquad \forall g \in G.$$

We deduce easily that u is a no-regret control related to u_0. ∎

Now we give the optimality system for the low-regret control.

PROPOSITION 2.9. *The low-regret control u_γ solution to* (2.4)–(2.5) *is characterized by the unique solution* $\{y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma\}$ *of the optimality system*

$$\begin{cases} A\,y_\gamma = f + Bu_\gamma, & A^*\zeta_\gamma = C^*C(y_\gamma - y(0,0)), \\ A\,\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\zeta_\gamma, & A^*p_\gamma = C^*(Cy_\gamma - z_d) + C^*C\rho_\gamma, \\ B^*p_\gamma + Nu_\gamma = 0 \quad in \quad \mathcal{U}. \end{cases}$$

Proof. Let u_γ be the solution of (2.4)–(2.5) on $\mathcal{U}$. The Euler–Lagrange necessary condition gives for every $w \in \mathcal{U}$

$$\langle C^*\left(Cy(u_\gamma,0) - z_d\right), y(w,0) - y(0,0)\rangle_{\mathcal{H}\times\mathcal{H}} + \langle Nu_\gamma, w\rangle_{\mathcal{U}\times\mathcal{U}} + 2\left\langle \frac{1}{\gamma}S(u_\gamma), S(w)\right\rangle_{G\times G} \geq 0.$$

Denoting $y_\gamma = y(u_\gamma, 0)$ and $\xi_\gamma(v) = \beta S(v)$ we have

$$A^*\xi_\gamma = C^*C(y_\gamma - y(0,0)).$$

Let ρ_γ be the solution to

$$A\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\xi.$$

And as it is classical, we introduce the adjoint state p_γ defined by

$$B^*p_\gamma = C^*(Cy_\gamma - z_d) + C^*C\rho_\gamma$$

so that we obtain

$$\langle B^*p_\gamma + Nu_\gamma, w\rangle \geq 0 \qquad \forall w \in \mathcal{U}.$$

But also we have $\langle B^*p_\gamma + Nu_\gamma, w\rangle \leq 0$ $\forall w \in \mathcal{U}$. The optimality system follows. $\square$

2.4. Singular optimality system. Now, we give the optimality system for the no-regret control.

As in [12] let $\mathcal{R}$ be an operator defined as follows.

We solve first

$$A\,\rho = \beta\,g, \quad g \in G, \quad \rho \in \mathcal{V},$$

then

$$A^*\sigma = C^*C\rho, \quad \sigma \in \mathcal{V},$$

and we set $\mathcal{R}\,g = B^*\,\sigma$. We suppose that

$$\left\|\mathcal{R}\,g\right\|_{\widehat{G}} \geq c\left\|g\right\|_{G}, \quad c > 0, \quad \text{for any} \quad g \in G, \tag{2.7}$$

where $\widehat{G}$ is the completion of G in F, containing the elements $\mathcal{R}g$.

Remark 6. The space $\widehat{G}$ is in fact the completion of G for a subspace $(H, \|.\|_{\|.\|})$ of F which can be bigger than G. This will be made precise in the applications below.

Remark 7. The hypothesis (2.7) is very useful theoretically but is not necessary in practice. We need only to make sure that the adjoint state p_γ of Proposition 2.9 is bounded in a suitable Hilbert space, which is the case in the applications given below.

THEOREM 2.10. *Suppose that* (2.7) *holds true. Then the no-regret control* u *related to* u_0*, solution to* (2.1)*, is characterized by the unique* $\{y, \lambda, \rho, p\}$ *solution to the singular optimality system*

$$\begin{cases} A\,y &= f + Bu, \\ A\,\rho &= \lambda, \\ B^*p + Nu &= 0, \end{cases} \qquad A^*p = C^*(Cy - z_d) + C^*C\rho,$$

with $\lambda \in \widehat{G}$.

Proof. From the relation (2.6) and Theorem 2.8, the sequence $\{u_\gamma\}$ weakly converges in $\mathcal{U}$ to u the unique no-regret control related to u_0. The operator B being continuous from $\mathcal{U}$ to $\mathcal{V}'$, $\{Bu_\gamma\}$ weakly converges in $\mathcal{V}'$ to Bu. Now, from the above optimality system of Proposition 2.9, the sequence $\{Ay_\gamma\}$ is bounded in $\mathcal{V}'$ and, as A is an isomorphism, weakly converges to Ay in $\mathcal{V}'$. Passing to the limit in the first equation we obtain $Ay = f + B\,u$. We also deduce from Proposition 2.9 that $B^*p_\gamma = -Nu_\gamma$ is bounded in $\mathcal{V}'$. According to the hypothesis (2.7), let $\mathcal{R}$ be the operator such that $\mathcal{R}(\frac{1}{\gamma}\beta^*\xi_\gamma) = B^*p_\gamma$. We deduce under (2.7) that $\{\frac{1}{\gamma}\beta^*\xi_\gamma\}$ is bounded in G subset of the Hilbert space F. Then it converges to $\lambda \in \widehat{G} \subset F$. Hence, $A\rho_\gamma = \frac{1}{\gamma}\beta^*\xi_\gamma$ is bounded, and then $\{\rho_\gamma\}$—also bounded thanks to the isomorphism of A—weakly converges to $\rho \in \mathcal{V}$. Consequently, $A\rho_\gamma \rightharpoonup A\rho$.

From the boundness of $\{\rho_\gamma\}$ and $\{y_\gamma\}$ we obtain that A^*p_γ is bounded. Then $\{p_\gamma\}$ converges to p. The optimality system follows. $\square$

Remark 8. The situation described by Theorem 2.10, as indicated by Lions in [12], is completely general, but with λ which should be in the completion of G. This will be made precise in the following applications.

3. Application. In this section, we apply the above method throughout the examples given below in different situations: control and uncertainty given in the interior domain, as well as on the boundary.

Example 1. A distributed control, uncertain boundary values, and a boundary cost function.

Let Ω be a bounded open domain of $\mathbb{R}^N$ of regular boundary Γ. We consider the distributed system

$$\begin{cases} -\Delta y + y = f + v & \text{in } \Omega, \\ \dfrac{\partial y}{\partial \nu} \quad = \quad g & \text{on } \Gamma, \end{cases} \tag{3.1}$$

where $v \in \mathcal{U} = L^2(\Omega)$, and where $g \in G \subset F = L^2(\Gamma)$, G a closed subspace of F. If $f \in L^2(\Omega)$, there exists a unique $y(v, g) \in H^{3/2}(\Omega)$ solution to (3.1).

We associate with the state $y(v, g)$ the cost function

$$J(v, g) = \Big| y(v, g) - z_d \Big|^2_{L^2(\Gamma)} + N \Big\| v \Big\|^2_{L^2(\Omega)}. \tag{3.2}$$

For $u_0 \in \mathcal{U}$, there exists a unique no-regret control u related to u_0. For simplicity, take $u_0 = 0$. The problem now is to give the optimality system for the no-regret control u.

Notice that

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\left(y(v,0) - y(0,0), y(0,g) - y(0,0)\right)_{L^2(\Gamma)}$$

and that the function $v \longmapsto y(v,0) - y(0,0)$ (resp., $g \longmapsto y(0,g) - y(0,0)$) is linear w.r.t v (resp., g) and is the solution to

$$\begin{cases} A\,z = v & \text{in } \Omega \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma \end{cases} \qquad \left(\text{resp., } \begin{cases} A\,z = 0 & \text{in } \Omega, \\ \frac{\partial z}{\partial \nu} = g & \text{on } \Gamma \end{cases}\right),$$

where $A = -\Delta + I$. Using the Green formula

$$(\varphi, A\psi)_{L^2(\Omega)} - (\psi, A\varphi)_{L^2(\Omega)} = \int_\Gamma \varphi \frac{\partial \psi}{\partial \nu_A} d\gamma - \int_\Gamma \psi \frac{\partial \varphi}{\partial \nu_A} d\gamma, \tag{3.3}$$

we find

$$0 = \int_\Gamma \left(y(0,g) - y(0,0)\right)\,\left(y(v,0) - y(0,0)\right)\,d\gamma - \int_\Gamma S(v)\,g\,d\gamma,$$

where $v \longmapsto S(v)$ is a linear function so that $AS = 0$, $\frac{\partial S}{\partial \nu} = y(v,0) - y(0,0)$.

Moreover, the following regularity result holds: We have $y(0,g) - y(0,0) \in H^{3/2}(\Omega)$ as $\frac{\partial}{\partial \nu_A}\left(y(0,g) - y(0,0)\right) \in L^2(\Gamma)$, and as $S(v) \in H^2(\Omega)$ we have also $\frac{\partial S}{\partial \nu} = y(v,0) - y(0,0) \in H^{3/2}(\Omega)$.

From section 2, the low-regret control method associated with (3.1)–(3.2) is defined by

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|S(v)\right\|^2_{L^2(\Gamma)}, \tag{3.4}$$

where $S(v) = \zeta(v)$ is the solution of

$$\begin{cases} A\,S(v) = 0 & \text{in} \quad \Omega, \\ \frac{\partial S}{\partial \nu_A} \quad = y(v,0) - y(0,0) & \text{on} \quad \Gamma. \end{cases}$$

The problem

$$\inf_{v \in \mathcal{U}} \mathcal{J}^\gamma(v) \tag{3.5}$$

admits a unique solution $v = u_\gamma$. Then the necessary condition of first order of Euler on $\mathcal{U}$ for every $w \in \mathcal{U}$ writes

$$\left(y(u_\gamma,0) - z_d, y(w,0) - y(0,0)\right) + (Nu_\gamma, w) + \left(\frac{1}{\gamma} S(u_\gamma), S(w)\right) \geq 0. \tag{3.6}$$

We have the following proposition.

Proposition 3.1. *The low-regret control u_γ solution to* (3.4)–(3.5) *is characterized by the unique $\{y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma\}$ solution to*

$$\begin{cases} A\,y_\gamma = f + u_\gamma, & A\zeta_\gamma = 0, & A\,\rho_\gamma = 0, & A\,p_\gamma = 0, \\ \dfrac{\partial y_\gamma}{\partial \nu} = 0, & \dfrac{\partial \zeta_\gamma}{\partial \nu} = y_\gamma - y(0,0), & \dfrac{\partial \rho_\gamma}{\partial \nu} = \frac{1}{\gamma}\zeta_\gamma, & \dfrac{\partial p_\gamma}{\partial \nu} = y_\gamma - z_d + \rho_\gamma, \\ p_\gamma + Nu_\gamma = 0 & & & \textit{in} \quad L^2(\Omega), \end{cases}$$

with

$$u_\gamma \in L^2(\Omega) \qquad \textit{and} \quad y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma \in H^{3/2}(\Omega).$$

Proof. Denote $y_\gamma = y(u_\gamma, 0)$, and let $\zeta(u_\gamma)$ be the solution of $A\zeta_\gamma = 0$, $\frac{\partial \zeta_\gamma}{\partial \nu} = y_\gamma - y(0,0)$. Let now ρ_γ be the solution of $A\rho_\gamma = 0$, $\frac{\partial \rho_\gamma}{\partial \nu} = \frac{1}{\gamma}\zeta_\gamma$. Then by the above Green formula

$$\begin{aligned}\left(\frac{1}{\gamma}\zeta_\gamma(u_\gamma), \zeta_\gamma(w)\right)_{L^2(\Gamma)} &= (A\rho_\gamma, \zeta)_{L^2(\Omega)} - (\rho_\gamma, A\zeta_\gamma)_{L^2(\Omega)} + \left(\rho_\gamma, \frac{\partial \zeta_\gamma}{\partial \nu}\right)_{L^2(\Gamma)} \\ &= \left(\rho_\gamma, \frac{\partial \zeta_\gamma}{\partial \nu}\right)_{L^2(\Gamma)}.\end{aligned}$$

The inequality (3.6) becomes

$$(y_\gamma - z_d + \rho_\gamma, y(w,0) - y(0,0)) + (Nu_\gamma, w) \leq 0.$$

Now, and as it is classical, we calculate the adjoint state p_γ such that $Ap_\gamma = 0$, $\frac{\partial p_\gamma}{\partial \nu} = y_\gamma - z_d + \rho_\gamma$.

This is for any w in the vector space $\mathcal{U}$. Then we have

$$p_\gamma \; + \; Nu_\gamma \; = 0. \qquad \square$$

Remark 9. The passage to the limit on γ for the no-regret control is an adaptation of the proof of the Theorem 2.10. Let us note that we do not need the hypothesis (2.7) as we have $B^* = B = Id$.

We obtain the following theorem.

THEOREM 3.2. *The no-regret control u related to $u_0 = 0$ of the problem* (3.1)–(3.2) *is characterized by the unique solution $\{y, \lambda, \rho, p\}$ of the optimality system*

$$\begin{cases} A\,y = f + u, \quad A\,\rho = 0, \quad A\,p = 0 & \textit{in } \Omega, \\ \dfrac{\partial y}{\partial \nu} = 0, \quad \dfrac{\partial \rho}{\partial \nu} = \lambda, \quad \dfrac{\partial p}{\partial \nu} = y - z_d + \rho & \textit{on } \Gamma, \\ p + Nu = 0 & \textit{in } L^2(\Omega), \end{cases}$$

with

$$\begin{cases} u \in L^2(\Omega),\ y \in H^{3/2}(\Omega), & p \in L^2(\Omega), \\ \lambda \in \widehat{G} \quad \textit{completion of } G \textit{ for the norm } H^{-5/2}(\Gamma), & \rho \in H^{-1}(\Omega). \end{cases} \qquad \square$$

Example 2. A boundary control, boundary uncertainty, boundary cost function.

Let Ω be an open domain from $\mathbb{R}^N$ of boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, with Γ_0 and Γ_1 being two regular boundaries such that $\Gamma_0 \cap \Gamma_1 = \emptyset$.

We consider the distributed parameter system

$$\begin{cases} -\Delta y + y = 0 & \text{in } \Omega, \\ \dfrac{\partial y}{\partial \nu} = v & \text{on } \Gamma_0, \\ \dfrac{\partial y}{\partial \nu} = g & \text{on } \Gamma_1. \end{cases} \tag{3.7}$$

For $v \in \mathcal{U} = L^2(\Gamma_0)$ and $g \in G \subset L^2(\Gamma_1)$, (3.7) admits a unique solution $y(v,g) \in H^{3/2}(\Omega)$.

We associate with the state $y(v,g)$ the cost function

$$J(v,g) = \Big|y(v,g) - z_d\Big|^2_{L^2(\Gamma_0)} + N\Big|v\Big|^2_{L^2(\Gamma_0)}. \tag{3.8}$$

For u_0 fixed in $\mathcal{U}$, there exists a unique no-regret control u related to u_0. We suppose that $u_0 = 0$.

The low-regret control associated is defined by the following cost function:

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\Big\|S(v)\Big\|^2_{L^2(\Gamma_1)}, \tag{3.9}$$

where $S(v) = \zeta(v)$ is the solution to

$$\begin{cases} A\,S(v) = 0 & \text{in } \Omega, \\ \dfrac{\partial S}{\partial \nu} \quad = y(v,0) & \text{on } \Gamma_0, \\ \dfrac{\partial S}{\partial \nu} \quad = 0 & \text{on } \Gamma_1 \end{cases} \tag{3.10}$$

and where $A = -\Delta + I$.

Indeed,

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2(y(v,0), y(0,g))_{L^2(\Gamma_0)\times L^2(\Gamma_0)}.$$

Then by the Green formula we obtain

$$(y(v,0), y(0,g))_{L^2(\Gamma_0)\times L^2(\Gamma_0)} = (S(v), g)_{L^2(\Gamma_0)\times L^2(\Gamma_0)},$$

with $S(.)$ the solution to (3.10). The problem

$$\inf_{v\in\mathcal{U}} \mathcal{J}^\gamma(v) \tag{3.11}$$

admits a unique solution u_γ called the low-regret control.

Proposition 3.3. *The low-regret control u_γ solution to* (3.9)–(3.11) *is characterized by the unique solution $\{y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma\}$ of the optimality system*

$$\begin{cases} A\,y_\gamma = 0, & A\zeta_\gamma = 0, & A\,\rho_\gamma = 0, & A\,p_\gamma = 0 & \textit{in} \quad \Omega, \\ \dfrac{\partial y_\gamma}{\partial \nu} = u_\gamma, & \dfrac{\partial \zeta_\gamma}{\partial \nu} = y_\gamma, & \dfrac{\partial \rho_\gamma}{\partial \nu} = 0, & \dfrac{\partial p_\gamma}{\partial \nu} = y_\gamma - z_d + \rho_\gamma & \textit{on} \quad \Gamma_0, \\ \dfrac{\partial y_\gamma}{\partial \nu} = 0, & \dfrac{\partial \zeta_\gamma}{\partial \nu} = 0, & \dfrac{\partial \rho_\gamma}{\partial \nu} = \frac{1}{\gamma}\zeta_\gamma, & \dfrac{\partial p_\gamma}{\partial \nu} = 0 & \textit{on} \quad \Gamma_1, \\ p_\gamma + N u_\gamma = 0 & & & & \textit{on} \quad \Gamma_0, \end{cases}$$

with, $u_\gamma \in L^2(\Gamma_0)$, and $y_\gamma \in H^{3/2}(\Omega), \zeta_\gamma \in H^{5/2}(\Omega), \rho_\gamma \in H^{7/2}(\Omega), p_\gamma \in H^{1/2}(\Omega)$.

Proof. The Euler condition gives

$$(y_\gamma - z_d, y(w,0))_{L^2(\Gamma_1)\times L^2(\Gamma_1)} + N\,(u_\gamma, w)_{L^2(\Gamma_0)\times L^2(\Gamma_0)} + \left(\frac{1}{\gamma}\xi(u_\gamma), \xi(w)\right)_{L^2(\Gamma_1)\times L^2(\Gamma_1)} \geq 0. \tag{3.12}$$

We first solve for ρ_γ: $A\,\rho_\gamma = 0$, with $\frac{\partial \rho_\gamma}{\partial \nu} = 0$ on Γ_0, and $\frac{\partial \rho_\gamma}{\partial \nu} = \frac{1}{\gamma}\xi(u_\gamma)$ on Γ_1. Hence

$$\left(\frac{1}{\gamma}\xi(u_\gamma), \xi(w)\right)_{L^2(\Gamma_1)\times L^2(\Gamma_1)} = (\rho_\gamma, y(w,0))_{L^2(\Gamma_0)\times L^2(\Gamma_0)}$$

so that (3.12) becomes

$$(y_\gamma - z_d + \rho_\gamma, y(w,0))_{L^2(\Gamma_0)\times L^2(\Gamma_0)} + N\,(u_\gamma, w)_{L^2(\Gamma_0)\times L^2(\Gamma_0)} \geq 0.$$

Let now p_γ be the solution of $A\,p_\gamma = 0$, with $\frac{\partial p_\gamma}{\partial \nu} = y_\gamma - z_d$ on Γ_0, and $\frac{\partial p_\gamma}{\partial \nu} = 0$ on Γ_1.

We have then

$$(y_\gamma - z_d + \rho_\gamma, y(w,0))_{L^2(\Gamma_0)\times L^2(\Gamma_0)} = (p_\gamma, w)_{L^2(\Gamma_0)\times L^2(\Gamma_0)}.$$

Finally, as $L^2(\Gamma_0)$ is a vector space, we have

$$p_\gamma \; + \; N u_\gamma \; = 0 \qquad \forall w \in L^2(\Gamma_0). \qquad \square$$

The passage to the limit on γ leads to the following theorem.

THEOREM 3.4. *The no-regret control u of the system* (3.7)–(3.8) *is characterized by the unique solution $\{y, \lambda, \rho, p\}$ of the optimality system*

$$\begin{cases} A\,y = 0, & A\,\rho = 0, \quad A\,p = 0 & \textit{in} \quad \Omega, \\ \dfrac{\partial y}{\partial \nu} = u, & \dfrac{\partial \rho}{\partial \nu} = 0, \quad \dfrac{\partial p}{\partial \nu} = y - z_d + \rho & \textit{on} \quad \Gamma_0, \\ \dfrac{\partial y}{\partial \nu} = 0, & \dfrac{\partial \rho}{\partial \nu} = \lambda, \quad \dfrac{\partial p}{\partial \nu} = 0 & \textit{on} \quad \Gamma_1, \\ p + Nu = 0 & & \textit{in} \quad L^2(\Gamma_0), \end{cases}$$

with

$$\begin{cases} u \in L^2(\Omega), & y \in H^{3/2}(\Omega), \qquad p \in H^{1/2}(\Omega), \\ \lambda \in \widehat{G} \textit{ completion of } G \textit{ in } H^{-2}(\Gamma), & \rho \in H^{-1/2}(\Omega). \end{cases} \qquad \square$$

4. The evolution case.

4.1. No-regret control for systems of parabolic type. In this section, $A \in \mathcal{L}(\mathcal{V}; \mathcal{V}')$ is an elliptic differential operator

$$(\,A\,v\,,\,v\,) \geq \alpha \left\|v\right\|^2, \quad \alpha > 0, \quad \left\|\;\right\| = \text{norm in} \quad \mathcal{V},$$

$B \in \mathcal{L}\left(\mathcal{U}; L^2(0,T;\mathcal{V}')\right)$, and F is the real Hilbert space of uncertainties such that

$$\mathcal{V} \subset F \subset \mathcal{V}'.$$

Let then G be the closed vector subspace of F.

For $f \in L^2(0,T;\mathcal{V}')$, the state equation that we consider is

$$\frac{\partial y}{\partial t} + A\,y = f + B\,v, \tag{4.1}$$

with

$$y(t=0,v,g) = y_0 + g, \tag{4.2}$$

where y_0 is a given data in F and where $g \in G$.

For chosen v and g, the problem (4.1)–(4.2) admits a unique solution noted $y(v,g) \in L^2(0,T;\mathcal{V})$.

For a fixed $t \in (0,T)$, and for any $g \in G$ we have then a possible state for which we attach a cost function given by

$$J(v,g) = \int_0^T \|Cy(v,g) - z_d\|^2_{\mathcal{H}}\, dt + N \int_0^T \|v\|^2_{\mathcal{U}}\, dt, \tag{4.3}$$

where

$$C \in \mathcal{L}\,(\,L^2(0,T;\mathcal{V});\mathcal{H}\,), \tag{4.4}$$

the set $\mathcal{H}$ is a Hilbert space, $z_d \in \mathcal{H}$ fixed, $N > 0$, and $\|.\|_X$ represents the norm defined on the Hilbert space X.

When $G = \{0\}$, a standard control problem is to find

$$\inf_{v\in\mathcal{U}} J(v,0). \tag{4.5}$$

We now develop the approach of the first part to this evolution case, when $G \neq \{0\}$.

4.1.1. Least regret control. Approached optimality system. Following the lines of [13] and using the notations in [17], we have then

$$J(v,g) - J(u_0,g) = J(v,0) - J(u_0,0) + 2\langle\, \xi(v-u_0), g\,\rangle_{G'\times G}, \tag{4.6}$$

where

$$S(v) = \zeta(t=0,v) \tag{4.7}$$

and where ζ is the solution to the backwards problem

$$\begin{cases} -\zeta' + A^*\,\zeta\; = C^*C(y(v,0) - y(0,0)), \\ \zeta\,(t=T,v) = 0, \end{cases} \tag{4.8}$$

with $\zeta' = \frac{\partial\zeta}{\partial t}$.

Then the low-regret control associated with the problem (4.1)–(4.3) is defined by

$$\inf_{v\in\mathcal{U}} \mathcal{J}^{\gamma}(v), \tag{4.9}$$

$$\mathcal{J}^{\gamma}(v) = J(v,0) - J(u_0,0) + \frac{1}{\gamma}\Big\|\zeta(0,v-u_0)\Big\|^2_{G'}, \tag{4.10}$$

where G' is the dual of G which can be identified to G. The problem (4.9)–(4.10) has a unique solution u_γ called low-regret control.

Proposition 4.1. *The low-regret control u_γ solution to* (4.9)–(4.10) *is characterized by the unique solution $\{y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma\}$ of the optimality system*

$$\begin{cases} y'_\gamma + A\,y_\gamma \quad = f + Bu_\gamma, & -\zeta'_\gamma + A^*\zeta_\gamma = C^*C(y_\gamma - y(0,0)), \\ \rho'_\gamma + A\rho_\gamma \quad = 0, & -p'_\gamma + A^*p_\gamma = C^*(Cy_\gamma - z_d) + C^*C\rho_\gamma, \\ y_\gamma(t=0) = y_0, \quad \rho_\gamma(0) = \frac{1}{\gamma}\zeta_\gamma, & \zeta_\gamma(T) = 0, \qquad p_\gamma(T) = 0, \\ B^*p_\gamma + Nu_\gamma \quad = 0 \quad in \quad \mathcal{U}. \end{cases}$$

Proof. Let u_γ be the solution of the problem (4.9)–(4.10). The Euler first order condition gives the following optimality system:

$$
\begin{aligned}
(C y(u_\gamma, 0) - z_d, C(y(w,0) - y(0,0)))_{\mathcal{H}\times\mathcal{H}} &+ (N u_\gamma, w)_{\mathcal{U}\times\mathcal{U}} \\
&+ \left(\frac{1}{\gamma}\zeta(0, u_\gamma - u_0), \zeta(0,w)\right)_{G\times G} \geq 0.
\end{aligned} \tag{4.11}
$$

With this in mind, let $y_\gamma = y(u_\gamma, 0)$, and look for $\zeta_\gamma = \zeta(0, u_\gamma - u_0)$ to be the solution of (4.8) and $\rho_\gamma \in \mathcal{V}$ the solution of

$$
\begin{cases} \rho_\gamma' + A\rho_\gamma = 0, \\ \rho_\gamma(t=0) = \frac{1}{\gamma}\zeta_\gamma. \end{cases}
$$

As it is classical, we introduce the adjoint state p_γ defined by

$$
-p_\gamma' + A^* p_\gamma = C^*(C y_\gamma - z_d) + C^* C \rho_\gamma, \qquad \text{with} \quad p_\gamma(T) = 0.
$$

Hence we deduce from (4.11)

$$
B^* p_\gamma + N u_\gamma = 0 \qquad \text{in} \quad \mathcal{U}. \tag{4.12}
$$

This ends the proof. □

4.1.2. Singular optimality system. We now give the optimality system for the no-regret control. We need a supplementary hypothesis. Let $\rho \in L^2(0,T;V)$ be defined by

$$
\rho' + A\,\rho = 0, \quad \rho(0) = g, \quad g \in G,
$$

and $\sigma \in L^2(0,T;V)$ as

$$
-\sigma' + A^*\sigma = C^* C\,\rho, \quad \sigma(T) = 0.
$$

Setting $R\,g = B^*\sigma$, then we define the continuous operator $g \mapsto Rg$ from F to $\mathcal{U}$, and we do the hypothesis

$$
\left\| R\,g \right\|_{\mathcal{U}} \geq c \left\| g \right\|_{F} \quad c > 0 \quad \forall g \in G. \tag{4.13}
$$

THEOREM 4.2. *We suppose that* (4.13) *holds true. Then the no-regret control u related to u_0, for the system* (4.1)–(4.3), *is characterized by the unique solution $\{y, \lambda, \rho, p\}$ to the optimality system*

$$
\begin{cases}
y' + A\,y \;= f + Bu, \qquad -\zeta' + A^*\zeta \;= C^*C(y - y(0,0)), \\
\rho' + A\rho \;= 0, \qquad\qquad\quad -p' + A^* p \;= C^*(Cy - z_d) + C^* C\rho, \\
y(0) = y_0, \quad \rho(0) = \lambda, \quad \zeta(T,u) = 0, \quad p(T) = 0, \\
B^* p + N u = 0 \quad \text{in} \quad \mathcal{U},
\end{cases}
$$

with $\lambda \in \widehat{G}$.

Proof. The proof holds from the approached optimality system of Proposition 4.1 for which a priori estimates allow us to pass to the limit when $\gamma \to 0$ as in section 2. □

4.2. No-regret control for well-posed systems of Petrowsky type. We now consider an elliptic differential operator A such as

$$A^* = A,$$

and to simplify we consider the state equation

$$y'' + A\,y = v, \tag{4.14}$$

with

$$y \in L^\infty(0,T;V), \qquad y' \in L^\infty(0,T;F), \tag{4.15}$$

$$y(0) = y_0 + g_0, \qquad y'(0) = y_1 + g_1, \tag{4.16}$$

where $\{y_0, y_1\}$ is bounded in $\mathcal{V} \times F$ and where

$$\begin{cases} g_0 \in G_0, & G_0 = \text{ closed vector subspace of } \mathcal{V}, \\ g_1 \in G_1, & G_1 = \text{ closed vector subspace of } F. \end{cases} \tag{4.17}$$

Let $y(v,g)$ be the solution of (4.14)–(4.16), $g = (g_0, g_1)$. Let C be defined by (4.4) and the cost function $J(v,g)$ be defined by (4.3). We look for the no-regret control related to $u_0 = 0$. We define $y = y(v,0)$ and $\zeta(t,v)$ (or $\zeta(t)$), respectively, by

$$y'' + A\,y = v, \qquad y(t=0) = y_0, \qquad y'(t=0) = y_1, \tag{4.18}$$

$$\zeta'' + A\,\zeta = C^*C\,y(v,0), \qquad \zeta(T) = 0, \qquad \zeta'(T) = 0. \tag{4.19}$$

Set $z = y(0,g) - y(0,0)$. Then z is the solution of

$$\begin{cases} z'' + A\,z = 0, \\ \quad z(0) \quad = g_0, \\ \quad z'(0) \quad = g_1. \end{cases}$$

Then by the Green formula we obtain

$$\begin{aligned} J(v,g) - J(0,g) &= J(v,0) - J(0,0) + 2\int_0^T (\,\zeta'' + \Delta\zeta, z\,)\,dt \\ &= J(v,0) - J(0,0) + 2\,(\,\zeta(0), g_1\,)_{G_0,G_1} - 2\,(\,\zeta'(0), g_0\,)_{G_1,G_0}\,. \end{aligned}$$

As the low-regret control solution is defined by the

$$\inf_{v\in\mathcal{U}} \left(\sup_{g\in G_0\times G_1} \left(J(v,g) - J(0,g) + \gamma\left\|g_0\right\|^2_{G_0} - \gamma\left\|g_1\right\|^2_{G_1} \right)\right),$$

the low-regret control method reads

$$\inf_{v\in\mathcal{U}} \mathcal{J}^\gamma(v) = \inf_{v\in\mathcal{U}} \left(J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|\zeta(0)\right\|^2_{G_1} - \frac{1}{\gamma}\left\|\zeta'(0)\right\|^2_{G_0} \right). \tag{4.20}$$

And we have for the Petrowsky systems the following result.

THEOREM 4.3. *The no-regret control u related to $u_0 = 0$ is characterized by the unique solution $\{y, \lambda_0, \lambda_1, \zeta, \rho, p\}$ to the optimality system*

$$\begin{cases} y'' + A\,y = 0, & \zeta'' + A\,\zeta = 0, & \rho'' + A\,\rho = 0, & p'' + A\,p = 0, \\ y(0) = y_0, & \zeta(T) = 0, & \rho(0) = \lambda_0, & p(T) = 0, \\ y'(0) = y_1, & \zeta'(T) = 0, & \rho'(0) = \lambda_1, & p'(T) = 0, \\ p + Nu = 0, \end{cases}$$

with

$$\begin{cases} \lambda_0 = -\lim_{\gamma\to 0} \dfrac{1}{\gamma}\zeta'(0), \; \lambda_0 \in \widehat{G_0} & \text{completion of} \quad G_0 \quad \text{for the norm} \quad \Big\|.\Big\|_{G_0}, \\ \lambda_1 = \lim_{\gamma\to 0} \dfrac{1}{\gamma}\zeta(0), \quad \lambda_1 \in \widehat{G_1} & \text{completion of} \quad G_1 \quad \text{for the norm} \quad \Big\|.\Big\|_{G_1}. \end{cases} \qquad \square$$

Remark 10. These results are also valid for well-posed problems of hyperbolic type.

5. Application. Hereafter, we discuss an example of parabolic type with boundary control, boundary uncertainty, and cost function.

Let Ω be an open set of $\mathbb{R}^N$ of boundary $\Gamma_0 \cup \Gamma_1$, with Γ_0 and Γ_1 being two regular boundaries of empty set intersection. We consider the distributed system

$$(5.1) \qquad \begin{cases} y' - \Delta\, y + y = 0 & \text{in } \Omega, \\ y(0, x, v, g) = 0 & \text{on } \{0\} \times \Omega, \\ \dfrac{\partial y}{\partial \nu} = v & \text{on }]0, T[\times \Gamma_0 = \Sigma_0, \\ \dfrac{\partial y}{\partial \nu} = g & \text{on }]0, T[\times \Gamma_1 = \Sigma_1. \end{cases}$$

For $v \in \mathcal{U} = L^2(\Sigma_0)$, $g \in G$, a vector closed subspace of $L^2(\Sigma_1)$, (5.1) has a unique solution $y(t, x, v, g)$ noted $y(v, g)$. We associate with the state $y(v, g)$ the cost function

$$(5.2) \qquad J(v, g) = \Big|y(v, g) - z_d\Big|^2_{L^2(\Sigma_0)} + N\Big|v\Big|^2_{L^2(\Sigma_0)}.$$

For u_0 fixed in $\mathcal{U}$, there exists a unique control u related to u_0. Take $u_0 = 0$. Then the associated low-regret control is defined by the following cost function:

$$(5.3) \qquad \mathscr{J}^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma}\Big|\zeta(v)\Big|^2_{L^2(\Sigma_1)},$$

where ζ is the solution of

$$\begin{cases} -\zeta' + A\,\zeta = 0 & \text{in } \Omega, \\ \zeta(T, v) = 0 & \text{on } \{T\} \times \Omega, \\ \dfrac{\partial \zeta}{\partial \nu_A} = y(v, 0) & \text{on } \Sigma_0, \\ \dfrac{\partial \zeta}{\partial \nu_A} = 0 & \text{on } \Sigma_1 \end{cases}$$

and where $A = -\Delta + I = A^*$.

The problem

$$\inf_{v\in\mathcal{U}} \mathcal{J}^{\gamma}(v) \tag{5.4}$$

has a unique solution: the low-regret control u_γ. We set $y_\gamma = y(u_\gamma, 0)$ and $\zeta_\gamma = \zeta(u_\gamma)$. We then have immediately the following proposition.

PROPOSITION 5.1. *The low-regret control u_γ is characterized by the unique solution $\{y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma\}$ of the optimality system*

$$\begin{cases} y'_\gamma + A\,y_\gamma = 0, & -\zeta'_\gamma + A\,\zeta = 0, \quad \rho_\gamma + A\,\rho_\gamma = 0, \quad -p'_\gamma + A\,p_\gamma = 0, \\ y_\gamma(0) = 0, & \zeta_\gamma(T) = 0, \quad \rho_\gamma(0) = 0, \quad p_\gamma(T) = 0, \\ on \quad \Sigma_0, \ \dfrac{\partial y_\gamma}{\partial \nu} = u_\gamma, & \dfrac{\partial \zeta_\gamma}{\partial \nu} = y(v,0), \quad \dfrac{\partial \rho_\gamma}{\partial \nu} = 0, \quad \dfrac{\partial p_\gamma}{\partial \nu} = y_\gamma - z_d + \rho_\gamma, \\ on \quad \Sigma_1, \ \dfrac{\partial y_\gamma}{\partial \nu} = 0, & \dfrac{\partial \zeta_\gamma}{\partial \nu} = 0, \quad \dfrac{\partial \rho_\gamma}{\partial \nu} = \tfrac{1}{\gamma}\zeta_\gamma, \quad \dfrac{\partial p_\gamma}{\partial \nu} = 0, \\ p_\gamma + N u_\gamma = 0 & in \quad L^2(\Sigma_0), \end{cases}$$

with,
$u_\gamma \in L^2(\Sigma_0)$, *and* $y_\gamma \in L^2((0,T); H^{3/2}(\Omega)), \zeta_\gamma \in L^2((0,T); H^{5/2}(\Omega)), \rho_\gamma \in L^2((0,T); H^{7/2}(\Omega))$,
$p_\gamma \in L^2((0,T); H^{1/2}(\Omega))$.

For the proof, we use the same technique as detailed for the stationary Example 2 in section 3. □

We also deduce easily the following theorem.

THEOREM 5.2. *The no-regret control u related to $u_0 = 0$ of the system* (5.1)–(5.2) *is characterized by the unique solution $\{y, \lambda, \rho, p\}$ of the optimality system*

$$\begin{cases} y' + A\,y = 0, \quad -\zeta' + A\zeta = 0, \quad \rho' + A\rho = 0, \quad -p' + A\,p = 0 & in \quad \Omega, \\ y(0) = 0, \quad \zeta(T) = 0, \quad \rho(0) = 0, \quad p(T) = 0, & \\ \dfrac{\partial y}{\partial \nu} = u, \quad \dfrac{\partial \zeta}{\partial \nu} = y, \quad \dfrac{\partial \rho}{\partial \nu} = 0, \quad \dfrac{\partial p}{\partial \nu} = y - z_d + \rho & on \quad \Sigma_0, \\ \dfrac{\partial y}{\partial \nu} = 0, \quad \dfrac{\partial \zeta}{\partial \nu} = 0, \quad \dfrac{\partial \rho}{\partial \nu} = \lambda, \quad \dfrac{\partial p}{\partial \nu} = 0 & on \quad \Sigma_1, \\ p + N u = 0 & in \quad L^2(\Sigma_0), \end{cases}$$

with

$$\begin{cases} u \in L^2((0,T); L^2(\Omega)), \quad y \in L^2((0,T); H^{3/2}(\Omega)), \\ \lambda \in \widehat{G} \ completion \ \ of \ G \ \ in \ L^2((0,T); H^{-2}(\Sigma_1)), \\ \rho \in L^2((0,T); H^{-1/2}(\Omega)), \quad p \in L^2((0,T); H^{1/2}(\Omega)). \end{cases} \qquad \square$$

Conclusion. As we have seen, the low-regret control method allows us to transform systematically a problem with uncertainty to a standard control problem. It is then easier to obtain optimality systems applying the Euler–Lagrange formula.

This method can be used for the control of singular distributed systems as in [4] (see also [5]). Here, the singularity of the backward heat equation is taken off by adding the needed data which may belong to the unknown vector closed subspace G of a given Hilbert space of uncertainties. The system becomes regular, but it contains incomplete data. We then give an optimality system to the no-regret control. In [4], the comparison with the classical penalization method for the control of the backward heat equation in Lions [11] is discussed.

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